MASTER'S DEGREE THESIS

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Title: An Introduction to Polytope Theory through Ehrhart's Theorem

Author: Filip Cano Córdoba

Advisor: Julian Pfeifle

Department: Mathematics

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UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

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An Introduction to Polytope Theory through Ehrhart's Theorem

With applications to Erhart positivity

Filip Cano Córdoba

Supervised by Julian Pfeifle

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I want to thank my colleagues from MAMME, specially Ander, Armando, Marc, Alberto and Marta for their support and for all the time spent together learning mathematics. I also want to thank another mathematician colleague, Zaira, who has been there in important times.

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Abstract

A classic introduction to polytope theory is presented, serving as the foundation to develop more advanced theoretical tools, namely the algebra of polyhedra and the use of valuations. The main theoretical objective is the construction of the so called Berline-Vergne valuation. Most of the theoretical development is aimed towards this goal. A little survey on Ehrhart positivity is presented, as well as some calculations that lead to conjecture that generalized permutohedra have positive coefficients in their Ehrhart polynomials. Throughout the thesis three different proofs of Ehrhart's theorem are presented, as an application of the new techniques developed.

Keywords

Ehrhart polynomials, Generalized permutohedron, Ehrhart positivity, Polytope theory, Algebra of Polyhedra, Brion's theorem

Contents

Introduction

Polytopes and polyhedra are fundamental geometric objects that appear in different areas of mathematics, from algebraic geometry to operative research.

The study of planar and 3-dimensional polytopes has been a fascinating area of study since ancient Greek mathematics, as they are appealing geometrical objects that can be represented in the real world and gave rise to a rich theory.

Higher dimensional polytopes have a much more recent history. Swiss mathematician Ludwig Schläfli was a pioneer in the study of higher dimensional polytopes in the 18th century, and opened a topic that is today an active area of research in mathematics.

We refer the reader to [7, 11] for a comprehensive text on polytopes.

Polytopes whose vertices lie in integer coordinates are of special interest for the discrete mathematician's eye. In particular, we are interested in the following problem: *Given an integral polytope, how much the number of integer points in it increases when we dilate it by an integer factor?* Georg Pick in [9] opened a crack proving the well known *Pick's formula* for the area of an integral polygon. But it was Eugène Ehrhart in [6] who proved in that the behavior is polynomial in the general case. The polynomials arising from the solution to this problem are thus known as Ehrhart polynomials.

The study of Ehrhart polynomials is a useful tool to gain intuition and understanding of how the discrete volume (number of integer points) and the continuous volume of a polytope are related through dilations of the polytope.

In this thesis, the reader starts a journey through the basics of polytope/polyhedra theory that ends in a conjecture about the coefficients of Ehrhart polynomials in a certain family of polytopes, namely generalized permutohedra. This conjecture was first stated in [5] by Castillo and Liu, and has remained open since then. In this journey we will introduce different technical tools, mainly valuations in the algebra of polyhedra, that are used to pose the aforementioned conjecture. Furthermore, throughout the thesis three different proofs of the mentioned Ehrhart's theorem are presented.

Most of the thesis is self-contained. Nevertheless, some results have only sketched proves, and a few of them, specially in the most advanced part, are presented with no proof at all, because doing so would have supposed a thesis extension way larger than the recommended maximum size.

This thesis has three objectives:

(i) Make a sound introduction to polytope/polyhedra theory, focusing in lattice polytopes. Furhter the introduction to the tools of valuations in the algebra of polyhedra.

(ii) Present three different proofs of one of the most relevant theorems in lattice polytope theory, namely Ehrhart's theorem.

(iii) Present recent developments that lead to active research topics in the area of the study of Ehrhart's polynomials.

The original objective of this master's project was to obtain some new knowledge about the conjecture that is presented in the last section, but unfortunately all attempts to further advance current knowledge resulted in failure.

Chapter 1

Polytopes, polyhedra and lattices

1.1 Basic concepts

1.1.1 Polytopes and polyhedra: Definitions

In this section we will introduce the main objects of study of this thesis: polytopes and polyhedra. Classically there are two equivalent definitions of polytopes, one in terms of convex hull and another in terms of half-spaces. We will give both and outline the reasons of why they are equivalent.

1.1.1.1 Polyhedra

Definition 1.1.1 (Convexity). A set $C \in \mathbb{R}^N$ is *convex* if for all $\boldsymbol{u}, \boldsymbol{v} \in C$ and all $0 \leq \tau \leq 1$, the point $\tau\boldsymbol{u} + (1-\tau)\boldsymbol{v}$ lies in C.
The *convex hull* of C is the set

$$\operatorname{conv}(C) = \bigcap_{A \supseteq C, A \text{ convex}} A.$$

Given $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{R}^N$, a *convex combination* of them is any point of the form

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i, \qquad \text{with } \lambda_i \geq 0 \text{ for all } i \text{ and } \lambda_1 + \cdots + \lambda_n = 1.$$

Remark 1. The convex hull of a set is the set of the convex combinations of its points.

Definition 1.1.2 (Polytope 1). A set $C \subseteq \mathbb{R}^N$ is a polytope if it is the convex hull of finitely many points.

Definition 1.1.3 (Half-space). A set $L \subseteq \mathbb{R}^N$ is a *half-space* if it is of the form

$$L = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ a(\boldsymbol{x}) \leq \alpha\},$$

for some $\alpha \in \mathbb{R}$ and $a : \mathbb{R}^N \to \mathbb{R}$ non-zero linear function. The hyperplane defined by $\{a(\boldsymbol{x}) = \alpha\}$ is called its *supporting hyperplane.*

Definition 1.1.4 (Polytope 2). A set $H \subseteq \mathbb{R}^N$ is a *polytope* if it is the bounded intersection of finitely many half-spaces.

Note that in both definitions we are considering the empty set to be a polytope.

While polytopes are our main interest, we will be working with "unbounded polytopes", that we call polyhedra.

Definition 1.1.5 (Polyhedron). A set $P \in \mathbb{R}^N$ is a *polyhedron* if it is the intersection of finitely many half-spaces. In other words, is a set of the form

$$P = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ a_i(\boldsymbol{x}) \leq \alpha_i, i \in I\}$$

for some finite set I, where a_i are linear functions and $\alpha_i \in \mathbb{R}$.
The *dimension* of a polyhedron is the dimension of its affine span.

A polyhedron is called *rational* if it can be written as

$$P = \left\{\boldsymbol{x} \in \mathbb{R}^d \ : \ \sum_{j=1}^{d} a_{i,j} x_j \leq \alpha_i, i \in I\right\}$$

with $a_{i,j}, \alpha_i \in \mathbb{Z}$ for all i and j.

Note that a polyhedron is always a convex set. Given two points $\boldsymbol{x}, \boldsymbol{y} \in P$ satisfying some inequalities of the form $a_i(\boldsymbol{x}) \leq \alpha_i$, then any convex combination also satisfies it because of linearity:

$$a_i(\gamma \boldsymbol{x} + (1 - \gamma \boldsymbol{y})) = \gamma a_i(\boldsymbol{x}) + (1 - \gamma) a_i(\boldsymbol{y}) \leq \gamma \alpha_i + (1 - \gamma)\alpha_i = \alpha_i.$$

Example 1. There are three families of polytopes that are widely known and serve as a good introduction to the concept. We will use $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ to denote the canonical basis in $\mathbb{R}^d$.

1. The standard simplex in d-dimensions Δ^d. Its definitions in terms of half-spaces and of convex hull are, respectively:

   $$\Delta^d = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ x_i \geq 0 \, \forall i, \ x_1 + \cdots + x_d = 1\}, \qquad \Delta^d = \operatorname{conv}(\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n).$$

   The standard simplex is the generalization of the triangle and the tetrahedron.

2. The standard d-dimensional (± 1)-cube $\square^d$. Its definitions are:

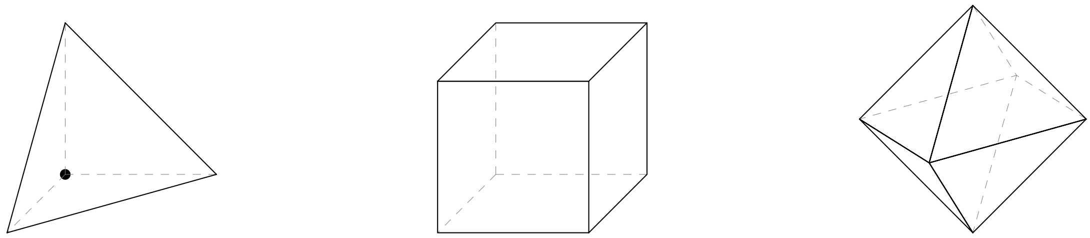
   $$\square^d = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ -1 \leq x_i \leq +1 \, \forall i\}, \qquad \square^d = \operatorname{conv}\left(\{\pm \boldsymbol{e}_1 \pm \cdots \pm \boldsymbol{e}_n\}\right).$$

   It is the generalization of the square and the cube.

3. The standard d-dimensional cross polytope $\diamond^d$. Its definitions are:

$$\Diamond^d = \{\boldsymbol{x} \in \mathbb{R}^d \;:\; \pm x_1 + \cdots + \pm x_n \leq 1\}, \qquad \Diamond^d = \operatorname{conv}\left(\{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j,\ \forall i, j\}\right).$$

The cross polytope is the generalization of the square and the octahedron.



(a) Simplex Δ^3 (b) Cube $\square^3$ (c) Cross polytope $\Diamond^3$

Figure 1.1: Three-dimensional version of the three basic families of polytopes.

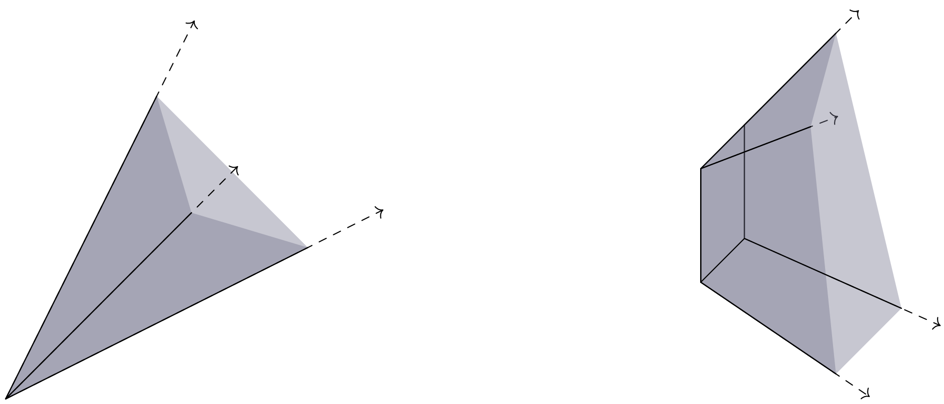


Figure 1.2: Two examples of 3-dimensional unbounded polyhedra.

1.1.1.2 Faces

Definition 1.1.6 (Face). Given a polyhedron P, a linear inequality of the type $\{a(\boldsymbol{x}) \leq \alpha\}$ is a *valid inequality* if it is satisfied for all $\boldsymbol{x} \in P$. Given a valid inequality, a *face F of P* is the subset of P where it is satisfied as an equality, $F = \{\boldsymbol{x} \in P \;:\; a(\boldsymbol{x}) = \alpha\}$.
The *dimension* of a face is the dimension of its affine span.

Some faces are specially relevant and have their own names. For a polyhedron of dimension d:

- Faces of dimension 0 are called *vertices*.
- Faces of dimension 1 are called *edges*.
- Faces of dimension $d-2$ are called *ridges*.

- Faces of dimension $d-1$ are called *facets*.

Remark 2. Some interesting properties of faces are:

- By Definition 1.1.5, any face of a polyhedron is in itself a polyhedron.
- The empty set and the polyhedron itself are always faces. A face is called *proper* if it is neither of them.
- All proper faces can be described with an inequality that is part of the defining inequalities of P. Since a polyhedron can be described with finitely many defining inequalities, the number of faces is finite.

A polyhedron is called *integral* if its vertices have all integer coordinates.

Definition 1.1.7 (Relative interior). Let P be a polyhedron and F a face of P. The *relative interior* of F is the interior of F as a subset of its affine span, denoted by $\text{relint}(F)$.

Note that a point $\boldsymbol{v}$ is in the relative interior of F if and only if non of the defining inequalities of F are active on $\boldsymbol{v}$. Note also that a point $\boldsymbol{v} \in P$ lies in no proper face if and only it lies in the relative interior.

Lemma 1.1.1. *A point $\boldsymbol{v} \in P$ is a vertex if and only if the only pair $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$ such that $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$ is $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}$.*

PROOF.
($\Longrightarrow$) Let $a(\boldsymbol{x}) = \alpha$ be the defining inequality of $\boldsymbol{v}$. Since $\dim(\text{span}(\boldsymbol{v})) = 0$, $\boldsymbol{v}$ is the only point in P satisfying its defining inequality. Now consider a pair $\boldsymbol{v}_1, \boldsymbol{v}_2$ as in the statement. If $\boldsymbol{v}_1 \neq \boldsymbol{v}$, then $a(\boldsymbol{v}_1) < \alpha$, and thus $a(\boldsymbol{v}) < \alpha$. Contradiction. Analogous for $\boldsymbol{v}_2$.
($\Longleftarrow$). We know that $\boldsymbol{v}$ lies in some faces. Let F be a face of minimum dimension where $\boldsymbol{v}$ lies. Let $d = \dim F$. If $d = 0$, we are done. If $d > 0$, since $\boldsymbol{v}$ lies in no proper faces of F, it lies in its relative interior, which is not empty because $d > 0$. Thus we can find $\boldsymbol{v}_1, \boldsymbol{v}_2 \neq \boldsymbol{v}$ satisfying $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$. Contradiction. □

1.1.1.3 Cones

Definition 1.1.8 (Cone, conic combination). A polyhedron K is called a *cone* or *polyhedral cone* if for any $\boldsymbol{x} \in P$ and $\lambda > 0$, we have $\lambda \boldsymbol{x} \in K$.
Given $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathbb{R}^N$, a *conic combination* of them is any point of the form

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i, \qquad \text{with } \lambda_i \geq 0.$$

Given a set A, the set of all its conic combinations is called its *conic hull* and denoted as $\text{co}(A)$.

Remark 3. A cone is defined by a system of *homogeneous* linear inequalities.

This is true because if $a(\boldsymbol{x}) \leq \alpha$ is a valid inequality for K, then

(i) $\alpha \geq 0$ and

(ii) $a(\boldsymbol{x}) \leq 0$ is also a valid inequality.

For the first property, take $\lambda = 0$, by linearity, $0 \leq \alpha$. For the second one, assume there exists $\boldsymbol{x} \in K$ such that $a(\boldsymbol{x}) = \beta > 0$. Take $\lambda > \alpha/\beta$. Applying the cone property we have $a(\lambda\boldsymbol{x}) = \lambda\beta > \alpha$. Contradiction.

A cone always contains the origin. Also, for any point $\boldsymbol{u}$ in a cone K, the ray through the origin in the direction of $\boldsymbol{u}$ is contained in K. Therefore, *only the origin may be a vertex of a cone.* We call them *pointed cones.*

Sometimes we will want to consider the translation of a cone.

Definition 1.1.9 (Shifted cone). Given a cone K and a point $\boldsymbol{v}$, a set of the form $K+\boldsymbol{v}$ will be called a *shifted cone.* If K is pointed, it will be called a *shifted pointed cone.* In a shifted pointed cone, the vertex is called its *apex.*

Given a polyhedron, there is a cone that plays a fundamental role in its structural decomposition, see Proposition 1.1.14.

Definition 1.1.10 (Recession cone). Let P be a polyhedron. Its *recession cone* is the set

$$K_P = \{\boldsymbol{u} \,:\, \boldsymbol{x} + \tau\boldsymbol{u} \in P, \quad \text{for some } \boldsymbol{x} \in P \quad \text{and all} \quad \tau \geq 0\}.$$

Alternatively, for a polytope defined by inequalities

$$P = \{\boldsymbol{x} \,:\, a_i(\boldsymbol{x}) \leq \alpha_i, \ i \in I\},$$

its recession cone can be defined as

$$K_P = \{\boldsymbol{x} \,:\, a_i(\boldsymbol{x}) \leq 0, \ i \in I\}.$$

This is true by the same argument we made for Remark 3.

Many geometrically meaningful cones can be constructed from lower dimensional polytope.

Definition 1.1.11 (Cone over a polytope). Let P be a d-dimensional polytope in $\mathbb{R}^d$ with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, we define the *cone over* P as the conic hull of

$$\boldsymbol{w}_1 = (\boldsymbol{v}_1, 1), \ldots, \boldsymbol{w}_n = (\boldsymbol{v}_n, 1).$$

We will denote it as cone(P). When a cone can be constructed as a cone over a polytope, the vectors $\boldsymbol{w}_i$ are called *generators.* A cone with exactly d linearly independent generators is called *simplicial.*

1.1.1.4 Minkowski sum

Definition 1.1.12 (Minkowski sum). Let $P, Q \in \mathbb{R}^N$ be sets. Their *Minkowski sum* is the set

$$P + Q = \{\boldsymbol{p} + \boldsymbol{q} \ : \ \boldsymbol{p} \in P, \ \ \boldsymbol{q} \in Q\}.$$

Example 2. The standard cube $\square^d$ can be regarded as a Minkowski sum of the intervals $[-\boldsymbol{e}_i, \boldsymbol{e}_i]$:

$$\square^d = [-\boldsymbol{e}_1, \boldsymbol{e}_1] + \cdots + [-\boldsymbol{e}_d, \boldsymbol{e}_d].$$

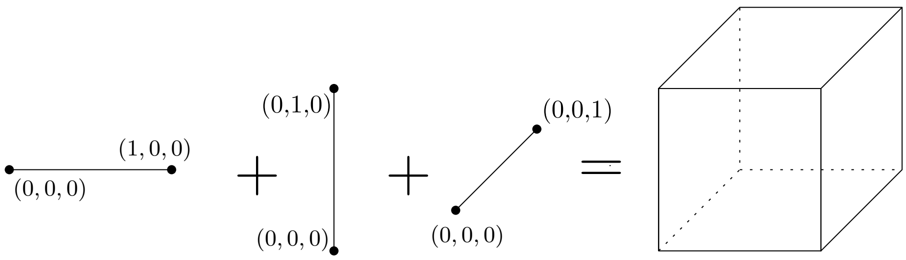


Figure 1.3: The cube is an example of Minkowski sum.

1.1.1.5 Polarity

Definition 1.1.13 (Polar). Let $A \subseteq \mathbb{R}^N$ be a set. The *polar of* A is the set

$$A^\circ = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \quad \text{for all} \quad \boldsymbol{y} \in A\}.$$

Remark 4. If K is a cone, by the same argument as in Remark 3, the polar K° is defined as

$$K^\circ = \{\boldsymbol{x} \ : \ \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0 \quad \text{for all} \quad \boldsymbol{y} \in K\}.$$

If $P = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$, then the polar is

$$P^\circ = \{\boldsymbol{x} \ : \ \langle \boldsymbol{x}, \boldsymbol{v}_i \rangle \leq 1 \quad \text{for all} \quad i = 1, \ldots, n\}.$$

1.1.2 Polytopes and polyhedra: Basic properties

Now we will define some important concepts and give fundamental properties of both polytopes and polyhedra. Since the former are a special case of the latter, many definitions will be stated in terms of polyhedra.

1.1.2.1 Rays and lines

Definition 1.1.14 (Line, ray, interval). Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, with $\boldsymbol{u} \neq 0$, the *line through* $\boldsymbol{v}$ *in the direction of* $\boldsymbol{u}$ is the set $L = \{\boldsymbol{v} + \tau\boldsymbol{u} \,:\, \tau \in \mathbb{R}\}$.
The *ray emanating from* $\boldsymbol{v}$ *in the direction of* $\boldsymbol{u}$ is the set $R = \{\boldsymbol{v} + \tau\boldsymbol{u} \,:\, \tau > 0\}$.
The *interval with endpoints* $\boldsymbol{u}, \boldsymbol{v}$ is the set $I = \{\tau\boldsymbol{v} + (1-\tau)\boldsymbol{u} \,:\, 0 \le \tau \le 1\}$.

Now that we know the basic language, we will prove some results that appear very basic and straightforward. They serve a double purpose. First, they are a sanity check, as we see that definitions make sense, and the properties we expected from them are indeed satisfied. And secondly, they give a taste on what working with polyhedra and polytopes look like, the basic techniques and difficulties.

They are also fundamental results that we will be using through the thesis, so we will need to mention them one way or the other.

Proposition 1.1.2. *A polyhedron is bounded if and only if it does not contain any ray.*

PROOF.
In both implications we will prove the contrapositive statement.
($\Longrightarrow$) Clearly a ray is unbounded, so if a polyhedron contains a ray, it cannot be bounded.
($\Longleftarrow$) Let P be an unbounded polyhedron. Then there exists a sequence of points $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ in P such that $\|\boldsymbol{u}_n\| \to \infty$. Consider the normalized sequence $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$, defined by $\boldsymbol{w}_n = \boldsymbol{u}_n/\|\boldsymbol{u}_n\|$. The normalized sequence is bounded, so it has partial limits. Let $\boldsymbol{w}$ one of its limit points. Since the original sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is in P, for any valid inequality in P of the form $a(\boldsymbol{x}) \le \alpha$, we have

$$a(\boldsymbol{w}_n) = a\left(\frac{\boldsymbol{u}_n}{\|\boldsymbol{u}_n\|}\right) = \frac{1}{\|\boldsymbol{u}_n\|}a(\boldsymbol{u}_n) \le \frac{1}{\|\boldsymbol{u}_n\|}\alpha \xrightarrow[n\to\infty]{} 0.$$

And therefore, $a(\boldsymbol{w}) \le 0$, so for any point $\boldsymbol{x} \in P$, the ray emanating from $\boldsymbol{x}$ in the direction of $\boldsymbol{w}$ is contained in P. □

Corollary 1.1.3. *A polyhedron is bounded if and only if its recession cone is* $\{0\}$.

1.1.2.2 Linear transformations

Lemma 1.1.4. *Let* $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ *be the projection defined by* $\pi(x_1, \ldots, x_N) = (x_1, \ldots, x_{N-1})$. *Let* $P \in \mathbb{R}^N$ *be a polyhedron. Then* $\pi(P) \in \mathbb{R}^{N-1}$ *is a polyhedron.*

PROOF.
Assume that P is defined as a system of linear inequalities,

$$P = \left\{ \boldsymbol{x} \in \mathbb{R}^N \,:\, \sum_{j=1}^{N} a_{i,j}x_j \le \alpha_i, \text{ for } i \in I \right\}.$$

Classify the defining inequalities with respect to the last coefficient

$$I_+ = \{i \in I \,:\, a_{i,N} > 0\}, \qquad I_- = \{i \in I \,:\, a_{i,N} < 0\}, \qquad I_0 = \{i \in I \,:\, a_{i,N} = 0\}.$$

Given $\boldsymbol{y} = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$, we know that $\boldsymbol{y} \in \pi(P)$, if there exists $x_N \in \mathbb{R}$ such that $\boldsymbol{x} = (\boldsymbol{y}, x_N) \in P$. For this to happen, the following $|I|$ inequalities must be satisfied

$$\sum_{j=1}^{N-1} a_{i,j} y_j \leq \alpha_i, \qquad \text{for all } i \in I_0. \tag{1.1.1}$$

$$\frac{\alpha_i}{a_{i,N}} - \sum_{j=1}^{N-1} \frac{a_{i,j}}{a_{i,N}} y_j \geq x_N, \qquad \text{for all } i \in I_+. \tag{1.1.2}$$

$$\frac{\alpha_i}{a_{i,N}} - \sum_{j=1}^{N-1} \frac{a_{i,j}}{a_{i,N}} y_j \leq x_N, \qquad \text{for all } i \in I_-. \tag{1.1.3}$$

$$\tag{1.1.4}$$

The inequalities with I_0 are already linear inequalities involving $\boldsymbol{y}$. For ones involving I_+ and I_-, note that the existence of such x_N is equivalent to the fact that every lower bound for x_N (the I_--type inequalities) does not exceed every upper bound (the I_+-type inequalities). Therefore, they are equivalent to

$$\frac{\alpha_{i_+}}{a_{i_+,N}} - \sum_{j=1}^{N-1} \frac{a_{i_+,j}}{a_{i_+,N}} y_j \geq \frac{\alpha_{i_-}}{a_{i_-,N}} - \sum_{j=1}^{N-1} \frac{a_{i_-,j}}{a_{i_-,N}} y_j, \qquad \text{for all pair } (i_+, i_-) \in I_+ \times I_-. \tag{1.1.5}$$

Now $\pi(P)$ is defined by the linear system of inequalities in Eqs. (1.1.1) and (1.1.5). □

The process we followed in this proof is called *Fourier-Motzkin elimintation*. It plays a key role in the algorithm to obtain a polytope in half-space presentation (Definition 1.1.4) to its convex-hull presentation (Definition 1.1.2) that we will see in the proof of Theorem 1.1.8.

