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MASTER IN ADVANCED MATHEMATICS AND MATHEMATICAL ENGINEERING

AMENABILITY AND THOMPSON'S GROUP F

Master thesis presented by

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Abstract

The following Master Thesis analyses a group theoretical property called amenability, which consists in the existence of a finite measure defined on all subsets of a group. This property is studied in a group of piecewise linear homeomorphisms of the interval $[0, 1]$, Thompson's group F, which to this date represents one of the biggest problems in the theory of amenability. The project consists of four chapters. The first chapter introduces the basic ideas in the theory of amenability. In particular, some criteria and characterizations of this concept are presented. The second chapter studies non-amenable groups. Namely, it introduces a weaker idea than amenability, which studies how far is a group from being amenable. The third chapter introduces Thompson's group, and several tools to work with it. It also presents its most important properties in relation to amenability. Finally, the last chapter contains some of the most relevant results with respect to Følner sets in F, along with some original computations for possible candidates to be Følner sequences in this group.

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CHAPTER

1

Introduction

An amenable group is a group that can be assigned a finitely additive finite measure which is also invariant through the action of the group on itself by left translation. The concept was introduced by John von Neumann [24] in 1929 , as a result of his study on the Banach-Tarski paradox. This is a well known paradox in modern mathematics, and states the following:

Theorem 1.1 (Banach, Tarski.[3])**.** *Let $\mathbb{B}$ be a ball in $\mathbb{R}^3$. There exists a partition of $\mathbb{B}$ in sets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset \mathbb{B}$, and isometries $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m$ of $\mathbb{R}^3$ such that*

$$\bigcup_{i=1}^{n} \sigma_i A_i = \mathbb{B} = \bigcup_{i=1}^{n} \tau_j B_j.$$

In other words, a ball can be decomposed into pieces, which can be rearranged into two balls equal to the original one. The proof of the paradox uses the Axiom of Choice, and thus, when it was presented, the result was quite controversial, and strongly criticized. Some of the sets appearing in the partition are non-measurable, so the theorem is not contradictory, but just counter intuitive. Indeed, the theorem could not be true if all subsets in the partition were measurable, as it would be

$$\mu(\mathbb{B}) = \sum \mu(A_i) + \sum \mu(B_i) = \sum \mu(\sigma_i A_i) + \sum \mu(\tau_j B_j) = 2\mu(\mathbb{B}),$$

and this contradicts the fact that $\mu(\mathbb{B}) > 0$.

Banach and Tarski also proved that the paradox is not true in the real line or the plane. Von Neumann noticed that the paradox does not depend so much on the underlying set, but on the group of transformations which acts on it. In particular, he realizes that the group of isometries of $\mathbb{R}^3$ contains a non-abelian free subgroup, but this does not happen in dimensions one or two; in fact, he finds new paradoxes in these low dimensions, by using larger groups of transformations which actually contain non-abelian free subgroups.

One can generalize the idea of the paradox via isometries acting on the ball to any group acting on any set, by just changing the corresponding terms in the statement of the theorem. This is called a **paradoxical decomposition**, and a set admitting a paradoxical decomposition with respect to a group G is called G**-paradoxical**. Any group acts on itself by left translation, and it is possible to see that a paradoxical decomposition on a group of transformations (with respect to this action) can be transferred to the set upon which it acts (with a small number of fixed points). The Banach-Tarski paradox relies on the fact that the free group in two generators is paradoxical. With a bit of effort, its decomposition can be transferred to the ball.

Thus, the study of these decompositions can be reduced to the case of paradoxical groups, and here is why Neumann introduces amenability: as we mentioned, a measure defined on all subsets of the group implies that there cannot be a paradoxical decomposition on this group. Tarski showed in 1936 that, in fact, this is a characterization of paradoxical decompositions:

Theorem 1.2 (Tarski)**.** *Let G be a group acting on a set X, and let $E \subseteq X$. The following are equivalent:*

1. *E is not G-paradoxical.*
2. *There exists a finitely additive measure $\mu\colon \mathcal{P}(X) \to [0, \infty]$ such that $\mu(E) = 1$, and $\mu(gA) = \mu(A)$ for any $g \in G, A \subseteq X$.*

As with paradoxes, a measure on a group G can be transferred to a G-invariant measure on a set upon which it acts, and so this set will neither be G-paradoxical. Thus, again, the study of paradoxical decompositions can be reduced to groups admitting this kind of finite measure on all subsets, i.e., amenable groups.

Amenability has many other equivalent definitions, which allows to use and mix many different techniques from distinct parts of mathematics. In this project we will mostly focus on a combinatorial characterization of amenability, introduced by the danish mathematician

Enrich Følner in 1955 [11]. Følner proves that amenability is equivalent to the existence of a sequence of subsets of the group which do not change too much when another element of the group acts on them. Even though Følner does not refer to it, it is remarkable that his ideas are quite similar to those introduced by L. Ahlfors on Riemannian geometry twenty years before (1935), with the concept of regularly exhaustible surfaces. Følner's work changes completely the perspective of the problem: instead of working with measures, all the effort is directed to finding these subsets in the group.

One of the biggest unsolved problems in the theory of amenability is to see whether a particular group, Thompson's group F, is amenable. This group is defined as a family of homeomorphisms of the interval $[0, 1]$, satisfying some properties that reduce it to a finitely generated group. It was introduced by Richard Thompson in the 1960's, along with other two groups, a bit bigger but defined similarly: T and V. These are all discrete groups which have really interesting properties, and were introduced as counterexamples to some existing conjectures. The original notes by Thompson are unpublished, but an article by Cannon, Floyd and Parry [8], which contains most of the basic definitions and properties of these groups, has become a classical reference to the topic.

All the existing criteria to check amenability fails when applied to the group F, and thus this group motivates the development of the theory of amenability. It is conjectured that F is not amenable, and its study seems to tell that this is correct, but there is not a formal proof of this fact yet. Several authors have claimed to solve the problem, but all the proofs presented contained mistakes which could not be avoided. In 2009, E. Shavgulidze [21] claimed to have a proof that F is amenable, and Moore [18] gave another proof of the amenability of F in 2012. On the other hand, A. Akhmedov [1] presented a proof that F is not amenable in 2013. Errors have been found in all these proofs, so the problem remains open. It is also remarkable that, in 2004, to celebrate the 40 years of Thompson's group, there was a conference treating all the big questions related to F, along with possible methods for solving these questions.

The project consists in four main chapters, and some conclusions. In the first chapter, we make an introductory approach to amenability, some of its most important characterizations and criteria, and its relation with other properties of the group, such as commutativity or growth. In the next chapter we use Følner's theorem to study non-amenable groups, and somehow measure how far they are from being amenable. From here on, we stop studying amenability as a general property, and we focus in Thompson's group F. In the third chapter, we introduce the group, along with its presentation and diagrams. We prove some of its most important properties and relate them to amenability. Finally, in the last chapter, we search for Følner sequences in F. We first introduce the best Følner sets to date, introduced by Belk and Brown, and then some original work: computations of the boundary ratios of different sequences of sets, mostly coming from the presentation of F, among others. Even though we achieved the goal of computing the ratios for the proposed sets, the results obtained are negative, and far from improving the bound obtained by Belk and Brown. Despite this fact, our work highlights the combinatorial difficulty in computing these ratios, even for simple sets.

The reader is encouraged to read the project linearly, although most of the results in the chapter about Thompson's group can be read without any background on the other chapters.

It is recommended for the reader to be familiar with group theory, and know some basics in graph and measure theory, but one does not need much more than this. Some comments will be made about the Banach-Tarski paradox, but the ideas presented in this introduction should be enough to understand these comments. The author will not develop this theory further, as he already did this in his Bachelor Thesis. This document is not available online, but the author will gladly sent it to anyone interested. It is in spanish, though. The reader interested in this topic is also invited to check the book by Wagon on the Banach-Tarski paradox [23], which contains pretty much all existing information on the topic.

Finally, remark that we will assume the Axiom of Choice throughout all this project, as the existence of non-measurable sets depends on this axiom, and thus without it, the problem we are considering might not make sense. In any case, all results which make explicit use of this axiom will be marked with (AC).

CHAPTER

2

Amenability

2.1 Definition

Note that a group acts on itself by left translation. An amenable group is one that admits a finite measure, defined on all subsets, and which is invariant by this action.

Definition 2.1. A group G is **amenable** if there exists a finite measure $\mu\colon \mathcal{P}(G) \to [0,1]$, such that

1. $\mu(G) = 1$.
2. μ is finitely additive, i.e., if $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.
3. μ is translation invariant, i.e., $\mu(gA) = \mu(A)$ for all $g \in G$, $A \subseteq G$.

In a more general fashion, one can define an **amenable action** analogously, but asking for the measure to be invariant by the given action. The importance of the amenability of the action by left translation is that it implies amenability for any other action of the group:

Proposition 2.2 (AC)**.** *Let G be an amenable group acting on a non-empty set X. Then, there is a finitely additive, G-invariant measure $\nu\colon \mathcal{P}(X) \to [0,1]$, such that $\nu(X) = 1$.*

Proof. Let μ be the measure in G. Choose $x \in X$ and define

$$\nu(A) = \mu(\{g \in G\colon gx \in A\}).$$

Then, $\nu(X) = \mu(G) = 1$ and, since $h \to gh$ is a bijection in X, if $A \cap B = \emptyset$, it is $\nu(A \cup B) = \nu(A) + \nu(B)$. Left invariance of ν follows from left invariance of μ. $\square$

There are some basic examples of amenable and non-amenable groups:

Example 2.3 (Finite groups are amenable)**.** Let G be a finite group. For any $A \subset G$, define $\mu(A) = \frac{|A|}{|G|}$. We have that $\mu(G) = 1$. Given $g \in G$, we know that $h \to gh$ is a bijection from A to gA, so $|gA| = |A|$, and thus $\mu(A) = \mu(gA)$. For the finite additivity, note that if $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

Example 2.4 (The free group on two generators is not amenable)**.** It is enough to find a paradoxical decomposition of $F_2 = \langle \sigma, \tau \rangle$, and apply the idea in the beginning of the introduction.

Let $W(\rho)$ be the set of words in F_2 starting with ρ, for $\rho \in \{\sigma^{\pm 1}, \tau^{\pm 1}\}$. Note that

$$\sigma^{-1}W(\sigma) = \{1\} \cup W(\tau) \cup W(\tau^{-1}) \cup W(\sigma),$$

and we can do this analogously with τ. Then,

$$W(\sigma^{-1}) \cup \sigma^{-1}W(\sigma) = F_2 = W(\tau^{-1}) \cup \tau^{-1}W(\tau),$$

and so, if there was a measure μ on F_2 with the desired properties, it would be

$$\begin{aligned}\mu(F_2) &= \mu(W(\sigma)) + \mu(W(\sigma^{-1})) + \mu(W(\tau)) + \mu(W(\tau^{-1}) = \\ &= \mu(\sigma^{-1}W(\sigma)) + \mu(W(\sigma^{-1})) + \mu(\tau^{-1}W(\tau)) + \mu(W(\tau^{-1}) = 2\mu(F_2).\end{aligned}$$

But $\mu(F_2) = 1$, so we reach a contradiction. Thus, F_2 is not amenable.

Now, what happens when we look for measures on other infinite groups? The reader is encouraged to try to find a left invariant measure on $\mathbb{Z}$, the "smallest" infinite group, to see that this is not an easy task. To solve this, we will introduce several characterizations of amenability, which provide easier criteria to check, and also different perspectives on the topic. Apart from this, we will study some ways to transfer amenability from one group to another. Some of the following results will depend on the Axiom of Choice, and so they will not provide explicit measures, but only their guarantee their existence.

2.2 Left Invariant Means

The idea of our first characterization will be that the existence of a measure is equivalent to the existence of some kind of "averaging operator", just like the Lebesgue measure and integral are related in $\mathbb{R}^n$. Let us do this formally:

Given an amenable group, we can define a linear functional with similar properties to the integral in $\mathbb{R}$. Let G be a group, and let $B(G) = \{f \colon G \to \mathbb{R} : f \text{ is bounded}\}$. Then,

$B(G)$ is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication. Since all subsets of G are measurable, all functions in $B(G)$ are also measurable. Finally, note that G acts on $B(G)$, by

$$(g.f)(x) \to {}_gf(x) = f(g^{-1}x).$$

The same process used to construct the Lebesgue integral in $\mathbb{R}$ turns out to be useful in this scenario. We define first the "integral" on simple functions, then on non-negative functions, using the supremum, and finally we extend this to any bounded function. Thus, we obtain a linear functional, $F\colon B(G) \to \mathbb{R}$ (that depends on the measure), which by construction verifies:

1. $F(f) \geq 0$ if $f \geq 0$.
2. $F(\chi_G) = 1$, where χ_G is the characteristic function of G.
3. $F(f) = F({}_gf)$, i.e., F is invariant by the action of G.

A linear functional with this properties is called a **Left Invariant Mean (LIM)**. It can be shown that

$$\inf\{f(g)\colon g \in G\} \leq F(f) \leq \sup\{f(g)\colon g \in G\},$$

so F works as some kind of average of the function.

Conversely, if we have a left invariant mean F, we can define a measure μ in $\mathcal{P}(G)$ as $\mu(A) = F(\chi_A)$. It is easy to see from the properties of F that μ is indeed a measure. This proves the following theorem:

Theorem 2.5. *A group G is amenable if and only if it admits a left invariant mean on $B(G)$.*

The importance of this idea is that it gives a new perspective on how to approach the problem of finding a measure, by focusing on the functions $G \to \mathbb{R}$, instead of the subsets in $\mathcal{P}(G)$. In particular, it allows us to use tools from functional analysis, as we will see soon in the proof of Følner's theorem.

2.3 Operations preserving amenability

Here we use several techniques which allow us to obtain amenable groups from other amenable groups. We will use Tychonoff's theorem:

Theorem 2.6 (Tychonoff. (AC)). *The arbitary product of compact spaces is compact.*

In particular, since $[0,1]$ is compact, the function space $[0,1]^X = \{f\colon X \to [0,1]\}$ is also compact with the product topology. This will be quite useful, due to the following property of compact spaces:

Definition 2.7. A collection of sets $\mathcal{F}$ has the **finite intersection property** if, for any finite subcollection $\{F_1, \dots, F_n\} \subset \mathcal{F}$, it is

$$\bigcap_{i=1}^{n} F_i \neq \emptyset.$$

Proposition 2.8. *Let X be a compact topological space. Then, for any collection $\mathcal{F}$ of closed subsets of X we have that, if $\mathcal{F}$ has the finite intersection property, then*

$$\bigcap_{F\in\mathcal{F}} F \neq \emptyset.$$

We call this the **compactness principle**. The idea will be to find families of closed sets of almost invariant measures. We will apply the compactness principle to these families, in order to find a measure in the intersection, which will end up being the desired invariant measure.

Let us also introduce direct unions of groups:

Definition 2.9. A **directed system of groups** is a family of groups, $\{G_\alpha\colon \alpha \in I\}$ such that for each $\alpha, \beta \in I$ there exists $\gamma \in I$ such that G_α, G_β are both subgroups of G_γ. The group $G = \bigcup_{\alpha\in I} G_\alpha$ is called the **direct union** of the directed system.

In particular, any group is the direct union of their finitely generated subgroups: notice that if A and B are finitely generated subgroups, then $\langle A \cup B\rangle \leq G$ is also finitely generated, so finitely generated subgroups form a directed system whose direct union is precisely G.

With these ideas, we are ready to present the most important operations preserving amenability:

Theorem 2.10.

1. *(AC) A subgroup of an amenable group is amenable.*
2. *If N is a normal subgroup of the amenable group G, then G/N is amenable.*
3. *If $N \to G \to H$ is a short exact sequence of groups, and N, H are amenable, then G is amenable.*

4. *(AC) If G is the direct union of a directed system of amenable groups $\{G_\alpha \colon \alpha \in I\}$, then G is amenable.*

5. *Abelian groups are amenable.*

Proof. 1. Let μ be a measure on G, and let H be a subgroup of G. Let M be a set which contains a unique element from each of the right cosets of H in G. Then, we can define $\nu \colon \mathcal{P}(H) \to [0,1]$ by

$$\nu(A) = \mu\left(\bigcup\{Ag \colon g \in M\}\right).$$

Then, $\nu(H) = \mu(G) = 1$. If $A \cap B = \emptyset$, then $\{Ag \colon g \in M\} \cap \{Bg \colon g \in G\} = \emptyset$, and since μ is finitely additive, so is ν. The left invariance of ν is deduced from the left invariance of μ.

2. Note that G acts on G/N. Using Proposition 2.2, we can obtain a G-invariant measure on G/N. It is straightforward to check that this measure is also G/N-invariant.

3. We identify H with G/N to make the proof simpler. Let ν_1, ν_2 be measures on $N, G/N$ respectively. Given $A \subseteq G$, let us define $f_A \colon G \to \mathbb{R}$ by

$$f_A(g) = \nu_1(N \cap g^{-1}A).$$

We see first that if $g_1N = g_2N$, then $f_A(g_1) = f_A(g_2)$, and so f_A induces a mapping in G/N. Indeed, if $g_1N = g_2N$, then $g_2^{-1}g_1 = h \in N$. Thus, $hN = N$, and we get

$$\begin{aligned} f_A(g_2) &= \nu_1(N \cap g_2^{-1}A) = \nu_1(N \cap hg_1^{-1}A) = \nu_1(h(N \cap g_1^{-1}A)) = \\ &= \nu_1(N \cap g_1^{-1}A) = f_A(g_1). \end{aligned}$$

Hence, f_A induces $\hat{f}_A \colon G/N \to \mathbb{R}$. Since G/N is amenable, it admits a left invariant mean, F_{ν_2}. Define $\mu \colon \mathcal{P}(G) \to [0,1]$ as $\mu(A) = F_{\nu_2}(\hat{f}_A)$. First of all, note that, for all $g \in G$,

$$f_G(g) = \nu_1(N \cap g^{-1}G) = \nu_1(N) = 1 = \chi_G(g),$$

and so $f_G = \chi_G$. Hence,

$$\mu(G) = F_{\nu_2}(\hat{f}_G) = F_{\nu_2}(\hat{\chi}_G) = F_{\nu_2}(\chi_{G/N}) = 1.$$

For the finite additivity, notice that if $A \cap B = \emptyset$, then $g^{-1}A \cap g^{-1}B = \emptyset$ for any $g \in G$, and $f_{A\cup B} = f_A + f_B$, so $\hat{f}_{A\cup B} = \hat{f}_A + \hat{f}_B$. Applying linearity of F_{ν_2} we obtain finite additivity of μ. Finally, for left invariance, notice that

$$f_{gA}(h) = f_A(g^{-1}h) = {}_g f_A(h).$$

Since F_{ν_2} is left invariant, so is μ.

4. Let $G = \bigcup_{\alpha \in I} G_\alpha$ be a directed union of amenable groups, and let μ_α be a measure in G_α. By Tychonoff's theorem, $[0,1]^{\mathcal{P}(G)}$ is compact. For each $\alpha \in I$, let $\mathcal{M}_\alpha$ be the set of finitely additive measures $\mu \colon \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$ and $\mu(gA) = \mu(A)$ for each $g \in G_\alpha$. We know that this set is not empty, since one can see that μ defined by $\mu(A) = \mu_\alpha(A \cap G_\alpha)$ is in $\mathcal{M}_\alpha$. It can be shown that $\mathcal{M}_\alpha$ is closed in $[0,1]^{\mathcal{P}(G)}$ (if a function does not verify a condition, then one can "change" the function a bit so that the condition is still not verified). The collection $\{\mathcal{M}_\alpha \colon \alpha \in I\}$ has the finite intersection property: by definition, given α, β, there exists γ such that $G_\alpha, G_\beta \subset G_\gamma$. If μ is invariant by G_γ, then μ is also invariant by G_α and G_β, and so $\emptyset \neq \mathcal{M}_\gamma \subseteq \mathcal{M}_\alpha \cap \mathcal{M}_\beta$. Finally, by Proposition 2.8, there exists $\mu \in \bigcap_{\alpha \in I} \mathcal{M}_\alpha$. This measure is G-invariant, finitely additive and $\mu(G) = 1$.

5. This proof is not finished, but it is convenient to place this result here. Since any group is the direct union of its finitely generated subgroups, using (4) we can reduce this case to finitely generated abelian groups. We have a complete classification of these groups: they are finite direct sums of copies of $\mathbb{Z}$ and $\mathbb{Z}/n_i\mathbb{Z}$, with $n_i \in \mathbb{N}$. Direct sums of amenable groups are amenable by (3), since $A \to A \oplus B \to B$ is a short exact sequence. $\mathbb{Z}/n\mathbb{Z}$ is finite for all $n \in \mathbb{N}$, so these groups are amenable. It is left to show that $\mathbb{Z}$ is also amenable. We will do this very soon using Følner's theorem, although it can also be done directly using the compactness principle.

□

Note that (1) is not trivial: at first sight, we could try to restrict the measure to the subgroup, or divide everything by its measure, but the subgroup could have measure 0. This is solved by "copying" a subset $A \subset H$ into each coset, and giving A the measure of the union of all copies in G.

The proof of (3) partitions a subset $A \subset G$ into the intersections with each coset, and then gives as a measure to A the average (using the LIM on G/H) of the measures on each coset (using the measure in H).

Finally, the proof of (4) provides a good example on how the compactness principle is applied to get a measure.

This theorem is really useful to prove amenability in most common groups. For instance, from (3) and (5), one gets:

Corollary 2.11. *Solvable groups are amenable.*

Statements (1) and (4) also imply that a group will be amenable if and only if all of its finitely generated subgroups are amenable, and so we can reduce our study to the finitely generated case without missing any information.

We can define a class of groups formed by those that can be obtained using the operations in the previous theorem. These are known as elementary groups:

Definition 2.12. The class of **elementary groups** EG is the smallest class of groups containing finite and abelian groups, and such that:

1. If $G \in EG$, and H is a subgroup of G, then $H \in EG$.
2. If $H \lhd G$ and $G \in EG$, then $G/H \in EG$.
3. If $H \lhd G$,and both H and G/H are in EG, then $G \in EG$
4. If $G = \bigcup G_\alpha$ is a direct union of groups, and $G_\alpha \in EG$ for each α, then $G \in EG$.

Theorem 2.10 states that the class of elementary groups is contained in the class of amenable groups, denoted by AG. It was a long open problem whether this inclusion was strict, that is, whether there was an amenable group which could not be obtained from a finite or abelian group through these operations. This was solved by Grigorchuk [13] in 1985 who found a group in $AG \setminus EG$. This group was found as a group of transformations of the (infinite) rooted binary tree, and it was introduced as a counter-example for another conjecture, namely the Milnor-Wolf Conjecture, about growth in groups.

Another long open conjecture was whether being amenable was equivalent to not containing a copy of the free group on two generators. This was known as the Day-Von Neumann conjecture, and it was proven false by Ol'Shanskii [20] in 1980. He found a non-amenable group in which every element has finite order, and so this group cannot contain a copy of a free group. Thus, both inclusions are strict:

$$EG \subsetneq AG \subsetneq NF.$$

Nevertheless, the Day-Von Neumann conjecture is true for groups of euclidean transformations, due to the following theorem of Tits:

Theorem 2.13 (Tits Alternative [22])**.** *Let G be a subgroup of $GL_n(K)$, the group of $n \times n$ matrices with coefficients in the field K. If K has characteristic 0, then either G contains a free subgroup of rank 2 or G is almost solvable (i.e., G contains a solvable normal subgroup with finite index).*

Remark that almost solvable groups are amenable: if $H \lhd G$ is solvable, there is a short exact sequence

$$H \to G \to G/H,$$

where H is solvable and G/H finite, and so both are amenable. This is relevant because, again, amenability was in fact motivated by the Banach-Tarski paradox, where the groups acting are groups of euclidean transformations.

2.4 Følner's characterization

The theorem in this section is the main tool used in Geometric Group Theory to work with amenability. We will first prove the theorem in a more general setting, and then restrict ourselves to the finitely generated case, where we can relate it to locally finite Cayley graphs. This will be the framework where we will work from here on, and it will allow us to change our mindset into a more combinatorial one.

2.4.1 Følner's theorem

We will assume for simplicity that the groups we are considering are discrete, and that they are equipped with the counting measure, given by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

Using this measure, we can construct normed spaces of linear mappings $G \to \mathbb{R}$, such as $l^1(G)$ or $l^\infty(G)$, and use tools from functional analysis. The following results and proofs can be generalized to locally compact groups, which can be equipped with the Haar measure. It is remarkable that this measure is only defined on the Borel algebra of G (algebra generated by closed sets in G), and not necessarily on $\mathcal{P}(G)$, so the existence of this measure does not guarantee that the group will be amenable. Moreover, neither the Haar measure nor the counting measure have total measure one (in general), so again, their existence does not guarantee that the group will be amenable.

Følner's idea is to find finite sets that do not change too much when a generator of the group acts on them. These sets will admit a measure (since they are finite), which is almost invariant by the action of the group. Then, we will use a compactness principle to extend these measures to an invariant one defined on the whole group. This idea is quite simple, but changes completely the point of view of the problem. Note that a measure which is invariant by generators will also be invariant by any element of the group, since the element can be written as a word in the generators.

Let us formalize all these ideas. Recall that the symmetric difference of A and B,

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

is the set of elements that are either in A or in B, but not in both.

Definition 2.14. Let G be a discrete group, and let $X \subset G$ be a generating set. A **Følner sequence** for X is a sequence of finite non-empty sets (F_n), such that, for any generator $g \in X$,

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

Equivalently, we have the following condition:

Definition 2.15. A discrete group G satisfies the **Følner condition** if for every finite $X' \subseteq X$ and $\epsilon > 0$, there exists a finite non-empty subset $F \subseteq G$ such that for each $g \in X'$ we have

$$\frac{|gF \triangle F|}{|F|} < \epsilon.$$

Proposition 2.16. *A group G with generators X contains a Følner sequence if and only if it satisfies the Følner condition.*

Proof. Suppose G satisfies the condition, and write $X = \bigcup_n X_n$ as an ascending union of finite sets, $X_1 \subset X_2 \subset \dots$ Set $\epsilon_n = 1/n, n \in \mathbb{N}$. By Følner's condition, for each $n \in \mathbb{N}$ there exists a finite subset F_n such that for every $g \in X_n$ we have $|gF_n \triangle F_n|/|F_n| \leq 1/n$. Now, given $g \in X$, there exists X_m such that $g \in X_k$ for $k \geq m$. By construction, $|gF_n \triangle F_n|/|F_n| \leq 1/n \to 0$.

Conversely, suppose that G has a Følner sequence, (F_n). Given $\epsilon > 0$, there exists n_0 such that, if $n \geq n_0$, then $|gF_n \triangle F_n|/|F_n| < \epsilon$ for all $g \in X$, so G satisfies Følner's condition. $\square$

We will treat both conditions equivalently. In the practice, finding a Følner sequence can be easier, but in some proofs it is more convenient to use Følner's condition.

The importance of these ideas is that they characterize amenability:

Theorem 2.17 (Følner [11])**.** *A group is amenable if and only if it satisfies Følner's condition.*

Proof. The backward implication is just a compactness principle argument. It is enough to find a measure which is invariant by generators.

Let $\{F_n : n \in \mathbb{N}\}$ be a Følner sequence. For each $n > 0$, let $\mathcal{M}_n$ be the set of finitely additive measures $\mu \colon \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$ and, for all $g \in X, A \subseteq G$, it is

$$|\mu(gA) - \mu(A)| \leq \frac{|gF_n \triangle F_n|}{|F_n|}.$$

Let us check that $\mathcal{M}_n \neq \emptyset$ for all $n > 0$:

Define $\mu_n \colon \mathcal{P}(G) \to [0,1]$ as $\mu_n(A) = \frac{|A \cap F_n|}{|F_n|}$. Then, clearly μ_n is finitely additive and $\mu_n(G) = 1$. For the other condition, observe that, given $g \in X, A \subseteq G$, we have that, since g defines a bijection in G, it is $|g(A \cap F_n)| = |A \cap F_n|$. Moreover,

$$g(A \cap F_n) = (g(A \cap F_n) \cap F_n) \cup (g(A \cap F_n) \setminus F_n) \subseteq$$
$$\subseteq (gA \cap F_n) \cup (gF_n \setminus F_n) \subseteq (gA \cap F_n) \cup (gF_n \triangle F_n).$$

Hence, $|A \cap F_n| \leq |gA \cap F_n| + |gF_n \triangle F_n|$, and so

$$|A \cap F_n| - |gA \cap F_n| \leq |gF_n \triangle F_n|.$$

A similar argument shows that

$$|gA \cap F_n| - |A \cap F_n| \leq |gF_n \triangle F_n|.$$

Thus, we obtain

$$|\mu_n(gA) - \mu_n(A)| = \left| \frac{|gA \cap F_n| - |A \cap F_n|}{|F_n|} \right| \leq \frac{|gF_n \triangle F_n|}{|F_n|}.$$

Hence, $\mu_n \in \mathcal{M}_n \neq \emptyset$.

Now, by Tychonoff's Theorem, $[0,1]^{\mathcal{P}(G)}$ is compact. It can be seen that each $\mathcal{M}_n$ is closed in $[0,1]^{\mathcal{P}(G)}$ (same idea as in Thm 2.10(4). Moreover, since $\frac{|gF_n \triangle F_n|}{|F_n|} \to 0$, we can choose a subsequence $(n_i)_{i \in \mathbb{N}}$, $n_i \to \infty$, such that $\mathcal{M}_{n_{i+1}} \subset \mathcal{M}_{n_i}$ for all $i \in \mathbb{N}$. In particular, the family $\mathcal{M}_{n_i}$ will have the finite intersection property, and so there exists $\mu \in \cap_{i>0} \mathcal{M}_{n_i}$. By construction, μ is G-invariant.

For the right implication, the proof will be based on the Hahn-Banach Separation Theorem.

Theorem 2.18 (Hahn-Banach Separation Theorem (AC))**.** *Let A and B be non-empty and convex subsets of a real normed vector space V. Furthermore, assume that A and B are disjoint and that A has an interior point. Then there is a hyperplane that separates A and B.*

Let

$$\Phi := \{f \in l^1(G) \colon f \geq 0 \text{ is finitely supported and } ||f||_1 = \sum_{g \in G} |f(g)| = 1\},$$

Claim. If G is amenable, then for every finite $A \subset G, \epsilon > 0$ there exists $f \in \Phi$ such that $||f - {}_a f||_1 \leq \epsilon$ for all $a \in A$.

Suppose this is not the case, and let be $A \subset X$, $\epsilon > 0$ such that for every $f \in \Phi$ there is some $a \in A$ with $||f - {}_af||_1 > \epsilon$. Now, the set $\{f - {}_af \colon f \in \Phi\}$ is convex, and it is bounded away from zero (since every element is at distance at least ϵ). By the Hahn-Banach Separation Theorem, there exists a linear functional $\tilde{m} \in l^1(G)^*$ and $t > 0$ such that $\tilde{m}(f - {}_af) \geq t > 0$ for all $f \in \Phi$. Now, since $l^1(G)^* \cong l^\infty(G)$, there exists $m \in l^\infty(G)$ such that $\langle f - {}_af, m\rangle = \sum_{x\in G}(f - {}_af)(x)m(x) \geq t$ for all $f \in \Phi$. Then, for any $y \in G$, if we choose $f = \chi_y$, it is

$$\begin{aligned}\langle \chi_y - {}_a\chi_y, m\rangle &= \sum_{x\in G} \chi_y(x)m(x) - \chi_y(a^{-1}x)m(x) = \\ &= m(y) - m(ay) = m(y) - {}_{a^{-1}}m(y) \geq t > 0.\end{aligned}$$

On the other hand, since G is amenable, there exists a left invarian mean $M \colon B(G) \to \mathbb{R}$, and M induces a left invariant mean on $l^\infty(G)$. By the properties of M, we have that $M(m - {}_{a^{-1}}m) \geq t > 0$, which contradicts the fact that M is left invariant. Hence, the claim holds.

Now, we see that G satisfies Følner's condition. Let $A \subset X, \epsilon > 0$. By the claim, there is $f \in \Phi$ such that, for all $a \in A$,

$$||f - {}_af||_1 = \sum_{g\in G} |f(g) - f(a^{-1}g)| \leq \frac{\epsilon}{|A|}.$$

Since f is finitely supported, we can write $f = \sum_{i=1}^n c_i\chi_{F_i}$, for a descending family $F_1 \supset F_2 \supset \cdots \supset F_n$ and $c_i > 0$. Then, it is $\sum c_i|F_i| = ||f||_1 = 1$. Finally, if $g \in aF_i \triangle F_i$, then $f(g)$ and $f(a^{-1}g)$ are in different layers of the representation of f. Hence, $|f(g) - {}_af(g)| \geq c_i$, and so for all $a \in A$,

$$\sum_{i=1}^n c_i|aF_i\triangle F_i| \leq \sum_{g\in G} |f(g) - {}_af(g)| = ||f - {}_af||_1 \leq \frac{\epsilon}{|A|} = \frac{\epsilon}{|A|}\sum_{i=1}^n c_i|F_i|.$$

Thus,

$$\sum_{i=1}^n \sum_{a\in A} c_i|aF_i\triangle F_i| \leq \epsilon \sum_{i=1}^n c_i|F_i|.$$

Hence, there is at least one term in the first sum (in i) such that $\sum_{a\in A} |aF_i\triangle F_i| \leq \epsilon|F_i|$, and so

$$|aF_i\triangle F_i| \leq \sum_{b\in A} |bF_i\triangle F_i| \leq \epsilon|F_i|$$

implies that $\frac{|aF_i\triangle F_i|}{|F_i|} \leq \epsilon$ for all $a \in A$, so G satisfies Følner's condition. $\square$

Corollary 2.19. *$\mathbb{Z}$ is amenable.*

Proof. $\mathbb{Z} = \langle 1 \rangle$ has a Følner sequence, given by $F_n = \{-n, \dots, n\}$. It is clear that $|(1 + F_n) \triangle F_n| = 2$, and $|F_n| = 2n + 1$. Hence,

$$\lim_n \frac{|(1 + F_n) \triangle F_n|}{|F_n|} = \lim_n \frac{2}{2n+1} = 0.$$

In particular, $\mathbb{Z}$ is amenable. This completes the proof of Theorem 2.10(5). $\square$

Notice that Følner sequences are defined for a group with a generating set, so one could ask whether having a Følner sequence depends on the generating set that we choose. Følner's theorem guarantees that these sequences are equivalent to the existence of a measure, so we know that if Følner's condition is satisfied for a generating set, then it is satisfied for any generating set. Nevertheless, we do not need Følner's theorem for this. We will prove that being amenable is quasi-isometry invariant.

2.4.2 Some comments on Cayley graphs and boundaries.

Følner sequences are intimately related with the Cayley graph of a group.

Recall that, for a finitely generated group, and a finite symmetric set of generators X, we can define the **Cayley graph** $\Gamma(G, X)$, whose vertices are the elements of G, and two vertices a, a' will share an edge, labelled $e = (a, a', g)$ if there exists a generator $g \in X$ such that $a' = ga$.