This lemma was in fact the difficult case in the following result.

Theorem 1.1.5. *Let $T : \mathbb{R}^N \to \mathbb{R}^M$ be a linear map, and $P \subseteq \mathbb{R}^N$ be a polyhedron. Then the image $T(P)$ is also a polyhedron.*

PROOF.
First note that if T is an isomorphism, the result is trivially true because the inequalities are linear. Also, if T is injective, it is also true because we have an isomorphism in the restriction $T : \mathbb{R}^N \to \operatorname{Im}(T)$. Moreover, if T is a surjective map, by applying the previous lemma $\dim(\operatorname{Ker}(T))$ times the result holds. Finally, any linear map can be decomposed into an injective and a surjective map, by

$$\begin{array}{ccccc} \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \times \mathbb{R}^M & \longrightarrow & \mathbb{R}^M \\ \boldsymbol{x} & \longmapsto & (\boldsymbol{x}, T(\boldsymbol{x})) & \longmapsto & T(\boldsymbol{x}). \end{array}$$

This concludes the proof. □

1.1.2.3 Weyl-Minkowski theorem

Lemma 1.1.6. *Let $P \in \mathbb{R}^d$ be a d-dimensional polyhedron defined by the inequalities $\{a_i(\boldsymbol{x}) \leq \alpha_i\}_{i \in I}$, let $\boldsymbol{v} \in P$ and let $I_{\boldsymbol{v}}$ be the set of indices for which the inequalities are active in $\boldsymbol{v}$. Then $\boldsymbol{v}$ is a vertex of P if and only if*

$$\operatorname{span}\{a_i \,:\, i \in I_{\boldsymbol{v}}\} = (\mathbb{R}^d)^*,$$

where $(\mathbb{R}^d)^*$ is the dual space (in the sense of linear algebra) of $\mathbb{R}^d$.

PROOF.
($\Longleftarrow$) Assume the set $\{a_i \,:\, i \in I_v\}$ spans $(\mathbb{R}^d)^*$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$ be such that $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$, we have

$$a_i(\boldsymbol{v}_1), a_i(\boldsymbol{v}_2) \leq \alpha_i, \qquad \text{for all } i \in I_{\boldsymbol{v}}.$$

And since $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$, we have

$$(a(\boldsymbol{v}_1) + a(\boldsymbol{v}_2))/2 = \alpha_i, \qquad \text{for all } i \in I_{\boldsymbol{v}}.$$

Therefore we have

$$a_i(\boldsymbol{v}_1) = a_i(\boldsymbol{v}_2) = \alpha_i, \qquad \text{for all } i \in I_{\boldsymbol{v}}.$$

Since the a_i's span all $(\mathbb{R}^d)^*$, the only solution to the system is $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}$. By Lemma 1.1.1 $\boldsymbol{v}$ is a vertex.
($\Longrightarrow$) Assume now that the set $\{a_i \,:\, i \in I_v\}$ does not span $(\mathbb{R}^d)^*$. Then there exists $\boldsymbol{u} \in \mathbb{R}^d$, $\boldsymbol{u} \neq 0$ such that

$$a_i(\boldsymbol{u}) = 0, \qquad \text{for all } i \in I_{\boldsymbol{v}}.$$

Then for sufficiently small $\varepsilon > 0$, we can set $\boldsymbol{v}_1 = \boldsymbol{v} + \varepsilon\boldsymbol{u}$ and $\boldsymbol{v}_2 = \boldsymbol{v} - \varepsilon\boldsymbol{u}$ with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$, so $\boldsymbol{v}$ is not a vertex.

□

Lemma 1.1.7. *Let P be a polyhedron, F a face of P and $\boldsymbol{v}$ a vertex of F. Then $\boldsymbol{v}$ is a vertex of P.*

PROOF.
Suppose $a(\boldsymbol{x}) \leq \alpha$ is a valid inequality for P and $F = \{\boldsymbol{x} \in P \,:\, a(\boldsymbol{x}) = \alpha\}$. Again, let us write $\boldsymbol{v}$ as $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$ with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$. Since $a(\boldsymbol{v}) = \alpha$ and $a(\boldsymbol{v}_1), a(\boldsymbol{v}_2) \leq \alpha$, we have that $a(\boldsymbol{v}_1) = a(\boldsymbol{v}_2) = \alpha$. But then $\boldsymbol{v}_1, \boldsymbol{v}_2 \in F$, and since $\boldsymbol{v}$ is a vertex of F, $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}$, so $\boldsymbol{v}$ is a vertex of P. □

Theorem 1.1.8 (Weyl-Minkowski). *Both definitions of polytope are equivalent. Concretely:*

(i) The convex hull of finitely many points is a bounded polyhedron.

(ii) A bounded polyhedron is the convex hull of its vertices.

PROOF.
(i) Let $P = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$, where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^N$. Let us consider the standard n-dimensional

simplex $\Delta^n = \operatorname{conv}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$, where $\boldsymbol{e}_i$'s are the vectors of the canonical basis in $\mathbb{R}^n$. Consider the linear map

$$\begin{array}{rccc} T: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^N \\ & \boldsymbol{e}_i & \longmapsto & \boldsymbol{v}_i. \end{array}$$

By construction $P = T(\Delta^n)$. Then P is a polyhedron by Theorem 1.1.5. Finally, by compacity of Δ^n and continuity of T, we have that P is compact, and thus bounded.
(ii) We proceed by induction on $d = \dim P$. If $d = 0$, the result is clear. Assume $d > 0$. Assume P is defined by the inequalities

$$P = \{\boldsymbol{x} \,:\, a_i(\boldsymbol{x}) \leq \alpha_i, \text{ for } i \in I\}.$$

Take an arbitrary point $\boldsymbol{y} \in P$. Either $\boldsymbol{y}$ lies in the relative interior of P or it lies in a proper face.
If $\boldsymbol{y}$ lies in a proper face F of P, by the induction hypothesis it is in the convex hull of the vertices of F, which are vertices of P by Lemma 1.1.7.
If $\boldsymbol{y}$ lies in the relative interior of P, there is a line through $\boldsymbol{y}$ in the span of P. Since P is bounded, it cannot contain any ray, so the line intersects with the boundary of P in two points, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. This way, $\boldsymbol{y}$ is a convex combination of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. By the induction hypothesis, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are convex combination of the vertices of the proper faces they lie in, that again by Lemma 1.1.7, are also vertices of P. □

1.1.2.4 Cones

Proposition 1.1.9. *A non-empty polyhedron contains a vertex if and only if it contains no lines.*

PROOF.
($\Longrightarrow$) We will prove the contrapositive statement. Suppose that P contains a line in the direction of a vector $\boldsymbol{u} \neq 0$. Then for any valid inequality of P of the form $a(\boldsymbol{x}) \leq \alpha$, we must have $a(\boldsymbol{u}) = 0$. This way, for any $\boldsymbol{v} \in P$, we can write it in the form of $\boldsymbol{v} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$ with $\boldsymbol{v}_1 = \boldsymbol{v} + u$ and $\boldsymbol{v}_2 = \boldsymbol{v} - u$, with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in P$.
($\Longleftarrow$) We will proceed by induction on $d = \dim P$. If $d = 0$, the result is clear. Suppose $d > 0$. Let $\boldsymbol{y} \in \operatorname{relint}(P)$ and consider a line through $\boldsymbol{y}$ that lies in the span of P. Since P contains no lines, it must intersect with the boundary of P in at least one point. Call such a point $\boldsymbol{z}$. Since it is in the boundary, $\boldsymbol{z}$ lies in a proper face of P, call it F. This proper face contains no lines, so by the induction hypothesis it contains a vertex, which is also a vertex of P by Lemma 1.1.7. □

Corollary 1.1.10. *A non-empty cone K is pointed if and only if it contains no lines.*

Corollary 1.1.11. *Let P be an unbounded polyhedron. Then its recession cone K_P is a pointed cone if and only if P contains no lines.*

PROOF.
($\Longleftarrow$) Recall that by the characterization we give of the recession cone (Definition 1.1.10), $K_P \subseteq P$. In particular, if P contains no lines, then K contains no lines and therefore it is a pointed cone.
($\Longrightarrow$) We will prove the contrapositive. Assume P contains a line $\{\boldsymbol{x} + \tau \boldsymbol{u} \,:\, \tau \in \mathbb{R}\}$. Then K_P contains not only $\boldsymbol{u}$, but $\{\tau \boldsymbol{u} \,:\, \tau \in \mathbb{R}\}$, which is a line. Thus K_P is not pointed. □

By the pictures we have seen and our general intuition, we expect a pointed cone to be the result of coning over some polytope. This is indeed the case.

Proposition 1.1.12. *Let $K \subset \mathbb{R}^d$ be a d-dimensional pointed cone of positive dimension. Then there exists an affine hyperplane $H \subset H$ and a polytope $P \subset H$ such that $K = \mathrm{co}(P)$.*

PROOF.
Let K be defined as a system of inequalities by

$$K = \{\boldsymbol{x} \,:\, a_i(\boldsymbol{x}) \leq 0, \quad \text{for} \quad i \in I\}.$$

By Lemma 1.1.6 the linear functions $\{a_i\}_{i\in I}$ span $(\mathbb{R}^d)*$. Let $a = \sum_{i\in I} a_i$ the hyperplane be $H = \{\boldsymbol{x} : a(\boldsymbol{x}) = -1\}$ and the polytope be $P = H \cap K$ (we know it is a polyhedron, we have yet to show that it is bounded). First we claim that $a \neq 0$. If it was, by the construction of K, for all $i \in I$ and all $\boldsymbol{x} \in K$, we would have $a_i(\boldsymbol{x}) = 0$. Since K is full dimensional, this would contradict Lemma 1.1.6.
For any $\boldsymbol{x} \in K \setminus \{0\}$ we know that $a(\boldsymbol{x}) < 0$ because K has only one vertex. Therefore there exists $\lambda \geq 0$ such that $\lambda x \in P$. With this we have that $K = \mathrm{co}(P)$ with P a polyhedron.
To prove it is bounded, suppose it is not. Then it contains a ray in a direction $\boldsymbol{u}$ parallel to H, so with $a(\boldsymbol{u}) = 0$. Since the ray is contained in K, we have $a_i(\boldsymbol{u}) \leq 0$, and thus $a_i(\boldsymbol{u}) = 0$ for all $i \in I$. But then the line $\{\tau\boldsymbol{u} \,:\, \tau \in \mathbb{R}\}$ is contained in K, contradicting K being a pointed cone. □

Remark 5. A direct consequence of this result is that any pointed cone can be constructed as the cone over a polytope.

1.1.2.5 Minkowski sum

Lemma 1.1.13. *Let $P_1, P_2 \subseteq \mathbb{R}^N$ be two polyhedra. Then their Minkowski sum $P_1 + P_2$ is a polyhedron.*

PROOF.
Consider the Cartesian product

$$P = P_1 \times P_2 = \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \,:\, \boldsymbol{x}_1 \in P_1,\ \boldsymbol{x}_2 \in P_2\}.$$

We can describe $P \subseteq \mathbb{R}^N \times \mathbb{R}^N$ joining the defining inequalities of P_1 and P_2. Now consider the linear transform

$$\begin{array}{rccc} T: & \mathbb{R}^N \times \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ & (\boldsymbol{x}, \boldsymbol{y}) & \longmapsto & \boldsymbol{x} + \boldsymbol{y}. \end{array}$$

Clearly, $P_1 + P_2 = T(P)$, so by Theorem 1.1.5 it is a polyhedron. □

We are now prepared for two fundamental results on the structure of polyhedra as Minkowski sums.

Proposition 1.1.14. *Let P be a non-empty polyhedron without lines. Let K be its recession cone and M be the convex hull of the set of vertices of P. Then*

$$P = K + M.$$

PROOF.
Note that by Corollary 1.1.3 and Theorem 1.1.8, we have already proved the result for polytopes, so we may assume that P is unbounded, and thus $K \neq \{0\}$.

($\supseteq$) Suppose P is defined by the inequalities $P = \{\boldsymbol{x} \ : a_i(\boldsymbol{x}) \leq \alpha, \ i \in I\}$. Recall that the recession cone (see Definition 1.1.10) can be equivalently defined as

$$K = \{\boldsymbol{x} \ : a_i(\boldsymbol{x}) \leq 0\}.$$

Then clearly $P \subseteq K + M$.
($\subseteq$) We will argue by induction on $d = \dim P$. For $d = 0$ the result is clear. Consider $d > 0$.
Let $\boldsymbol{y} \in P$, we want to write it as $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{z}$, where $\boldsymbol{u} \in K$ and $\boldsymbol{z} \in M$. Either $\boldsymbol{y}$ lies in a proper face or it lies in the relative interior of P.

If it lies in a face, by the induction hypothesis it can be written as $\boldsymbol{y} = \boldsymbol{u}_F + \boldsymbol{z}_F$, where $\boldsymbol{u}_F \in K_F$ and $\boldsymbol{z}_F \in M_F$. By Lemma 1.1.7 $M_F \subseteq M_P$, and by definition $K_F \subseteq K_P$, so we are done.

If $\boldsymbol{y}$ lies in the relative interior, consider $\boldsymbol{u} \in K \setminus \{0\}$ (exists because K is unbounded). Consider now the line $\{\boldsymbol{y} + \tau \boldsymbol{u} \ : \ \tau \in \mathbb{R}\}$. Since P contains no lines, it will intersect a proper face F of P at a point $\boldsymbol{y}'$. This point is of the form $\boldsymbol{y} + \tau_0 \boldsymbol{u}$, and since $\boldsymbol{u} \in K_P$, we have $\tau < 0$ and thus $-\tau_0 \boldsymbol{u} \in K_P$. so we can write $\boldsymbol{y} = \boldsymbol{y}' + \boldsymbol{u}'$, where $\boldsymbol{u} \in K_P$ and, by the induction hypothesis, $\boldsymbol{y}' \in K_F + M_F$. We conclude the proof applying the same argument as in the previous case.

□

Proposition 1.1.15. *Let $P \subseteq \mathbb{R}^N$ be a non-empty polyhedron. Then it can be decomposed as*

$$P = L + K + M,$$

where L is a linear subspace of $\mathbb{R}^N$, K is a pointed cone and M is a polytope.

PROOF.
Suppose P is defined by the inequalities $P = \{\boldsymbol{x} \in \mathbb{R}^N \ : a_i(\boldsymbol{x}) \leq \alpha, \ i \in I\}$. If P contains a line in the direction $\boldsymbol{u} \in \mathbb{R}^N$, then $a_i(\boldsymbol{u}) = 0$ for all $i \in I$. This justifies the following construction.

Let $L = \{\boldsymbol{u} \ : \ l_i(\boldsymbol{u}) = 0, \ i \in I\}$. It is clearly a linear subspace, so we can find a decomposition $\mathbb{R}^N = L \oplus W$, where $W \subseteq \mathbb{R}^N$ is a linear subspace. Then consider the projection map

$$\begin{array}{rccl} \pi & : & L \oplus W & \longrightarrow & W \\ & & \boldsymbol{u} + \boldsymbol{w} & \longmapsto & \boldsymbol{w}, \end{array}$$

and the image of P by π, $Q = \pi(P)$. We have $P = L + Q$. Since $\ker \pi = L$, Q contains no lines, and by Proposition 1.1.14 it can be decomposed as $Q = K + M$. □

Note that in the characterization of polyhedra with lines we lost most of the information of what the polytope M and the pointed cone K mean.

1.1.2.6 Polarity

Lemma 1.1.16 (Separation lemma). *Let $A \subseteq \mathbb{R}^N$ be a closed convex set and $\boldsymbol{a} \notin A$ be a point. Then there exists a non-zero vector $\boldsymbol{u} \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ such that*

$$\langle \boldsymbol{u}, \boldsymbol{a} \rangle < 0 \quad \text{and} \quad \langle \boldsymbol{u}, \boldsymbol{x} \rangle > \alpha \qquad \text{for all} \quad \boldsymbol{x} \in A.$$

PROOF.
If $\boldsymbol{a} \neq \mathbf{0}$, we can consider the shifted set $A - \boldsymbol{a}$ and find $\boldsymbol{u}$ and α there, so without loss of generality we may assume $\boldsymbol{a} = \mathbf{0}$.

Since A is closed and $\mathbf{0} \notin A$, there exists (maybe more than one) a point $\boldsymbol{y} \in A$ that is *the closest* point of A to $\mathbf{0}$. Consider $\boldsymbol{u} = \boldsymbol{y}$, $\alpha = \|y\|^2/2$ and the half-space

$$H = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq \alpha\}.$$

We claim that H contains no points in A. To prove it, suppose there exists $\boldsymbol{z} \in A \cap H$. Since A is convex, for all $0 \leq \varepsilon \leq 1$, the point $\boldsymbol{y}_\varepsilon = (1-\varepsilon)\boldsymbol{y} + \varepsilon \boldsymbol{z}$ lies in A. We will see that, for some ε, the point $\boldsymbol{y}_\varepsilon$ is closer to $\mathbf{0}$ that $\boldsymbol{y}$:

$$\begin{aligned}
\|\boldsymbol{y}_\varepsilon\| &= (1-\varepsilon)^2\|y\|^2 + \varepsilon^2\|z\|^2 + 2\varepsilon(1-\varepsilon)\langle \boldsymbol{y}, \boldsymbol{z} \rangle \\
&\leq (1-\varepsilon)^2\|y\|^2 + \varepsilon^2\|z\|^2 + 2\varepsilon(1-\varepsilon)\alpha && \text{(by } \boldsymbol{z} \in H) \\
&= (1-\varepsilon)^2\Big[\|y\|^2 + \varepsilon\left(\|y\|^2 - 2\alpha\right)\Big] + \varepsilon^2\|z\|^2 && \text{(rearranging terms)} \\
&= (1-\varepsilon)\|y\|^2 + \varepsilon^2\|z\|^2. && \text{(by } \alpha = \|\boldsymbol{y}\|^2/2)
\end{aligned}$$

Choosing $\varepsilon < (\|z\|/\|y\|)^2$ we have $\|y\|_\varepsilon < \|y\|$, which contradicts $\boldsymbol{y}$ being the closest point on A to $\mathbf{0}$.
□

Theorem 1.1.17. *Let $A \subseteq \mathbb{R}^N$ be a non-empty set. Then*

(i) The polar set A° is closed, convex and contains the origin.

(ii) If A is closed and convex, then $(A^\circ)^\circ = A$.

(iii) If A is a polyhedron, then A° is a polyhedron.

PROOF.
(i) By definition A° is the intersection of (possibly infinitely many) closed half-spaces, so it is closed and convex. Also by definition $\mathbf{0} \in A^\circ$.

(ii) $(\supseteq)$ This inclusion is always true (even if A is not closed or convex). Points $\boldsymbol{x} \in A^\circ$ are defined as $\langle \boldsymbol{x}, \boldsymbol{a} \rangle \leq 1$ for all $\boldsymbol{a} \in A$. Thus, given $\boldsymbol{a} \in A$, clearly $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq 1$ for all $\boldsymbol{u} \in A^\circ$, which is the definition of $\boldsymbol{a} \in (A^\circ)^\circ$.
$(\subseteq)$ We will prove this inclusion by contradiction. Assume there exists $\boldsymbol{a} \in (A^\circ)^\circ$, $\boldsymbol{a} \notin A$. Since A is closed and convex, by the separation lemma there exists $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{\alpha} \in \mathbb{R}$ such that

$$\langle \boldsymbol{u}, \boldsymbol{a} \rangle > \alpha \quad \text{and} \quad \langle \boldsymbol{u}, \boldsymbol{x} \rangle < \alpha, \quad \text{for all} \quad \boldsymbol{x} \in A.$$

Since $\mathbf{0} \in A$, we have $\alpha > 0$. Consider the vector $\boldsymbol{u}' = \alpha^{-1}\boldsymbol{u}$. Now $\langle \boldsymbol{u}', \boldsymbol{x} \rangle < 1$ for all $\boldsymbol{x} \in A$, so $\boldsymbol{u}' \in A^\circ$. Since $\boldsymbol{a} \in (A^\circ)^\circ$, we must have $\langle \boldsymbol{u}', \boldsymbol{a} \rangle \leq 1$, which contradicts $\langle \boldsymbol{u}, \boldsymbol{a} \rangle > \alpha$.

(iii) Recall that by Proposition 1.1.15 we can decompose any polyhedron as $P = L + K + M$, where M is a polytope, K a pointed cone and L a subspace. If $M = \mathrm{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$, then M° is defined as intersection of half-spaces:

$$M^\circ = \{\boldsymbol{x} \,:\, \langle \boldsymbol{x}, \boldsymbol{v}_i \rangle \leq 1, \quad \text{for} \quad i = 1, \ldots, n\}.$$

By Proposition 1.1.12, we can write K as $K = \mathrm{co}(\boldsymbol{u}_1, \ldots, u_m)$, thus K° can be defined as intersection of half-spaces:

$$K^\circ = \{\boldsymbol{x} \,:\, \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \leq 1, \quad \text{for} \quad i = 1, \ldots, m\}.$$

We will prove that

$$P^\circ = M^\circ \cap K^\circ \cap L^\perp.$$

($\subseteq$) Take an arbitrary $\boldsymbol{x} \in P^\circ$. Since $M \subseteq P$, we have $\boldsymbol{x} \in M^\circ$. Choose $\boldsymbol{u} \in K$ and $\boldsymbol{y} \in P$. Then $\boldsymbol{y} + \lambda\boldsymbol{u} \in P$ for all $\lambda > 0$, so we must have $\langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq 0$, and thus $\boldsymbol{x} \in K^\circ$. Finally, for $\boldsymbol{u} \in L$ and $\boldsymbol{y} \in P$, we have $\boldsymbol{y} + \lambda\boldsymbol{u} \in P$ for all $\lambda \in \mathbb{R}$, so we must have $\langle \boldsymbol{u}, \boldsymbol{x} \rangle = 0$, and thus $\boldsymbol{x} \in L^\perp$.
($\supseteq$) Let $\boldsymbol{x} \in M^\circ \cap K^\circ \cap L^\perp$. Any point in P can be written as

$$\boldsymbol{p} = \boldsymbol{l} + \boldsymbol{k} + \boldsymbol{m}, \quad \text{where} \quad \boldsymbol{l} \in L,\, \boldsymbol{k} \in K,\, \boldsymbol{m} \in M.$$

Thus $\langle \boldsymbol{p}, \boldsymbol{x} \rangle = \langle \boldsymbol{l}, \boldsymbol{x} \rangle + \langle \boldsymbol{k}, \boldsymbol{x} \rangle + \langle \boldsymbol{m}, \boldsymbol{x} \rangle \leq 1$. □

1.1.2.7 Triangulations

Definition 1.1.15 (Simplex). A polytope P is called a simplex if it is the convex hull of $d+1$ points and has dimension d. Namely $P = \mathrm{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1})$ and $\dim P = d$.

Definition 1.1.16 (Triangulation of a polytope). Given a polytope $\mathcal{P}$, a *triangulation* of $\mathcal{P}$ is a set T of simplices such that

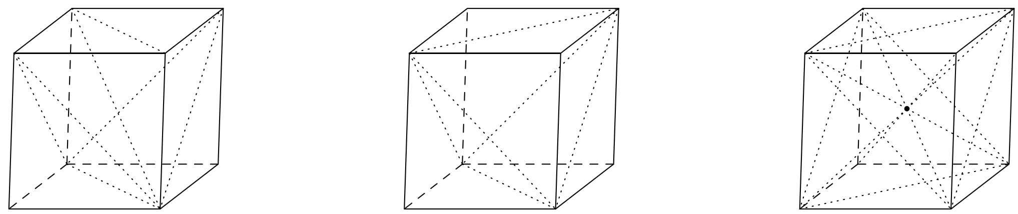
1. The simplices cover the polytope, that is $\bigcup_{\Delta \in T} \Delta = \mathcal{P}$.
2. For any pair of simplices $\Delta, \Delta' \in T$, the intersection $\Delta \cap \Delta'$ is a face of both Δ and Δ'.