Any graph is a metric space, and so this provides a distance (and thus a topology) to our finitely generated group G. Nevertheless, there are several ways to define the boundary of a set in Γ (apart from the topological one). We present here the ones that we will use:

The **inner boundary** of a subset $C \subseteq \Gamma$, denoted $\partial_X C$, is the set of elements that can leave C when multiplied by an element in X. Formally,

$$\partial_X C = \{a \in C : \text{there exists } g \in X \text{ such that } ga \notin C\}.$$

The **outer boundary** of $C \subset \Gamma$ is the set of elements outside C that can enter C when multiplied by an element in X:

$$\partial'_X C = \partial_X (\Gamma \setminus C) = \{a \in \Gamma \setminus C : \text{there exists } g \in X \text{ such that } ga \in C\}.$$

In particular,

$$\bigcup_{x \in X} (xA \triangle A) = \partial_X A \cup \partial'_X A.$$

We also have the **Cheeger boundary**, as the set of edges joining C to $\Gamma \setminus C$:

$$\partial_X^{\#} = \{(a, a', g) \in E : a \in C, a' \in \Gamma \setminus C\}.$$

and the k**-boundary**: the points of C at distance at most k from $\Gamma \setminus C$:

$$\partial_X^k C = \{a \in C : \text{there exist } g_1, \ldots, g_k \in X \cup \{e\} \text{ such that } g_1 \ldots g_k a \notin C\}.$$

Nevertheless, these are all equivalent for our purposes:

Proposition 2.20. *Let G be a finitely generated group with generators X, and $|X| = N$. Let $A \subset G$ be finite. Then,*

1. $\frac{1}{2N}|\partial_X A| \leq |\partial'_X A| \leq 2N|\partial_X A|$.
2. $|\partial_X A| \leq |\partial_X^{\#} A| \leq 2N|\partial_X A|$.
3. *For every $k \in \mathbb{N}$, there exists a constant $C = C(k) > 0$ such that $C|\partial_X^k A| \leq |\partial_X A| \leq |\partial_X^k A|$.*

Proof.

1. By definition, any point in the inner boundary is connected to a point in the outer bundary. Since a point can have up to $2N$ neighbours, we obtain the right inequality. The left inequality is deduced from the fact that $\partial'_X A = \partial_X(\Gamma \setminus A)$.

2. For the right inequality, again, if an edge is in $\partial_X^{\#} A$, then one of its ending points must be in $\partial_X A$, and there can be up to $2N$ edges for each point in $\partial_X A$. For the left one, any point in $|\partial_X A|$ must have at least one edge connecting it to $\Gamma \setminus C$.

3. Clearly $\partial_X A \subseteq \partial_X^k A$, so we have the right inequality. For the left one, remark that a path going from A to $\Gamma \setminus A$ must go through $\partial_X A$ at some point. Thus, the k-boundary is contained in the union of all the balls of radius k centered at some point in $\partial_X A$ and, by transitivity, all these balls have the same cardinality. Letting C be this cardinality, the inequality follows.

□

We can define Følner sequences for any of these boundaries, as sequences of finite sets (F_n) such that, for some finite generating set X,

$$\lim_{n \to \infty} \frac{|\partial_X F_n|}{|F_n|} = 0,$$

and the previous proposition guarantees that being a Følner sequence for a particular boundary implies being Følner for any of the other boundaries. Thus, we may use any boundary in Følner's theorem to check amenability, and it will apply the same.

2.4.3 Quasi-isometry invariance

To see how to work with Følner's idea and Cayley graphs, let us show one of the properties that are most relevant to group theorists about amenability: it is invariant through quasi-isometries.

Definition 2.21. Let $(X, d), (X', d')$ be metric spaces. A **quasi-isometry** is a map $f\colon X \to X'$ such that there exists $g\colon X' \to X$ (called quasi-inverse) and a constant $C > 0$ so that, for any $x, y \in X, x', y' \in X'$,

1. $d'(f(x), f(y)) \leq Cd(x, y) + C$,
2. $d(g(x'), g(y') \leq Cd'(x', y') + C$,
3. $d(x, g(f(x))) \leq C$,
4. $d'(x', f(g(x'))) \leq C$.

We say that X and X' are **quasi-isometric**, or **coarse equivalent**, if there exists a quasi-isometry $f\colon X \to X'$. It is straightforward to check that being quasi-isometric is an equivalence relation.

Proposition 2.22. *Given a group G and two symmetric generating sets, X, X', (G, d_X) and $(G, d_{X'})$ are quasi-isometric.*

Proof. Write the elements in X as words in elements of X'. Let L be the longest length of any of these words. Then, $id\colon G \to G$ satifies $d(g, h) \leq Ld'(g, h)$. In a similar fashion, there is $L' > 0$ such that $d'(g, h) \leq L'd(g, h)$. Clearly, $d(g, id(g)) = 0$. Using $C = \max\{L, L'\}$ we have all conditions. □

It would be desirable that Følner sequences were preserved through quasi-isometries, but this does not happen in general: the image of a Følner could be really different from the original sequence. For instance, connectivity need not be preserved, so we could get very big boundaries. To solve this connectivity problem, we use balls around the images: given a set A, we will use

$$Z = \bigcup_{x \in A} B(f(x), C).$$

Let us first introduce a short lemma:

Lemma 2.23. *Given a quasi-isometry $f\colon G \to H$, and $y \in H$,*

$$f^{-1}(y) \subset B(g(y), C).$$

Proof. If $f(x) = y$, then $d(x, g(y)) = d(x, gf(x)) \leq C$. □

Theorem 2.24. *Let G, H be finitely generated groups, and let $f\colon G \to H$ be a quasi-isometry. If G is amenable then H is amenable.*

We use inner boundaries, although as we mention, this is not restrictive.

Proof. Let $K = |B(x, C)|$ be the cardinal of a radius C in G (which does not depend on $x \in G$). By lemma 2.23, at most K elements in our finite set A have the same image, so

$$|f(A)| \geq \frac{|A|}{K},$$

and then also

$$|Z| \geq \frac{|A|}{K}.$$

Let us look at the boundaries now. Given $y \in \partial Z$, by definition there exists $y' \notin Z$ such that $d(y, y') = 1$ (they are joined by a generator.) Then, for all $x \in A$, since $y' \notin Z$,

$$d(f(x), y') \geq C + 1.$$

Also, as $y \in Z$, there is $x_0 \in A$ such that $d(f(x_0), y) \leq C$. Using triangle inequality,

$$d(f(x_0, y')) \leq d(f(x_0), y) + d(y, y') = C + 1,$$

so it must be $d(f(x_0), y')) = C + 1$ and so $d(f(x_0), y) = C$.

Now, let us go back to G. Let $x = g(y), x' = g(y')$. We claim that $x' \notin A$. To see this, notice that

$$d(f(x'), y') = d(f(g(y')), y') \leq C,$$

and $d(y, f(A)) \geq C + 1$, so $f(x') \notin f(A)$. Nevertheless,

$$\begin{aligned} d(x_0, x') \leq d(x_0, x) + d(x, x') \leq d(x_0, x) + d(g(y), g(y')) \leq \\ \leq d(x_0, x) + Cd(y, y') + C = d(x_0, x) + 2C, \end{aligned}$$

and

$$\begin{aligned} d(x_0, x) = d(x_0, g(y)) \leq d(x_0, g(f(x_0))) + d(g(f(x_0)), g(y)) \leq \\ \leq C + Cd(f(x_0), y) + C \leq C^2 + 2C. \end{aligned}$$

Thus, $d(x_0, x') \leq C^2 + 4C$. Let $L = C^2 + 4C, M = |B(f(x_0), C)|$. From our work, we get

$$\frac{|\partial Z|}{M} \leq |\partial^L A|.$$

And so,

$$\frac{|\partial Z|}{|Z|} \leq \frac{|\partial^L A|}{M|Z|} \leq \frac{K|\partial^L|}{M|A|}$$

So now, if F_n is a Følner sequence in G, with ratio $\frac{K|\partial^L A|}{M|A|}$ small enough (which we can find, because of Proposition 2.20), we can make the quotient as small as we need. □

2.5 Growth, cogrowth and amenability

The growth rate of a group is a measure of how many words appear when we consider longer and longer words from an alphabet of generators. If this rate is not too large, then the balls in our Cayley graph will be Følner sets, and so our group will be amenable. Nevertheless, the converse is not true: there are amenable groups with large growth rate. To solve this problem, there is a refinement of the concept of growth, called cogrowth, which characterizes amenability: The idea is to count the number of words of the free group that vanish when they are interpreted in G. We state the main results of this theory, but we will not prove them.

We will only work with finitely generated groups, but as we mentioned this is not restrictive.

2.5.1 Growth

Given a finitely generated group G and a set of generators X, define the **length** of a reduced word ω as the word length of the shortest representative of ω as a word in X, i.e., $l_X = d_X(g, e)$. This length defines a metric in G, so we can consider balls $B_X(n) = \{\omega \colon l_X(\omega) \leq n\}$. With this in mind, let us define the growth function:

Definition 2.25. Let X be a finite symmetric subset of generators of a group G. The **growth function** for G with respect to X, $\gamma_X^G \colon \mathbb{N} \to \mathbb{N}$, is defined by $\gamma_X^G(n) = |B_X(n)|$, the number of words in G of length at most n.

Recall that a function f is said to be **dominated** by a function g if there are constants $\alpha, \beta > 0$ such that $f(x) \leq \alpha g(\beta x)$ for all $x \in \mathrm{Dom} f$. We will denote this by $f \preceq g$. If $f \preceq g$, and $g \preceq f$, we will say that f and g are **asymptotically equivalent**, and write $f \sim g$. It is easy to show that this is an equivalence relation. The first thing we show is that all growth functions of a group are equivalent.

Proposition 2.26. *Let X, X' be two finite symmetric generating subsets of G. Then, $\gamma_X^G \sim \gamma_{X'}^G$.*

Proof. Since changing generators is a quasi-isometry, $l_X(\omega) \leq C \cdot l_{X'}(\omega)$ for some $C > 0$. In particular, $B_{X'}(n) \subseteq B_X(Cn)$, and $\gamma_{X'}(n) \leq \gamma_X(Cn)$. Symmetry yields the equivalence. □

Thus, asymptotically, it does not matter which generating subset we choose, and so we can drop the subscript X.

Another important property is that γ is submultiplicative, that is,

$$\gamma(m+n) \leq \gamma(m)\gamma(n).$$

This happens because a word of length $n+m$ can be seen as a concatenation of two words of lengths m and n. This property implies that $\gamma(n) \leq \gamma(1)^n$, and so γ is always bounded by an exponential function. We will say that G **has exponential growth** if γ_X^G is equivalent to an exponential function for some generating set X (and thus for every). Otherwise, we will say that G **has subexponential growth**. The following proposition characterizes exponential growth:

Proposition 2.27. *1.* $\lim_n \gamma(n)^{1/n}$ *exists, and* $\lim_n \gamma(n)^{1/n} = \inf_t a_t^{1/t}$

2. G *has exponential growth if and only if* $\lim \gamma(n)^{1/n} > 1$.

Proof. Fix $t \in \mathbb{N}$ and, for each $n > 0$, write $n = qt + r$, $0 \leq r < t$. Remark that $q = q(n), r = r(n)$ depend on n. Since γ is submultiplicative,

$$\gamma(n)^{1/n} \leq \gamma(t)^{q/n}\gamma(r)^{1/n}.$$

Since $r < t$, and t is fixed, we deduce from $n = qt + r$ that

$$\lim_n \frac{q}{n} = \frac{1}{t}.$$

And thus, for all $t > 0$,

$$\limsup_n \gamma(n)^{1/n} \leq a_t^{1/t}.$$

In particular, $\limsup \gamma(n)^{1/n} \leq \inf_t a_t^{1/t}$. On the other hand, $\inf \gamma(n)^{1/n} \leq \liminf \gamma(n)^{1/n}$, so the limit exists and the first statement is proven.

For the second part, let $\lambda = \lim_n \gamma(n)^{1/n}$. We know that

$$\gamma(n)^{1/n} > \inf_n \gamma(n)^{1/n} = \lambda,$$

so if $\lambda > 1$, then $\gamma(n) \geq \lambda^n$ and γ has exponential growth. Conversely, if $\gamma \sim \exp(n)$, there exist $c, c' > 0$ such that $\exp(n) \leq c\gamma(c'n)$ for all $n > 0$. Then, one can see that

$$\lambda = \lim_n \gamma(n)^{1/n} \geq \lim_n \frac{1}{c}\exp(\frac{n}{c'n}) > 1.$$

□

Definition 2.28. The **growth rate** of a group G with generators X is defined as

$$\omega_X(G) = \lim_n \gamma_X^G(n)^{1/n}$$

The **uniform growth rate** is defined as

$$\omega(G) = \inf_X \omega_X(G).$$

In groups with subexponential growth, the balls around the identity form a Følner sequence:

Theorem 2.29. *Groups with subexponential growth are amenable.*

Proof. First, we claim that for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $|B_{k+1}| - |B_k| \leq 1 + \epsilon$. We argue by contradiction: suppose there is $\epsilon > 0$ such that for all $k \in \mathbb{N}$ it is $|B_{k+1}| \geq |B_k|(1+\epsilon)$. Then, for any $n \in \mathbb{N}$, we have that $|B_n| \geq |B_1|(1+\epsilon)^n$, and so $\gamma(n)$ dominates an exponential function, so we reach a contradiction.

Now, given $\epsilon = \frac{1}{n} > 0$, let $k \in \mathbb{N}$ be such that $|B_{k+1}| - |B_k| \leq 1 + \frac{1}{n}$. Then,

$$0 \leq \frac{|\partial_X B_k|}{|B_k|} \leq \frac{|B_{k+1}| - |B_k|}{|B_k|} = \frac{|B_{k+1}|}{|B_k|} - 1 \leq \frac{1}{n},$$

and so $(B_k)_k$ is a Følner sequence. □

This also gives us a proof that abelian groups are amenable (and in fact the Følner sets that we used in Corollary 2.19 were these balls in $\mathbb{Z}$). To see this, we just need to check that finitely generated abelian groups have subexponential growth. Note that, if we have a generating set $X = \{g_1, \ldots, g_r\}$, then the number of words of length n are the words of the form $g_1^{m_1}, \ldots, g_r^{m_r}$ such that $\sum m_i \leq n$. In particular, each $m_i \in [-n, n]$, so we have $2n+1$ options for each m_i. This implies that $\gamma_X(n) \preceq (2n+1)^r$, so abelian groups have polynomial growth, and so again, they are amenable.

Groups of subexponential growth are not only amenable. They satisfy a stronger definition:

Definition 2.30. A group G is supramenable if, for any nonempty $A \subseteq G$, there is a finitely additive, left-invariant measure, $\mu \colon \mathcal{P}(G) \to [0, \infty]$ with $\mu(A) = 1$.

In particular, choosing $A = G$ we deduce that supramenability implies amenability. It can be seen that groups containing a free semigroup cannot be supramenable, and just like in the amenability case, there are examples of groups not containing free semigroups that are neither supramenable.

Again, we may ask whether there are groups that are amenable but not supramenable. Grigorchuk found p-groups which are amenable, do not contain free semigroups and are not supramenable.

By Tarski's Theorem, a group G will be supramenable if and only if no non-empty subset of G is paradoxical.

Proposition 2.31. *If G has subexponential growth, G acts on X, and A is a nonempty subset of X, then E is not G-paradoxical.*

Proof. We do this by contrapositive. Suppose E is G-paradoxical, with $E = \cup g_i A_i = \cup h_j B_j$. We can define transformations $F_1 \colon E \to \bigcup_i A_i \subset E$, given by $F_1(g) = g_i^{-1} g$ if $g \in g_i A_i$, and $F_2 \colon E \to \bigcup_j B_j \subset E$, given by $F_2(g) = h_j^{-1} g$ if $g \in h_j B_j$. By the definition of a paradoxical decomposition, F_1, F_2 are well defined, and $F_1(E) \cap F_2(E) = \emptyset$. Now, we can obtain 2^n different functions H_i, as strings of length n of compositions of F_1 and F_2. These functions have the property that, if $i \neq j$, then $H_i(E) \cap H_j(E) = \emptyset$. To see this, let p be the first position in the string of F_1's and F_2's in which H_i and H_j differ. Since F_1 and F_2 have disjoint images, the images of H_i and H_j after the pth position are also disjoint. Now, for each $x \in E$, the set $\{H_i(x)\}$ has 2^n different elements, and each $H_i(x)$ is of the form ωx, where ω is a word of length n composed by elements $g_1, \dots, g_n, h_1, \dots, h_m$. Thus, $\gamma(n)$ dominates an exponential function, and so G has exponential growth. □

Corollary 2.32. *If G is exponentially bounded, then G is supramenable.*

Proof. Since G acts on itself by left translation, we can apply the previous theorem to check that no non-empty subset of G is paradoxical. By Tarski's Theorem, for each nonempty subset E, there exists a finite G-invariant measure normalizing E. □

2.5.2 Cogrowth

Let $G = \langle a_1, \dots, a_r \colon b_1, \dots, b_s \rangle$ be a finitely presented group. Then, $G \cong F_r/N$, where N is the normal closure of $\{b_1, \dots, b_s\}$. From a word of length n, we can obtain $2r-1$ words of length $n+1$ (we are not allowed to use the inverse of the first symbol of the word). Let E_n be the set of words in F_r of length at most n. Then,

$$|E_n| = 1 + \sum_{k=1}^{n} (2r)(2r-1)^{k-1} = \frac{r(2r-1)^n - 1}{r-1},$$

and so we have that $|E_n|^{1/n} \xrightarrow{n} 2r-1$.

Notice also that for the set $S = \{a_1, \dots, a_r\}$, we have that $\pi(E_n) \subset B_S(n)$ (the length of a word can only decrease when π is applied), and so $|\pi(E_n)| \leq \gamma_S(n)$. All these ideas motivate the following definition:

Definition 2.33. Let $G = \langle S \colon R \rangle$ be a finitely presented group, and let N be the normal closure of R. We define the cogrowth function of the presentation, $\tilde{\gamma}(n) = |N \cap E_n|$.

Thus, $\tilde{\gamma}$ counts the number of words in E_n that vanish when they are interpreted in G. As with the growth function, the best way to work with this concept is to use the sequence $\tilde{\gamma}(n)^{1/n}$, and consider its limit. If $G = F_r$, then $N = \{e\}$, so $\tilde{\gamma}(n) = 1$ for all n.

On the other hand, if G has subexponential growth, then there exists some $g \in G$ such that $|\pi^{-1}(g) \cap E_n| \geq |E_n|/|\pi(E_n)|$. Indeed, if not,

$$|E_n| < \sum_{g \in \pi(E_n)} \frac{|E_n|}{|\pi(E_n)|} = |E_n|.$$

This g represents a coset in F_r/N, so there are at least this many words vanishing when applying $\pi(E_n)$, and so $|N \cap E_n| \geq \frac{|E_n|}{|\pi(E_n)|} \geq \frac{|E_n|}{\gamma(n)}$. Thus, we obtain:

$$|E_n| \geq \tilde{\gamma}(n) \geq \frac{|E_n|}{\gamma(n)}.$$

Since G has subexponential growth, $\gamma(n) \to 1$, and we have that

$$\lim \tilde{\gamma}(n)^{1/n} = \lim |E_n|^{1/n} = 2r - 1.$$

This only ilustrates the behavior of $\tilde{\gamma}$ in the extreme cases. In general, there is the following result:

Theorem 2.34. *Let G be a group with a fixed presentation using $2 \leq r < \infty$ generators. Then,*

1. $\lim \tilde{\gamma}(n)^{1/n}$ *exists.*
2. $1 \leq \lim \tilde{\gamma}(n)^{1/n} \leq 2r - 1.$
3. $\lim \tilde{\gamma}(n)^{1/n} = 1$ *if and only if there are no relations in the presentation ($G = F_r$.)*
4. *If* $\lim \tilde{\gamma}(n)^{1/n} > 1$*, then* $\sqrt{2r-1} < \lim \tilde{\gamma}(n)^{1/n} \leq 2r - 1.$

Definition 2.35. Let G be a finitely presented group. Given a presentation of G, the **cogrowth** of the presentation is defined as

$$\eta = \frac{\log\left(\lim \tilde{\gamma}(n)^{1/n}\right)}{\log(2r-1)}.$$

The previous theorem shows that $\eta \in \{0\} \cup (1/2, 1]$, $\eta = 0$ if and only if there are no relations on the presentation, and $\eta = 1$ if G has subexponential growth. This last value of η gives another characterization for amenability:

Theorem 2.36. *Let G be a finitely generated group. Then, G is amenable if and only if $\eta = 1$ for all presentations of G with finitely many generators.*

Finally, as a Corollary, we can define $\eta(G)$ as the infimum of η over all presentations with finitely many generators. Then, the following theorem reunites generalizes many ideas that we already proved:

Corollary 2.37. *A group G is amenable if and only if $\eta(G) = 1$. G has no free subgroups of rank 2 if and only if $\eta(G) \geq 1/2$. G has a free subgroup of rank 2 if and only if $\eta(G) = 0$.*

CHAPTER

3

Uniform non-amenability

From now on, we will focus on amenability using Følner sets. As we learnt in last chapter, given a particular set of generators, a group will be amenable if and only if we can find a Følner sequence for these generators. For non-amenable groups we can never find Følner sequences, but we can still compute the ratio of a finite set and its boundary, and study how close to zero we can get. This will give us an idea on "how close to being amenable" our group is. We will define the Følner constant as the infimum of this ratio among all possible finite sets and families of generators. We will study how it behaves through some operations and compute it for some particular cases. Finally, we will find groups that are not amenable but have uniform Følner constant equal to zero.

Throughout all this chapter we will be working with the inner boundary, but working with other boundaries is similar. This chapter is based on the article [2].

3.1 The Følner constant

Let us formalize the ideas in the introduction.

Definition 3.1. Given a finitely generated group G, and a finite generating set X of G, we define the **Følner constant**

$$\text{Føl}_X G = \inf\left\{\frac{|\partial_X A|}{|A|} : A \subseteq G \text{ finite.}\right\},$$

and the **uniform Følner constant**, as

$$\text{Føl } G = \inf_X \text{Føl}_X G.$$

Følner's theorem states that a group G will be amenable if and only if $\text{Føl}_X G = 0$ for some (and thus for every) generating set X. If the group is not amenable, then having a lower bound on $\text{Føl}_X G$ for some X does not give us any information, since a different set of generators could achieve a lower bound. To get a number which only depends on the group, we introduce Føl G. Remark that if a group is amenable, then it will be Føl $G = 0$, but the converse need not be true: it could be possible to get a sequence of generators, $(X_k)_k$ such that $\text{Føl}_{X_k} G \to 0$ when $k \to \infty$, and so this would mean that Føl $G = 0$, even if for each set of generators $\text{Føl}_X G$ is positive. This idea provides a stronger property than non-amenability:

Definition 3.2. A finitely generated group G is called **uniformly non-amenable** if Føl $G > 0$.

At the end of this chapter we will see that, indeed, there are non-amenable groups which are not uniformly non-amenable.

Recall that, as in Theorem 2.29, the balls of radius n can be used as Følner sets in groups with subexponential growth. This same idea can be applied in general to get an upper bound on Føl G:

Proposition 3.3. *Let G be a finitely generated group, and let X be a finite generating set. Then,*

$$\text{Føl}_X G \leq 1 - \frac{1}{\omega_X(G)},$$

and so

$$\text{Føl } G \leq 1 - \frac{1}{\omega(G)}.$$

Proof. Note that $\partial_X B_X(n) \subseteq B_X(n) \setminus B_X(n-1)$. Hence, for all $n \in \mathbb{N}$,

$$\text{Føl}_X G \leq \frac{|B_X(n)| - |B_X(n-1)|}{|B_X(n)|} \leq 1 - \frac{|B_X(n-1)|}{|B_X(n)|}.$$

On the other hand,

$$|B_X(n)|^{1/n} = \left(\frac{|B_X(n)|}{|B_X(n-1)|} \cdot \frac{|B_X(n-1)|}{|B_X(n-2)|} \cdot \ldots \cdot \frac{|B_X(1)|}{|B_X(0)|} \right)^{1/n} \geq \inf_{k=1}^{n} \frac{|B_X(k)|}{|B_X(k-1)|},$$

and taking limits,

$$\inf_n \frac{|B_X(n)|}{|B_X(n-1)|} \leq \lim_n |B_X(n)|^{1/n}.$$

Joining both results, we deduce

$$\text{Føl}_X G \leq 1 - \frac{1}{\omega_X(G)}.$$

This holds for all X, so we also get the bound for Føl G. □

Just as we saw before, if the growth rate is 1 (subexponential growth) for some generating set, then our group will be amenable. Now we also get a bound for groups of exponential growth.

We will now study how these constants behave through subgroups and quotients, but before this, let us introduce a lemma explaining how the ratio $\frac{|\partial A|}{|A|}$ behaves when dividing our set into connected components:

Lemma 3.4. *Let Γ be a Cayley graph of a finitely generated group, and let A be a finite subset of G. If we have a partition $A = A_1 \cup \cdots \cup A_n$ such that $\partial A = \partial A_1 \cup \cdots \cup \partial A_n$, then it is*

$$\min_{i=1}^{n} \frac{|\partial A_i|}{|A_i|} \leq \frac{|\partial A|}{|A|} \leq \max_{i=1}^{n} \frac{|\partial A_i|}{|A_i|}.$$

Proof. Recall the following property of real numbers: If $\frac{a}{b} \leq \frac{c}{d}$, then

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.$$

Using fractions $\frac{|\partial A_i|}{|A_i|}$ the proof follows. □

It is also remarkable that if $|A| = 1$, then $\frac{|\partial A|}{|A|} = 1$. These two results simplify our study to connected Følner sets, since getting bounds for each connected component also gives a bound for the whole set.

3.1.1 Subgroups

We will study subgroups in two parts: we first consider the case in which the generators of H can be extended to a system of generators of G, and then we will generalize this to work with any system of generators of H.

First of all, notice that adding a generator adds edges to the Cayley graph, and so it can only increase the boundary of a finite set. This provides a trivial bound:

Lemma 3.5. *Let G be a finitely generated group, X finite generating system and $g \in G$. Let $Y = X \cup \{g\}$. Then,*

$$\text{Føl}_X G \leq \text{Føl}_Y G.$$

Proof. For any finite $A \subset G$, we have $\partial_X A \subseteq \partial_Y A$, and so $\frac{|\partial_X A|}{|A|} \leq \frac{|\partial_Y A|}{|A|}$. □

This idea can be used for subgroups whose generating set extends to a generating set of the whole group. The idea for this is to abuse the fact that a group admits a partition into cosets. In our case, the Cayley graph of the group will be made of copies of the Cayley graph of the subgroup, and thus will allow us to compare the Følner constants.

Theorem 3.6 (First Subgroup Theorem)**.** *Let G be a group, and let $X = \{x_1, \ldots, x_n\}$ be a finite generating system of G. Let $m < n$, and let H be the subgroup of G generated by $Y = \{x_1, \ldots, x_m\}$. Then,*

$$\text{Føl}_X G \geq \text{Føl}_Y H.$$

Proof. We need to show that for any finite $A \subset G$, it is $\frac{|\partial_X A|}{|A|} \geq \text{Føl}_Y H$.

Let A be a finite subset of G, and let $y_1, \ldots, y_k$ be representatives of the cosets intersecting A. Define $A_i = A \cap y_i H$, the intersection with each of these cosets. Note that $A_i \neq \emptyset$.

The Cayley graph $\Gamma(H, Y)$ is a subgraph of $\Gamma(G, X)$. Moreover, if we only consider edges labelled in Y, then $\Gamma(G, X)$ consists of disjoint copies of $\Gamma(H, Y)$ (one for each coset). Note that $y_i^{-1} A_i$ is a finite subgraph in H, and applying y_i^{-1} maintains the size of the boundary. Hence,

$$\frac{|\partial_Y A_i|}{|A_i|} = \frac{|\partial_Y (y_i^{-1} A_i)|}{|y_i^{-1} A_i|} \geq \text{Føl}_Y H.$$

Now, by Lemma 3.5,

$$\frac{|\partial_X A|}{|A|} \geq \frac{|\partial_Y A|}{|A|} = \frac{\sum_i |\partial_Y A_i|}{\sum_i |A_i|} \geq \text{Føl}_Y H.$$

□

This simple result is already quite useful to obtain lower bounds. We have the following:

Proposition 3.7. *Let G be a finitely generated group that admits a surjective homomorphism $\phi\colon G \to F_2$. Then,*

$$\text{Føl}\, G \geq \text{Føl}\, F_2.$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ generate G. Given $1 \leq i \neq j \leq n$, the subgroup $\langle \phi(x_i), \phi(x_j) \rangle$ in F_2 is either cyclic or free of rank 2. As ϕ is surjective, we can find generators x_i, x_j such that $\langle \phi(x_i), \phi(x_j) \rangle$ is free of rank 2, and so $\langle x_i, x_j \rangle$ must be a free non-abelian subgroup of G (if there was any relation in $\langle x_i, x_j \rangle$, it would also appear in $\langle \phi(x_i), \phi(x_j) \rangle$, but this is free). The result follows from the First Subgroup Theorem. $\square$

We will soon compute Føl F_2, and check that it is bigger than 0, so that this bound is not trivial. This will tell us that, in particular, G is not amenable, but we already knew this, since such a G must contain a free non-abelian subgroup, as shown in the proof.

Remark that the theorem does not give us bounds on the uniform Følner constants, since we could have generating systems of G for which no subset generates H.

In general, this theorem does not allow us to compare the Følner constants of any sets of generators in H and G. The same proof does not work in the general case, since the Cayley graphs are not directly related to each other. The solution will be to write generators of H in terms of the generators of G, and the price to pay is getting a slightly worse bound.

Theorem 3.8 (Second Subgroup Theorem)**.** *Let G be a finitely generated group, $X = \{x_1, \ldots, x_n\}$ a generating system for G. Let $H \leq G$ be a subgroup, generated by $Y = \{y_1, \ldots, y_m\}$. Let L be the maximum length of the y_i as words in X. Then,*

$$F\o l_X G \geq \frac{1}{1+mL} F\o l_Y H.$$

Proof. Let $A \subset G$ be finite. We can follow the proof just like in the last theorem, to see that

$$\text{Føl}_Y H \leq \frac{|\partial_Y A|}{|A|}$$

The problem is that we can not directly compare this to $\text{Føl}_X G$. Nevertheless, every element in $\partial_Y A_i$ can be joined with a point outside A_i using some y_j. Let us write y_j as a word w_j in the generators X (i.e., a path in $\Gamma(G, X)$), and let $L = \max l(w_j)$. This path must go through some vertex in $\partial_X A$ (because it leaves A at some point), and this is not the final vertex of the path. Let $z \in \partial_X A$. It may be that $z \in \partial_Y A$; otherwise, there are at most $l(\omega_j) - 1 \leq L - 1$ ways in which a path ω_j may pass through z withouth z being the initial or final vertex.

Hence, a vertex $z \in \partial_X A$ corresponds at most to

$$1 + \sum_{j=1}^{m} (l(\omega_j) - 1) \leq 1 + mL,$$

different vertices in $\partial_Y A$, and each vertex in $\partial_Y A$ has one corresponding vertex in $\partial_X A$. This implies that

$$|\partial_Y A| \leq 1 + \sum_{j=1}^{m} (l(\omega_j) - 1)|\partial_X A| \leq (1 + mL)|\partial_X A|.$$

This is true for any finite subset of G, so the statement follows. □

Again, remark that this does not give us a bound on the uniform Følner constants. This is due to the fact that, in general, the number and the maximum length of y_i as words need not be bounded for different sets of generators of G.

It is important to notice that these two theorems also prove that amenability is preserved through subgroups, since $\text{Føl}_X G = 0$ will imply $\text{Føl}_Y H = 0$ for any generating Y.

3.1.2 Quotients

Let us now study the behavior on quotients.

Before stating the main theorem, let us prove a helping lemma. We are going to divide a subset $\pi(A) = B \subseteq G/N$ on layers, depending on how many elements of A go to each coset. Formally, given $A \subset G$, $B = \pi(A)$, define the **i-level subset** of B as

$$B_i = \{b \in G/N : |\pi^{-1}(b) \cap A| \geq i\} \subseteq B$$

Clearly, this is a descending sequence with $B = B_1$ and $B_i = \emptyset$ for $i > |A|$.

Lemma 3.9. *Let $b, c \in G/N$ such that $d_{X'}(b, c) = 1$, i.e., such that there exists $x' = \pi(x)$ with $bx' = c$. Suppose $b \in B_i$, $c \in B_j \setminus B_{j+1}$ for some $i > j \geq 0$. Then,*

$$|\partial_X A \cap \pi^{-1}(b)| \geq i - j.$$

The following lemma will help to provide a lower bound for the boundary of A based on neighbours that are in different B_i.

Proof. Let $a_1, \dots, a_i$ be i different points in $\pi^{-1}(b) \cap A$. Then, the points $a_1 x, \dots, a_i x$ are all in $\pi^{-1}(c)$, but at most j of them can be also in A, because $c \in B_j \setminus B_{j+1}$. Hence, at least $i - j$ points in $\pi^{-1}(b) \cap A$ must be in $\partial_X A$. □

Theorem 3.10. *Let G be a finitely generated group with generators X. Let N be a normal subgroup of G, with projection $\pi\colon G \to G/N$, and define $X' = \pi(X)$. Then,*

$$\text{Føl}_X G \geq \text{Føl}_{X'} G/N,$$

and also

$$\text{Føl}\, G \geq \text{Føl}\, G/N.$$

Proof. The second statement is easily deduced from the first: for any generating system $X \subset G$,

$$\text{Føl}_X G \geq \text{Føl}_{X'} G/N \geq \text{Føl}\, G/N,$$

and so

$$\text{Føl}\, G = \inf_X \text{Føl}_X G \geq \text{Føl}\, G/N.$$

For the first statement, note that

$$\text{Føl}_{X'} G/N \leq \frac{|\partial_{X'} B_i|}{|B_i|}$$

for all non-empty B_i. Also, $|A| = \sum_i i(|B_i| - |B_{i-1}|) = \sum_i |B_i|$, so

$$\text{Føl}_{X'} G/N \leq \frac{\sum_i |\partial_{X'} B_i|}{\sum_i |B_i|} = \frac{\sum_i |\partial_{X'} B_i|}{|A|}$$

Now it is only left to prove that

$$\sum_i |\partial_{X'} B_i| \leq |\partial_X A|.$$

Let $b \in \partial_{X'} B_i$, and let $i_0 = |\pi^{-1}(b) \cap A| \geq i$ be the number of preimages of b. In order to apply the previous lemma, we look for the neighbour of b that has the least number of preimages: let

$$i_1 = \min_{x' \in X'} \{j : bx' \in B_j \setminus B_{j+1}\}.$$

Then, since b has i_0 preimages in A, and it has a neighbour with exactly i_1 preimages, b contributes to $\sum_i |\partial_{X'} B_i|$ either zero if $i_0 \leq i_1$ (all its neighbours are in B_i), or $i_0 - i_1$ otherwise (since b appears in all sets $\partial_{X'} B_j$ for all $j = i_1 + 1, \ldots, i_0$). Now, by Lemma 3.9, the preimage of b must have at least $i_0 - i_1$ points in $\partial_X A$, so

$$\sum_i |\partial_{X'} B_i| \leq |\partial_X A|.$$

□

Again, this proves in particular that amenability is preserved through quotients, as we had already shown, but it is a stronger result, since it gives a more general bound.

3.2 The Følner constant on the free group

Any finitely generated group is a quotient of a free group of finite rank, so computing the Følner constant of these groups will provide upper bounds for the Følner constant, using Theorem 3.10.

First of all, we have a direct upper bound using the growth rate (Proposition 3.3). The growth rate of the free group is $2k-1$, as it was computed at the beginning of Section 2.5.2.

Lemma 3.11. *Let $F_k = \langle x_1, \dots, x_k \rangle$ be a free group of rank k, and let $X_k = \{x_1, \dots, x_k\}$. Then,*

$$\text{Føl } F_k \leq \text{Føl}_{X_k} F_k \leq \frac{2k-2}{2k-1}.$$

We are going to show that these are actually equalities. First we show the second equality, for X_k, and then we show that this generating set always achieves the minimum value for the følner constant. Recall that $\Gamma(F_k, X_k)$ is an infinite tree in which every vertex has degree $2k$.