We say that a triangulation *adds no new vertices* if for all $\Delta \in T$, the vertices of Δ are vertices of $\mathcal{P}$.

Definition 1.1.17 (Triangulation of a cone). Given a cone K, a *triangulation* is a set T of simplicial cones such that

1. The simplicial cones cover the original cone, that is $\bigcup_{S\in T} S = K$.
2. For any pair of cones $S, S' \in T$, the intersection $S \cap S'$ is a face of both S and S'.

We say that a triangulation *adds no new generators* if for all $S \in T$, the generators of S are also generators of K.



(a) $S = 4$, $E = 7$, $V = 0$ (b) $S = 4$, $E = 6$, $V = 0$ (c) $S = 12$, $E = 10$, $V = 1$

Figure 1.4: Three different triangulations of the cube $\square^3$. The first number (S) indicates the number of simplices in which the cube is divided, the second one (E) is the number of edges used in the triangulation, and the third one (V) indicates the number of new vertices added.

Theorem 1.1.18 (Existence of triangulations). *Every polytope admits a triangulation with no new vertices.*

We refer to [2, Appendix] for a complete proof.

Corollary 1.1.19. *Every shifted pointed cone admits a triangulation with no new generators.*

PROOF.
Just observe that a triangulation of a polytope P with no new vertices corresponds to a triangulation of cone(P) with no new generators. $\square$

1.1.3 Lattices and lattice polyhedra

Definition 1.1.18 (Lattice). A set $\Lambda \in \mathbb{R}^d$ is a *d-dimensional lattice* if it satisfies three conditions:

(i) *Subgroup:* Λ is a subgroup of $(\mathbb{R}^d, +)$.

(ii) *Discrete:* for any bounded set $B \subset \mathbb{R}^d$, the intersection $\Lambda \cap B$ is finite.

(iii) *Spans:* the linear span of Λ is $\mathbb{R}^d$.

Example 3. The first examples of lattices that we have:

- The typical example of a lattice in $\mathbb{R}^d$ is $\mathbb{Z}^d$.
- We know that as an additive group, $\mathbb{Z}^d$ is generated by the basis vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$. For any vector $\boldsymbol{\lambda} \in \mathbb{R}^d$, the group generated by $\lambda_1 \boldsymbol{e}_1, \dots, \lambda_d \boldsymbol{e}_d$ is also a lattice.
- To make things more interesting, we can take any invertible matrix in the integers, $A \in \mathrm{GL}_d(\mathbb{Z})$ and take the group generated by $\lambda_1 A \cdot \boldsymbol{e}_1, \dots, \lambda_d A \cdot \boldsymbol{e}_d$.

We will see in Theorem 1.1.22 that these are essentially the only possible lattices.

Notation. We will be using $\lfloor \alpha \rfloor$ and $\{\alpha\}$ to denote the floor integer part and the fractional part of the real number α respectively. So we have $\alpha = \lfloor \alpha \rfloor + \{\alpha\}$, with $\lfloor \alpha \rfloor \in \mathbb{Z}$ and $0 \leq \{\alpha\} < 1$.

Lemma 1.1.20. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $L \subseteq \mathbb{R}^d$ be a proper linear subspace spanned by points in Λ. Then there exists $\boldsymbol{v} \in \Lambda \setminus L$ such that*

$$\operatorname{dist}(\boldsymbol{v}, L) \leq \operatorname{dist}(\boldsymbol{w}, L) \quad \textit{for all} \quad \boldsymbol{w} \in \Lambda \setminus L. \tag{1.1.6}$$

In other words, there exists a closest point of Λ to L.

PROOF.

Let $k = \dim L$ and $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ a basis of L consisting of lattice points $\boldsymbol{u}_i \in \Lambda$. Consider the parallelepiped

$$\Pi = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{u}_i, \, : \, 0 \leq \lambda_i \leq 1, \quad \text{for} \quad i = 1, \dots, k \right\},$$

and its neighborhoods $\Pi_\rho = \{\boldsymbol{x} \in \mathbb{R}^d \, : \, \operatorname{dist}(\boldsymbol{x}, \Pi) \leq \rho\}$. For sufficiently large ρ, the set Π_ρ contains points of $\Lambda \setminus L$. Since Λ is discrete, the set $\Pi_\rho \cap \Lambda$ is finite, so there exists $\boldsymbol{v} \in \Lambda \setminus L$ that is the closest to Π, i.e.

$$\operatorname{dist}(\boldsymbol{v}, \Pi) \leq \operatorname{dist}(\boldsymbol{a}, \Pi) \quad \text{for all} \quad \boldsymbol{a} \in \Lambda \setminus L. \tag{1.1.7}$$

We will prove that in fact $\boldsymbol{v}$ satisfies Eq. (1.1.6). Suppose that there exists $\boldsymbol{w} \in \Lambda \setminus L$ such that $\operatorname{dist}(\boldsymbol{w}, L) < \operatorname{dist}(\boldsymbol{v}, L)$. Let $\boldsymbol{x} \in L$ be a point that satisfies $\operatorname{dist}(\boldsymbol{w}, L) = \operatorname{dist}(\boldsymbol{w}, \boldsymbol{x})$. Since $\boldsymbol{x} \in L$, it can be written as

$$\boldsymbol{x} = \sum_{i=1}^{k} \alpha_k \boldsymbol{u}_k.$$

We can decompose it further into $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{y}$, where

$$\boldsymbol{u} = \sum_{i=1}^{k} \lfloor \alpha_k \rfloor \boldsymbol{u}_k \, \in \Lambda \cap L, \qquad \text{and} \qquad \boldsymbol{y} = \sum_{i=1}^{k} \{\alpha_k\} \boldsymbol{u}_k \, \in \Pi.$$

Consider $\boldsymbol{w} - \boldsymbol{u} \in \Lambda \setminus L$. By construction of $\boldsymbol{x}$, we have

$$\operatorname{dist}(\boldsymbol{w} - \boldsymbol{u}, \Pi) \leq \operatorname{dist}(\boldsymbol{w} - \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{u}) = \operatorname{dist}(\boldsymbol{w}, \boldsymbol{x}) < \operatorname{dist}(\boldsymbol{v}, L) \leq \operatorname{dist}(\boldsymbol{v}, \Pi).$$

This contradicts Eq. (1.1.7) by taking $\boldsymbol{a} = \boldsymbol{w} - \boldsymbol{u}$. □

Lemma 1.1.21. *Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ be a basis of $\mathbb{R}^d$. The additive group generated by them is a lattice.*

PROOF.
Consider the linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ defined by $T(\alpha_1, \ldots, \alpha_d) = \alpha_1 \boldsymbol{u}_1 + \ldots \alpha_d \boldsymbol{u}_d$. Then $T(\mathbb{Z}^d)$ is the additive group generated, is discrete because it is the image of $\mathbb{Z}^d$ and spans $\mathbb{R}^d$ because T is a bijective map. □

Theorem 1.1.22. *Let $\Lambda \in \mathbb{R}^d$ be a lattice. Then there exists a basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\} \subset \Lambda$ of $\mathbb{R}^d$ such that for every point $\boldsymbol{u} \in \Lambda$, there is a unique decomposition of the form*

$$\boldsymbol{u} = \sum_{i=1}^{d} m_i \boldsymbol{u}_i, \quad \textit{where} \quad m_i \in \mathbb{Z}.$$

We call $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ a basis *of Λ.*

PROOF.
We will proceed by induction on d. For $d = 1$, since Λ is discrete, there exists (maybe more than one) the smallest (in absolute value) non-zero element of Λ. We call it $\boldsymbol{a}$. Then $\Lambda = \{m\boldsymbol{a} : m \in \mathbb{Z}\}$.

For $d > 1$, choose $d-1$ linearly independent points of Λ and let L be the $(d-1)$-dimensional subspace they span. Consider the induced lattice $\Lambda_1 = \Lambda \cap L$. By the induction hypothesis, there exists a basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d-1}\}$ of Λ_1. By Lemma 1.1.20 there exists $\boldsymbol{u}_d \in \Lambda \setminus L$ such that

$$\operatorname{dist}(\boldsymbol{u}_d, L) \leq \operatorname{dist}(\boldsymbol{w}, L) \quad \text{for all} \quad \boldsymbol{w} \in \Lambda \setminus L. \tag{1.1.8}$$

We will prove that $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d-1}, \boldsymbol{u}_d\}$ is a basis of Λ. We already know that it constitutes a basis of $\mathbb{R}^d$, so for a given $\boldsymbol{u} \in \Lambda$, there exists a unique decomposition

$$\boldsymbol{u} = \sum_{i=1}^{d} \alpha_i \boldsymbol{u}_i, \quad \alpha_i \in \mathbb{R}.$$

Consider as usual the decomposition of the real number α_d into its integer and fractional part, $\alpha_d = \lfloor \alpha_d \rfloor + \{\alpha_d\}$. Suppose that $\{\alpha_d\} > 0$. Then the point

$$\boldsymbol{v} = \boldsymbol{u} - \lfloor \alpha_d \rfloor \boldsymbol{u}_d = \{\alpha_d\} \boldsymbol{u}_d + \sum_{i=1}^{d-1} \alpha_i \boldsymbol{u}_i$$

lies in $\Lambda \setminus L$ and satisfies

$$\operatorname{dist}(\boldsymbol{v}, L) = \operatorname{dist}\left(\{\alpha_d\} \boldsymbol{u}_d, L\right) = \{\alpha_d\} \operatorname{dist}(\boldsymbol{u}_d, L) < \operatorname{dist}(\boldsymbol{u}_d, L).$$

This contradicts Eq. (1.1.8). We have just proved $\alpha_d \in \mathbb{Z}$. To prove that the rest of coefficients are also integers, we just consider $\boldsymbol{w} = \boldsymbol{u} - \alpha_d \boldsymbol{u}_d$. We have $\boldsymbol{w} \in \Lambda \cap L = \Lambda_1$, so by the induction hypothesis $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{Z}$. □

Definition 1.1.19 (Fundamental parallelepiped). Let Λ be a lattice and $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ a basis of Λ. The *fundamental parallelepiped of* of Λ associated to $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ is the set

$$\Pi(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d) = \left\{ \sum_{i=1}^{d} \alpha_i \boldsymbol{u}_i \ : \ 0 \leq \alpha_i < 1 \right\}.$$

Lemma 1.1.23. *Let $\Lambda \in \mathbb{R}^d$ be a lattice and Π a fundamental parallelepiped. Then every point $\boldsymbol{x} \in \mathbb{R}^d$ can be uniquely written as*

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{v}, \quad \textit{where} \quad \boldsymbol{y} \in \Pi \quad \textit{and} \quad \boldsymbol{v} \in \Lambda.$$

PROOF.
Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ be the basis of Λ associated to Π. We know that for all $\boldsymbol{x} \in \mathbb{R}^d$, there is a unique decomposition

$$\boldsymbol{x} = \sum_{i=1}^{d} \alpha_i \boldsymbol{u}_i, \qquad \alpha_i \in \mathbb{R}.$$

As in the proof of Lemma 1.1.20 we can further decompose it into $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{y}$ where

$$\boldsymbol{v} = \sum_{i=1}^{k} \lfloor \alpha_k \rfloor \boldsymbol{u}_k \ \in \Lambda \cap L, \qquad \text{and} \qquad \boldsymbol{y} = \sum_{i=1}^{k} \{\alpha_k\} \boldsymbol{u}_k \ \in \Pi.$$

To prove that this decomposition is unique, suppose we have

$$\boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{y}_1 = \boldsymbol{v}_2 + \boldsymbol{y}_2, \quad \text{where} \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in \Lambda \quad \text{and} \quad \boldsymbol{y}_1, \boldsymbol{y}_2 \in \Pi.$$

Since $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \Pi$, they are expressed as

$$\boldsymbol{y}_1 = \sum_{i=1}^{d} \beta_i \boldsymbol{u}_i, \quad \boldsymbol{y}_2 = \sum_{i=1}^{d} \gamma_i \boldsymbol{u}_i, \quad \text{where} \quad 0 \leq \beta_i, \gamma_i < 1, \quad \text{for all} \quad i = 1, \ldots, d.$$

On the other hand, $\boldsymbol{y}_2 - \boldsymbol{y}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_2 \in \Lambda$, so there exists a unique decomposition

$$\boldsymbol{y}_2 - \boldsymbol{y}_1 = \sum_{i=1}^{d} \delta_i \boldsymbol{u}_i, \quad \text{where} \quad \delta_i \in \mathbb{Z}.$$

Since the decomposition is unique, for all $i = 1, \ldots, d$ we have $\delta_i = \gamma_i - \beta_i$. The only way that $0 \leq \beta_i, \gamma_i < 1$ and $\beta_i - \gamma_i \in \mathbb{Z}$ can be satisfied is if $\gamma_i = \beta_i$. □

Theorem 1.1.24. *Let Λ be a lattice and Π a fundamental parallelepiped. Let $B_r \subset \mathbb{R}^d$ be the ball of radius r centered at the origin. Then*

$$\lim_{r \to \infty} \frac{|B_r \cap \Lambda|}{\operatorname{vol} B_r} = \frac{1}{\operatorname{vol} \Pi}. \tag{1.1.9}$$

In particular, the volume of Π is an invariant of Λ. We call it the determinant of Λ, *denoted* $\det \Lambda$.

PROOF. (Sketch only)
Choose a particular fundamental parallelepiped Π of Λ. By Lemma 1.1.23, its lattice translates cover all $\mathbb{R}^d$. Suppose $\Pi \subset B_\alpha$ for some $\alpha > 0$. Then for $r > \alpha$ consider

$$X_r = \bigcup_{\boldsymbol{u} \in B_r} (\Pi + \boldsymbol{u}).$$

We can check that $B_{r-\alpha} \subset X_r \subset B_{r+\alpha}$. Therefore

$$\operatorname{vol} B_{r-\alpha} \leq \operatorname{vol} X_r = |B_r \cap \Lambda| \operatorname{vol} \Pi \leq \operatorname{vol} B_{r+\alpha}.$$

Applying the limit $r \to \infty$ in the previous equation we get Eq. (1.1.9). □

Definition 1.1.20 (Lattice polyhedron). Given a lattice Λ and a polyhedron P, we will say that P is a Λ-*lattice polyhedron* (or simply *lattice polyhedron* when the lattice is understood) if the vertices of P are in Λ.

The said concept is just a generalization of integral polyhedron, but we will see in Section 2.4 that it will be useful to think of general lattice polytopes, and not only integral ones.

Definition 1.1.21 (Normalized volume). Given a lattice $\Lambda \subset \mathbb{R}^d$ and a measurable set $S \subset \mathbb{R}^d$, we define the *normalized volume* of S as

$$\operatorname{nvol}(S) = \operatorname{vol}(S)/\det \Lambda.$$

1.2 Ehrhart's original proof

In this section we will give the first proof of Ehrhart's theorem, using the same method as the original proof in [6].

Theorem 1.2.1 (Ehrhart, 1962). *Let $P \subseteq \mathbb{R}^d$ be a d-dimensional integral polytope, and let tP be its dilation by an integer factor t. Then*

$$\left| tP \cap \mathbb{Z}^d \right| = p(t), \qquad \text{where } p(t) \text{ is a polynomial of degree } d.$$

To do this, we first need to reformulate the problem with generating functions and cones.

Recall that by Proposition 1.1.12, any pointed cone can be constructed as a cone over a polytope. We will be using shifted pointed cones in this section, so they will look like

$$K = \{\boldsymbol{v} + \lambda_1 \boldsymbol{w}_1 + \ldots, \lambda_n \boldsymbol{w}_n \ : \ \lambda_1, \ldots, \lambda_n \geq 0\},$$

where $\boldsymbol{v}$ is the apex and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ are the generators.

As we will see, cones are convenient to us because we can have a richer theory that allows us to perform computations that we cannot do otherwise. Recall that given a polytope, there is a natural construction of a shifted pointed cone (Definition 1.1.11).

Remark 6. If K is the cone over P, we can recover the original polytope when we intersect with the hyperplane $\{x_{d+1} = 1\}$. Also in general, the intersection

$$\operatorname{cone}(P) \cap \{x_{d+1} = t\}$$

gives us the dilation of $\mathcal{P}$ by a factor t. See Fig. 1.5 for an example.

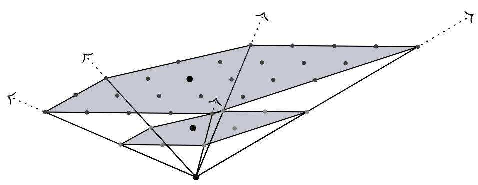


Figure 1.5: Cone over a pentagon.

We will prove Ehrhart's theorem for simplices, and then we will extend the result for general polytopes by triangulations. We have introduced the concept of simplicial cone because the cone over a simplex is always simplicial.

With the spirit of generating functions in mind, we define the integer-point enumerator of a set.

Definition 1.2.1 (Integer-point enumerator). Given a set $S \subseteq \mathbb{R}^d$, we define the *integer-point enumerator* of S as the function

$$\sigma_S(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{m}},$$

with the notation $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{m} = (m_1, \ldots, m_d)$ and $\boldsymbol{x}^{\boldsymbol{m}} = x_1^{m_1} \cdot \cdots \cdot x_d^{m_d}$.

When the set S is bounded, it makes sense to evaluate at $\boldsymbol{x} = (1, \ldots, 1)$, and this evaluation gives us how many integer points lie in our set S.

When the set S is actually a simplicial cone, we have the following lemma:

Lemma 1.2.2. *Let K be a simplicial cone with apex $\boldsymbol{v}$ and integer generators $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_d$, then*

$$\sigma_K(\boldsymbol{x}) = \frac{\sigma_\Pi(\boldsymbol{x})}{(1 - \boldsymbol{x}^{w_1}) \cdots (1 - \boldsymbol{x}^{w_d})}, \tag{1.2.1}$$

where Π is the half-open fundamental parallelepiped, defined as

$$\Pi = \{\boldsymbol{v} + \lambda_1 \boldsymbol{w}_1 + \cdots + \lambda_d \boldsymbol{w}_d \;:\; 0 \leq \lambda_1, \ldots, \lambda_d < 1\}.$$

The geometric idea behind this proof is that we can tile our original cone K with copies of its fundamental parallelepiped Π in directions defined by the generators $\boldsymbol{w}_i$'s. The proof, however, is purely algebraic.

We are interested in this decomposition because Π is bounded.

PROOF.
Expanding the denominators in the RHS of Eq. (1.2.1) we get

$$\left(\sum_{\boldsymbol{p} \in \Pi \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{p}} \right) \left(\sum_{k_1 \geq 0} \boldsymbol{x}^{k_1 \boldsymbol{w}_1} \right) \cdots \left(\sum_{k_N \geq 0} \boldsymbol{x}^{k_d \boldsymbol{w}_d} \right).$$

Therefore we have terms of the form $\boldsymbol{x}^{\boldsymbol{p} + k_1 \boldsymbol{w}_1 + \cdots + k_d \boldsymbol{w}_d}$ for some $\boldsymbol{p} \in \Pi$ and $k_1, \ldots, k_d$ nonnegative integers. On the other hand, in the LHS we have the terms of the form $\boldsymbol{x}^{\boldsymbol{m}}$, where $\boldsymbol{m} \in K$. By definition, points in K can be written as $\boldsymbol{m} = \boldsymbol{v} + \lambda_1 \boldsymbol{w}_1 + \dots \lambda_d \boldsymbol{w}_d$, with nonnegative real λ_i's. Since the cone is simplicial, this decomposition is unique. Now we break down each λ_i in its floor integer part $\lfloor \lambda_i \rfloor$ and its fractional part $\{\lambda_i\}$. Therefore each point $\boldsymbol{m} \in K$ is uniquely written as

$$\boldsymbol{m} = \boldsymbol{p} + \lfloor \lambda_1 \rfloor \boldsymbol{w}_1 + \dots + \lfloor \lambda_1 \rfloor \boldsymbol{w}_d, \qquad \text{where} \quad \boldsymbol{p} = \boldsymbol{v} + \{\lambda_1\} \boldsymbol{w}_1 + \dots + \{\lambda_d\} \boldsymbol{w}_d \ \in \Pi.$$

Comparing with the previous expression and by the uniqueness of the coefficients λ_i's, we only need to check that if a choice of λ_i's produces a point $\boldsymbol{m} \in K \cap \mathbb{Z}^d$, the corresponding $\boldsymbol{p}$ lies not only in Π but in $\Pi \cap \mathbb{Z}^d$. This is indeed true because the vectors $\lfloor \lambda_i \rfloor_i \boldsymbol{w}_i$ are integer vectors. □

In the integer-point enumerator language, Remark 6 translates to

$$\sigma_{\text{cone}(\mathcal{P})}(x_1, \dots, x_{d+1}) = \sum_{t \geq 0} \sigma_{t\mathcal{P}}(x_1, \dots, x_d) x_{d+1}^t.$$

If we evaluate this equation into $x_1 = \dots = x_d = 1$, we get $\sum_{t \geq 0} \left| t\mathcal{P} \cap \mathbb{Z}^d \right| x_{d+1}^t$, a series whose coefficients count the integer points in dilations of $\mathcal{P}$. This is called in the literature the *Ehrhart series* of $\mathcal{P}$.

Lemma 1.2.3. *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be functions satisfying*

$$\sum_{t \geq 0} f(t) x^t = \frac{g(x)}{(1-x)^{d+1}}.$$

Then f is a polynomial of degree d if g is a polynomial of degree at most d and $g(1) \neq 0$.

PROOF.
Let $g(x) = b_0 + \dots + b_{d+1} x^{d+1}$, satisfying $g(1) = b_0 + \dots + b_{d+1} \neq 0$. Using the expansion $\frac{1}{(1-x)^{d+1}} = \sum_{t \geq 0} \binom{t+d}{d} x^t$, we can write

$$\begin{aligned} \frac{g(x)}{(1-x)^{d+1}} = \\ & \sum_{t \geq 0} \binom{t+d}{d} \left(b_0 + \dots + b_{d+1} x^{d+1} \right) x^t \\ & \sum_{k=0}^{d+1} \sum_{t \geq 0} \binom{t+d}{d} b_k x^{t+k}. \end{aligned}$$

Now we have a sum of $d+1$ series, in each of them we perform a change of indexes $\tau_k = t + k$. This way we get

$$\sum_{k=0}^{d+1} \sum_{t \geq 0} \binom{t+d}{d} b_k x^{t+k} = \sum_{k=0}^{d+1} b_k \sum_{\tau_k \geq k} \binom{\tau_k - k + d}{d} b_k x^{\tau_k}.$$

Finally, using $\binom{n}{k} = 0$ for $n < k$, we can write the previous sum as $\sum_{k=0}^{d+1} b_k \sum_{\tau \geq 0} \binom{\tau - k + d}{d} x^\tau$. When we regard this as a series in x, the coefficient of degree d is

$$[x^\tau] \frac{g(x)}{(1-x)^{d+1}} = \sum_{k=0}^{d+1} b_k \binom{\tau - k + d}{d},$$

which is clearly a polynomial in τ of degree at most d. It is indeed of degree d because the leading term is $\frac{1}{d!}(b_0 + \cdots + b_{d+1}) \neq 0$ by hypothesis. □

Now we have all the ingredients to prove Ehrhart's theorem for simplices.

PROOF. (Of Theorem 1.2.1 for simplices.)
Let Δ be a d-dimensional simplex in $\mathbb{R}^d$ with vertices $\boldsymbol{v}_1, \ldots, v_{d+1}$. By Lemma 1.2.3 it suffices to prove that there exists some polynomial g of degree at most d and $g(1) \neq 0$ satisfying

$$\sigma_{\text{cone}(\Delta)}(1, \ldots, 1, x) = \frac{g(x)}{(1-x)^{d+1}}.$$

The cone over a simplex is simplicial with apex in the origin and generators of the form $\boldsymbol{w}_i = (\boldsymbol{v}_i, 1)$, so we can apply Lemma 1.2.2. When we evaluate the first d variables in order to count integer points, we note that the denominator in $\frac{\sigma_\Pi(\boldsymbol{x})}{(1-\boldsymbol{x}^{\boldsymbol{w}_1})\cdots(1-\boldsymbol{x}^{\boldsymbol{w}_{d+1}})}$ is indeed $(1-x_{d+1})^{d+1}$, so we are only interested in the numerator as a polynomial in the variable x_{d+1}. In other words, we propose $g(x) = \sigma_\Pi(1, \ldots, 1, x)$.