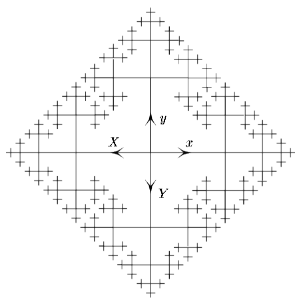


Figure 3.1: Cayley graph of a Free group of rank 2, $F_2 = \langle x, y \rangle$.

Let us define some concepts that will provide a new way to study the boundary:

Definition 3.12. Let A be a finite subset of F_k, and let $a \in A$. The **valence** of $a \in A$ is defined as the number of neighbours of a in A, i.e., the value of $|A \cap N_a|$.

A finite subset $A \subseteq \Gamma$ is **complete** if all its vertices have valence $2k$ or 1. Equivalently, A is complete if all the points in its boundary have valence 1.

Informally, A is complete if, for any vertex in the boundary, there is only a way back to A, and for a non-boundary vertex, all its neighbours in Γ are in A. Let us first prove the result for complete graphs, since it allows for a better separation between boundary and interior.

Lemma 3.13. *If A is a complete finite subset of F_k, then*

$$\frac{|\partial_{X_k} A|}{|A|} > \frac{2k-2}{2k-1}.$$

Proof. By Lemma 3.4, we may assume that A is connected. Let V_i be the number of vertices of A of valence i. Since A is a tree, $|E(A)| = |V(A)| - 1$, and so $(V_1 + V_{2k}) - (V_1 + 2kV_{2k})/2 = 1$. Hence, $V_1 = 2+(2k-2)V_{2k}$. Then, $|A| = V_1 + V_{2k} = 2+(2k-1)V_{2k}$. And since $|\partial_{X_k} A| = V_1$, we deduce

$$\frac{|\partial_{X_k} A|}{|A|} = \frac{2+(2k-2)V_{2k}}{2+(2k-1)V_{2k}} > \frac{2k-2}{2k-1}.$$

□

In the case that our subset is not complete, we can just extend it to get a complete one, and this will not worsen the bound:

Proposition 3.14. *For any finite subset $A \subset F_k$, it is*

$$\frac{|\partial_{X_k} A|}{|A|} > \frac{2k-2}{2k-1}.$$

Proof. Again, we may assume that $A \subset F_k$ is connected. Denote $N(A)$ as the number of vertices in A with valence different from 1 and $2k$. Let us argue by induction on $N(A)$. If $N(A) = 0$, then A is complete, and the result holds.

Suppose $N(A) = K$ and the statement is true for all sets B with $N(B) < K$. Let v be a vertex of valence l, with $1 < l < 2k$. Let A' be the set obtained by adding to A all the other $2k-l$ neighbours of v that are not in A, along with their respective edges. Then, these new vertices belong to $\partial_{X_k} A'$, so $|\partial_{X_k} A'| = |\partial_{X_k} A| + 2k - l - 1$ and $|A'| = |A| + 2k - l$. Moreover, now $N(A') = N(A) - 1$, and so by induction hypothesis

$$\frac{|\partial_{X_k} A'|}{|A'|} = \frac{|\partial_{X_k} A| + 2k - l - 1}{|A| + 2k - l} > \frac{2k-2}{2k-1},$$

and we conclude

$$\frac{|\partial_{X_k} A|}{|A|} > \frac{2k-2}{2k-1}.$$

□

This gives a lower bound on $\text{Føl}_{X_k} F_k$, and it is the same as the upper bound given by the growth rate. Thus, we get

Corollary 3.15.

$$\textit{Føl } F_k = \frac{2k-2}{2k-1}.$$

Let us now compute Føl F_k. To do this, we just need to study how changing the base affects to the Følner constant.

Proposition 3.16. *One has*
$$\text{Føl } F_k = \frac{2k-2}{2k-1}.$$

Proof. Let Y be a finite generating system for F_k. Let $\pi\colon F_k \to \mathbb{Z}^k$ be the abelianization map, and $Y^{ab} = \pi(Y)$. Since Y^{ab} generates $\mathbb{Z}^k$, there exist $y_1, \ldots, y_k \in Y$ such that $\pi(y_1), \ldots, \pi(y_k) \in Y^{ab}$ are linearly independent, and so they generate a subgroup of $\mathbb{Z}^k$ which is isomorphic to $\mathbb{Z}^k$. Then, $H = \langle y_1, \ldots, y_k \rangle$ is a subgroup of F_k which must be isomorphic to F_k, since it maps onto $\mathbb{Z}^k$, which cannot be generated by less than k elements. By the First Subgroup Theorem,

$$\text{Føl}_Y F_k \geq \text{Føl}_{\{y_1,\ldots,y_k\}} H = \frac{2k-2}{2k-1}$$

This happens for any generating set Y in F_k, and we know that the bound can be achieved, by Corollary 3.15, so the statement follows. □

Proposition 3.17. *Let G be a finitely generated group which admits a generating system with $k \geq 2$ elements. Then, Føl$_X G \leq \frac{2k-2}{2k-1}$, and equality holds if and only if G is free with basis X.*

Analogously, Føl $G \leq \frac{2k-2}{2k-1}$ and equality holds if and only if G is free of rank k.

Proof. Since G has k generators, clearly $\omega_X(G) \leq 2k-1$, and so we get the bound in the statement from Proposition 2.20.

For the second part, we have already proven that if G is free on X, then the equality holds. For the converse, if $\text{Føl}_X G = \frac{2k-2}{2k-1}$, it needs to be $\omega_X(G) \geq 2k-1$, and so $\omega_X(G) = 2k-1$. A result from Koubi [17] states that this is only possible if G is free with basis X. □

Combining this with the bounds given by subgroups and quotients, the Følner constant gives some information about the rank of G and the maximum free quotients that it may have . These ideas also provide bounds on some families of groups, such as virtually free groups, hyperbolic groups and Burnside groups. We do not include them here, but the curious reader is referred to the original article [2] to find more information.

3.3 Non-amenable groups that are not uniformly non-amenable

Define $\mathcal{Q}_m$, the set of **marked m-generated groups**, as the family of all quotients of the free group $F(X)$, where X is a fixed free generating set with $|X| = m$. This can also be seen as the set of normal subgroups of $F(X)$, or the Cayley graphs $\Gamma(F(X)/N, X)$.

We can define a metric on $\mathcal{Q}_m$, thus making it a topological space. Let $N_1, N_2 \lhd F(X)$ be two normal subgroups, and let C_i be the Cayley graph for $F(X)/N_i$, $i = 1, 2$. Define a metric

$$D(N_1, N_2) = \inf\{\frac{1}{n+1} : B_{\mathcal{C}_1}(n) \text{ is isometric to } B_{C_2}(n)\}.$$

In this sense, two groups will be close to each other if we can find large isometric balls in their corresponding Cayley graphs.

This is a metric indeed: two groups will be at distance 0 if and only if the balls are the same for all $n \in \mathbb{N}$, i.e., if the corresponding Cayley graphs are isomorphic (and thus the groups). For the triangle inequality, one just needs to notice that if we have isomorphisms $\phi_1 \colon B_{\mathcal{C}_1}(n_1) \to B_{\mathcal{C}_2}(n_1)$ and $\phi_2 \colon B_{\mathcal{C}_2}(n_2) \to B_{\mathcal{C}_3}(n_2)$, then for $n = \min\{n_1, n_2\}$ there is an isomorphism $\phi \colon B_{\mathcal{C}_1}(n) \to B_{\mathcal{C}_3}(n)$. Choosing n_1, n_2 so that they define the distances,

$$D(N_1, N_3) \leq \frac{1}{n+1} \leq \frac{1}{n_1+1} + \frac{1}{n_2+1} = D(N_1, N_2) + D(N_2, N_3).$$

This distance allow us to define limits on $\mathcal{Q}_m$. For instance, if $N_1 \leq N_2 \leq \dots$ is an ascending sequence, then the limit group will be $G/(\cup_{k=1}^{\infty} N_i)$. This is due to the fact that more and more relations are being added, and so paths are closing in the Cayley graph of G; the limit group will be the one having all these relations and no more. Similarly, if $N_1 \geq N_2 \geq \dots$ is decreasing, then the limit group will be $G/(\cap_{k=1}^{\infty} N_i)$. In this case, we are taking relations away at each step, so the limit group will be the one containing only those relations that belong to every N_i.

These limits do not exist in general, but there is a bound depending on the upper limit:

Proposition 3.18. *Let $G = F(X)/N$ be the limit group of a sequence $\{G_k\}_{k\in\mathbb{N}}$, with $G_k = F(X)/N_k$, for $k \geq 1$. Then,*

$$\text{Føl}_X G \geq \limsup_k \text{Føl}_X G_k.$$

Proof. Let C, C_k be the Cayley graphs of G, G_k relative to X, and let $\epsilon > 0$. Since $\text{Føl}_X G$ is an infimum, there exists a finite set $A \subset G$ such that $\text{Føl}_X G \leq \frac{|\partial_X A|}{|A|} < \text{Føl}_X G + \epsilon$. Since A is finite, it is bounded, and so there is $n = n(A) \in \mathbb{N}$ such that $A \subset B_C(n)$. Again, since $G_k \to G$, there is $K = K(n) > 0$ such that for any $k > K(n)$ it is $D(N_k, N) < \frac{1}{n+2}$. Thus, for $k > K(n)$, the balls $B_C(n+1)$ and $B_{C_k}(n+1)$ are isometric. Let ϕ_k be the corresponding isometry. Let $A_k = \phi_k(A) \subset G_k$. Then,

$$\text{Føl}_X G \leq \frac{|\partial_X A_k|}{|A_k|} = \frac{|\partial_X A|}{|A|} < \text{Føl}_X G + \epsilon.$$

From this we get $\text{Føl}_X G_k < \text{Føl}_X G + \epsilon$ for any $k > K(n)$. Thus, $\limsup_k \text{Føl}_X G_k \leq \text{Føl}_X G$. $\square$

This proposition provides a way to study the Følner constants of a very well known family of groups: the **Baumslag-Solitar groups**, with presentation

$$BS(p,q) = \langle a, t : ta^p t^{-1} = a^q \rangle.$$

For instance, $BS(1,1) = \mathbb{Z}2$. These are the simplest examples of a construction that can be performed on a group: given a group G with presentation $G = \langle S : R \rangle$, subgroups $J, K \leq G$ and an isomorphism $\alpha\colon J \to K$, then an **HNN-extension** of G is a group with presentation

$$G*_\alpha = \langle S, t : R, tjt^{-1} = \alpha(j)\, \forall j \in J \rangle.$$

In the case of $BS(p,q)$, the underlying group is $\mathbb{Z} = \langle a \rangle$, and the subgroups are $J = \langle a^p \rangle, K = \langle a^q \rangle$.

One of the most useful lemmas involving this construction is Britton's lemma: using the relations, we can write any element $\omega \in G*_\alpha$ as

$$\omega = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_n} g_n,\ g_i \in G,\ \epsilon_i = \pm 1.$$

Britton's lemma states the following:

Lemma 3.19 (Britton)**.** *Let $\omega \in G*_\alpha$ be a word written as before. Suppose that*

1. $n = 0$ *and* $g_0 \neq 1 \in G$, *or*
2. $n > 0$ *and there are no subwords of the form* tjt^{-1}, *with* $j \in J$ *nor of the form* $t^{-1}kt$, *with* $k \in K$.

Then, $\omega \neq 1$ *in* $G*_\alpha$.

This lemma turns out to be really powerful when checking whether we can do cancellations in an HNN-extension, or when trying to prove that a subgroup is free:

Proposition 3.20. *If $p, q > 1$, then $BS(p,q)$ contains a free group on two generators.*

Proof. We claim that $\langle t, a^{-1}ta \rangle$ is free on two generators.

Let $\omega = (a^{-1}ta)^{n_1} t^{n_2} \dots t^{n_{k-1}} (a^{-1}ta)^{n_k}$ be a word in $\langle t, a^{-1}ta \rangle$. We may assume it is of this form by conjugating if necessary. Britton's lemma states that $\omega = 1$ if and only if ω is trivial, or it contains some subword of the form $t^{-1}a^{px}t$ or $ta^{qy}t^{-1}$ for some $x, y \in \mathbb{Z}$. But this is never possible, since all the a appearing have exponent ± 1. Hence, ω cannot contain any subword of the given forms, and so $\omega \neq 1$. Thus, the subgroup is free. $\square$

In particular, if $p, q > 1$, then $BS(p, q)$ is not amenable. Nevertheless, not all Baumslag-Solitar groups are amenable:

Proposition 3.21. $BS(1, 2)$ *is amenable:*

Proof. There is a short exact sequence

$$1 \to \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{\alpha} BS(1,2) \xrightarrow{\beta} \mathbb{Z} \to 1$$

where α sends $\frac{1}{2^n} \to t^n a t^{-n}$, and β sends $a \to 1$ and $t \to t$. Both $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are abelian, and thus amenable, so, by Theorem 2.10(3), $BS(1, 2)$ is also amenable. $\square$

When p, q are relatively prime, $BS(p, q)$ is non-hopfian, i.e., it has a proper quotient which is isomorphic to itself. Abusing this property we can find a sequence of generators whose Følner constants tend to zero, and so this proves that the uniform constant must be zero:

Proposition 3.22. *If* $p, q > 1$ *are relatively prime, then the group* $BS(p, q)$ *is a non-amenable group with Føl* $BS(p, q) = 0$.

Proof. We deduce from Lemma 3.20 that $BS(p, q)$ is not amenable. We want to apply Proposition 3.18 with $G_k = BS(p, q)$ for all $k \in \mathbb{N}$, in order to get an upper bound for Føl $BS(p, q)$.

Denote $G = BS(p, q)$. Consider the homomorphism $\phi\colon G \to G$ sending $\phi(a) = a^p$ and $\phi(t) = t$. We claim that this homomorphism is surjective. Clearly the elements t, a^p are in the image, and so also a^q is. By Bezout's lemma, there exist $x, y \in \mathbb{Z}$ such that $xp + yq = 1$, and thus, $a = a^{px+yq} = (a^p)^x (a^q)^y$ is also in the image. Since both generators are in the image, the map is surjective.

Now, let $N_i = \ker \phi^i, i \geq 1$. Then, $N_1 \leq N_2 \leq \dots$, and by first isomorphism theorem, $G/N_k \cong G$ for all $k \geq 1$. Let $L = G/\cup_{k=1}^{\infty} N_k$ be the limit group. We claim that L is amenable. Consider the homomorphism $\psi\colon L \to \mathbb{Z} = \langle t \rangle$ sending $a \to 1$ and $t \to t$. A word is in the kernel if and only if the total sum of the exponents of the t's in the word is zero, hence the kernel is generated by $\{t^{-n} a t^n : n \in \mathbb{Z}\}$. These elements commute in $BS(p, q)$ and thus also in L, so $\ker \psi$ is abelian. We then have a short exact sequence

$$0 \to \ker \psi \to L \to \mathbb{Z} \to 0,$$

so L is an extension of an abelian group by a cyclic one, and thus L is amenable. In particular, $\text{Føl}_X L = 0$, and by the previous theorem,

$$0 \geq \text{Føl}_X L \geq \limsup_k \text{Føl}_X F_2/N_k \geq 0$$

But as we mentioned, $F_2/N_k \cong G$ for all $k \in \mathbb{N}$, so Føl $G = 0$. $\square$

This theorem can be generalized to HNN-extensions, with a proof that works essentially the same:

Theorem 3.23. *Let $A = \langle x_1, \dots, x_m : \mathcal{R}\rangle$ be an amenable group with relations $\mathcal{R}$. Let $\mu, \nu \colon A \to A$ be injective homomorphisms such that:*

1. *$\mu \circ \nu = \nu \circ \mu$;*
2. *$\mu(A) \cup \nu(A)$ generate A.*

Then, for the group $G = \langle t, A : t^{-1}\mu(x_i)t = \nu(x_i)\rangle$ it is Følner $G = 0$.
Moreover, if $\mu(A) \cup \nu(A) \neq A$, then G is a non-amenable group with Følner $G = 0$.

Proof. Define $\Phi \colon G \to G$ by $\Phi(t) = t, \Phi(x_i) = \mu(x_i)$ for each $i = 1, \dots, m$. Using (i) we can prove that Φ is a homomorphism:

$$\Phi(\nu(x_i)) = \mu(\nu(x_i)) = \nu(\mu(x_i)) = t^{-1}\mu^2(x_i)t = \Phi(t^{-1}\mu(x_i)t).$$

By (ii), the generators appear in the image, and thus Φ is surjective.

Define $N_i = \ker \Phi^i$, and consider the limit $L = G/\cup_{i=1}^{\infty} N_i$. We claim that L is amenable: as before, we can consider the map $L \to \mathbb{Z}$ sending $x_i \to 1$ and $t \to t$. Proving that the kernel of this map is amenable suffices, since then L is an extension of an amenable group by $\mathbb{Z}$ and thus it will be amenable. Denote K to be the kernel of this map. Let us show that any finitely generated subgroup of K is amenable.

Any finitely generated $H \lhd K$ is generated by conjugates of elements of the form $t^{-n}x_i t^n$. Using the relation $t^{-1}\mu(x_i)t = \nu(x_i) \in A$, we can find $l \in \mathbb{N}$ so that $\Phi(l)(h) \in A$ for all generating $h \in H$. Since A is amenable, $\Phi^l(H) \leq A$, and Φ is an isomorphism in L, H must be amenable, and by Theorem 2.10(4) so will be K.

For the second statement, it is clear that G is an HNN-extension of A, and using Britton's lemma one can see that the subgroup $\langle b, b^{-1}tb\rangle$ is free for any $b \in A \setminus (\mu(A) \cup \nu(A))$. □

This finishes our study of amenability as a general property. From here on, we will introduce a particular group, Thompson's group F, study how to apply the theory on this group, and get our hands dirty trying to look for Følner sets here.

CHAPTER

4

Thompson's group F

Thompson introduced in the $1960's$ three different discrete groups of piecewise linear bijections of the interval $[0,1]$, with several interesting properties. One of the biggest unsolved problems in the theory of amenability is whether one of Thompson's groups, F, is amenable or not. In this chapter we are going to introduce this group, along with two presentations of it, and some different interpretations of its elements, which will provide us with different ways to study the group. With these tools, we will talk about its properties, mostly in relation to amenability, and some of its subgroups. Finally, we will make a couple of comments about the other two Thompson groups, T and V.