First we note that it is certainly a polynomial: since Π is bounded, it has finitely many integer points, thus its integer-point enumerator has a finite number of terms. Each term is of the form $\boldsymbol{x}^{\lambda_1 \boldsymbol{w}_1 + \cdots + \lambda_{d+1}\boldsymbol{w}_{d+1}}$ for some $0 \leq \lambda_1, \ldots, \lambda_{d+1} < 1$. For each of them, we want to bound the exponent of x_{d+1}. Since the $(d+1)$-th coordinate of all vectors $\boldsymbol{w}_i$'s is 1, the exponent of x_{d+1} is $\lambda_1 + \cdots + \lambda_{d+1} < d+1$. Since we are counting integer points, the exponent needs to be an integer, therefore it is at most d.

When we evaluate $g(1)$ we are counting integer points in Π. This is non-zero, because the origin is an integer point and always belongs to Π. This concludes the proof. □

Now we need to extend the result to general polytopes. To do this, we will use the fact that we can always triangulate any polytope into simplices (recall Theorem 1.1.18).

This is the tool we need to finish our first proof of Ehrhart's theorem.

PROOF. (Of Theorem 1.2.1)
Let $\mathcal{P}$ be a d-dimensional polytope, and T a triangulation of $\mathcal{P}$ that adds no new vertices. Then by the inclusion-exclusion principle,

$$\left|t\mathcal{P} \cap \mathbb{Z}^d\right| = \sum_{\Delta \in T} \left|t\Delta \cap \mathbb{Z}^d\right| - \sum_{(\Delta_1, \Delta_2 \in T)} \left|t(\Delta_1 \cap \Delta_2) \cap \mathbb{Z}^d\right| + \cdots + (-1)^{|T|+1} \left|t(\Delta_1 \cap \cdots \cap \Delta_{|T|}) \cap \mathbb{Z}^d\right|.$$

All the terms in the previous sum are polynomials of degree at most d by the simplex case, and only the first one is of degree exactly d, so there can be no cancellations and thus the result is a polynomial of degree exactly d. □

1.3 Algebra of polyhedra

We have already seen that in order to study integer points in polytopes it can be useful to introduce more general structures and then find a way to specialize the results for the case we in which are interested.

In this section we introduce the algebra of polyhedra, which is useful to think as a structure we give to polyhedra to define basic algebraic operations with them.

Definition 1.3.1 (Indicator function). Let $A \subseteq \mathbb{R}^N$ be any set. Its *indicator function* is the function $[A] : \mathbb{R}^N \to \mathbb{R}$ defined by

$$[A](\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in A \\ 0 & \text{if } \boldsymbol{x} \notin A \end{cases}$$

One interesting property of identifying sets with their indicator functions is that some algebraic operations with indicator functions do have geometric meaning. For multiplication we have $[A]\cdot[B] = [A\cap B]$, and for addition we have $[A] + [B] = [A \cup B] + [A \cap B]$. In fact, for addition we actually have an inclusion-exclusion formula that will be useful later, namely

$$\left[\bigcup_{i\in I} A_i\right] = \sum_{J\subseteq I} (-1)^{|J|+1} \left[\bigcap_{j\in J} A_j\right] \tag{1.3.1}$$

With this concept we can define the linear spaces we will be working on for a major part of the thesis, as the span of indicator functions of certain families of sets.

Definition 1.3.2 (Algebra of polyhedra). Given an ambient space $\mathbb{R}^N$, the *algebra of polyhedra* is the linear span of indicator functions of polyhedra, we denote it by

$$\mathcal{P}(\mathbb{R}^N) = \operatorname{span}\left\{[A] \ : \ A \subseteq \mathbb{R}^N \text{ polyhedron } \right\}.$$

Similarly, the *algebra of polytopes* is the linear span of polytopes, we denote it with a subscript b, accounting for *bounded*:

$$\mathcal{P}_b(\mathbb{R}^N) = \operatorname{span}\left\{[A] \ : \ A \subseteq \mathbb{R}^N \text{ polytope } \right\}.$$

Note that the indicator functions of polytopes or polyhedra are far from being a basis of their corresponding algebras, since the property we have just described for addition of indicator functions is in itself a linear relation.

Remark 7. Since all points in a polytope lie in the relative interior of a unique face, we have

$$[P] = \sum_{F\,:\,\text{face of } P} [\operatorname{rel\,int} F]$$

Also note that the algebra of polyhedra contains the indicator functions of open polyhedra as well. In particular, for a polyhedron P of dimension d we have

$$[\operatorname{rel\,int} P] = \sum_{F\,:\,\text{face of } P} (-1)^{d-\dim F}[F]. \tag{1.3.2}$$

Eq. (1.3.2) is a consequence of the inclusion-exclusion principle applied to faces of P.

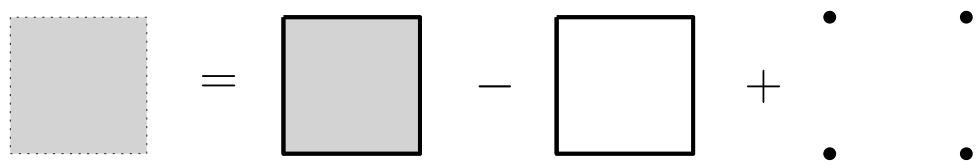


Figure 1.6: An open square as sum of polytopes.

Remark 8. The algebra of polyhedra $\mathcal{P}(\mathbb{R}^N)$ is spanned by the indicator functions of polyhedra without lines.

This is easy to prove by induction. For $N = 0$ the result is clear. For $N > 0$, consider P a polyhedron, if it has a line, cut the polyhedron in two halves by a hyperplane. Repeating this process a finite amount of times we get some polyhedra without lines and in the intersection some polyhedra of smaller dimension, to which we apply the induction hypothesis.

1.3.1 Valuations

Now that we have given polyhedra the structure of a vector space, we may be able to find useful results studying linear maps.

Definition 1.3.3 (Valuation). A *valuation* is a linear map $T : \mathcal{P}, \mathcal{P}_b \to V$, where V is a vector space.

We typically think of valuations as a way of *assigning value* to a polyhedron. In that case, $V = \mathbb{R}$. But it will be useful for more convoluted constructions to consider valuations that map to some functions spaces, so we keep a general vector space V in our definition.

Theorem 1.3.1 (Euler characteristic). *There exists a unique valuation $\chi : \mathcal{P}(\mathbb{R}^N) \to \mathbb{R}$ such that*

$$\chi([A]) = 1, \quad \textit{for all polyhedron } A.$$

This is called the Euler valuation *or* Euler characteristic.

PROOF.
First observe that we are giving the value of χ on a set of generators, so if it exists is necessarily unique.

We will prove by induction on N. If $N = 0$, clearly $\chi(f) = f(0)$ satisfies the theorem. Consider $N > 0$.

Let $a : \mathbb{R}^N \to \mathbb{R}$ be a linear function, $\boldsymbol{a} \neq 0$, and consider for all $\tau \in \mathbb{R}$ the hyperplanes

$$H_\tau = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ a(\boldsymbol{x}) = \tau\}.$$

By the induction hypothesis, there exists a valuation $\chi_\tau : \mathcal{P}(H_\tau) \to \mathbb{R}$ such that $\chi_\tau(A) = 1$ for all polyhedron $A \in H_\tau$. We will first define χ in the algebra of bounded polyhedra, and the we will extend the definition to unbounded polyhedra too. Let us choose an element $f \in \mathcal{P}_b(\mathbb{R}^N)$. By construction, it can be written as

$$f = \sum_{i \in I} \alpha_i [A_i], \quad \text{where } A_i \text{ are polytopes.}$$

Consider also the restrictions $f_\tau = f\big|_{H_\tau} \in \mathcal{P}_b(H_\tau)$. We can write them as $f_\tau = \sum_{i\in I} \alpha_i [A_i \cap H_\tau]$, so

$$\chi_\tau(f_\tau) = \sum_{i \in I_\tau} \alpha_i, \qquad \text{where} \quad I_\tau = \{i \ : \ A_i \cap H_\tau \neq \emptyset\}.$$

This value will be constant in τ except when we find boundaries of sets A_i. In particular, observe that

$$\lim_{\varepsilon \to 0^+} \chi_{\tau-\varepsilon}\left(f_{\tau-\varepsilon}\right) = \sum_{i \in J_\tau} \alpha_i, \qquad \text{where} \quad J_\tau = \left\{ i \in I_\tau \ : \ \min_{\boldsymbol{x} \in A_i} a(\boldsymbol{x}) \neq \tau \right\}. \tag{1.3.3}$$

This way, we can define $\chi : \mathcal{P}_b(\mathbb{R}^N) \to \mathbb{R}$ as

$$\chi(f) = \sum_{\tau \in \mathbb{R}} \left(\chi_\tau\left(f_\tau\right) - \lim_{\varepsilon \to 0^+} \chi_{\tau-\varepsilon}\left(f_{\tau-\varepsilon}\right) \right). \tag{1.3.4}$$

By Eq. (1.3.3) only finitely many terms of the sum are non-zero, so it is well defined. Linearity is preserved because all the operations we have used to define χ (sums, limits, restriction to hyperplane) preserve linearity. Finally, for a polytope, only one term is non-zero, namely the one corresponding to the only proper face of the form $\{a(\boldsymbol{x}) = \tau\}$ (see Fig. 1.7).

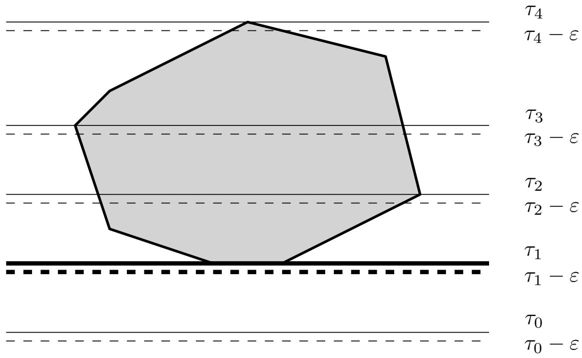


Figure 1.7: Sketch of the limit in Eq. (1.3.3) for polytopes.

To extend χ to $\mathcal{P}(\mathbb{R}^N)$, consider for $t \geq 0$, the dilated (± 1)-cube $t\square^N$. Since $\mathbf{0} \in \operatorname{rel\,int} \square^N$, we have that $\mathbb{R}^N = \bigcup_{t \geq 0} t\square^N$. We define

$$\chi(f) = \lim_{t \to \infty} \chi\left(f \cdot \left[t\square^N\right]\right).$$

This is well defined because for any polyhedron $[A]$, we have $[A] \cdot [t\square^N] = [A \cap (t\square^N)] \in \mathcal{P}_b(\mathbb{R}^N)$, and for sufficiently large t we have $A \cap (t\square^N) \neq \emptyset$, thus the limit is well defined. □

Observe that the proof gives us a formula for computing χ in terms of hyperplanes H_τ. This will be useful later.

We know that the algebra of polyhedra contains also open polyhedra, see Fig. 1.6. We can recover the Euler-Poincaré formula by studying χ on open polytopes.

Theorem 1.3.2. *Let P be a d-dimensional polytope. Then*

$$\chi([\operatorname{rel\,int} P]) = (-1)^d.$$

PROOF.
Let $a : \mathbb{R}^N \to \mathbb{R}$ be a linear function, $a \neq 0$. Consider for all $\tau \in \mathbb{R}$ the hyperplanes

$$H_\tau = \{\boldsymbol{x} \in \mathbb{R}^N \ : \ a(\boldsymbol{x}) = \tau\}$$

and the functions $f_\tau = [\operatorname{rel\,int} P] \cdot [H_\tau]$. By Eq. (1.3.4) we know that

$$\chi([\operatorname{rel\,int} P]) = \sum_{\tau \in \mathbb{R}} \left(\chi_\tau\left(f_\tau\right) - \lim_{\varepsilon \to 0^+} \chi_{\tau-\varepsilon}\left(f_{\tau-\varepsilon}\right) \right),$$

and as in Fig. 1.7, only one term of the sum is non-zero, corresponding to τ such that $\min_{\boldsymbol{x} \in A_i} a(\boldsymbol{x}) \neq \tau$. In this case, $P \cap H_\tau = \emptyset$ and $P \cap H_{\tau-\varepsilon}$ is a $(d-1)$-dimensional polytope, so we can finish the proof using the induction hypothesis: $\chi([\operatorname{rel\,int} P]) = 0 - (-1)^{d-1} = (-1)^d$. □

Corollary 1.3.3 (Euler-Poincaré formula). *Let P be a d-dimensional polytope and let f_k be the number of k-dimensional faces of P, for $k = 0, \ldots, d$. Then*

$$\sum_{k=0}^{d} (-1)^k f_k = 1.$$

PROOF.
Recall Remark 7 and apply the previous theorem and definition of χ for polytopes:

$$1 = \chi([P]) = \sum_{F\,:\,\text{face of } P} [\operatorname{rel\,int} F] = \sum_{F\,:\,\text{face of } P} (-1)^{\dim F} = \sum_{k=0}^{d} f_k.$$

□

1.3.2 Basic properties

In this section we will basically study what can we say about the results we have seen through Chapter 1 when we regard not only polyhedra (resp. polytopes) but their whole algebra.

Proposition 1.3.4. *Let $T : \mathbb{R}^N \to \mathbb{R}^M$ be a linear transformation. Then there exists a unique valuation $\mathcal{T} : \mathcal{P}(\mathbb{R}^N) \to \mathcal{P}(\mathbb{R}^M)$ such that $\mathcal{T}([P]) = [T(P)]$ for every polyhedron P.*

PROOF.
As in Theorem 1.3.1, if such a valuation exists, it must be unique.

Given $\boldsymbol{x} \in \mathbb{R}^M$, its preimage $T^{-1}(\boldsymbol{x}) \subseteq \mathbb{R}^N$ is an affine subspace, so for every polyhedron $P \subseteq \mathbb{R}^N$, the set $P \cap T^{-1}(\boldsymbol{x})$ is also a polyhedron. Therefore $f \cdot [T^{-1}(\boldsymbol{x})] \in \mathcal{P}(\mathbb{R}^N)$ for every $f \in \mathcal{P}(\mathbb{R}^N)$. With this observation, we will use the Euler valuation to construct $\mathcal{T}$:

$$\begin{array}{rccccccc} \mathcal{T}: & \mathcal{P}\left(\mathbb{R}^N\right) & \longrightarrow & \mathcal{P}\left(\mathbb{R}^M\right) & & & & \\ & f & \longmapsto & \mathcal{T}(f): & \mathbb{R}^M & \longrightarrow & \mathbb{R} & \\ & & & & \boldsymbol{x} & \longmapsto & h(\boldsymbol{x}) & = \chi\left(f \cdot [T^{-1}(\boldsymbol{x})]\right). \end{array}$$

Now note that if $f = P$, we have $P \cap T^{-1}(\boldsymbol{x}) \neq \emptyset$ if and only if $\boldsymbol{x} \in P$. Since it is always a polyhedron, this means that

$$\chi\left(\left[P \cap T^{-1}(\boldsymbol{x})\right]\right) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in P, \\ 0 & \text{otherwise.} \end{cases}$$

□

Proposition 1.3.5. *There exists a unique bilinear operation* $* : \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \to \mathcal{P}(\mathbb{R}^N)$ *such that*

$$[P_1] * [P_2] = [P_1 + P_2] \quad \textit{for every pair of polyhedra} \quad P_1,\ P_2.$$

PROOF. (Sketch only)
As in Theorem 1.3.1 and previous proposition, if such a valuation exists, it must be unique. For $f, g \in \mathcal{P}(\mathbb{R}^N)$, define the function $f \times g : \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}$ as $(f \times g)(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x})g(\boldsymbol{y})$. Note that $f \times g \in \mathcal{P}(\mathbb{R}^N \oplus \mathbb{R}^N)$. Let $\mathcal{T} : \mathcal{P}(\mathbb{R}^N \oplus \mathbb{R}^N) \to \mathcal{P}(\mathbb{R}^N)$ be the linear transform of Proposition 1.3.4. Then define $f * g = \mathcal{T}(f \times g)$ and check that the properties are satisfied. □

1.3.3 Decomposition modulo polyhedra with lines

One of the most powerful properties of working with the algebra of polyhedra is that we will be able to describe a polyhedron as a certain combinations of its cones from its vertices, modulo some equivalence relation.

Definition 1.3.4 (Equivalence modulo polyhedra with lines). Let $f, g \in \mathcal{P}(\mathbb{R}^N)$, we say that

$$f \equiv g \qquad \text{modulo polyhedra with lines}$$

if $f - g$ can be expressed as a linear combination of polyhedra that contain some line.

To understand why this definition is useful, we need to introduce tangent cones.

Definition 1.3.5 (Tangent cone, feasible cone). Let P be a polyhedron and $\boldsymbol{v} \in P$ a point. Then the *tangent cone of* P *at* $\boldsymbol{v}$ *is*

$$\operatorname{tcone}(P, \boldsymbol{v}) = \{\boldsymbol{v} + \boldsymbol{x} \ : \ \boldsymbol{v} + \varepsilon \boldsymbol{x} \in P, \ \text{for some } \varepsilon > 0\}.$$

Analogously, the *feasible cone* or *cone of feasible directions* is

$$\operatorname{fcone}(P, \boldsymbol{v}) = \{\boldsymbol{x} \ : \ \boldsymbol{v} + \varepsilon \boldsymbol{x} \in P, \ \text{for some } \varepsilon > 0\}.$$

It is clear from the definition that $\operatorname{tcone}(P, \boldsymbol{v}) = \boldsymbol{v} + \operatorname{fcone}(P, \boldsymbol{v})$. It is useful to characterize the tangent and feasible cones with respect to the defining inequalities of a polyhedron.

Given a polyhedron P defined by inequalities $\{a_i(\boldsymbol{x}) \leq \alpha_i\}_{i \in I}$ and a point $\boldsymbol{v} \in P$, we say that a particular inequality $i_0 \in I$ is *active on* $\boldsymbol{v}$ if $a_{i_0}(\boldsymbol{v}) = \alpha_{i_0}$. We denote the set of indices of active equations as at a given point as $I_{\boldsymbol{v}}$. Then our cones have the following description:

$$\operatorname{tcone}(P, \boldsymbol{v}) = \{\boldsymbol{x} \ : \ a_i(\boldsymbol{x}) \leq \alpha_i, \ \text{for } i \in I_{\boldsymbol{v}}\},$$

$$\operatorname{fcone}(P, \boldsymbol{v}) = \{\boldsymbol{x} \ : \ a_i(\boldsymbol{x}) \leq 0, \ \text{for } i \in I_{\boldsymbol{v}}\}.$$

These cones are well behaved with respect to linear transforms and intersections:

Lemma 1.3.6. *Let $T : \mathbb{R}^N \to \mathbb{R}^M$ be a linear transformation, $P \subseteq \mathbb{R}^N$ and polyhedron and $\boldsymbol{v} \in P$. Then*

$$\operatorname{tcone}(T(P), T(\boldsymbol{v})) = T(\operatorname{tcone}(P, \boldsymbol{v})).$$

PROOF.

We know that the linear image of a polyhedron is also a polyhedron by Proposition 1.3.4. Let $Q = T(P)$ and $\boldsymbol{w} = T(\boldsymbol{v})$.

($\subseteq$) Let $\boldsymbol{w} + \boldsymbol{y} \in \operatorname{tcone}(Q, \boldsymbol{v})$. Then there exists $\varepsilon > 0$ such that $\boldsymbol{w} + \varepsilon\boldsymbol{y} \in Q = T(P)$, so there exists $\boldsymbol{v} + \boldsymbol{x} \in P$ such that $T(\boldsymbol{v} + \boldsymbol{x}) = \boldsymbol{w} + \varepsilon\boldsymbol{y}$. But then $T(\varepsilon\boldsymbol{y}) = \boldsymbol{x}$, so we have that $\boldsymbol{v} + \varepsilon^{-1}\boldsymbol{x} \in \operatorname{tcone}(P, \boldsymbol{v})$ and $T(\boldsymbol{v} + \varepsilon^{-1}\boldsymbol{x}) = \boldsymbol{w} + \boldsymbol{y}$.

($\supseteq$) Let $\boldsymbol{x} \in \operatorname{tcone}(P, \boldsymbol{v})$. Then $\boldsymbol{v} + \delta\boldsymbol{x} \in P$ for some δ, so $\boldsymbol{w} + \delta T(\boldsymbol{x}) \in Q$, and thus $T(\boldsymbol{v} + \boldsymbol{x}) \in \operatorname{tcone}(Q, \boldsymbol{w})$. □

Lemma 1.3.7. *Given a finite collection of polyhedra $\{P_i\}_{i\in I}$, and a point $\boldsymbol{v} \in \bigcap_{i\in I} P_i$, the tangent cones satisfy*

$$\operatorname{tcone}\left(\bigcap_{i\in I} P_i, \boldsymbol{v}\right) = \bigcap_{i\in I} \operatorname{tcone}(P_i, \boldsymbol{v}).$$

PROOF.

For each $i \in I$, let J_i denote the indices of the active inequalities of P_i on $\boldsymbol{v}$. The tangent cone $\operatorname{tcone}(P_i, \boldsymbol{v})$ is defined precisely by the inequalities indexed by J_i, and the intersection of tangent cones in therefore defined by the inequalities indexed in $\cup_i J_i$.

On the other hand, the inequalities defining $\cap_i P_i$ are the reunion of the defining inequalities of all P_i's, and an inequality is active on a point independently of the ambient polytope, so the set of defining inequalities of $\operatorname{tcone}[\cap_i P_i, \boldsymbol{v}]$ is precisely $\cup_i J_i$. □

The following is the main result that motivates the introduction of this equivalence relation. It was essentially first proved by Brion in [4] albeit in a different setting, similar to Eq. (2.1.11).

Theorem 1.3.8 (Brion). *Let P be a polyhedron and $\operatorname{vert}(P)$ its set of vertices. Then*

$$[P] \equiv \sum_{\boldsymbol{v}\in\operatorname{vert}(P)} [\operatorname{tcone}(P, \boldsymbol{v})] \qquad \textit{modulo polyhedra with lines.}$$

PROOF.

We will prove the theorem first for simplices, then for polytopes and finally for a general polyhedron. For simplices, by Lemma 1.3.6 we can restrict ourselves to the standard simplex Δ^d, defined by

$$\Delta^d = \{\boldsymbol{x} : x_1 + \cdots + x_d = 1, x_1, \ldots, x_d \geq 0\}, \text{ and equivalently as } \Delta^d = \operatorname{conv}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the vectors of the canonical basis in $\mathbb{R}^d$. Let H be the hyperplane defined by the equation $x_1 + \cdots + x_d = 1$, and let for $i \in \{1, \ldots, d\}$ let $H_i \subseteq H$ be the half-hyperplane defined by the inequality $\{x_i \geq 0\}$. We have that $\bigcup_{i=1}^d H_i = H$, so we can apply Eq. (1.3.1) to get

$$[H] = \sum_{J\subseteq\{1,\ldots,d\}} (-1)^{|J|+1} \left[\bigcap_{j\in J} H_j\right].$$

Now we look at this equation with the *modulo-polyhedra-with-lines* glasses. The LHS vanishes, because a hyperplane clearly contains lines. In the RHS we have for $J = \{1, \dots, d\}$ that $\bigcap_{j\in J} H_j = \Delta^d$. Also, by the characterization of tangent cones with active inequalities, for each $i \in \{1, \dots, d\}$, the term corresponding with $J_i = \{1, \dots, d\}\backslash\{i\}$ satisfies $\bigcap_{j\in J_i} H_j = \text{tcone}(\Delta^d, \boldsymbol{e}_i)$. And then the rest of the terms in the RHS vanish, because in those there are at least two indices, i_1, i_0 out of the intersection, and thus the aforementioned intersection contains the line that goes through $\boldsymbol{e}_{i_1}$ and $\boldsymbol{e}_{i_2}$. This concludes the proof for simplices.