4.1 Definition and realizations of F.

Thompson's group F is introduced as a group of piecewise linear homeomorphisms of the interval $[0,1]$. The points where a map is not differentiable will be called **breakpoints**. The set of breakpoints will always be discrete, and so, in the case that our interval is compact, also finite.

Definition 4.1. Thompson's group F is the group of piecewise linear homeomorphisms of the interval $[0,1]$ such that:

1. They are orientation-preserving.

2. In the pieces where the maps are linear, the slope is always a power of 2.

3. The breakpoints are dyadic, i.e., they belong to the set $D \times D$, where $D = [0,1] \cap \mathbb{Z}[\frac{1}{2}]$.

An element of F can be defined by a finite set of breakpoints, and extended linearly on the intervals between them. For instance, the breakpoints

$$(a_1, b_1), \ldots (a_k, b_k)$$

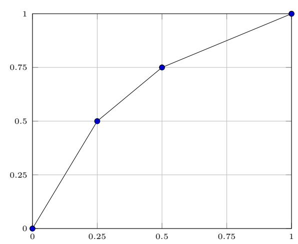
represent the element sending linearly each interval (a_i, a_{i+1}) to (b_i, b_{i+1}), $i = 1, \ldots, k-1$. The three conditions in the definition can be translated into conditions on these breakpoints:

1. Condition 1 implies that the breakpoints must be in ascending order (in both coordinates).
2. Condition 2 imposes that $\frac{b_{i+1}-b_i}{a_{i+1}-a_i}$ must be a power of 2 for each $i = 1, \ldots, k-1$.
3. Condition 3 states that each a_i, b_i must be of the form $\frac{m}{2^n}$ for some $m, n \in \mathbb{Z}$.

It is straightforward to check that F is in fact a group: the identity is in F, it is closed under composition and inverses can be obtained using breakpoints $(b_1, a_1), \ldots, (b_k, a_k)$.

Example 4.2. The set of breakpoints $[(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4})]$ represents the element

$$f(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{4}, \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$



Definition 4.3. Given $f \in F$, we define the **support** of f as the set of points $t \in [0,1]$ such that $f(t) \neq t$.

In the previous example, the support is the whole open interval $(0,1)$. The support will always be a disjoint union of open intervals, so it has a relatively simple shape, and that makes it easy to work with. For instance, any two elements will commute on the complement of the support, and this set will behave nicely.

One of the beautiful things about this group is that it can be interpreted in several different ways, and so there are many approaches to working with it. Here we will introduce some of them, namely the binary tree representations (along with other similar versions), and two presentations of F.

4.1.1 Binary trees.

As we already mentioned, an element of f is defined by its breakpoints. These breakpoints define two subdivisions of the unit interval (one with the $[a_i, a_{i+1}]$, and another with the $[b_i, b_{i+1}]$), in such a way that the element sends the first subdivision into the other. Thus, we can also define elements as two dyadic subdivisions of the interval having the same number of pieces. Binary trees are a nice tool for working with subdivisions and, as we will see, they also behave very well with the operation in F.

Definition 4.4. A **rooted binary tree** is a tree starting with a distinguished vertex, called the **root**, so that, if the tree has more than one vertex, then the root is the only vertex with valence 2, and the other vertices either have valence 3 (nodes) or 1 (leaves).

A **caret** is a rooted binary subtree with only two edges. It consists of a node (or the root) along with the two vertices coming down from it and the corresponding edges.

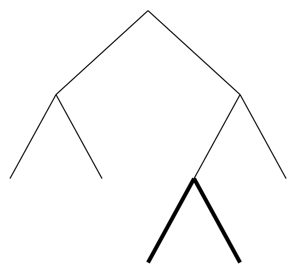


Figure 4.2: A rooted binary tree with a bold caret.

Any dyadic subdivision of the interval can be represented using a (deep enough) binary tree, where each leaf represents an interval in the subdivision. This can be seen in the following figure:

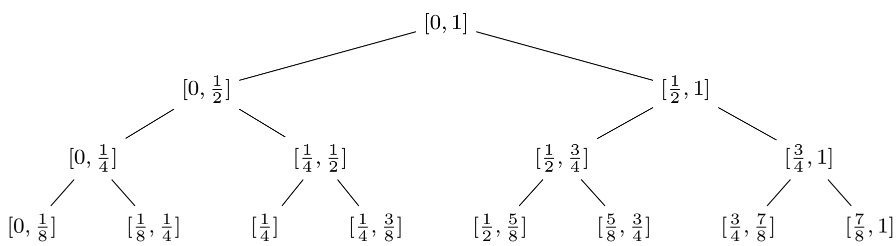
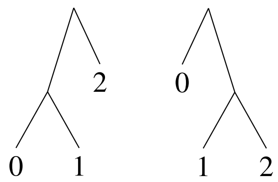


Figure 4.3: Dyadic subdivision of the interval in 8 subintervals of the same size.

Definition 4.5. A **tree pair diagram** is a pair of rooted binary trees with the same number of leaves.

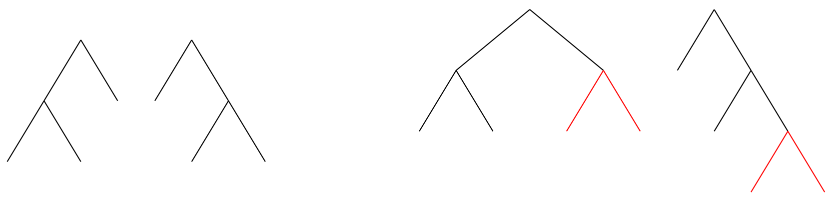
Using the idea at the beginning of the section, any tree pair diagram represents an element in F: the subintervals in the left tree are sent to the subintervals in the right tree having the same leaf number.

Example 4.6. The element in Example 4.2 can be represented by the tree pair diagram



Here the interval $[0, \frac{1}{4}]$, in leaf 0, is sent to the interval $[0, \frac{1}{2}]$. Analogously, $[\frac{1}{4}, \frac{1}{2}]$ is sent to $[\frac{1}{2}, \frac{3}{4}]$, and $[\frac{1}{2}, 1]$ goes to $[\frac{3}{4}, 1]$.

Remark. If we have two carets hanging from the same leaf in both trees, this caret can be deleted without changing the element. This is due to the fact that, even if having the caret represents a deeper subdivision of the interval, this happens in both the original and target intervals, so the images are the same. For instance, the following pair tree diagrams represent the same element:



In particular, a tree pair diagram having the same tree in both sides must be the identity. This makes sense, since it sends linearly each subinterval to itself.

Tree diagrams exhaust all possible elements in F:

Proposition 4.7. *Given an element in F, there is a tree diagram representing it.*

Proof. Consider $f \in F$ with breakpoints $[(a_1, b_1), \dots (a_k, b_k)]$. Let $n \in \mathbb{N}$ be sufficiently big so that each a_i can be written as a dyadic number with denominator 2^n. Let $m \in \mathbb{N}$ work the same way for the b_i.

Then, we can subdivide the interval in 2^n pieces of the same length, and this subdivision will be represented by a balanced tree of depth n, i.e., a tree where all the leaves are at distance n from the root. We do the same for m, but now we have a different number of leaves in both trees, so we need to either add or cut some carets.

Suppose that the interval $[a_i, a_{i+1}]$ has length $\frac{l_i}{2^n}$. If f has slope 2^{r_i} in this interval, then the length of $[b_i, b_{i+1}]$ will be $\frac{2^{r_i} l_i}{2^n}$. Writing $s_i = r_i + m - n$, then a quick computation will convince us that

$$b_{i+1} - b_i = \frac{2^{r_i} l_i}{2^n} = \frac{2^{s_i} l_i}{2^m}.$$

So now, it is the sign of s_i that will tell us which subtree has more leaves, and so where we need to add carets:

1. If $s_i > 0$, then $[b_i, b_{i+1}]$ has more leaves. Subdividing s_i times each leaf in $[a_i, a_{i+1}]$, we get $l_i 2^{s_i}$ new leaves, as in the target interval, and all at the same depth, so that the slope stays the same in the whole subinterval.
2. If $s_i = 0$, then both subtrees will have the same number of leaves, and there is no need to subdivide.
3. If $s_i < 0$, then we can proceed as in 1, but now subdividing $(-s_i)$ times the leaves in $[b_i, b_{i+1}]$.

$\square$

An example of this construction will clarify the proof:

Example 4.8. Let $f \in F$ be defined by the breakpoints $[(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{8}), (\frac{3}{4}, \frac{7}{8})]$.

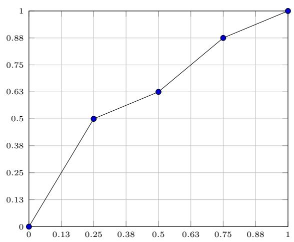
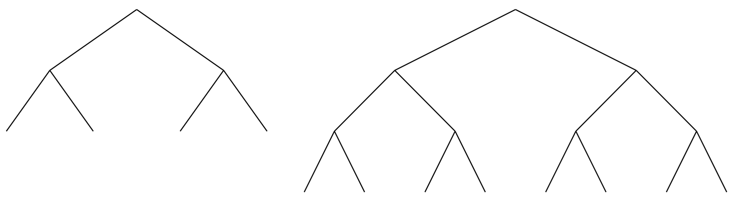


Figure 4.4: Graph of f.

Using the notation of the previous proof, we have $n = 2, m = 3$, so we begin with two balanced trees with these depths:



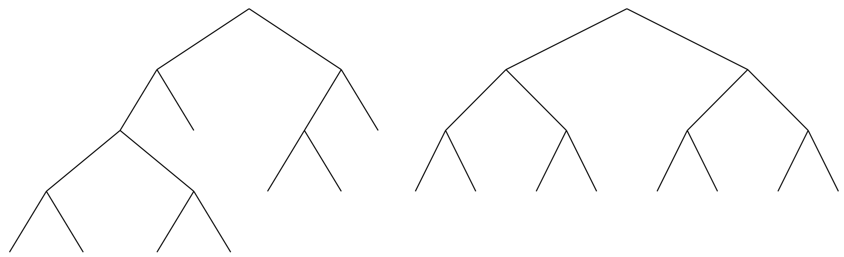
First of all, the interval $[0, \frac{1}{4}]$ is sent to $[0, \frac{1}{2}]$. The domain interval consists of 1 leaf in the left tree, and the target interval has 4 leaves. Hence, we need to subdivide the first leaf in the left tree twice.

The second intervals, $[\frac{1}{4}, \frac{1}{2}] \to [\frac{1}{2}, \frac{5}{8}]$, have one leaf in each tree, so there is no need to subdivide.

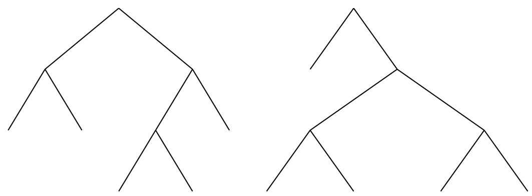
The third intervals, $[\frac{1}{2}, \frac{3}{4}] \to [\frac{5}{8}, \frac{7}{8}]$ have one leaf in the leaf tree and two in the right, so we have to subdivide once the corresponding leaf in the left tree.

Finally, the last intervals $[\frac{3}{4}, 1] \to [\frac{7}{8}, 1]$ have one leaf in each tree, so we do not need to subdivide.

Hence, we obtain the following tree diagram:



Finally, we can reduce this diagram to obtain:



As mentioned in the remark, several tree diagrams can represent the same element of F. Nevertheless, if we restrict to reduced diagrams, this representation is unique:

Proposition 4.9. *Every element of F has a unique reduced tree diagram.*

Proof. Note that the tree diagram of an element $f \in F$ is determined by the domain tree. Moreover, if a domain tree is contained in one another, $T \subset T'$, then the tree diagram with domain tree T must be a reduction of the tree diagram with domain tree T'. Thus, it is enough to show that the set of all possible domain trees f has a minimum element under inclusion.

Define as standard dyadic intervals those which can be obtained as a leaf, i.e., those of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$. Let us call a standard dyadic interval *regular* if f maps it linearly onto a standard dyadic interval (that is, if it is mapped to a leaf). Then, a tree T is a possible domain tree for f if and only if all its leaves are regular. Note that this property is preserved through intersections: if some tree has a regular leaf, then all other trees can delete whatever they have under that same node. We conclude that the set of possible trees is closed under intersections, and thus it has a minimum element. $\square$

Working with tree diagrams makes some operations easier than using functions. For composition, given $f, g \in F$ with pair diagrams $[L, R], [L'R']$, notice that if $R = L'$, then the composition $g \circ f$ is simply represented by $[L, R']$, as it sends the subdivision in L to $R = L'$, and then g sends precisely this subdivision to R'.

In the case that $R \neq L'$, we just need to put these trees into a bigger one. To do this, we use a clarifying lemma:

Lemma 4.10. *Given two binary trees, there exists a least common multiple, that is, a unique minimal tree containing both of them. As a result, any two dyadic subdivisions of* $[0, 1]$ *always contain a common subdivision.*

Proof. Add to each tree the missing carets from the other one. □

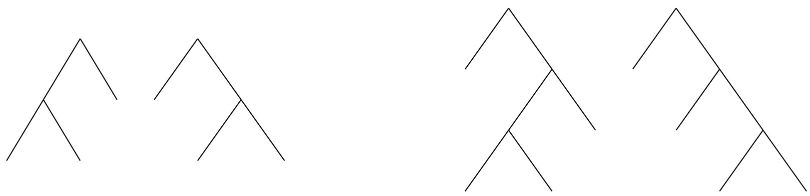
Adding carets on the same leaf of both trees in a tree diagram will not change the element, so we can always do this without any trouble.

Notation. For simplicity's sake, due to the fact that, if $f = [L, R], g = [L', R']$, then the product $[L, R][L', R']$ represents the composition $g \circ f$, we will denote the product

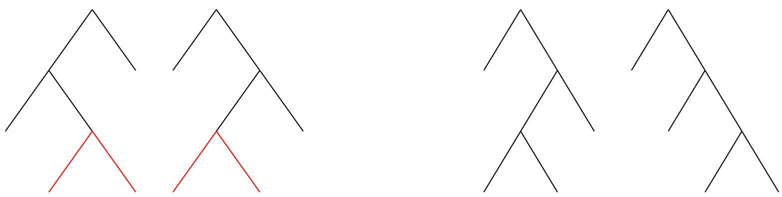
$$fg = g \circ f$$

so that the group F acts on $[0, 1]$ on the right, as $(x)(fg) = g(f(x))$.

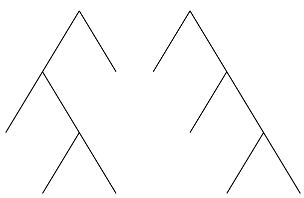
Example 4.11. Let f, g be given by the following tree pair diagrams:



We can add carets to obtain the least common multiple of the middle trees without changing the elements:



And the product fg will be given by the trees outside:



With this in mind, the inverse is quite easy to do: if f has diagram $[L, R]$, then f^{-1} has diagram $[R, L]$. This is simply because $ff^{-1} = [L, L]$ has the same tree in both sides, so it must be the identity.

4.1.2 Presentations of F

We are about to introduce an infinite and a finite presentation for F. The proofs are a bit technical, and we are mostly interested in finding the final presentation, so we will just state the results, talk about the ideas behind, put some example and sketch the most interesting proofs.

Theorem 4.12. *F has a presentation*

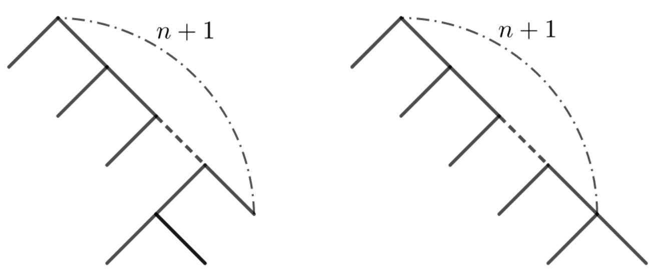
$$\langle x_0, x_1, x_2, \dots : x_k^{-1} x_n x_k = x_{n+1} \textit{ for } k < n. \rangle$$

To prove the theorem, let G be the group defined by the presentation, and let us find an isomorphism $\Theta\colon G \to F$.

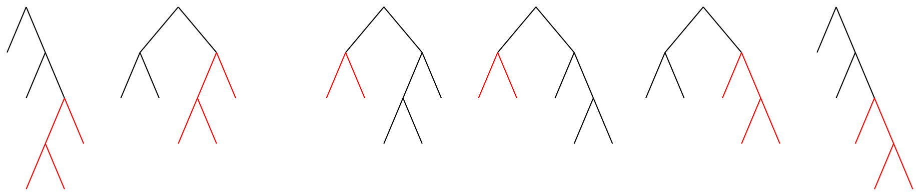
There exists a group homomorphism $\Theta\colon G \to F$ sending each generator x_n to the element f_n defined by the breakpoints

$$\left[\left(1-\frac{1}{2^n}, 1-\frac{1}{2^n}\right), \left(1-\frac{3}{2^{n+2}}, 1-\frac{1}{2^{n+2}}\right), \left(1-\frac{1}{2^{n+1}}, 1-\frac{1}{2^{n+2}}\right)\right]$$

This element can be represented using the following tree diagram:



The element in Examples 4.2 and 4.6 is precisely f_0. It is straightforward to see that F satisfies the relations in G. This can be seen in the following Figure 4.6 for the case $f_2 = f_0^{-1} f_1 f_0$:



(a) f_0^{-1} **(b)** f_1 **(c)** f_0

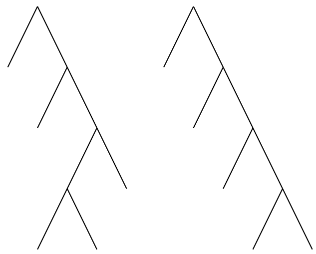


Figure 4.6: The relation $f_2 = f_0^{-1} f_1 f_0$.