For a general polytope P, by Theorem 1.1.18 we can find a triangulation T that adds no new vertices. Assume $T = \{\Delta_1, \dots, \Delta_n\}$. For each Δ_k we know that $[\Delta_k] \equiv \sum_{\boldsymbol{v}_k\in\text{vert}(\Delta_k)}[\text{tcone}(\Delta_k, \boldsymbol{v}_k)]$. Adding all together we get

$$\sum_{k=1}^{n}[\Delta_k] = \sum_{\boldsymbol{v}\in\text{vert}(P)} \sum_{k_{\boldsymbol{v}}\in K_{\boldsymbol{v}}} [\text{tcone}(\Delta_{k_{\boldsymbol{v}}}, \boldsymbol{v})], \tag{1.3.5}$$

where $K_{\boldsymbol{v}}$ is the set that indexes the simplices that have $\boldsymbol{v}$ as a vertex. Now we can expand the LHS of Eq. (1.3.5) using inclusion-exclusion:

$$\sum_{k=1}^{n}[\Delta_k] = \left[\bigcup_{k=1}^{n} \Delta_k\right] + \sum_{J\subseteq 1,\dots,n,|J|>1} (-1)^{|J|} \left[\bigcap_{j\in J} \Delta_j\right].$$

The first term is $[P]$, and we can expand the second term using the theorem for simplices:

$$\sum_{J\subseteq 1,\dots,n,|J|>1} (-1)^{|J|} \left[\bigcap_{j\in J} \Delta_j\right] = \sum_{J\subseteq 1,\dots,n,|J|>1} (-1)^{|J|} \sum_{\boldsymbol{v}_J\in\text{vert}(\cap_J \Delta_j)} \left[\text{tcone}\left(\bigcap_{j\in J} \Delta_j, \boldsymbol{v}_J\right)\right]. \tag{1.3.6}$$

Now we apply the inclusion-exclusion principle to the RHS of Eq. (1.3.5). For each vertex $\boldsymbol{v} \in \text{vert}(P)$:

$$\sum_{k_{\boldsymbol{v}}\in K_{\boldsymbol{v}}} [\text{tcone}(\Delta_{k_{\boldsymbol{v}}}, \boldsymbol{v})] = \left[\bigcup_{k_{\boldsymbol{v}}\in K_{\boldsymbol{v}}} \text{tcone}(\Delta_{k_{\boldsymbol{v}}}, \boldsymbol{v})\right] + \sum_{J_{\boldsymbol{v}}\subseteq K_{\boldsymbol{v}},|J_{\boldsymbol{v}}|>1} (-1)^{|J_{\boldsymbol{v}}|} \left[\bigcap_{j\in J_{\boldsymbol{v}}} \text{tcone}(\Delta_j, \boldsymbol{v})\right].$$

The first term is just $[\text{tcone}(P, \boldsymbol{v})]$, and adding for all $\boldsymbol{v}$ we get the desired result, so hopefully the second summation in the previous equation will vanish with Eq. (1.3.6). Indeed, if we compare them, the sums run over the same elements, they have the same sign with the $(-1)^{|J|}$, and the indicator functions are the same thanks to Lemma 1.3.7. This concludes the proof for polytopes.

For an arbitrary polyhedron, it it contains a line, then it contains no vertices so the result is clear. If it contains a line, by Proposition 1.1.14 it can be decomposed as a Minkowski sum $P = Q + K$, where Q is the convex hull of the vertices of P and K is the recession cone of P. We already know that $[Q] \equiv \sum_{\boldsymbol{v}\in\text{vert}(Q)}[\text{tcone}(Q, \boldsymbol{v})]$, and by Proposition 1.3.5 we know that Minkowski sum preserves linear relations, so we have

$$[Q + K] \equiv \sum_{\boldsymbol{v}\in\text{vert}(P)} [\text{tcone}(Q, \boldsymbol{v}) + K] \qquad \text{modulo polyhedra with lines.}$$

Now we just need to note that for the recession cone $\text{tcone}(Q, \boldsymbol{v}) + K = \text{tcone}(Q + K, \boldsymbol{v})$ and we are done. □

Corollary 1.3.9. *Let $\{P_i\}_{i\in I}$ be a family of polytopes such that $\mathbf{0} \in P_i$ for all $i \in I$ and $\{\alpha_i\}_{i\in I}$ be real numbers. Then, modulo polyhedra with lines,*

$$\sum_{i\in I} \alpha_i [P_i] \equiv 0 \quad \Longleftrightarrow \quad \sum_{i\in I} \alpha_i [P_i^\circ] \equiv 0.$$

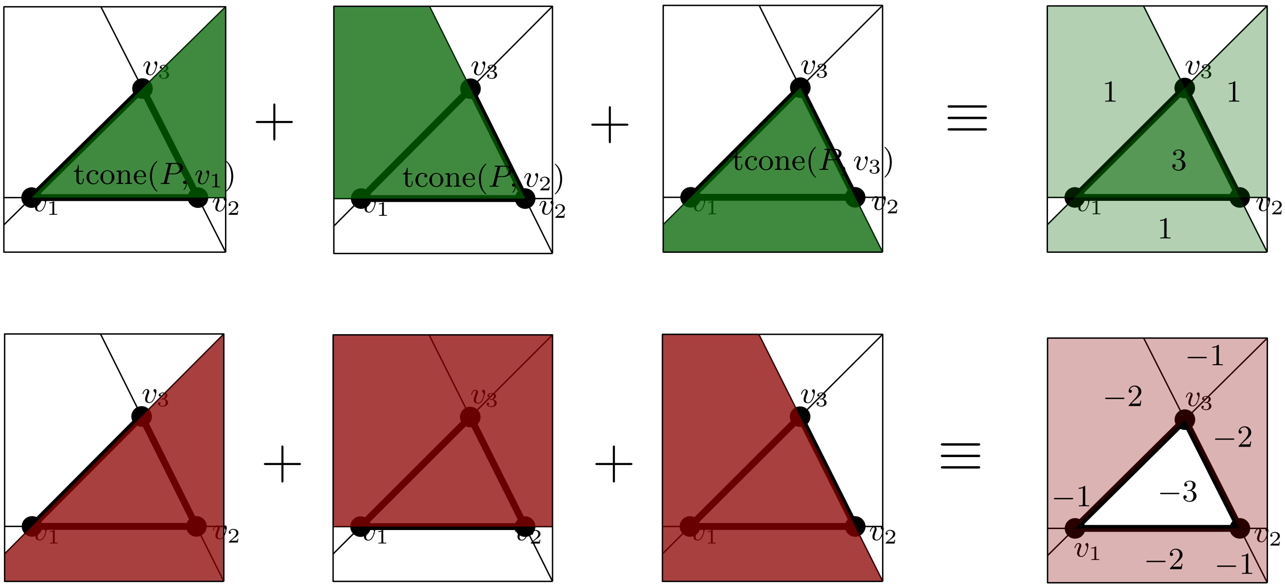


Figure 1.8: Illustration of Brion's theorem for a triangle. In the first line we add the tangent cones. In the second line, we half-planes, that are polyhedra with lines. Adding both results together plus the indicator function of the whole plane (which is also a polyhedron with lines) yields the indicator of the triangle.

We have a similar result for the cones of feasible directions.

Corollary 1.3.10 (Brion's theorem with feasible cones). *Let P be a polyhedron,* $\operatorname{vert}(P)$ *its set of vertices, and K_P its recession cone. Then*

$$[K_P] \equiv \sum_{\boldsymbol{v}\in \operatorname{vert}(P)} [\operatorname{fcone}(P, \boldsymbol{v})] \qquad \textit{modulo polyhedra with lines.}$$

We also have an analogous result for open polyhedra, but we first need to prove an auxiliary equation.

Lemma 1.3.11. *Let P be a d-dimensional cone. Then*

$$[P] \equiv (-1)^d[-\operatorname{rel\,int} K] \quad \textit{modulo polyhedra with lines.}$$

Corollary 1.3.12 (Brion's theorem for open polytopes). *Let P be a polytope and* $\operatorname{vert}(P)$ *its set of vertices. Then*

$$[\operatorname{rel\,int} P] \equiv \sum_{\boldsymbol{v}\in \operatorname{vert}(P)} [\operatorname{rel\,int} \operatorname{tcone}(P, \boldsymbol{v})] \qquad \textit{modulo polyhedra with lines.}$$

We will not prove these last results. A proof can be found in [1, Ch. 7].

Chapter 2

Local formula for Ehrhart polynomials

2.1 The exponential valuation

Recall that, informally, a valuation is a rule to assign a value to objects in our vector spaces. The exponential valuation we will define in this section, following [1, Ch. 8] is motivated by the idea of extending the concept of *volume of a polytope* to any element in the algebra of polyhedra, as defined in Section 1.3.

The natural way to define the volume of a polytope is through its integral. This is useful for polytopes, and even for the algebra of polytopes, but we immediately reach a problem when the domains of the functions we want to *measure* are not bounded.

It turns out that the right extension is to assign a function instead of a number, that when evaluated at the origin, if the region is bounded, will let us recover the volume. Let's get our hands dirty.

Lemma 2.1.1. *Let $K \subseteq \mathbb{R}^N$ be a pointed cone, then for every $\boldsymbol{c} \in \operatorname{rel\,int} K^\circ$, the integral*

$$\int_K e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x}$$

converges absolutely and uniformly on compact subsets of $\operatorname{rel\,int} K^\circ$ to a rational function of the form

$$\phi(K, \boldsymbol{c}) = \sum_I \alpha_I \prod_{i \in I} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle},$$

where $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \mathbb{R}^N$ are non-zero vectors, $\alpha_I \in \mathbb{R}_+$ and $I \subseteq \{1, \dots, n\}$.

PROOF.
Without loss of generality, we can assume that $\dim K = N$, otherwise the measure of K is null.

We will first prove it for simplicial cones. If K is simplicial it can be written as $K = \operatorname{co}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_N)$ for some $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ that for a basis of $\mathbb{R}^N$. Let U be the corresponding basis transformation. In that case we can compute the integral via a change of variables. Let $\alpha = \operatorname{vol}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_N)$, then

$$\int_K e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = \alpha \int_{\mathbb{R}^N_{\geq 0}} e^{\langle \boldsymbol{c}, U\boldsymbol{y} \rangle} \mathrm{d}\boldsymbol{y} = \alpha \prod_{i=1}^{m} \int_0^\infty e^{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle y_i} \mathrm{d}y_i = \alpha \prod_{i=0}^{m} \frac{1}{-\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}.$$

In the last step we are using that the integrals converge precisely because $\boldsymbol{c} \in \operatorname{rel\,int} K^\circ$.

By Proposition 1.1.12 we can represent K as the cone over a $(N-1)$-dimensional polytope. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be the vertices of such a polytope, then $K = \operatorname{co}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$. By Corollary 1.1.19 we can triangulate K using no new generators. Let $\mathcal{I}$ be the set indexing the vertices of such a triangulation. For every $I \in \mathcal{I}$, the corresponding cone $K_I = \operatorname{co}\{u_i : i \in I\}$ is simplicial. We can perform the integral using the result for simplicial cones. Let $\alpha_I = \operatorname{vol}(u_i : i \in I)$, then

$$\int_K e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = \sum_{I \in \mathcal{I}} \int_{K_I} e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = \sum_{I \in \mathcal{I}} \alpha_I \prod_{i \in I} \frac{1}{-\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}.$$

□

Remark 9. If we want to allow this same result for a shifted cone $K + \boldsymbol{v}$, with a change of variables we get

$$\int_{K+\boldsymbol{v}} e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = e^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} \int_K e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x}.$$

This observation, along with the previous lemma, justify the choice of the following space of functions:

$$\mathcal{M}(\mathbb{R}^N) = \operatorname{span}\left\{ f(\boldsymbol{c}) = e^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} \prod_{i=1}^{N} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle} \ : \ \boldsymbol{u}_1, \ldots, \boldsymbol{u}_N \text{ form a basis and } \boldsymbol{v} \in \mathbb{R}^N \right\}.$$

Theorem 2.1.2 (Exponential valuation). *There exists a valuation $\Phi : \mathcal{P}(\mathbb{R}^N) \to \mathcal{M}(\mathbb{R}^N)$ such that*

(1) For any polyhedron P without lines, and for all $\boldsymbol{c} \in \operatorname{int} K^\circ$, where K is its recession cone, the integral $\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x}$ converges absolutely and uniformly on compact subsets of $\operatorname{int} K^\circ$ to $\Phi([P])$.

(2) For any polyhedron with lines, $\Phi([P]) = 0$.

PROOF.
We will prove the result by induction on N. If $N = 0$, the result is clear. Assume $N > 0$.

(1) Since polyhedra without lines span $\mathcal{P}(\mathbb{R}^N)$, we just need to prove two different things:

(a) The integral $\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x}$ converges to a function in $\mathcal{M}(\mathbb{R}^N)$.

(b) The assignation defined in (a) is linear.

(a) As we did before, we can assume $\dim P = N$, otherwise the integral is null. By Proposition 1.1.14, we can write $P = K + Q$, where K is the recession cone of P and Q is the convex hull of the vertices of P. Let $\boldsymbol{u} \in \operatorname{int} K^\circ$ such that $\|u\| = 1$ and $\boldsymbol{u}$ is not orthogonal to any proper face of P. We partition $\mathbb{R}^N$ into hyperplanes

$$H_\tau = \{\boldsymbol{x} \in \mathbb{R}^N \;:\; \langle \boldsymbol{x}, \boldsymbol{u} \rangle = \tau\}, \qquad for \qquad \tau \in \mathbb{R},$$

and consider $P_\tau = P \cap H_\tau$. The induction hypothesis holds for P_τ in H_τ.

To use it, first note that there exists τ_1 such that $P_\tau = \emptyset$ for $\tau > \tau_1$. This is true because given $\boldsymbol{x} \in P$, we have $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{v}$, where $\boldsymbol{y} \in Q$ and $\boldsymbol{v} \in K$. Then

$$\langle \boldsymbol{x}, \boldsymbol{u} \rangle = \langle \boldsymbol{y}, \boldsymbol{u} \rangle + \langle \boldsymbol{v}, \boldsymbol{u} \rangle \leq \langle \boldsymbol{y}, \boldsymbol{u} \rangle,$$

where $\langle \boldsymbol{v}, \boldsymbol{u} \rangle \leq 0$ holds because $u \in K^\circ$ and Remark 4. Therefore

$$\max_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle = \max_{\boldsymbol{x} \in Q} \langle \boldsymbol{x}, \boldsymbol{u} \rangle.$$

Since Q is compact, this max does exist. This max value is τ_1 and thus we can write

$$\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = \int_{-\infty}^{\tau_1} \left(\int_{P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{y} \rangle} \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\tau, \tag{2.1.1}$$

where $\mathrm{d}\boldsymbol{y}$ is the standard Lebesgue measure in H_τ.

Since faces are defined by valid inequalities, the faces of P_τ correspond to faces of P of greater or equal dimension. In particular, the vertices of P_τ correspond to 4 either vertices or edges of P.

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be the vertices of P, ordered in such a way that

$$\tau_m \geq \cdots \geq \tau_1, \qquad \text{where} \qquad \tau_i = \langle \boldsymbol{v}_i, \boldsymbol{u} \rangle.$$

For notation purposes, we may consider $\tau_{m+1} = -\infty$, and thus we can rewrite Eq. (2.1.1) as

$$\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \mathrm{d}\boldsymbol{x} = \sum_{i=1}^{m} \int_{\tau_{i+1}}^{\tau_i} \left(\int_{P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{y} \rangle} \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\tau. \tag{2.1.2}$$

The induction hypothesis holds in for P_τ in H_τ. Since $K_{P_\tau} \subseteq K_P$, we have $K^\circ_{P_\tau} \supseteq K^\circ_P$ and thus by item (1) of the statement

$$\Phi([P_\tau]) = \int_{P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{y} \rangle} \mathrm{d}\boldsymbol{y}. \tag{2.1.3}$$

Item (2) of the statement is also part of our induction hypothesis, and it gives us permission to use Brion's decomposition of P_τ into its tangent cones (see Theorem 1.3.8), so

$$\Phi([P_\tau]) = \sum_{\boldsymbol{v} \in \operatorname{vert} P_\tau} \Phi\big(\,[\operatorname{tcone}(P_\tau, \boldsymbol{v})]\,\big).$$

We know that $\operatorname{tcone}(P_\tau, \boldsymbol{v}) = \boldsymbol{v} + \operatorname{fcone}(P_\tau, \boldsymbol{v})$, and Lemma 2.1.1 applies to the feasible cone, so there exist functions $f^{\boldsymbol{c}}_\tau \in \mathcal{M}(H_\tau)$ such that $\Phi([\operatorname{fcone}(P_\tau, \boldsymbol{v})]) = f^{\boldsymbol{v}}_\tau(\boldsymbol{c})$, and by Remark 9

$$\Phi([\operatorname{tcone}(P_\tau, \boldsymbol{v})]) = e^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} f^{\boldsymbol{v}}_\tau(\boldsymbol{c}).$$

This last equation jointly with Brion's decomposition and Eq. (2.1.3) gives us for all τ:

$$\int_{P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{y} \rangle} \mathrm{d}\boldsymbol{y} = \sum_{v \in \operatorname{vert} P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} f^{\boldsymbol{v}}_\tau(\boldsymbol{c}), \qquad \text{where} \qquad f^{\boldsymbol{c}}_\tau \in \mathcal{M}(H_\tau). \tag{2.1.4}$$

The next step we need is to show that in the open intervals (τ_{k+1}, τ_k) the functions $f_\tau^{\boldsymbol{v}}$ remain constant with respect to τ. First observe that for $\tau \in (\tau_{k+1}, \tau_k)$, the vertices of P_τ come all from edges of P. This way, if a vertex $\boldsymbol{v}$ of some P_τ comes from the edge $[\boldsymbol{v}_i, \boldsymbol{v}_j]$ of P, then its variation with respect to τ is

$$\boldsymbol{v}_{i,j}(\tau) = \frac{\tau - \tau_j}{\tau_i - \tau_j}\boldsymbol{v}_i + \frac{\tau_i - \tau}{\tau_i - \tau_j}\boldsymbol{v}_j.$$

For $\tau \in (\tau_{k+1}, \tau_k)$, the vertices $\boldsymbol{v}_{i,j}(\tau)$ of P_τ come from specific edges $[\boldsymbol{v}_i, \boldsymbol{v}_j]$ of P.

The defining inequalities of P_τ are the same of P plus $\langle \boldsymbol{x}, \boldsymbol{u}\rangle = \tau$. Therefore, the active inequalities of P_τ on the vertex $\boldsymbol{v}_{i,j}(\tau)$ are the union of the active inequalities of $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$. Hence, by the description of the feasible cone with active inequalities (see Definition 1.3.5) we conclude that indeed the functions $f_\tau^{\boldsymbol{v}}$ remain constant with respect to τ in the open intervals (τ_{k+1}, τ_k). We will modify our notation accordingly: we will denote $f_\tau^{\boldsymbol{v}_{i,j}(\tau)}$ as $f_k^{i,j}$ when $\tau \in (\tau_{k+1}, \tau_k)$. As a convention, if the edge $[\boldsymbol{v}_i, \boldsymbol{v}_j]$ does not correspond to any vertex in τ or if said edge simply does not exist, we will say $f_k^{i,j} = 0$.

With this last observation and new notation, we can compute for the integrals in Eq. (2.1.2) as

$$\int_{\tau_{k+1}}^{\tau_k} \left(\int_{P_\tau} e^{\langle \boldsymbol{c}, \boldsymbol{y}\rangle} \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\tau = \sum_{i,j=1,\, i<j}^{m} f_k^{i,j} \int_{\tau_{k+1}}^{\tau_k} e^{\langle \boldsymbol{c}, \boldsymbol{v}_{i,j}(\tau)\rangle} \mathrm{d}\tau, \tag{2.1.5}$$

$$\int_{\tau_{k+1}}^{\tau_k} e^{\langle \boldsymbol{c}, \boldsymbol{v}_{i,j}(\tau)\rangle}\mathrm{d}\tau = e^{\left\langle \frac{-\tau_j \boldsymbol{v}_i + \tau_i \boldsymbol{v}_j}{\tau_i - \tau_j}, \boldsymbol{c}\right\rangle} \int_{\tau_{k+1}}^{\tau_k} e^{\frac{\langle \boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{c}\rangle}{\tau_i - \tau_j}\tau} \mathrm{d}\tau = e^{\left\langle \frac{-\tau_j \boldsymbol{v}_i + \tau_i \boldsymbol{v}_j}{\tau_i - \tau_j}, \boldsymbol{c}\right\rangle} \frac{\tau_i - \tau_j}{\langle \boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{c}\rangle} \left(e^{\frac{\langle \boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{c}\rangle}{\tau_i - \tau_j}\tau_k} - e^{\frac{\langle \boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{c}\rangle}{\tau_i - \tau_j}\tau_{k+1}} \right).$$

Since $f_k^{i,j}$ are rational functions of degree $-(d-1)$ and the integrals in Eq. (2.1.5) add only one degree in the denominator, we conclude that the integral $\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x}$ converges to a function in $\mathcal{M}(\mathbb{R}^N)$. We will denote this function by $\phi(P, \boldsymbol{c})$. Note that $\phi(P, \boldsymbol{c})$ is defined in all except for a finite number of values of $\boldsymbol{c}$, while $\int_P e^{\langle \boldsymbol{c}, \boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x}$ may diverge for values of $\boldsymbol{c} \notin \operatorname{int} K^\circ$.

(b) To prove linearity of Φ, we only need to prove it with indicators of polyhedra without lines, since they span the whole algebra of polyhedra (see Remark 8). So we want to prove that given a linear relation

$$\sum_{i \in I} \alpha_i [P_i] = 0, \tag{2.1.6}$$

where $\alpha_i \in \mathbb{R}$ and P_i are polyhedra without lines, we have

$$\sum_{i \in I} \alpha_i \Phi([P_i]) = 0. \tag{2.1.7}$$

Consider a decomposition

$$\left[\mathbb{R}^N\right] = \sum_{j \in J} \beta_j \left[Q_j\right],$$

where $\beta_j \in \mathbb{R}$ and Q_j are polyhedra without lines (such a decomposition exists on behalf of Remark 8). Then for every polyhedron P_i in Eq. (2.1.6) we have

$$[P_i] = [P_i] \cdot \left[\mathbb{R}^N\right] = \sum_{j \in J} \beta_j [P_i \cap Q_j]. \tag{2.1.8}$$

Now we can apply part (a) to each polyhedron $P_i \cap Q_j$ because they do not contain lines, so for all $\boldsymbol{c} \in \operatorname{int} K_{P_i \cap Q_j}$ we have $\phi(P_i \cap Q_j, \boldsymbol{c}) = \int_{P_i \cap Q_j} e^{\langle \boldsymbol{c}, \boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x}$, and $\phi(P_i \cap Q_j, \boldsymbol{c}) \in \mathcal{M}(\mathbb{R}^N)$. Since the recession cones satisfy $K_{P_i \cap Q_j} \subseteq K_{P_i}$ and

$$\int_{P_i} e^{\langle \boldsymbol{c},\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x} = \sum_{j\in J} \beta_j \int_{P_i\cap Q_j} e^{\langle \boldsymbol{c},\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x},$$

we have for all $\boldsymbol{c} \in \operatorname{int} K_{P_i}$

$$\phi(P_i, \boldsymbol{c}) = \sum_{j\in J} \beta_j \phi(P_i \cap Q_j, \boldsymbol{c}). \tag{2.1.9}$$

Equation (2.1.9) is an equation between two meromorphic functions that holds in a set with non-empty interior, so it holds on all their domain. By an analogous argument, if instead of Eq. (2.1.8) we consider for a given $j \in J$ the identity

$$0 = \sum_{i\in I} \alpha_i [P_i \cap Q_j],$$

we get

$$\sum_{i\in I} \alpha_i \phi(P_i \cap Q_j, \boldsymbol{c}) = 0 \tag{2.1.10}$$

for all $\boldsymbol{c} \in \operatorname{int} K^{\circ}_{Q_j}$, and thus for all $\boldsymbol{c} \in \mathbb{R}^N$.

Now combining Eqs. (2.1.9) and (2.1.10) we get what we wanted to prove (Eq. (2.1.7)):

$$\sum_{i\in I} \alpha_i \Phi([P_i]) = \sum_{i\in I} \alpha_i \left(\sum_{j\in J} \beta_j \Phi([P_i \cap Q_j]) \right) = \sum_{j\in J} \beta_j \left(\sum_{i\in I} \alpha_i \Phi([P_i \cap Q_j]) \right) = 0.$$

(2) Given a polyhedron without lines P, and a translation of it, $P + \boldsymbol{v}$, observe that the recession cones are the same, so by Remark 9 we have for all $\boldsymbol{c} \in K_P$

$$\int_{P+\boldsymbol{v}} e^{\langle \boldsymbol{c},\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x} = e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle} \int_{P} e^{\langle \boldsymbol{c},\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x}.$$

Since we have constructed Φ as extending these integrals, we have $\Phi([P + \boldsymbol{v}]) = e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle}\Phi([P])$ for all polyhedron P without lines and all vector $\boldsymbol{v}$. Moreover, the identity holds also for polyhedra with lines because they can be expressed as linear combinations of polyhedra with lines (see Remark 8) and we know that part (1) holds.