4.1.2.1 **Positive words**

We may rewrite the given presentation so that it does not involve inverses:

$$\langle x_0, x_1, x_2, \ldots : x_n x_k = x_k x_{n+1} \text{ for } k < n. \rangle$$

This defines a monoid, called the **monoid of positive words**, which we will denote as P, consisting of words that can be obtained using only the generators x_i, and not any inverses.

Notice that, in a positive word, the relations allow us to move the generators with higher indices to the right, at the cost of increasing its index by one, and so this allows us to write elements of P in a particular way:

Theorem 4.13. *Any element of P can be written as a word of the form*

$$x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n}$$

for some $n \in \mathbb{N}$ and non-negative integers a_i.

Tree diagrams of positive words can be easily constructed. To do this, let us introduce the concepts of all-right tree and leaf exponent:

Definition 4.14. An **all-right tree** is a binary tree where each caret hangs from the right tree of the previous one.

We define the **right stalk** of a binary tree as the maximal all-right subtree hanging from the root.

Definition 4.15. Given a binary tree, enumerate the leaves from left to right starting at 0. For each leaf i, the ***i*-th exponent** of the tree is defined as the number of times we can go up right from that leaf, before reaching the right stalk.

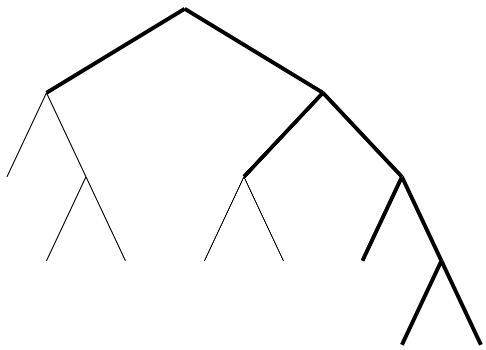


Figure 4.7: A binary tree with bold right stalk.

Example 4.16. In Figure 4.7, the right stalk of the tree is the bold all-right tree, and counting how many times we can go to the right from each leaf before arriving to the right stalk, we see that the leaf exponents are

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 1, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 0, \quad a_7 = 0.$$

The importance of these ideas lays on the fact that leaf exponents determine positive elements in F:

Proposition 4.17. *An element in $\Theta(P)$, of the form $f_0^{a_0} f_1^{a_1} \dots f_n^{a_n}$ can be represented by a tree diagram $[L, R]$, where L is a tree with leaf exponents $a_0, \dots, a_n$ and R is an all-right tree with the same number of carets.*

Proof. One can check that the definition of the generators f_i satisfies this condition. Then, multiplying by a generator adds a caret on the i-th leaf of the left tree, and so it increases a_i by 1. Arguing by induction concludes the result. $\square$

And having a way to represent tree diagrams of positive elements, we have essentially shown that Θ is surjective, i.e., that the f_i generate F:

Theorem 4.18. $\Theta\colon G \to F$ *is surjective. More in detail, every element of* F *admits an expression of the form*

$$f_0^{a_0} \dots f_n^{a_n} f_n^{-b_n} \dots f_0^{-b_0}.$$

where a_i, b_i *are non-negative integers.*

Proof. Let $f \in F$ have tree diagram $[L, R]$, where both L, R have n leaves. Let T be an all-right tree with n leaves. Then, f can be interpreted as the product of the elements with tree diagrams $[L, T]$ and $[T, R]$. From the previous proposition, we deduce that the first of these is a positive element, $p = f_0^{a_0} \dots f_n^{a_n}$, and the second is the inverse of another positive element, $q = f_0^{b_0} \dots f_n^{b_n}$, so $f = pq^{-1}$ is the image of:

$$x_0^{a_0} \dots x_n^{a_n} x_n^{-b_n} \dots x_0^{-b_0}.$$

□

This gives us a method for finding a formal expression of an element from its tree diagram. The problem here is that this form is not unique, due to the fact that an element can admit different tree diagrams. Nevertheless, as we proved in Proposition 4.9, there is a always a unique reduced diagram, and it can be seen that this corresponds to a normal form of the elements of F:

Theorem 4.19. *Every element of F can be expressed uniquely as*

$$f_0^{a_0} \dots f_n^{a_n} f_n^{-b_n} \dots f_0^{-b_0}.$$

where a_i, b_i *are non-negative integers, exactly one of* a_n, b_n *is non-zero, and*

$$a_i \neq 0, b_i \neq 0 \Rightarrow a_{i+1} \neq 0 \textit{ or } b_{i+1} \neq 0.$$

We do not formalize the proof, but give some ideas on why this works:

Proof. The presented form can always be achieved using the relations since, if none of $a_i, b_i, a_{i+1}, b_{i+1} \neq 0$, we can use the relation

$$x_k x_n \to x_{n-1} x_k$$

to bring an x_i, x_i^{-1} to the middle of the word, so that they cancel, at the cost of reducing by one the higher coefficients.

Then, one can check that the correspondence between tree pair diagrams and seminormal forms sends reduced diagrams to normal forms, and so by the uniqueness of the reduced diagram, the normal form must be unique.

To see this correspondence, consider a tree with an exposed caret whose leaves are labelled i and $i + 1$. If we look at the leaf exponents of this tree, they will be $a_i \neq 0$, but $a_{i+1} = 0$. Then, in a non-reduced diagram, both trees will have an exposed caret with the same labels (i.e., if this caret can be deleted), and so it will be $a_i, b_i \neq 0$ and $a_{i+1} = b_{i+1} = 0$, so the word can be reduced using the previous idea, and thus it does not correspond to the normal form. Conversely, it is possible to see that if the word is in normal form, then no reductions can be done in the tree diagram. $\square$

Example 4.20. Consider the element $f_0^3 f_2 f_5 f_7 f_6^{-1} f_2^{-1} f_0^{-1} \in F$. Since $a_0, b_0 \neq 0$, but $a_1 = b_1 = 0$, using the relations we get

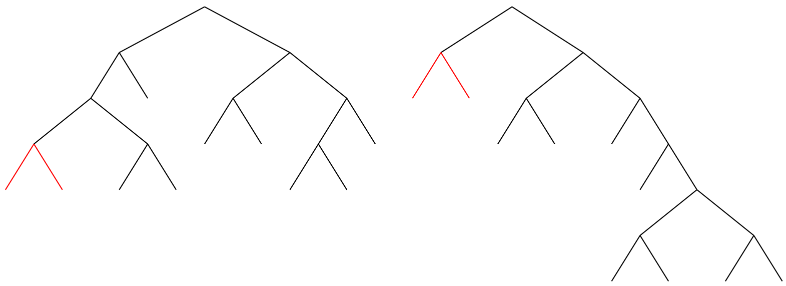
$$f_0^3 f_2 f_5 f_7 f_6^{-1} f_2^{-1} f_0^{-1} = f_0^2 f_1 f_4 f_6 f_0 f_0^{-1} f_5^{-1} f_1^{-1} = f_0^2 f_1 f_4 f_6 f_5^{-1} f_1^{-1}.$$

Now, it is $a_1, b_1 \neq 0$ and $a_2 = b_2 = 0$, so again,

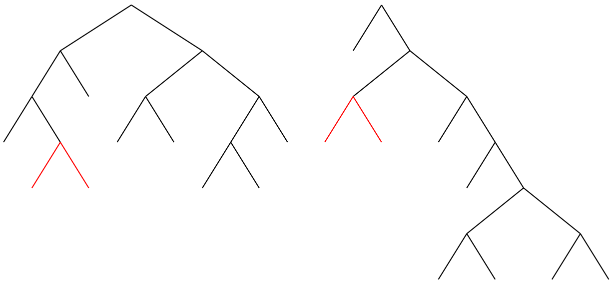
$$f_0^2 f_1 f_4 f_6 f_5^{-1} f_1^{-1} = f_0^2 f_3 f_5 f_1 f_1^{-1} f_4^{-1} = f_0^2 f_3 f_5 f_4^{-1}$$

and this corresponds to the normal form of the word.

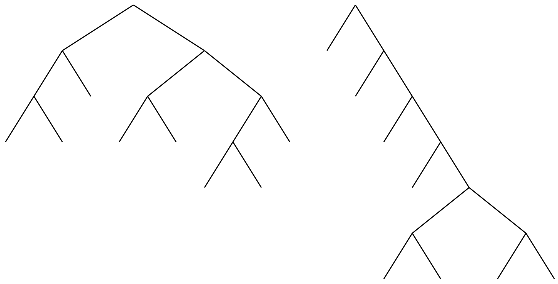
If we look at the tree diagrams for each of these expressions, then the initial expression corresponds to the following diagram, whose first caret is exposed:



We can reduce it to another one, where again, another caret is exposed



and reducing it again we get



which is a reduced diagram. Returning a word from the leaf exponents, as in Theorem 4.17, we obtain

$$f_0^2 f_2 f_3 f_5 f_4^{-1}.$$

This is the normal form that was found at the beginning of the example.

This finishes the proof of Theorem 4.12.

Theorem 4.21. $\Theta \colon G \to F$ *is injective.*

Proof. If two elements in G had the same image, we could obtain the normal form of this image in F. Then, we could use the relators in G to get to the same normal form (now in G) for these two elements, and so this allows to go from one element to the other using only the relators in G. Thus, they are the same element. □

Notation. From now on we drop the notation of the f_i in F, and we will simply name the generators $x_i \in F$.

4.1.3 The finite presentation

The former presentation has both infinite generators and relations, but there is also a finite presentation for F:

$$\langle x_0, x_1 : [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

The idea of reducing the number of generators is clear: using the relations enough times, we can write

$$x_n = x_0^{-1} x_{n-1} x_0 = x_0^{-2} x_{n-2} x_0^2 = \cdots = x_0^{-(n-1)} x_1 x_0^{n-1}.$$

The relations might still look a bit weird though. Nevertheless, using the notation of the previous generators for clarity, then this presentation is just

$$\langle x_0, x_1 : x_2 x_0 = x_0 x_3, x_3 x_0 = x_0 x_4 \rangle.$$

So what it says is that it is enough to keep the first two relations using x_0, and then the remaining relations are deduced from these two.

We will not prove here that this really is another presentation for F, because it does not introduce any new ideas, but the existence of a finite presentation will be quite relevant in our next chapter, in order to talk about the ideas introduced in Chapter 2. In any case, this can be done by finding an isomorphism between the two presentations, which corresponds to checking that all relations in each presentation can be achieved using the ones in the other presentation. This is trivial in one direction, and the other direction can be done inductively.

4.1.4 Forest diagrams

Forest diagrams are another realization of F, introduced by Belk and Brown in [4]. The idea comes from considering F as a family of piecewise homeomorphisms on infinite intervals, instead of $[0,1]$.

Definition 4.22. The group $PL_2(\mathbb{R}_+)$ is the family of all piecewise-linear, orientation-preserving, homeomorphisms f of $[0,\infty)$ satisfying:

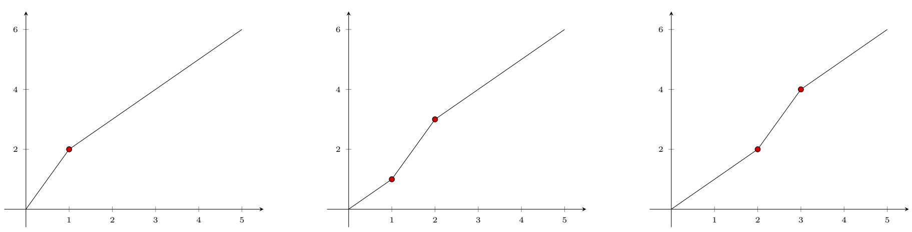
1. In the pieces where f is linear, the slope is always a power of 2.
2. f has only finitely many breakpoints, each of which has dyadic coordinates.
3. The rightmost segment of f is of the form $f(t) = t + m$, for some $m \in \mathbb{Z}$.

Proposition 4.23. *$PL_2(\mathbb{R}_+)$ is isomorphic to F.*

Proof. Let $\psi\colon [0,\infty) \to [0,1]$ be defined piecewise linearly sending the interval $[k, k+1]$ to $[\frac{2^k-1}{2^k}, \frac{2^{k+1}-1}{2^{k+1}}]$. It is straightforward to check that the map $f \to \psi^{-1} f \psi$ is an isomorphism $F \to PL_2(\mathbb{R}_+)$ (it is well defined, preserves identity and composition, and has an inverse map $f \to \psi f \psi^{-1}$. $\square$

Under this compositions, the generators $x_n \in F$ are mapped to functions $x_n\colon [0,\infty] \to [0,\infty]$ such that:

1. x_n is the identity in $[0,n]$.
2. x_n sends $[n, n+1]$ linearly onto $[n, n+2]$.
3. $x_n(t) = t + 1$ for $t \geq n + 1$.



(a) Image of x_0. **(b)** Image of x_1. **(c)** Image of x_2.

Definition 4.24. A binary forest is a sequence $(T_0, T_1, \dots)$ of finite binary trees. A binary forest is bounded if only finitely many of the trees T_i are non-trivial.

A bounded binary forest corresponds to a dyadic subdivision of the positive real line. Each tree T_i represents an interval $[i, i+1]$, and each leaf represents an interval in the subdivision. Just like with tree diagrams, a pair of binary forests represents an element in $PL_2(\mathbb{R}_+)$, where the interval in the i-th leaf of the upper forest is sent linearly to the interval corresponding to the ith leaf in the lower forest.

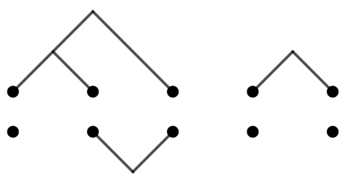


Figure 4.9: A binary forest diagram.

In the previous figure, the interval correspondance of the leaves is the following:

$$[0, \tfrac{1}{4}] \to [0, 1]$$

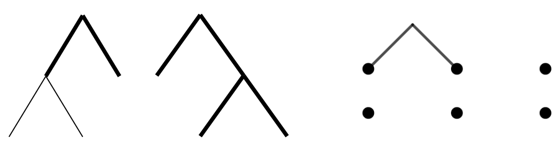
$$[\tfrac{1}{4}, \tfrac{1}{2}] \to [1, \tfrac{3}{2}]$$

$$[\tfrac{1}{2}, 1] \to [\tfrac{3}{2}, 2]$$

$$[1, \tfrac{3}{2}] \to [2, 3]$$

$$[\tfrac{3}{2}, 2] \to [3, 4]$$

It is quite easy to change from tree diagrams to forest diagrams and viceversa: we just need to remove the right stalk of the tree diagram and turn around the right tree.

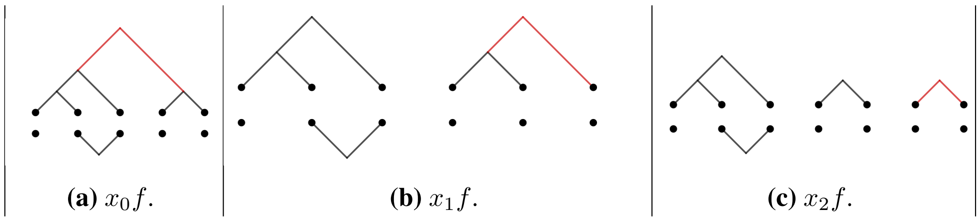


(a) Tree diagram of x_0. **(b)** Forest diagram of x_0.

Just like before, pairs of opposing carets can be deleted without changing the element, and the reduced forest diagram is unique.

The way generators act on forest diagrams is the following: given an element $f \in F$, a forest diagram for $x_n f$ can be obtained by attaching a caret to the roots of trees n and $n+1$ in the top forest of f.

Example 4.25. If f is the element in Figure 4.9, then we have forest diagrams:



(a) $x_0 f$. **(b)** $x_1 f$. **(c)** $x_2 f$.

Apart from these diagrams, Belk and Brown also use pointed forest diagrams. Now, instead of $PL_2(\mathbb{R}_+)$, we consider the whole real line.

Definition 4.26. Let $PL_2(\mathbb{R})$ be the group of piecewise-linear, orientation-preserving self-homeomorphims f of $\mathbb{R}$ such that

1. In the pieces where f is linear, the slope is a power of 2.
2. f has finitely many breakpoints, each of which is dyadic.
3. The leftmost and rightmost segments of f are respectively of the form $f(t) = t + m$ and $f(t) + n$, for some $m, n \in \mathbb{Z}$.

Proposition 4.27. *$PL_2(\mathbb{R})$ is isomorphic to F.*

Proof. Let $\Psi \colon \mathbb{R} \to (0, 1)$ be defined piecewise linearly, sending

$$k \to \frac{2^{k+1} - 1}{2^{|k|+1}}.$$

That is, Ψ sends 0 to $1/2$, 1 to $3/4$, 2 to $7/8$, etc.

Then, $f \to \Psi f \Psi^{-1}$ is a group isomorphism. $\square$

We can find new diagrams for the elements of $PL_2(\mathbb{R})$.

Definition 4.28. A pointed binary forest is a sequence of finite binary trees $(T_i)_{i\in\mathbb{Z}}$.

We mark the tree T_0 with an arrow. Here we only care about bounded pointed forests, as we will always have finitely many breakpoints. The (infinitely many) trees not appearing in the figures are trivial.

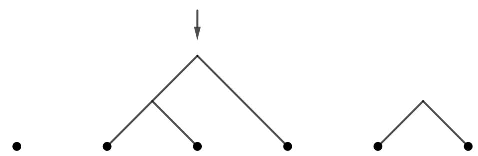


Figure 4.12: A pointed binary forest.

Any pointed binary forest represents a subdivision of the real line, where the tree T_i represents a subdivision of the interval $[i, i+1]$. Hence, any pair of bounded pointed binary forests represents an element in $PL_2(\mathbb{R})$. We call this representation the **pointed binary forest diagram**.