If P contains a line in the direction of $\boldsymbol{v} \neq \boldsymbol{0}$, then $P = P + \boldsymbol{v}$, but then for the valuations we have

$$\Phi([P]) = \Phi([P + \boldsymbol{v}]) = e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle}\Phi([P]).$$

We conclude the proof by noting that $e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle} \neq 1$ because $\boldsymbol{v} \neq \boldsymbol{0}$. □

Note that by the second statement of the theorem and Theorem 1.3.8, we can write

$$\Phi([P]) = \sum_{\boldsymbol{v}\in \operatorname{vert}(P)} \Phi([\operatorname{tcone}(P, \boldsymbol{v})]). \tag{2.1.11}$$

2.2 The counting valuation

In the spirit of Sections 1.2 and 2.1, we want to extend the concept of counting integer points in a polytope to the algebra of rational polyhedra. In particular, we want to extend the integer-point enumerator $\sigma_S(\boldsymbol{x})$ (Definition 1.2.1) to a valuation in the algebra of rational polytopes. Also, we want our valuation to assign values not in the integer or real numbers, but in a certain function space that we will define, similar to how we defined the exponential valuation, because our ultimate goal is to mix both valuations. We will follow mainly [1, Ch. 13] for this section.

Recall that we defined the integer-point enumerator of a set $S \subseteq \mathbb{R}^N$ as

$$\sigma_S(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^N} \boldsymbol{x}^{\boldsymbol{m}}.$$

Lemma 2.2.1. *Let $K \subseteq \mathbb{R}^N$ be a pointed rational cone. Then $K = \operatorname{co}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ where $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathbb{Z}^N$. Define the set $W_K = \{\boldsymbol{x} \in \mathbb{C}^N : \|\boldsymbol{x}^{\boldsymbol{w}_i}\| < 1 \quad \text{for} \quad i = 1, \ldots, n\}$. Then the series*

$$\sum_{\boldsymbol{m} \in K \cap \mathbb{Z}^N} \boldsymbol{x}^{\boldsymbol{m}}$$

converges for every $\boldsymbol{x} \in W_K$ and uniformly on compact subsets of W_K to a function of the type

$$f(K, \boldsymbol{x}) = \sum_{i \in I} \varepsilon_i \frac{\boldsymbol{x}^{\boldsymbol{v}_i}}{(1 - \boldsymbol{x}^{\boldsymbol{u}_{i,1}}) \cdot \cdots \cdot (1 - \boldsymbol{x}^{\boldsymbol{u}_{i,N}})},$$

where $\varepsilon_i = \pm 1$ and $\boldsymbol{v}_i, \boldsymbol{u}_{i,j} \in \mathbb{Z}^N$ for all i and j.

PROOF. (Sketch only)
First we prove the case $K = \mathbb{R}^N_{\geq 0}$. In this case using the geometric sum formula, when $\boldsymbol{x} \in W_K$ we have

$$\sum_{\boldsymbol{m} \in K \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{m}} = \sum_{m_1, \ldots, m_N \geq 0} x_1^{m_1} \cdots x_N^{m_N} = \frac{1}{1 - x_1} \cdots \frac{1}{1 - x_N}.$$

Then, if K is a simplicial cone $K = \operatorname{co}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N)$, for a given $\boldsymbol{m} \in \mathbb{Z}^d$, we have $\boldsymbol{m} = \sum_{i=1}^N \beta_i \boldsymbol{w}_i$ for some $\beta_i \in \mathbb{R}_{\geq 0}$. We can decompose it into $\boldsymbol{m} = \boldsymbol{n} + \boldsymbol{l}$ by

$$\boldsymbol{l} = \sum_{i=1}^N \lfloor \beta_i \rfloor \boldsymbol{w}_i, \qquad \text{and} \qquad \boldsymbol{l} = \sum_{i=1}^N \{\beta_i\} \boldsymbol{w}_i.$$

Thus getting

$$\sum_{\boldsymbol{m} \in K \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{n}} = \left(\sum_{\boldsymbol{n} \in \Pi \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{m}} \right) \left(\sum_{l_1, \ldots, l_N \geq 0} \boldsymbol{x}^{l_1 \boldsymbol{w}_1 + \cdots + l_N \boldsymbol{w}_N} \right) = \left(\sum_{\boldsymbol{n} \in \Pi \cap \mathbb{Z}^d} \boldsymbol{x}^{\boldsymbol{n}} \right) \prod_{i=1}^N \frac{1}{1 - \boldsymbol{x}^{\boldsymbol{w}_i}}. \tag{2.2.1}$$

Now, for a general (not necessarily simplicial) cone, we argue as in Lemma 2.1.1: triangulate K, apply the simplicial case to each element and add all together. □

In order to use the same notation as in Section 2.1, we will consider the change of variables $\boldsymbol{x} = e^{\boldsymbol{c}}$. With this notation, the condition $\boldsymbol{x} \in W_K$ of Lemma 2.2.1 translates to $\boldsymbol{c} \in \operatorname{rel\,int} K^\circ$.

Remark 10. In the spirit of Remark 9, if we want to allow this same result for a shifted cone $K + \boldsymbol{v}$, we get

$$\sum_{\boldsymbol{u}\in(K+\boldsymbol{v})\cap\mathbb{Z}^N} e^{\langle \boldsymbol{c},\boldsymbol{u}\rangle} = \sum_{\boldsymbol{m}\in K\cap\mathbb{Z}^N} e^{\langle \boldsymbol{c},\boldsymbol{m}+\boldsymbol{v}\rangle} = e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle} \sum_{\boldsymbol{m}\in K\cap\mathbb{Z}^N} e^{\langle \boldsymbol{c},\boldsymbol{m}\rangle}.$$

This observation, along with the previous lemma, justify the choice of the following space of functions, for a fixed lattice Λ in $\mathbb{R}^N$:

$$\mathcal{N}(\mathbb{R}^N) = \operatorname{span}\left\{ f(\boldsymbol{c}) = e^{\langle \boldsymbol{c},\boldsymbol{v}\rangle} \prod_{i=1}^{N} \frac{1}{1-e^{\langle \boldsymbol{c},\boldsymbol{u}_i\rangle}} \ :\ \boldsymbol{v},\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N \in \Lambda \right\}.$$

The main result of this section is the following:

Theorem 2.2.2 (Counting valuation). *There exists a valuation $\mathcal{F} : \mathcal{P}(\mathbb{Q}^N) \to \mathcal{N}(\mathbb{R}^N)$ such that*

1. *For any polyhedron P without lines, and for all $\boldsymbol{c} \in \operatorname{rel\,int} K^\circ$, where K is its recession cone, the series $\sum_{\boldsymbol{u}\in P\cap\Lambda} e^{\langle \boldsymbol{c},\boldsymbol{u}\rangle}$ converges absolutely and uniformly on compact subsets of $\operatorname{int} K^\circ$ to $\mathcal{F}([P])$.*
2. *For a polyhedron with lines, $\mathcal{F}([P]) = 0$.*

PROOF. (Sketch only)
The proof is, as usual, by induction on N. For $N = 0$ the result is clear. Assume $N > 0$.

First we have to prove that given a polyhedron P without lines and, K its recession cone and $\boldsymbol{c} \in \operatorname{int} K^\circ$, the series

$$\sum_{\boldsymbol{u}\in P\cap\Lambda} e^{\langle \boldsymbol{c},\boldsymbol{u}\rangle}$$

converges to a function in $\mathcal{N}(\mathbb{R}^N)$. Since P contains no lines, K is pointed and thus $\operatorname{int} K^\circ$ is a non-empty open set. Consider the dual lattice to Λ, namely

$$\Lambda^* = \{\boldsymbol{x} \ :\ \langle \boldsymbol{x},\boldsymbol{y}\rangle \in \mathbb{Z} \quad \text{for all} \quad \boldsymbol{y} \in \Lambda\}.$$

We can choose $\boldsymbol{w} \in -\operatorname{int} K^\circ \cap \Lambda^*$. We choose it primitive, that is, such that $\boldsymbol{w}$ is a basis of $\Lambda^* \cap \operatorname{span}(\boldsymbol{w})$. We can extend $\boldsymbol{w}$ to a basis of Λ^*. Let $\{\boldsymbol{w}_1 = \boldsymbol{w}, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_N\}$ be such basis. Then consider the basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ defined by

$$\langle \boldsymbol{u}_i, \boldsymbol{w}_j\rangle = \begin{cases} 1 & \text{if} \quad i + j = N + 1, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Now we can define the hyperplanes

$$H_k = \{\boldsymbol{x} \in \mathbb{R}^N \ :\ \langle \boldsymbol{x}, \boldsymbol{w}\rangle = k\} \qquad \text{for } k \in \mathbb{Z},$$

and a lattice Λ_k in each hyperplane

$$\Lambda_k = \Lambda \cap H_k = \left\{ k\boldsymbol{u}_d + \sum_{i=1}^{N-1} m_i \boldsymbol{u}_i \ :\ m_i \in \mathbb{Z} \right\}.$$

As we argued in the proof of Theorem 2.1.2, the restricted polyhedra $P_k = P \cap H_k$ are bounded and there exists k_0 such that $P_k = \emptyset$ for $k < k_0$. Thus we can write

$$\sum_{\boldsymbol{u} \in P \cap \Lambda} e^{\langle \boldsymbol{c}, \boldsymbol{u} \rangle} = \sum_{k=k_0}^{\infty} \left(\sum_{\boldsymbol{u} \in P_k \cap \Lambda} e^{\langle \boldsymbol{c}, \boldsymbol{u} \rangle} \right) \tag{2.2.2}$$

If P is bounded the result is clear, so we may assume P is unbounded. As in the proof of Theorem 2.1.2, in the previous sum, for sufficiently large k, all the vertices of P_k correspond to intersections with unbounded edges of P. This is the part that may diverge. To prove that it indeed converges, first we write the vertices of P_k for sufficiently large k as

$$\boldsymbol{v}_i(k) = \boldsymbol{a}_i + k\boldsymbol{b}_i, \qquad \text{for some} \quad \boldsymbol{a}_i, \boldsymbol{b}_i \quad \Lambda\text{-rational vectors.}$$

Being Λ-rational implies that there exists $M \in \mathbb{Z}$ such that $M\boldsymbol{b}_i \in \Lambda$ for all i, making $\boldsymbol{v}_i(k+M)$ a lattice translation of $\boldsymbol{v}_i(k)$. And thus, for sufficiently large k, we can write the sum in Eq. (2.2.2) as

$$\sum_{j=0}^{\infty} \left(\sum_{\boldsymbol{u} \in P_{k+jM} \cap \Lambda} e^{\langle \boldsymbol{c}, \boldsymbol{u} \rangle} \right). \tag{2.2.3}$$

Now applying the induction hypothesis in each polytope P_{k+jM} and Brion's theorem we have

$$\sum_{\boldsymbol{u} \in P_{k+jM} \cap \Lambda} e^{\langle \boldsymbol{c}, \boldsymbol{u} \rangle} = \sum_{i \in I} \mathcal{F}([\operatorname{tcone}(P_{k+jM}, \boldsymbol{v}_i(k+jM))]).$$

By the translation property (see Remark 10) we have

$$\mathcal{F}([\operatorname{tcone}(P_{k+jM}, \boldsymbol{v}_i(k+jM))]) = e^{\langle \boldsymbol{c}, j\boldsymbol{z}_i \rangle} F_i(\boldsymbol{c}), \quad \text{for some} \quad \boldsymbol{z}_i \in \Lambda, \qquad \text{where} \quad F_i = \mathcal{F}([\operatorname{tcone}(P_k, \boldsymbol{v}_i(k))]),$$

so finally we can write Eq. (2.2.3) as

$$\sum_{j=0}^{\infty} \left(\sum_{i \in I} e^{\langle \boldsymbol{c}, j\boldsymbol{z}_i \rangle} F_i(\boldsymbol{c}) \right) = \left(\sum_{i \in I} F_i \sum_{j=0}^{\infty} e^{\langle \boldsymbol{c}, j\boldsymbol{z}_i \rangle} \right) = \sum_{i \in I} \frac{F_i(\boldsymbol{c})}{1 - e^{\langle \boldsymbol{c}, \boldsymbol{z}_i \rangle}}.$$

As in the proof of Theorem 2.1.2, we now need to prove that the correspondence that we have defined respects linear relations. The proof is analogous as the one in Theorem 2.1.2.

Finally, to prove part (2) of the statement, we look again at the proof of Theorem 2.1.2, and argue analogously to get that if P contains a line, then for some vector $\boldsymbol{u} \in \Lambda \setminus \{0\}$ we have

$$\mathcal{F}([P]) = e^{\langle \boldsymbol{c}, \boldsymbol{u} \rangle} \mathcal{F}([P]),$$

and thus $\mathcal{F}([P]) = 0$. □

Definition 2.2.1 (Unimodular cone). A cone $K \subseteq \mathbb{R}^N$ is *unimodular* with respect to a lattice Λ if it can be written as $K = \operatorname{co}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N)$ with $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ being a basis of Λ.

An equivalent definition of unimodular cone is that the fundamental parallelepiped of K associated to $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$, that is

$$\Pi = \left\{ \sum_{i=1}^{N} \alpha_i \boldsymbol{u}_i \; : \; 0 \leq \alpha_i < 1 \right\},$$

contains only one point of Λ.

Remark 11. We want to point out two important properties of this cones.

1. If K is unimodular, then $\mathcal{F}(K)$ has a simple expression as in Eq. (2.2.1), because the only point in $\Pi \cap \Lambda$ is the origin with $\boldsymbol{n} = \mathbf{0}$.
2. We can always decompose a lattice cone into unimodular cones. This is a non-trivial result that we will need in Section 2.4. Chapter 16 of [1] is dedicated to it.

2.3 Generalizing Ehrhart's theorem

We do a little stop in our way on the land of valuations to present our second proof of Ehrhart's theorem. In fact, we will present a more general result, that specializes to Ehrhart's in a concrete case.

When we make dilations of a polytope, the vertices move, and so do the tangent cones, but the cones of feasible directions stay the same.

So now we ask ourselves what happens to the number of integer points in a polytope when we move the vertices in a way that the feasible cones are not modified. It turns out that the behavior is still polynomial, but as we can expect, a more complex kind of polynomial. We will follow mainly [1, Ch. 18].

We will also prove the result for open polytopes, using the results deduced in Section 1.3.

Theorem 2.3.1. *Let $\{P_\alpha : \alpha \in \mathbb{Z}\}$ be a family of integer N-dimensional polytopes with n vertices $\boldsymbol{v}_1(\alpha), \ldots, \boldsymbol{v}_n(\alpha) \in \mathbb{R}^N$, such that, for each family of vertices $\{\boldsymbol{v}_i(\alpha)\}_\alpha$, the cone of feasible directions* $\operatorname{fcone}(P_\alpha, \boldsymbol{v}_i(\alpha))$ *does not depend on α. Then there exists a polynomial $p : \mathbb{R}^{nN} \to \mathbb{R}$ such that*

1. $\left|P_\alpha \cap \mathbb{Z}^N\right| = p(\boldsymbol{v}_1(\alpha), \ldots, \boldsymbol{v}_n(\alpha))$.
2. $\left|\operatorname{int} P_\alpha \cap \mathbb{Z}^N\right| = (-1)^N p(-\boldsymbol{v}_1(\alpha), \ldots, -\boldsymbol{v}_n(\alpha))$.
3. $p(0, \ldots, 0) = 1$.

PROOF.

1. Recall that by Theorem 1.3.8, we have that, modulo polyhedra with lines, $[P_\alpha] = \sum_{i=1}^n [\operatorname{tcone}(P_\alpha, \boldsymbol{v}_i(\alpha))]$, so by the definition of the counting valuation (see Theorem 2.2.2), $\mathcal{F}([P_\alpha]) = \sum_{i=1}^n \mathcal{F}([\operatorname{tcone}(P_\alpha, \boldsymbol{v}_i(\alpha))])$. We know that the cone of feasible directions is not dependent on α, so we have the following decomposition
$$\operatorname{tcone}(P_\alpha, \boldsymbol{v}_i(\alpha)) = \boldsymbol{v}_i(\alpha) + K_i, \tag{2.3.1}$$

where K_i is the cone of feasible directions. This translates into the counting valuation as

$$\mathcal{F}([P_\alpha]) = \sum_{i=1}^{n} e^{\langle \boldsymbol{c}, \boldsymbol{v}_i(\alpha)\rangle} F(K_i, \boldsymbol{c}). \tag{2.3.2}$$

To actually *count* with the counting valuation, we have to evaluate at $\boldsymbol{c} = 0$, so to do this, let us fix a generic $\boldsymbol{c} \in \mathbb{R}^N$ and let us define for every vertex, the function $f_i : \mathbb{R} \to \mathbb{R}$ as:

$$f_i(\tau) = F(K_i, \tau\boldsymbol{c}), \quad \tau \in \mathbb{R}.$$

By the space to which the counting valuation maps (see Theorem 2.2.2), each f_i admits a Laurent expansion in a neighborhood of $\tau = 0$ of the form

$$f_i(\tau) = \sum_{j=-N}^{\infty} a_{i,j}\tau^j,$$

for some coefficients $a_{i,j} \in \mathbb{R}$. We also have an expansion for the exponential around $\tau = 0$:

$$e^{\langle \tau\boldsymbol{c}, \boldsymbol{v}_i(\alpha)\rangle} = \sum_{j=0}^{\infty} \frac{\langle \boldsymbol{c}, \boldsymbol{v}_i(\alpha)\rangle^j}{j!}\tau^j.$$

With all this setup, number of integer points we want to count is the constant term in τ of

$$\sum_{i=1}^{n} \left(\sum_{j=0}^{\infty} \frac{\langle \boldsymbol{c}, \boldsymbol{v}_i(\alpha)\rangle^j}{j!}\tau^j \right) \left(\sum_{j=-N}^{\infty} a_{i,j}\tau^j \right),$$

which turns out to be a polynomial on $\boldsymbol{v}_i(\alpha)$'s:

$$\left| P_\alpha \cap \mathbb{Z}^N \right| = \sum_{i=1}^{n}\sum_{j=1}^{N} \frac{\langle \boldsymbol{c}, v_i(\alpha)\rangle^j}{j!} a_{i,-j}. \tag{2.3.3}$$

2. By Corollary 1.3.12 we know that, modulo polyhedra with lines,

$$[\operatorname{rel\,int} P_\alpha] \equiv \sum_{i=1}^{n} [\operatorname{rel\,int} \operatorname{tcone}(P_\alpha, \boldsymbol{v}_i(\alpha))].$$

Using Eq. (2.3.1) we get an analogous result as Eq. (2.3.2). Now we apply that $[\operatorname{rel\,int} K] \equiv (-1)^d[-K]$ modulo polyhedra with lines (see Lemma 1.3.11) and get that

$$F(\operatorname{rel\,int} P_\alpha, \boldsymbol{c}) = (-1)^N \sum_{i=1}^{n} e^{\langle \boldsymbol{c}, \boldsymbol{v}_i(\alpha)} F(K_i, -\boldsymbol{c}).$$

Using an analogous argument as in (1) we get the desired result.

3. Recall Brion's theorem for the cones of feasible directions:

$$\sum_{i=1}^{n} [K_i] \equiv [0] \qquad \text{modulo polyhedra with lines.}$$

This together with Eqs. (1.1.8) and (2.3.1) implies that $\sum_{i=1}^{n} F(K_i, \boldsymbol{c}) = 1$, therefore for the coefficients $a_{i,j}$ for all $j = 0$ we have $a_{1,0} + \cdots + a_{n,0} = 1$. Looking at Eq. (2.3.3), we check that the constant term corresponds precisely to $a_{1,0} + \cdots + a_{n,0}$.

□

The situation in which we study dilations of a polytope fits clearly in the hypothesis of the previous theorem. Take an original polytope $P = \mathrm{conv}(v_1, \ldots, v_n)$, and move the vertices as $\boldsymbol{v}_i(\alpha) = \alpha, v_i$, for $\alpha \in \mathbb{N}$. In this case, Ehrhart polynomial is, as in Eq. (2.3.3)

$$p(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{N} \frac{\langle \boldsymbol{c}, \boldsymbol{v}_i \rangle^j}{j!} a_{i,-j} \alpha^j.$$

With this proof, however, it seems much more difficult to prove that this polynomial has indeed degree N, without having to compute the coefficients $a_{i,-N}$.

2.4 Berline-Vergne valuation

Our goal in this section is to use the exponential (Φ) and counting ($\mathcal{F}$) valuations that we have previously defined to construct a valuation that counts the number of integer points in a polyhedron, somehow decomposing the contribution of each face.

This result was first proved by N. Berline and M. Vergne in [3]. We will follow the exposition on [1, Ch. 19].

Notation. Given a linear subspace $L \subseteq \mathbb{R}^N$, we will denote $\mathbb{R}^N/L$ to the orthogonal complement of L in $\mathbb{R}^N$. In general, given a set $S \subseteq \mathbb{R}^N$ and a subspace $L \subseteq \mathbb{R}^N$, we will denote S/L to the orthogonal projection of S into $\mathbb{R}^N/L$.

Note that, in particular, if Λ is a lattice and L a subspace, $\Lambda' = \Lambda/L$ is also a lattice, typically finer than $\Lambda \cap (L/\mathbb{R}^N)$. See Fig. 2.1.

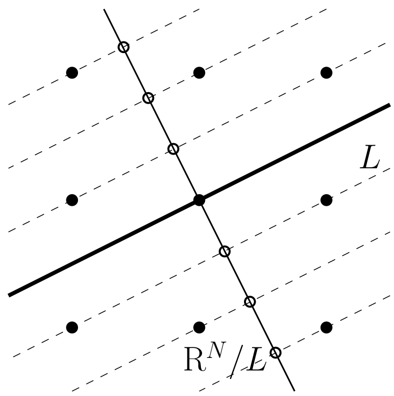


Figure 2.1: A lattice subspace L and the corresponding lattice Λ/L compared to the coarser lattice $\Lambda \cap (L/\mathbb{R}^N)$.

We want to define ψ such that the following identity holds for a given rational polytope P:

$$\mathcal{F}([P]) = \sum_{L \subseteq \mathbb{R}^N \text{ : lattice subspace}} \Psi([P/L]) \Phi_L([P \cap L]),$$

where $\mathcal{F}$ is the counting valuation and Φ_L is the exponential valuation with a measure $\mathrm{d}x_L$ such that $\det(L \cap \Lambda) = 1$. This way we may be able to decompose the sum into subspaces L of each dimension. Since $\mathcal{F}$ encodes the number of integer points and Φ encodes the volume, that behaves as $t^{\dim L}$ when dilating the polytope P, this seems a reasonable approach.

We will be able to define Ψ recursively from a formula similar to the one we have just written, because there is only one zero-dimensional subspace, and we will define that $\Psi([A]) = 1$ when $\dim A = 0$. This way, we have it for dimension zero, and we will be able to define for dimension n using only the definition of smaller dimensions. It will become more clear in the proof of Theorem 2.4.2.

We will see that we cannot construct such valuation Ψ, but we will be able to construct a valuation in shifted cones *around* each vertex of P, and the use Brion's theorem to get a formula for the whole polytope.

Definition 2.4.1 (Algebra of $\boldsymbol{v}$-shifted rational cones). Given a rational point $\boldsymbol{v} \in \mathbb{Q}^N$, we define the *algebra of $\boldsymbol{v}$-shifted rational cones* as

$$\mathcal{A}_{\boldsymbol{v}} = \operatorname{span}\left\{[K + \boldsymbol{v}] \ : \ K \quad \text{rational cone}\right\}.$$

We will prove that there exists a valuation in $\mathcal{A}_{\boldsymbol{v}}$ for any $\boldsymbol{v}$. First, we will prove that only the most essential identities need to be checked.

Lemma 2.4.1. *Let $\boldsymbol{v} \in \mathbb{Q}^N$, W be a vector space. Let $\mathcal{K} + \boldsymbol{v} = \{K + \boldsymbol{v} \subseteq \mathbb{R}^N : K \quad \text{rational cone}\}$ and $\theta : \mathcal{K} \to W$. Given $H \subseteq \mathbb{R}^N$ be a hyperplane containing $\boldsymbol{v}$, H_+, H_- the corresponding closed half-spaces and define $A_+ = A \cap H_+$, $A_- = A \cap H_-$, $A_0 = A \cap H$. If the identity*

$$\theta(A) = \theta(A_+) + \theta(A_-) - \theta(A_0) \tag{2.4.1}$$

holds for every hyperplane H, then there exists a valuation $\Theta : \mathcal{A}_v \to W$ that extends θ, that is, $\Theta([K + \boldsymbol{v}]) = \theta(K + \boldsymbol{v})$ for all $K + \boldsymbol{v} \in \mathcal{K} + \boldsymbol{v}$.

PROOF. (Sketch only)
Without loss of generality, we may assume that $\boldsymbol{v} = \mathbf{0}$. We have to prove that given a collection of cones $K_i \in \mathcal{K}$ and numbers $\alpha_i \in \mathbb{R}$ satisfying $\sum_{i \in I} \alpha_i [K_i] = 0$, we have that

$$\sum_{i \in I} \alpha_i \theta(K_i) = 0. \tag{2.4.2}$$

For every cone K_i, consider the supporting hyperplanes of its facets. Let J_i be a set of indices that indexes the facets of K_i, and $H_{i,j}$ be the supporting hyperplane of the corresponding facet of K_i. Let $\mathcal{H}_i$ be the set containing such hyperplanes, so

$$\mathcal{H}_i = \{H_{ij} \ : \ j \in J_i\}, \qquad \text{and} \qquad \mathcal{H} = \bigcup_{i \in I} \mathcal{H}_i.$$

Considering all hyperplanes in $\mathcal{H}$, they decompose $\mathbb{R}^N$ into finitely many polyhedral cones. Let $\mathcal{C} = \{C_j \ : \ j \in J\}$ be the set of all such cones and their non-empty faces.

First we observe that arguing about the dimensions of the different cones, we have that the indicators $\{[C_j] \ : \ j \in J\}$ are linearly independent.

Next, observe that we can write, for some coefficients $\beta_{i,j}$:

$$[K_i] = \sum_{j \in J} \beta_{i,j}[C_j], \quad \text{and thus} \quad \theta(K_i) = \sum_{j \in J} \beta_{i,j}\theta(C_j) \tag{2.4.3}$$

by iterating the identities

$$[K] = [K \cap H_+] + [K \cap H_-] - [K \cap H], \quad \text{and} \quad \theta(K) = \theta(K \cap H_+) + \theta(K \cap H_-) - \theta(K \cap H)$$

for hyperplanes $H \in \mathcal{H}$ and cones $K = K_i$.

Using the fat that C_j's are independent and Eq. (2.4.3) we get that for all $j \in J$, the sum $\sum_{i \in I} \alpha_i \beta_{i,j} = 0$, and we apply it to prove Eq. (2.4.2). □

We will define Ψ locally at every point $\boldsymbol{v}$. Since we will be using the exponential and counting valuation, we need to pick a vector space that contains the functions in $\mathcal{N}(\mathbb{R}^N)$ and $\mathcal{M}(\mathbb{R}^N)$ in Sections 2.1 and 2.2. The easy choice is the space of meromorphic functions, that we will denote $\mathcal{Q}(\mathbb{R}^N)$.

Theorem 2.4.2 (Berline-Vergne, 2007). *Let $\boldsymbol{v} \in \mathbb{Q}^N$, $\Lambda \in \mathbb{R}^N$ a lattice there exists a valuation*

$$\Psi : \mathcal{A}_{\boldsymbol{v}} \to \mathcal{Q}(\mathbb{R}^N)$$

that satisfies, for every shifted rational cone $A = K + \boldsymbol{v}$, the following properties:

1. *The formula*
$$\mathcal{F}([A]) = \sum_{L} \Psi([A/L])\Phi_L([A \cap (L + \boldsymbol{v})]) \tag{2.4.4}$$
is satisfied, where the sum is taken over all subspaces $L \subseteq \mathbb{R}^N$ parallel to faces of A.
2. *If A contains a line, then $\Psi([A]) = 0$.*
3. *The function $\Psi([A]) \in \mathcal{Q}(\mathbb{R}^N)$ is analytic at $\boldsymbol{c} = 0$.*

PROOF. (Sketch only)
The proof is by induction on N. For $N = 0$ the result is clear, so we may assume $N > 0$. Since there is only one subspace of dimension zero, we can isolate $\Psi([A])$ in Eq. (2.4.4):

$$\Psi([A]) = e^{-\langle \boldsymbol{c}, \boldsymbol{v} \rangle}\mathcal{F}([A]) - e^{-\langle \boldsymbol{c}, \boldsymbol{v} \rangle} \sum_{L \,:\, \dim L > 0} \Psi([A/L])\Phi_L([A \cap (L + \boldsymbol{v})]). \tag{2.4.5}$$

We observe that if K contains a line, $\Psi([A]) = 0$, because both the counting and the exponential valuation vanish when K contains a line. This proves (2).

Next, we observe that we can allow the sum in Eq. (2.4.4) to range over all lattice subspaces of L of positive dimension. This is because

- If L lies in a face F of K with $\dim F > L$, then A/L is a translation of a lower-dimensional cone K/L containing a line, thus $\Psi([A/L]) = 0$.

- If $\dim(L \cap K) < \dim L$, then by dimensions $\Phi_L([A \cap (L + \boldsymbol{v})]) = 0$.

In conclusion, the only contributions correspond to spaces parallel to faces of P. Also, note that in such case, if L is parallel to a face F, we have $A/L = \operatorname{tcone}(A, F)/L$.

Next we want to prove that Ψ as defined by Eq. (2.4.4) satisfies the hypothesis of Lemma 2.4.1. Let H be a hyperplane through $\boldsymbol{v}$ and H_+, H_-, A_+, A_- and A_0 as in Lemma 2.4.1. We want to prove that

$$\Psi([A]) = \Psi([A_+]) + \Psi(A_-) - \Psi([A_0]) \tag{2.4.6}$$

We already know that

$$\mathcal{F}([A]) = \mathcal{F}([A_+]) + \mathcal{F}(A_-) - \mathcal{F}([A_0]) \qquad \text{and}$$

$$\Phi_L([A \cap (L + \boldsymbol{v})]) = \Phi_L([A_+ \cap (L + \boldsymbol{v})]) + \Phi_L(A_- \cap (L + \boldsymbol{v})) - \Phi_L([A_0 \cap (L + \boldsymbol{v})])$$

because $\mathcal{F}$ and Φ_L are valuations. Also, since projections preserve linear relations among indicators of polyhedra, we have

$$[A/L] = [A_+/L] + [A_-/L] - [A_0/L].$$

We can thus apply the induction hypothesis to get

$$\Psi([A/L]) = \Psi([A_+/L]) + \Psi(A_-/L) - \Psi([A_0/L]). \tag{2.4.7}$$

By Eq. (2.4.4), we need to check that, for every subspace L, the term

$$\Xi(A, L) = \Psi([A/L])\Phi_L([A \cap (L + \boldsymbol{v})])$$

satisfies the relation

$$\Xi(A, L) = \Xi(A_+, L) + \Xi(A_-, L) - \Xi(A_0, L). \tag{2.4.8}$$

This is verified on a case-by-case basis. For a non-empty face F of A, either

(i) F lies in H, or

(ii) F lies in one of H_+ or H_- but not in both, or

(iii) H passes through an interior point of F and is transversal to the span of F.

We note that every face A_+ or A_- is either the intersection of a face of A with H_+ (or H_-), or the intersecton of a face of A with H. Also, every face of A_0 is the intersection of a face of A with H. Thus, to define $\Psi([A])$, $\Psi([A_+])$, $\Psi([A_-])$ and $\Psi([A_0])$ we can reduce it to the following five cases:

(a) Let L be a subspace parallel to F such that $F \subset H$. Then

$$A \cap (L + \boldsymbol{v}) = A_+ \cap (L + \boldsymbol{v}) = A_- \cap (L + \boldsymbol{v}) = A_0 \cap (L + \boldsymbol{v}),$$

and by Eq. (2.4.7), Eq. (2.4.6) holds.

(b) Let L be a subspace parallel to F such that F lies in H_+ but not in H_-. Then $A/L = A_+/L$ and $\dim(A_- \cap (L + \boldsymbol{v})), \dim(A_0 \cap (L + \boldsymbol{v})) < \dim L$, thus we have

$$\Xi(A, L) = \Xi(A_+, L), \qquad \text{and} \qquad \Xi(A_-, L) = \Xi(A_0, L) = 0,$$

so clearly Eq. (2.4.8) holds.

(c) Let L be a subspace parallel to F such that F lies in H_- but not in H_+. Then by an analogous argument as in (b) Eq. (2.4.8) holds.

(d) Let L be a subspace parallel to F such that H passes through an interior point of F and intersects the span of F transversally. Then $A/L = A_+/L = A_-/L$ and $\dim(A_0 \cap (L + \boldsymbol{v})) < \dim L$. Therefore $\Psi([A/L]) = \Psi([A_+/L]) = \Psi([A_-/L])$, and $\Phi_L([A_0 \cap (L+\boldsymbol{v})]) = 0$, so Eq. (2.4.8) holds.

(e) Let L be a subspace parallel to a face $G = F \cap H$ of A_0 where F is a face of A, H passes through an interior point of F and intersects the affine span of F transversally. Then

$$A \cap (L + \boldsymbol{v}) = A_+ \cap (L + \boldsymbol{v}) = A_- \cap (L + \boldsymbol{v}) = A_0 \cap (L + \boldsymbol{v}),$$

and therefore

$$\Phi_L([A \cap (L + \boldsymbol{v})]) = \Phi_L([A_+ \cap (L + \boldsymbol{v})]) = \Phi_L([A_- \cap (L + \boldsymbol{v})]) = \Phi_L([A_0 \cap (L + \boldsymbol{v})]),$$

concluding that Eq. (2.4.8) holds.

This concludes part (1) of the proof.

To prove part (3), we do it again by induction on N. By part (2) of Remark 11, it suffices to prove that $\psi(A, \boldsymbol{c}) = \Psi([A])$ is analytic at $\boldsymbol{c} = \boldsymbol{0}$ for A a full dimensional unimodular cone. Hence, we may suppose that $A = K + \boldsymbol{v}$, with $K = \operatorname{co}(\boldsymbol{u}_1, \ldots, u_N)$ and $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ constitute a basis of Λ. Since the cone is unimodular (recall part (1) of Remark 11) we can write

$$\mathcal{F}([A]) = e^{\langle \boldsymbol{c}, \boldsymbol{w} \rangle} \prod_{i=1}^{N} \frac{1}{1 - e^{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}},$$

where $\boldsymbol{w} \in \Lambda$ is defined as

$$\boldsymbol{v} = \sum_{i=1}^{N} \alpha_i \boldsymbol{u}_i, \quad \text{then} \qquad \boldsymbol{w} = \sum_{i=1}^{N} \lceil \alpha_i \rceil \boldsymbol{u}_i.$$

Also, for L parallel to faces of K such that

$$L = L_I = \operatorname{span}(\boldsymbol{u}_i \ : \ i \in I) \qquad \text{where} \qquad I \subset \{1, \ldots, N\},$$

by Lemma 2.1.1 we have

$$\Phi_L([A \cap (L + \boldsymbol{v})]) = e^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} \prod_{i \in I} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}.$$

By the induction hypothesis, all functions $\Psi([A/L])$ are analytic at $\boldsymbol{c} = 0$. Therefore, the only poles that $\Psi([A])$ may have are located in the hyperplanes $\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle = 0$. So we have just to prove that for all i, the function $\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle \Psi([A])$ is identically zero on the hyperplane $\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle = 0$.

Note that by Taylor expansion, we have

$$\frac{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}{1 - e^{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}} = -1, \qquad \text{on the hyperplane} \quad \langle \boldsymbol{c}, \boldsymbol{u}_i \rangle = 0. \tag{2.4.9}$$

Therefore, on the hyperplane $\langle \boldsymbol{c}, \boldsymbol{u}_N \rangle$ we have

$$\langle \boldsymbol{c}, \boldsymbol{u}_N \rangle \Psi([A]) = -e^{\langle \boldsymbol{c}, \boldsymbol{w} - \boldsymbol{v}} \prod_{i=1}^{N-1} \frac{1}{1 - e^{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}} + \sum_{I \subset [N],\, I \ni N} \Psi([A/L_I]) \prod_{i \in I \setminus \{N\}} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle}. \tag{2.4.10}$$

Now we want to apply the induction hypothesis. Therefore we consider $A' = A/\operatorname{span}(\boldsymbol{u}_N)$, that is $A' = \boldsymbol{v}' + \operatorname{co}(\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_{N-1})$, where $\boldsymbol{v}' = \boldsymbol{v}/\operatorname{span}(\boldsymbol{u}_N)$ and $\boldsymbol{u}'_i = \boldsymbol{u}_i/\operatorname{span}(\boldsymbol{u}_N)$. Let $K' = \operatorname{co}(\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_{N-1})$. It is a $(d-1)$-dimensional cone with respect to the lattice $\Lambda' = \Lambda/\operatorname{span}(\boldsymbol{u}_N)$, generated by $\{\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_{N-1}\}$. We have $\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle = \langle \boldsymbol{c}, \boldsymbol{u}'_i \rangle$, so with $\boldsymbol{w}' = \boldsymbol{w}/\operatorname{span}(\boldsymbol{u}_N)$ we can rewrite Eq. (2.4.10) as

$$\Psi([A']) - e^{\langle \boldsymbol{c}, \boldsymbol{w}' - \boldsymbol{v}'} \prod_{i=1}^{N-1} \frac{1}{1 - e^{\langle \boldsymbol{c}, \boldsymbol{u}'_i \rangle}} + \sum_{I \subset [N-1],\, I \neq \emptyset} \Psi([A'/L_I]) \prod_{i \in I} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}'_i \rangle},$$

which in terms of the original valuations is identically zero, since

$$\Psi([A']) - e^{\langle \boldsymbol{c}, \boldsymbol{v}' \rangle} \mathcal{F}([A']) + e^{-\langle \boldsymbol{c}, \boldsymbol{v}' \rangle} \sum_{L \,:\, \dim L > 0} \Psi([A'/L]) \Phi_L([A' \cap (L + \boldsymbol{v})]) = 0.$$

□

2.5 Ehrhart's theorem: third proof

In this section we present our third and last proof of Ehrhart's theorem (Theorem 1.2.1), using the Berline-Vergne valuation Ψ. In fact, we have already done all the work, we just need to interpret Theorem 2.4.2 in the correct way.

Since the valuation Ψ is analytic at $\boldsymbol{c} = \boldsymbol{0}$, we can evaluate at $\boldsymbol{c} = 0$. Note that for a given face F in $L = \operatorname{span} F$, of dimension δ we have $\phi_L(tF, \boldsymbol{c} = \boldsymbol{0}) = \operatorname{nvol}(tF) = t^\delta \operatorname{nvol}(F)$, and if we define $\alpha(P, F) = \psi(\operatorname{tcone}(P, F)/L, \boldsymbol{c} = \boldsymbol{0})$ we have

$$\left| tP \cap \mathbb{Z}^N \right| = \sum_{F \,:\, \text{face of } P} \alpha(P, F) \operatorname{nvol}(F) t^{\dim F}. \tag{2.5.1}$$

Chapter 3

Ehrhart positivity

3.1 Known results

Understanding the coefficients of the Ehrhart polynomials is quite an open problem. We have proved in Section 2.3 that the constant coefficient is 1.

The leading term can also be computed easily. If we consider a d-dimensional polytope $P \in \mathbb{R}^d$, its volume is the integral $\operatorname{vol} P = \int_P \mathrm{d}x$. We can approximate this integral with Riemannian sums. In particular, considering the lattices $\Lambda_t = (\frac{1}{t}\mathbb{Z}^d)$ for $t \in \mathbb{N}$. The corresponding Riemannian sum in each is $|\Lambda_t \cap P|$, and their fineness increases arbitrarily with t, so in the limit we have

$$\operatorname{vol} P = \lim_{t\to\infty} \left| \left(\frac{1}{t}\mathbb{Z}^d\right) \cap P \right| = \lim_{t\to\infty} \frac{1}{t^d} \left| \mathbb{Z}^d \cap P \right|.$$

And therefore, the leading term is $\operatorname{vol} P$.

The second leading coefficient can also be interpreted as counting something. In this case, the coefficient equals one half of the sum of the normalized volumes of the facets of P. To prove it, consider Eq. (2.5.1) for faces of dimension $\dim P - 1$. Then the second leading term of the Ehrhart polynomial is

$$\sum_{F \text{ facet}} \alpha(P, F)\,\mathrm{nvol}(F).$$

In this case, $\mathrm{tcone}(P, F)/L$ is 1-dimensional, so $\alpha(P, F) = 1/2$ by Eq. (3.2.4).

A more elementary proof, that is, without the need of constructing Ψ, can be found in [2, Sec. 5.3].

One natural question to ask is whether the rest of the coefficients count something, and the first step to an answer is to ask if they are at least positive.

Definition 3.1.1 (Ehrhart positive). Let P be an integral polytope and $p(t) = a_d t^d + \cdots + a_1 t + 1$ its Ehrhart polynomial. We say that P is Ehrhart positive if $a_i > 0$ for all i.

The answer in general is no. Since we know the constant and the two leading coefficients of the polynomial are positive, we need at least a polynomial with four coefficients, which means a 3-dimensional polytope.

The first known example was discovered by John Reeve in [10].

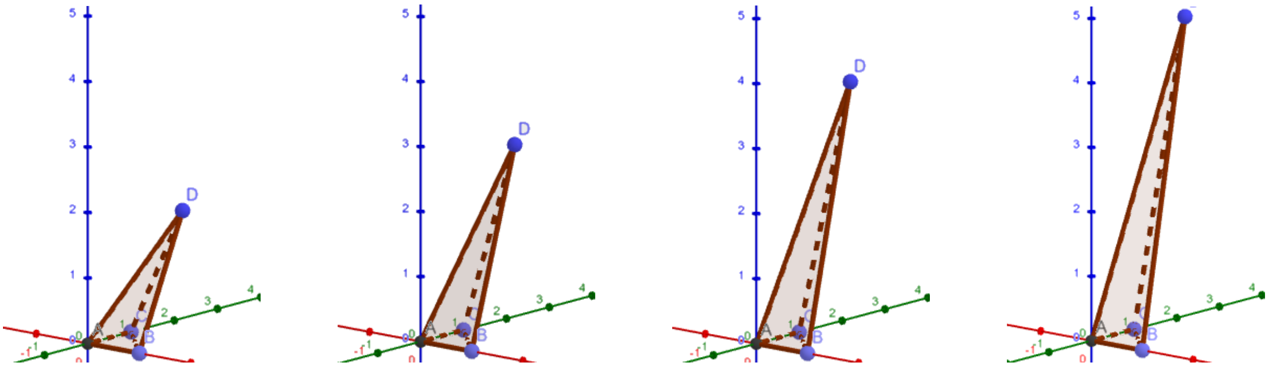
Example 4 (Reeve's tethrahedra). For $m \in \mathbb{N}$, we define the m-th Reeve tetrahedron in $\mathbb{R}^3$ as

$$T_m = \operatorname{conv}(\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2 + m\boldsymbol{e}_3).$$

Let $p_m(t) = a_3t^3 + a_2t^2 + a_1t + 1$ be its Ehrhart polynomial. We can check that the only integer points in T_m are its vertices, so we have for all m that $p_m(1) = 4$. The volume is one sixth of the corresponding cube, so $a_3 = \frac{m}{6}$. Each facet has normalized volume $1/2$, thus $a_2 = 1$. Now using $p_m(1) = 4$, we get $\frac{m}{6} + 1 + a_1 + 1 = 4$, and we can isolate $a_1 = \frac{12-m}{6}$. Putting all the coefficients together we have

$$p_m(t) = \frac{m}{6}t^3 + t^2 + \frac{12-m}{6}t + 1.$$

Clearly, $a_1 < 0$ for $m > 12$, and in fact it can be arbitrarily small for t arbitrarily large.



$m = 2$ $m = 3$ $m = 4$ $m = 5$

Figure 3.1: Reeve's tetrahedra for different values of m.

There are many families of polyhedra that are known to be Ehrhart positive, and others that are known to have some negative coefficients. We refer to [8] for a survey on what is currently known about Ehrhart positivity.

3.2 Computing the Berline-Vergne valuation

In this section we will compute the valuation Ψ for unimodular low-dimensional cones. In particular, we will compute up to 3-dimensional cones.

These computations serve two purposes. First, they will give an idea on how Ψ looks like and how difficult it is to compute. Secondly, they motivate the results presented in Section 3.3, where we expose the work of Castillo and Liu in [5], conjecturing that generalized permutohedra are Ehrhart positive.

To make the notation more understandable, we will use a subscript to denote the dimension of the cone in which we are computing ψ. So, for example, if K is a two dimensional cone, we will denote $\psi(K)$ as $\psi_2(K)$. Also, in this section we will make an exception to our general rule of writing vectors in boldface, because there appear a lot of vectors, and we find more useful to use boldface characters to point out the most essential parts of our formulas.

3.2.1 General formulation

The formula that defines ψ in Theorem 2.4.2 for a unimodular cone $K = \operatorname{co}(u_1, \ldots, u_n)$ reads

$$\psi_n(K, c) = \frac{1}{\prod_{i=1}^{n}(1 - e^{\langle c, u_i \rangle})} - \sum_{k=1}^{n} \sum_{I \in \binom{[n]}{k}} \psi_{n-k}(K/u_I) \frac{(-1)^k}{\prod_{i \in I} \langle c, u_i \rangle} \tag{3.2.1}$$

where $u_I = \{u_i\}_{i \in I}$, and K/u_I is the orthogonal projection of K onto $\operatorname{span}(u_I)^\perp$.

In general, this projection will also be a unimodular cone. The projected lattice will be generated by the orthogonal projection of the rest of vectors u_i that are not in u_I.

Lemma 3.2.1. *This projection is a linear mapping, with matrix*

$$\operatorname{Id} - A(A^T A)^{-1} A^T \tag{3.2.2}$$

where A is the matrix that contains vectors in u_I as columns.

PROOF.
Given a linear subspace F, any vector can be uniquely decomposed as $x = x_F + x_{F^\perp}$, where $x_F \in F$ and $x_{F^\perp} \in F^\perp$. Here, F is the colspan of A and the matrix $\Pi = A(A^T A)^{-1} A^T$ is the orthogonal projection of x into F, that is, $x_F = \Pi x$. Then $x_{F^\perp} = x - x_F = x - A(A^T A)^{-1} A^T x$. □

3.2.1.1 On the limit $c \to 0$

We know that ψ is analytic at $c = 0$, and in fact, we are interested in the limit at said point.

To compute ψ_n recursively, we can see in Eq. (3.2.1) that we need the coefficient of degree k of ψ_{n-k}. So, it may be interesting to compute $\psi_n(K, c)$ *as a power series* and try to deduce some behavior from it.

3.2.2 Dimension 1 and 2

For dimension 1, Eq. (3.2.1) reads

$$\psi_1(K) = \frac{1}{1 - e^{\langle c, u_1 \rangle}} + \frac{1}{\langle c, u_1 \rangle}.$$

We can see this as a function of $\tau = \langle c, u_1\rangle$, $\psi(\tau) = \frac{1}{1-e^\tau} + \frac{1}{\tau}$, and is very similar to the generating function of Bernoulli numbers: [1]

$$\frac{\tau}{e^\tau - 1} = \sum_{m=0}^{\infty} \frac{B_m \tau^m}{m!}$$

so we know that

$$\psi_1(\tau) = \frac{1}{\tau} - \frac{1}{\tau}\sum_{m=0}^{\infty} \frac{B_m \tau^m}{m!}. \tag{3.2.3}$$

The first Bernoulli numbers are $B_0 = 1$ and $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$ and $B_5 = 0$, so the first terms of ψ_1 are

$$\psi_1(\tau) = \frac{1}{2} - \frac{1}{12}\tau + \frac{1}{30}\tau^3 + o(\tau^4). \tag{3.2.4}$$

In particular, we know that $\psi_1(\tau = 0) = 1/2$.

For dimension 2, Eq. (3.2.1) reads for $K = \mathrm{co}(u_1, u_2)$

$$\psi_2(K) = \frac{1}{(1 - e^{\langle c,u_1\rangle})(1 - e^{\langle c,u_2\rangle})} + \psi_1(\mathrm{co}(v_1))\frac{1}{\langle c, u_1\rangle} + \psi_1(\mathrm{co}(v_2))\frac{1}{\langle c, u_1\rangle} - \frac{1}{\langle c, u_1\rangle\langle c, u_2\rangle}, \tag{3.2.5}$$

where v_1 is the orthogonal projection of v_2 onto $\langle v_1\rangle^\perp$, explicitly $v_1 = u_2 - \frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} u_1$ and analogous for v_2.

As before, we want to see this as a function of $\tau_1 = \langle c, u_1\rangle$ and $\tau_2 = \langle c, u_2\rangle$. But first we need to deal with ψ_1, because we have them as a function of $\langle c, v_i\rangle$, and not $\langle c, u_i\rangle$.

3.2.2.1 Computation of ψ_1

We define an auxiliary variable $t = \langle c, v_1\rangle$. We know how to express $\psi_1(\mathrm{co}(v_1))$ in therms of t as in Eq. (3.2.3). Since we know that ψ is analytical at $c = 0$, when we express it as a power series, we know that coefficients of negative degree will vanish, so we will not care of them. In some of the equations, negative degree terms will be ignored, we will use the notation $\psi \overset{0}{=} \phi$, whenever the nonnegative degree terms of ψ and ϕ are equal.

For $\psi_1(t)$ we have:

$$\psi_1(t) \overset{0}{=} -\frac{1}{\tau}\sum_{m=0}^{\infty} \frac{B_m t^m}{m!} \overset{0}{=} -\sum_{m=0}^{\infty} \frac{B_{m+1} t^m}{(m+1)!}.$$

Now using $t = \tau_2 - \frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\tau_1$ we get

$$t^m = \sum_{k=0}^{m} (-1)^k \left(\frac{\langle u_1, u_2\rangle}{\langle u_1, u_1\rangle}\right)^k \binom{m}{k} \tau_1^k \tau_2^{m-k}.$$

In Eq. (3.2.5) we need $\frac{1}{\tau_1}\psi_1(t)$. Mixing the two previous equations yields:

[1] For the Bernoulli numbers, we will be using the convention that $B_1 = -1/2$.

$$\frac{1}{\tau_1}\psi_1(\mathrm{co}(v_1)) \overset{0}{=} \sum_{m\geq 0}\frac{B_{m+2}}{(m+2)!}\sum_{k=0}^{m}(-1)^k\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^{k+1}\binom{m+1}{k+1}\tau_1^k\tau_2^{m-k}. \tag{3.2.6}$$

And analogously

$$\frac{1}{\tau_2}\psi_1(\mathrm{co}(v_2)) \overset{0}{=} \sum_{m\geq 0}\frac{B_{m+2}}{(m+2)!}\sum_{k=0}^{m}(-1)^{m-k}\left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^{m+1-k}\binom{m+1}{k}\tau_1^k\tau_2^{m-k}. \tag{3.2.7}$$

With this notation, we can easily compute the sums in Eq. (3.2.5):

$$\frac{1}{\tau_1}\psi_1(\mathrm{co}(v_1)) + \frac{1}{\tau_2}\psi_1(\mathrm{co}(v_2)) \overset{0}{=}$$
$$\sum_{m\geq 0}\frac{B_{m+2}}{(m+2)!}\sum_{k=0}^{m}\left[(-1)^k\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^{k+1}\binom{m+1}{k+1} + (-1)^{m-k}\left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^{m+1-k}\binom{m+1}{k}\right]\tau_1^k\tau_2^{m-k}$$

Since all terms are multiplied by a Bernoulli number B_{m+2}, and for m odd they are zero, we may assume m even, and therefore $(-1)^k = (-1)^{m-k}$. With this and playing with the binomial coefficients we get

$$\frac{1}{\tau_1}\psi_1(\mathrm{co}(v_1)) + \frac{1}{\tau_2}\psi_1(\mathrm{co}(v_2)) \overset{0}{=}$$
$$\sum_{m\geq 0}\frac{B_{m+2}}{(m+2)!}\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k}\left[(m+1-k)\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^{k+1} + (k+1)\left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^{m+1-k}\right]\tau_1^k\tau_2^{m-k}.$$

Finally, Eq. (3.2.5) reads:

$$\psi_2(\mathrm{co}(u_1,u_2)) =$$
$$\sum_{m\geq 0}\frac{1}{(m+2)!}\sum_{k=0}^{m}\binom{m+2}{k+1}B_{k+1}B_{m+1-k}\tau_1^k\tau_2^{m-k} +$$
$$\sum_{m\geq 0}\frac{B_{m+2}}{(m+2)!}\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k}\left[(m+1-k)\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^{k+1} + (k+1)\left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^{m+1-k}\right]\tau_1^k\tau_2^{m-k}.$$

We can check this formula for first degrees:

$$[deg0]\psi_2 = \frac{1}{2!}\binom{2}{1}B_1B_1 + \frac{B_2}{2!}\cdot 1\cdot\left[\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} + \frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right] = \frac{1}{4} + \frac{1}{12}\langle u_1,u_2\rangle\left[\frac{1}{\langle u_1,u_1\rangle} + \frac{1}{\langle u_1,u_2\rangle}\right]. \tag{3.2.8}$$

$$[deg1]\psi_2 = \frac{1}{3!}\left(\binom{3}{1}B_1B_2\tau_2 + \binom{3}{2}B_2B_1\tau_1\right) = \frac{-1}{24}(\tau_1+\tau_2). \tag{3.2.9}$$

$$\begin{aligned}
&[deg2]\psi_2 = \\
&\frac{1}{4!}\left(\binom{4}{1}B_1B_3\tau_2^2 + \binom{4}{2}B_2B_2\tau_1\tau_2 + \binom{4}{3}B_3B_1\tau_1^2\right) + \frac{B_4}{4!}\left\{\frac{1}{1}\binom{3}{0}\left[3\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} + \left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^3\right]\tau_2^2 + \right.\\
&\left. +\frac{1}{3}\binom{3}{2}\left[\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^3 + 3\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right]\tau_1^2 - \frac{1}{2}\binom{3}{1}\left[2\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^2 + 2\left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^2\right]\tau_1\tau_2\right\} \\
&= \frac{\tau_1\tau_2}{144} - \frac{1}{720}\left\{\left[\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^3 + 3\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right]\tau_1^2 + \right.\\
&\left.\left[3\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} + \left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^3\right]\tau_2^2 - 3\left[\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\right)^2 + \left(\frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right)^2\right]\tau_1\tau_2\right\}.
\end{aligned}$$

Observe that for the first degree term, only the first term in the sum is computed, because for m odd, the second term yields directly zero. The first term will also vanish for m odd, except for $m = 1$, because Bernoulli numbers B_m for m odd vanish all except for B_1.

3.2.3 Dimension 3

First of all, we need to compute the orthogonal projections as in Lemma 3.2.1.

For the projection onto $\langle u_i\rangle^\perp$ the following vectors v_{ij} are the projection of u_j onto $\langle u_i\rangle^\perp$:

$$v_{12} = \boldsymbol{u}_2 - \frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\boldsymbol{u}_1, \qquad v_{23} = \boldsymbol{u}_3 - \frac{\langle u_2,u_3\rangle}{\langle u_2,u_2\rangle}\boldsymbol{u}_2, \qquad v_{31} = \boldsymbol{u}_1 - \frac{\langle u_3,u_1\rangle}{\langle u_3,u_3\rangle}\boldsymbol{u}_3,$$

$$v_{13} = u_3 - \frac{\langle u_1,u_3\rangle}{\langle u_1,u_1\rangle}\boldsymbol{u}_1, \qquad v_{21} = u_1 - \frac{\langle u_2,u_1\rangle}{\langle u_2,u_2\rangle}\boldsymbol{u}_2, \qquad v_{32} = u_2 - \frac{\langle u_3,u_2\rangle}{\langle u_3,u_3\rangle}\boldsymbol{u}_3.$$

For the projections onto $\langle u_j,u_k\rangle^\perp$, the following vectors w_i are the projection of u_i onto $\langle u_j,u_k\rangle^\perp$ (see Section 3.2.3.1 to check calculations):

$$w_1 = \boldsymbol{u}_1 - \frac{1}{u_2^Tu_2u_3^Tu_3 - (u_2^Tu_3)^2}\Big((\langle u_3,u_3\rangle\langle u_2,u_1\rangle - \langle u_2,u_3\rangle\langle u_3,u_1\rangle)\cdot\boldsymbol{u_2} + (\langle u_2,u_2\rangle\langle u_3,u_2\rangle - \langle u_2,u_3\rangle\langle u_2,u_1\rangle)\cdot\boldsymbol{u_3}\Big),$$

$$w_2 = \boldsymbol{u}_2 - \frac{1}{u_3^Tu_3u_1^Tu_1 - (u_3^Tu_1)^2}\Big((\langle u_1,u_1\rangle\langle u_3,u_2\rangle - \langle u_3,u_1\rangle\langle u_1,u_2\rangle)\cdot\boldsymbol{u_3} + (\langle u_3,u_3\rangle\langle u_1,u_2\rangle - \langle u_3,u_1\rangle\langle u_3,u_2\rangle)\cdot\boldsymbol{u_1}\Big),$$

$$w_3 = \boldsymbol{u}_3 - \frac{1}{u_1^Tu_1u_2^Tu_2 - (u_1^Tu_2)^2}\Big((\langle u_2,u_2\rangle\langle u_1,u_3\rangle - \langle u_1,u_2\rangle\langle u_2,u_3\rangle)\cdot\boldsymbol{u_1} + (\langle u_1,u_1\rangle\langle u_2,u_3\rangle - \langle u_1,u_2\rangle\langle u_1,u_3\rangle)\cdot\boldsymbol{u_2}\Big).$$

With this notation, $\psi_3(\mathrm{co}(u_1,u_2,u_3))$ can be easily computed as

$$\psi_3(\mathrm{co}(u_1,u_2,u_3)) = \tag{3.2.10}$$
$$\frac{1}{(1-e^{\langle c,u_1\rangle})(1-e^{\langle c,u_2\rangle})(1-e^{\langle c,u_3\rangle})} \tag{3.2.11}$$
$$+\psi_2(\mathrm{co}(v_{31},v_{32}))\frac{1}{\langle c,u_3\rangle}+\psi_2(\mathrm{co}(v_{12},v_{13}))\frac{1}{\langle c,u_1\rangle}\psi_2(\mathrm{co}(v_{23},v_{21}))\frac{1}{\langle c,u_2\rangle} \tag{3.2.12}$$
$$-\psi_1(\mathrm{co}(w_1))\frac{1}{\langle c,u_2\rangle\langle c,u_3\rangle}-\psi_1(\mathrm{co}(w_2))\frac{1}{\langle c,u_3\rangle\langle c,u_1\rangle}-\psi_1(\mathrm{co}(w_3))\frac{1}{\langle c,u_1\rangle\langle c,u_2\rangle} \tag{3.2.13}$$
$$+\frac{1}{\langle c,u_1\rangle\langle c,u_2\rangle\langle c,u_3\rangle}. \tag{3.2.14}$$

As before, we want to consider $\psi_3=\psi_3(\tau_1,\tau_2,\tau_3)$, where $\tau_i=\langle c,u_i\rangle$.

For Eq. (3.2.11) we use the expansion $\frac{1}{1-e^t}\overset{0}{=}-\sum_{m\geq 0}\frac{B_{m+1}}{(m+1)!}t^m$, that in our 3-dimensional case becomes

$$\frac{1}{(1-e^{\tau_1})(1-e^{\tau_2})(1-e^{\tau_3})}\overset{0}{=}-\sum_{m\geq 0}\frac{1}{(m+3)!}\sum_{\boldsymbol{m}\,:\,|\boldsymbol{m}|=m}\binom{m+3}{m_1+1,m_2+1,m_3+1}B_{m_1+1}B_{m_2+1}B_{m_3+1}\tau_1^{m_1}\tau_2^{m_2}\tau_3^{m_3}. \tag{3.2.15}$$

Before going to the full expansion of ψ_3, observe that we can directly compute the constant term, which is the one we are most interested, and the one we can check with [5].

$$[deg0](Eq.\ (3.2.11))=-\frac{1}{3!}\binom{3}{1,1,1}\left(-\frac{1}{2}\right)^3=\frac{1}{8}. \tag{3.2.16}$$

Since ψ_1 has vanishing degree 2 term,

$$[deg2](Eq.\ (3.2.13))=0.$$

So we are only left with $[deg1](Eq.\ (3.2.12))$, for which we use our computations in Eq. (3.2.9). Let us define the auxiliary variables $\sigma_{ij}:=\langle c,v_{ij}\rangle$. From definition of v_{ij} we can see that for all i,j holds $\sigma_{ij}=\tau_j-\frac{\langle u_i,u_j\rangle}{\langle u_i,u_i\rangle}\tau_i$, so we have:

$$[deg1](Eq.\ (3.2.12))=-\frac{1}{24}\left(\frac{\sigma_{12}+\sigma_{13}}{\tau_1}+\frac{\sigma_{21}+\sigma_{23}}{\tau_2}+\frac{\sigma_{31}+\sigma_{32}}{\tau_3}\right) \tag{3.2.17}$$
$$=-\frac{1}{24}\left(\frac{\tau_2-\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}\tau_1+\tau_3-\frac{\langle u_1,u_3\rangle}{\langle u_1,u_1\rangle}\tau_1}{\tau_1}+\frac{\tau_1-\frac{\langle u_2,u_1\rangle}{\langle u_2,u_2\rangle}\tau_2+\tau_3-\frac{\langle u_2,u_3\rangle}{\langle u_2,u_2\rangle}\tau_2}{\tau_2}+\frac{\tau_1-\frac{\langle u_3,u_1\rangle}{\langle u_3,u_3\rangle}\tau_3+\tau_2-\frac{\langle u_3,u_2\rangle}{\langle u_3,u_3\rangle}\tau_3}{\tau_3}\right) \tag{3.2.18}$$
$$=\frac{1}{24}\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}+\frac{\langle u_1,u_3\rangle}{\langle u_1,u_1\rangle}+\frac{\langle u_2,u_1\rangle}{\langle u_2,u_2\rangle}+\frac{\langle u_2,u_3\rangle}{\langle u_2,u_2\rangle}+\frac{\langle u_3,u_1\rangle}{\langle u_3,u_3\rangle}+\frac{\langle u_3,u_2\rangle}{\langle u_3,u_3\rangle}-\left(\frac{\tau_2+\tau_3}{\tau_1}+\frac{\tau_1+\tau_3}{\tau_2}+\frac{\tau_1+\tau_2}{\tau_3}\right)\right) \tag{3.2.19}$$
$$\overset{0}{=}\frac{1}{24}\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle}+\frac{\langle u_1,u_3\rangle}{\langle u_1,u_1\rangle}+\frac{\langle u_2,u_1\rangle}{\langle u_2,u_2\rangle}+\frac{\langle u_2,u_3\rangle}{\langle u_2,u_2\rangle}+\frac{\langle u_3,u_1\rangle}{\langle u_3,u_3\rangle}+\frac{\langle u_3,u_2\rangle}{\langle u_3,u_3\rangle}\right). \tag{3.2.20}$$

3.2.3.1 Computation of w_i's

We will compute w_3, and the rest are analogous. We will denote column vectors as u_i, and row vectors as u_i^T. With this notation, $A = [u_1\ u_2]$ and $A^T = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$. The scalar product is written $\langle u, v\rangle = u^T v = v^T u$. So:

$$A^T A = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} [u_1\ u_2] = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 \\ u_2^T u_1 & u_2^T u_2 \end{bmatrix} \implies (A^T A)^{-1} = \frac{1}{u_1^T u_1 u_2^T u_2 - (u_1^T u_2)^2} \begin{bmatrix} u_2^T u_2 & -u_1^T u_2 \\ -u_2^T u_1 & u_1^T u_1 \end{bmatrix}.$$

To write less, we will denote the denominator $\det(A)$ as ζ.

$$(A^T A)^{-1} A^T = \frac{1}{\zeta} \begin{bmatrix} u_2^T u_2 & -u_1^T u_2 \\ -u_2^T u_1 & u_1^T u_1 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \frac{1}{\zeta} \begin{bmatrix} \langle u_2, u_2\rangle u_1^T - \langle u_1, u_2\rangle u_2^T \\ \langle u_1, u_1\rangle u_2^T - \langle u_1, u_2\rangle u_1^T \end{bmatrix}.$$

$$\begin{aligned} A(A^T A)^{-1} A^T =& \frac{1}{\zeta}[u_1\ u_2] \begin{bmatrix} \langle u_2, u_2\rangle u_1^T - \langle u_1, u_2\rangle u_2^T \\ \langle u_1, u_1\rangle u_2^T - \langle u_1, u_2\rangle u_1^T \end{bmatrix} \\ =& \frac{1}{\zeta}\left(\langle u_2, u_2\rangle u_1 u_1^T - \langle u_1, u_2\rangle u_1 u_2^T + \langle u_1, u_1\rangle u_2 u_2^T - \langle u_1, u_2\rangle u_2 u_1^T\right). \end{aligned}$$

$$A(A^T A)^{-1} A^T \cdot \boldsymbol{u}_3 = \frac{1}{\zeta}\Big((\langle u_2, u_2\rangle \langle u_1, u_3\rangle - \langle u_1, u_2\rangle \langle u_2, u_3\rangle)\cdot \boldsymbol{u_1} + (\langle u_1, u_1\rangle \langle u_2, u_3\rangle - \langle u_1, u_2\rangle \langle u_1, u_3\rangle)\cdot \boldsymbol{u_2}\Big)$$

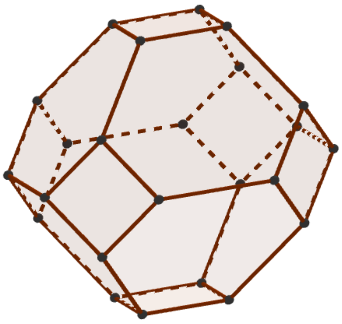
3.3 Generalized permutohedron

In this section, we study Ehrhart positivity of a specific family of polyhedra, namely generalized permutohedra. We will mainly follow [5].

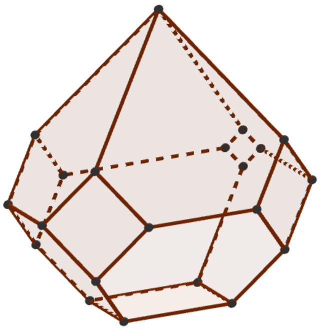
Definition 3.3.1 (Permutohedron). Let $\boldsymbol{v} = (v_1, \ldots, v_{d+1})$ with all its coordinates different, the *regular permutohedron* in $\boldsymbol{v}$ is the polytope

$$\Pi_{\boldsymbol{v}}^d = \operatorname{conv}\left\{ \left(v_{\sigma(1)}, \ldots, v_{\sigma(d+1)}\right) \ : \ \sigma \in \mathfrak{S}_{d+1} \right\}.$$

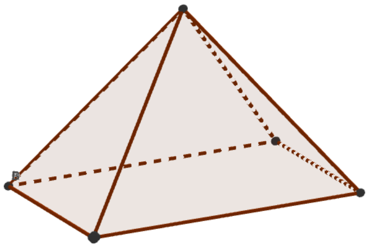
In the case $\boldsymbol{v} = (1, 2, \ldots, d)$ we call it the *usual* permutohedron.
A *generalized permutohedron* is any polytope that can be obtained from a regular permutohedron by moving the vertices while preserving edge directions. Some faces may collapse into lower dimensional faces.



(a) Regular permutohedron



(b) Generalized permutohedron. One face has collapsed into a vertex.



(c) A square pyramid as a generalized permutohedron. A lot of faces have collapsed.

Figure 3.2: It is well known that the 3-dimensional permutohedron is a truncated octahedron. In these figures we show a regular 3-permutohedron and two generalized permutohedra obtained from it.

Since we have in Eq. (2.5.1) a decomposition of the Ehrhart polynomial of a polytope in terms of the values $\alpha(P, F)$, we can define the concept of α-*positivity*, that refines Ehrhart positivity, as having $\alpha(P, F) > 0$ for all face F.

In [5, Th. 1.4] Castillo and Liu prove that regular permutohedra being α-positive implies that generalized permutohedra being α-positive, and thus Ehrhart positive and conjecture that indeed regular permutohedra are α-positive. They prove it for permutohedra up to dimension 6, using the formulas for ψ_1, ψ_2 and ψ_3 derived in the previous section. The problem they encounter is that the computations leading to ψ_n for larger values of n are too complicated. In fact, they claim that ψ_4 has way more than 1000 terms. An important part of this thesis has been understanding and reproducing these computations, and trying to simplify the case of ψ_4, but unfortunately we were not able to obtain a manipulable formula.

In this last section, we sketch the main ideas on how the values for $\alpha(P, F)$ can be computed for a regular permutohedron, following [5].

First of all, observe that the tangent cones of a regular permutohedron indeed unimodular, so we may use the formulas from the previous section. The result is independent of which vector $\boldsymbol{v}$ we use to define the permutohedron $\Pi^N = \Pi^N_{\boldsymbol{v}}$, as long as all its coordinates are different.

In [5, Lem. 4.9] prove that the problem is reduced to computing, for faces of dimension $N - l$, for every subset $S \subseteq [N]$, $|S| = N - l$, with $[N] \setminus S = \{i_1 < \cdots < i_l\}$, the value of $\psi(\mathrm{co}(\boldsymbol{R}_{i_1}, \dots, \boldsymbol{R}_{i_l}))$, where

$$\boldsymbol{R}_{i_j} = \left(\underbrace{0, \dots, 0}_{i_{j-1}}, \underbrace{\frac{1}{i_j - i_{j-1}}, \dots, \frac{1}{i_j - i_{j-1}}}_{i_j - i_{j-1}}, \underbrace{\frac{-1}{i_{j+1} - i_j}, \dots, \frac{-1}{i_{j+1} - i_j}}_{i_{j+1} - i_j}, \underbrace{0, \dots, 0}_{N+1-i_{j+1}} \right).$$

Since we know that Ehrhart positivity is always satisfied for $l = 0$, $l = 1$ and $l = N$, we can check for $l = 2, 3$, using the formulas for ψ_2 and ψ_3 of the previous section.

1. For $[N] \setminus S = \{i, j\}$ with $i < j$, we have

$$\boldsymbol{R}_i = \left(\frac{1}{i}, \dots, \frac{1}{i}, \frac{1}{i-j}, \dots, \frac{1}{i-j}, 0, \dots, 0\right), \quad \boldsymbol{R}_j = \left(0, \dots, 0, \frac{1}{j-i}, \dots, \frac{1}{j-i}, \frac{-1}{N+1-j}, \dots, \frac{-1}{N+1-j}\right).$$

Thus

$$\langle \boldsymbol{R}_i, \boldsymbol{R}_i \rangle = \frac{1}{i} + \frac{1}{j-i}, \quad \langle \boldsymbol{R}_j, \boldsymbol{R}_j \rangle = \frac{1}{j-i} + \frac{1}{N+1-j}, \quad \langle \boldsymbol{R}_i, \boldsymbol{R}_j \rangle = -\frac{1}{j-i},$$

applying Eq. (3.2.8) we get

$$\frac{1}{4} - \frac{1}{12}\frac{1}{j-i}\left(\frac{j-i}{i} + \frac{(N+1-j)(j-i)}{N+1-i}\right) = \frac{1}{4} - \frac{1}{12}\left(\frac{1}{i} + \frac{N+1-j}{N+1-i}\right) > 0.$$

2. For $[N] \setminus S = \{i, j, k\}$ with $i < j < k$, we have

$$\boldsymbol{R}_i = \left(\frac{1}{i}, \dots, \frac{1}{i}, \frac{1}{i-j}, \dots, \frac{1}{i-j}, 0, \dots, 0\right),$$

$$\boldsymbol{R}_j = \left(0, \dots, 0, \frac{1}{j-i}, \dots, \frac{1}{j-i}, \frac{-1}{k-j}, \dots, \frac{-1}{k-j}, 0, \dots, 0\right),$$
$$\boldsymbol{R}_k = \left(0, \dots, 0, \frac{1}{k-j}, \dots, \frac{1}{k-j}, \frac{-1}{N+1-k}, \dots, \frac{-1}{N+1-k}\right).$$

Computing the scalar products $\langle \boldsymbol{R}_\alpha, \boldsymbol{R}_\beta \rangle$ as in the previous case and applying this time Eqs. (3.2.16) and (3.2.20) we get

$$\frac{1}{8} - \frac{1}{24}\left(\frac{i}{j} + 1 + \frac{N+1-k}{N+1-j}\right) > 0.$$

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