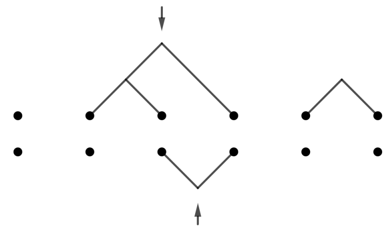
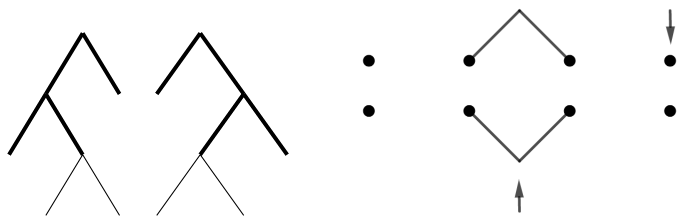


Figure 4.13: A pointed forest diagram.

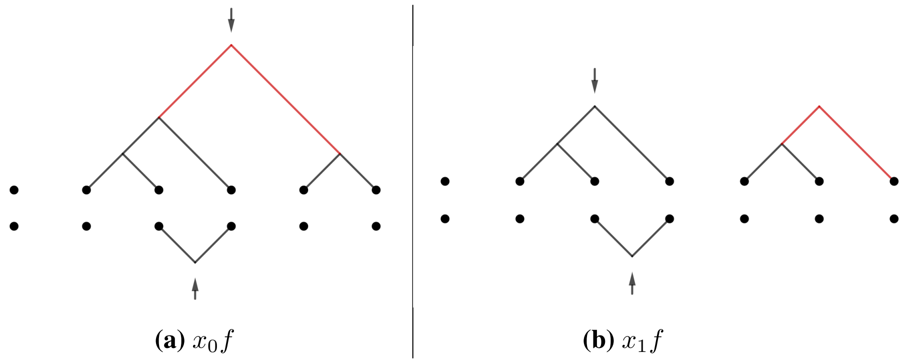
Again, it is easy to go from tree diagrams to pointed forest diagrams: in this case, we remove the whole stalk, both left and right, and put the arrow in the first subtree hanging to the right of the root:



(a) Tree diagram. **(b)** Pointed forest diagram.

The generators act on these diagrams in the following way: x_0 changes the position of the arrow in the top tree to the right, and x_1 puts a caret between the arrow tree and the next one, and puts the arrow in this new tree.

Example 4.29. Let f be the element in Figure 4.13 Then, diagrams for $x_0 f$ and x_1 look like this:



(a) $x_0 f$ **(b)** $x_1 f$

In general, x_i puts a caret between trees $i-1$ and i. This makes sense from the relations: as $x_i = x_0^{1-i} x_1 x_0^{i-1}$, the arrow moves $i-1$ positions forward, then x_1 puts a caret in this position, and the arrow goes back to the inital position.

4.2 Properties of F

Now that we have different tools for working with F, let us study some of the properties that make it an interesting group. To begin with, let us take a look at the torsion in F:

Proposition 4.30. *F is torsion free.*

Proof. If $f \neq id$, let $P \in [0,1]$ be the smallest point in the support of f. The right slope of f at P must be 2^k for some $k \neq 0$, and so f^n has right slope at P equal to 2^{nk}, and thus $f^{nk} \neq id$. □

This, along with the fact that elements with disjoint support commute, imply that F contains $\mathbb{Z}^n$ as a subgroup for all $n \in \mathbb{N}$, and even the infinite direct sum, $\oplus_{i=1}^{\infty}\mathbb{Z}$. The study of the subgroups in F is relevant in relation to amenability: finding a non-amenable subgroup would answer negatively our question on amenability, and put an end to the problem.

4.2.1 Growth

Since F is finitely generated, we can study its growth. Recall that, even if the growth function is different for different sets of generators, these are all asymptotically equivalent.

F grows exponentially, due to the fact that it contains a free submonoid.

Proposition 4.31. *The monoid generated by $\{x_0^{-1}, x_1\}$ is free.*

Proof. A general word in this monoid is of the form

$$x_1^{a_1} x_0^{-1} x_1^{a_2} \dots x_0^{-1} x_1^{a_n},$$

where $a_i \geq 0$, $i = 1, \dots, n$. We can put this word in the normal form in F, moving the x_0^{-1} to the right, and getting

$$x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} x_0^{-(n-1)}.$$

This normal form depends only on the values of the a_i, so it is different for different words in the monoid. Hence, it follows that the monoid is free. □

Corollary 4.32. *F has exponential growth.*

Proof. Using generators x_0, x_1, the free submonoid contains 2^n words of length n in the generators, so the growth function of F is lower bounded by an exponential function. □

Another way to prove this fact is showing that the norm of an element is asymptotically equivalent to the number of carets in the reduced tree diagram representation. It is a well known fact that this is a Catalan number, and this sequence grows as 4^n. A deeper explanation can be found in [7].

The exact value of the growth rate in F is still unknown, although there is a lower bound on the growth rate for the generating set $\{x_0, x_1\}$.

Theorem 4.33 (Guba [15])**.** *The growth rate of the group F in generators $\{x_0, x_1\}$ is not less than* $\frac{3+\sqrt{5}}{2} \approx 2.618034$

To obtain this bound, Guba constructs an automata in F, and counts the number of words of a given length expelled by the automata.

In particular, since F is not subexponential, this result does not give any information on its amenability.

4.2.2 n-Transitivity

The idea here is that the "fractal" structure of the dyadic intervals in $[0, 1]$ makes it so that F has many disjoint subgroups which are isomorphic to the whole F. Let us formalize these ideas.

Lemma 4.34. *Let $a/2^n$ be a dyadic number, with $a > 0$, and let b be another integer with $a \leq b$. Then, the interval $[0, a/2^n]$ can be divided into b intervals, all of them having as length a power of 2 (although the power may be different for each interval).*

Proof. We can divide $[0, a/2^n]$ into a subintervals of length $1/2^n$. Now since $b \geq a$, we can subdivide this intervals (by the middle) as many times as necessary, and the length will remain a power of 2. $\square$

Theorem 4.35. *Let $0 = x_0 < x_1 < \dots x_n = 1$, $0 = y_0 < y_1 < \dots y_n = 1$, be dyadic numbers in $[0, 1]$. Then, there exists a map $f \in F$ such that $f(x_i) = y_i$ for all $i = 1, \dots, n$.*

Proof. Assume the intervals $[x_i, x_{i+1}], [y_i, y_{i+1}]$ have lengths $a_i/2^{n_i}$ and $b_i/2^{m_i}$. We may assume without loss of generality that $a_i \leq b_i$. We can subdivide $[y_i, y_{i+1}]$ into b_i subintervals of length $1/2^{m_i}$, and by the previous lemma, we can also subdivide $[x_i, x_{i+1}]$ into b_i subintervals with length a power of 2. Now, we can construct our element linearly on each of these subintervals, for each $i = 1, \dots, n$. The element obtained will have dyadic breakpoints, and the slopes will be powers of 2, so it will be in F. $\square$

This proves that, essentially, all the dyadic subintervals of $[0, 1]$ look the same, and one can move between them using elements of F. We can refer to this property by saying that F acts n-transitively on $[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$.

This has important consequences. Using this we can show that F contains many disjoint subgroups which are isomorphic to itself, namely, the subgroups of elements with support in a dyadic subinterval. Denote $F[a, b]$ the set of elements with support in $[a, b]$.

Theorem 4.36. *Let $a < b$ be dyadic numbers in $[0, 1]$, then $F[a, b] \cong F$.*

Proof. The details are a bit technical, but the idea is the following: we can always find an orientation preserving affine function α sending bijectively $[0, 1] \to [a, b]$ and preserving dyadics. Then, we can define a map $\Phi_\alpha \colon F \to F[a, b]$ as

$$\Phi_\alpha f(t) = \begin{cases} \alpha^{-1} f \alpha(t) & \text{if } t \in [a, b] \\ t & \text{otherwise .} \end{cases}$$

By construction, $\Phi_\alpha(f) \in F[a, b]$, it is a homomorphism (due to the fact that $f \in F[a, b]$ sends $[a, b] \to [a, b]$) and $\Phi_\alpha^{-1} = \Phi_{\alpha^{-1}}$, so it is an isomorphism. $\square$

As a corollary, using the fact that elements with disjoint support commute, F also contains as subgroups all its finite direct products, $F^n, n \geq 1$, and the direct sum $\bigoplus_{k=1}^{\infty} F$.

One could think that this could be useful for finding a paradoxical decomposition of F, and prove that it is not amenable. It is a nice exercise to try finding one, and see how this is not as simple as it seems. This is due to the fact that, even if the elements of F can stretch and contract parts of $[0, 1]$, the total length is preserved, in the sense that $f([0, 1]) = [0, 1]$ for any $f \in F$.

4.2.3 Abelianization and the commutator subgroup

Let us study now some properties of F related to commutativity.

Proposition 4.37. *The abelianization of F is $\mathbb{Z}^2$. Moreover, $F' = [F, F]$ is the set of elements having slope 1 at the endpoints of $[0, 1]$.*

Proof. Since F can be generated by two elements, the abelianization must be a quotient of $\mathbb{Z}^2$. Thus, if we find a surjective map $\phi\colon F \to \mathbb{Z}^2$, then $\ker \phi$ must be the commutator $[F, F]$, and $\mathbb{Z}^2$ the abelianization.

Define $\phi\colon F \to \mathbb{Z}^2$ by $f \to (k, l)$, where k is the right slope of f at 0, and l is the left slope at 1. ϕ is a homomorphism, because the slope is multiplicative, and it is surjective, because any pair of slopes can be achieved. The kernel of this map is precisely the set described in the statement. □

In particular, this shows that F is not simple. Nevertheless,

Theorem 4.38. *The commutator of F is simple.*

This has very important consequences for F, but the proof is long and technical, so we decide to skip it. The curious reader can check two different proofs in [7] and [8]. The first thing we deduce is the following:

Corollary 4.39. *F is not solvable nor nilpotent.*

So we cannot use the criteria in Theorem 2.10 to get information about amenability. In fact, at the beginning of the next chapter we will prove that F is not elementary amenable, which will complete this idea.

Another corollary of the simplicity of the commutator is related to proper quotients of F. Before this, let us introduce a proposition regarding the structure of the center $Z(F)$.

Proposition 4.40. *The center of F is trivial.*

Proof. Let $f \in Z(F)$. Let S be the set of fixed points of x_1, $S = [0, \frac{1}{2}] \cup \{1\}$. We claim that $S = f(S)$. This is because, if $t \in S$,

$$(x_1 \circ f)(t) = (f \circ x_1)(t) = f(t),$$

and so $f(t) \in S$. Conversely, if $f(t) \in S$, since f is injective,

$$(f \circ x_1)(t) = (x_1 \circ f)(t) = f(t) \Rightarrow x_1(t) = t,$$

and so $t \in S$.

Because of this, it must be $f(\frac{1}{2}) = \frac{1}{2}$. By the same argument, every element in F commuting with f stabilizes the fixed point set of f. In fact, as $f \in Z(F)$, this happens for all elements. But now, by n-transitivity, given any two intervals there is an element taking one to the other, and so the fixed point set of f must contain both intervals. This happens for any two intervals, so the fixed point set of f must be the whole $[0,1]$, and f must be the identity. $\square$

Theorem 4.41. *Let $\Phi\colon F \to G$ be a surjective homomorphism. Then either Φ is an isomorphism or G is abelian.*

Proof. Suppose Φ is not an isomorphism. Let $x \neq 1, x \in \ker \Phi$. Since $x \notin Z(F) = \{1\}$, there exists $y \in F$ such that $[x,y] \neq 1$. Now, $x \in \ker \Phi$ and $yx^{-1}y^{-1} \in \ker \Phi$, so $[x,y] = x(yx^{-1}y^{-1}) \in \ker \Phi \cap F'$. Then, $\{1\} \neq \ker \Phi \cap F' \lhd F'$, and since F' is simple, it must be $F' \subset \ker \Phi$, so $Q \cong F/\ker \Phi$ is abelian. $\square$

Note that G corresponds to a quotient of F, so this theorem states that that all proper quotients of F are abelian. In particular, F is hopfian, so we can not use the idea in Theorem 3.22 to study amenability on F, or even its Følner constant.

We also get another property of F as a corollary, even though it is not related to amenability:

Corollary 4.42. *F is not residually finite.*

Proof. Let $x \in F'$, and $\Phi\colon F \to G$ with G finite. By the previous result, G is abelian, so $F' \subset \ker \Phi$ and so it must be $\Phi(x) = 1$. Hence, x can never be separated from 1. $\square$

4.2.4 Absence of free subgroups

Let us now prove that F has no free non-abelian subgroups. This result follows from the initial definition of F, as a group of piecewise linear bijections. We will prove it in full generality, because it is a consequence of the absence of free non-abelian subgroups in a bigger group of this type of mappings.

Definition 4.43. We define $PLF(\mathbb{R})$ as the set of piecewise-linear maps of $\mathbb{R}$ with finitely many breakpoints.

Note that slopes can take any value, and the breakpoints need not be dyadic. It is clear that F is a subgroup of $PLF(\mathbb{R})$, so showing that there are no free subgroups in $PLF(\mathbb{R})$ suffices to show it for F. We will divide the proof in some lemmas:

Lemma 4.44. *Let $f, g \in PLF(\mathbb{R})$. Then,*

1. *$[f,g]$ has slope 1 at $\pm\infty$.*
2. *If f and g have slope 1 at $\pm\infty$, then $[f,g]$ has bounded support.*
3. *If $f(x) = x = g(x)$ for some $x \in \mathbb{R}$, then $[f,g]$ is trivial in a neighbourhood of x.*

Proof.

1. This follows from the fact that slope is multiplicative.
2. The hypothesis implies that $f(x) = x + a, g(x) = x + b$ for some $a, b \in \mathbb{R}$ near infinity, so here it is $[f,g](x) = x$.
3. The slope of $[f,g]$ is 1 around x, and x is fixed for f and g, so $[f,g] = Id$ in a neighbourhood of x.

□

Lemma 4.45. *Let $f \in PLF(\mathbb{R})$, and let $a, b \in \mathbb{R}$ such that $[a,b]$ is contained in the support of f. Then, there exists $n \in \mathbb{Z}$ such that $f^n(a) > b$.*

What this says is that, by applying enough times either f or f^{-1}, we can surpass b starting from a.

Proof. Suppose the connected component of the support of f containing $[a,b]$ is (c,d). Then, for $t \in (c,d)$,

1. If $f(t) > t$, then $f^n(t) \to d$, $f^{-n}(t) \to c$.
2. If $f(t) < t$, then $f^n(t) \to c$, $f^{-n}(t) \to d$.

So, in particular, either f^n or f^{-n} must go beyond b at some point. □

This lemma extends to the case where we use different elements in F.

Lemma 4.46. *Let $f, g \in PLF(\mathbb{R})$, and let $[a,b] \subset$ supp $f \cup$ supp g. Then, there exists a word in f, g such that $w(f,g)(a) > b$.*

Proof. The idea here is that, repiting the previous lemma several times, we can move through connected components of each f and g, and so combining both f and g we can move through the union of the supports.

Let $(c_i, d_i), i = 1, \ldots, k$ be intervals in $\mathbb{R}$ that correspond to components of either supp f or supp g, and suppose that

$$[a,b] \subset \bigcup_{i=1}^{k} (c_i, d_i).$$

and $c_1 < c_2 < \cdots < c_k$ and $d_1 < d_2 < \cdots < d_k$. Then, $a \in (c_1, d_1)$, and assume wlog that this is a component of supp f. By the previous lemma, there is n_1 such that $f^{n_1}(a) > c_2$. Now, (c_2, d_2) is a component in supp g, and $f^{n_1}(a) \in (c_2, d_2)$, so we can find n_2 such that $g^{n_2} f^{n_1}(a) > c_3$. Applying this repeatedly, we can jump through intervals and reach b in a finite number of steps. This will give us the desired word in f, g. $\square$

Theorem 4.47. *Let G be a subgroup of $PLF'(\mathbb{R})$. Then, either G is abelian or it contains a copy of $\mathbb{Z}^2$.*

Proof. If G is not abelian, let $f, g \in G$ such that $z = [f, g] \neq 1$. Then,

$$\text{supp } z \subseteq \text{supp } f \cup \text{supp } g.$$

Suppose that the connected components of supp $f \cup$ supp g are intervals $(a_i, b_i), i = 1, \ldots, k$. Then all a_i, b_j are common fixed points of f and g, so by Lemma 4.44, z is trivial in a neighbourhood of these points. Hence, if supp $z \cap (a_i, b_i) \neq \emptyset$, there is $[c, d]$ such that

$$\text{supp } z \cap (a_i, b_i) \subset [c, d] \subset (a_i, b_i).$$

Now, let $W \subseteq \langle f, g \rangle$ be the set of non-identity words in f and g whose support is contained in a compact subset of supp $f \cup$ supp g. In particular, $z \in W$, so $W \neq \emptyset$.

Let $\omega \in W$ be a word whose support has non-empty intersection with a minimal number of components of supp $f \cup$ supp g. By the definition of W, there is $i \in \{1, \ldots, k\}$ such that

$$\text{supp } \omega \cap (a_i, b_i) \subseteq [c, d] \subseteq (a_i, b_i),$$

and by Lemma 4.46, there is $u \in \langle f, g \rangle$ such that $b_i > u(c) > d$.

Then, if $t \in$ supp $\omega \cap (a_i, b_i)$, it is $u(t) \geq u(c) > d$, and so $u(t) \notin$ supp ω. Hence, $u^{-1}\omega u(t) = t$, and so

$$\text{supp } \omega \cap \text{supp } u^{-1}\omega u \cap (a_i, b_i) = \emptyset.$$

But then, the support of the commutator,

$$\text{supp } [\omega, u^{-1}\omega u] \subseteq \text{supp } \omega \cap \text{supp } u^{-1}\omega u$$

and it also does not interesect (a_i, b_i), contradicting the minimality condition on ω. Thus, it must be the identity, and so these two elements commute, thus generating a copy of $\mathbb{Z}^2$. $\square$

Corollary 4.48. *Let G be a subgroup of $PLF(\mathbb{R})$. Then, either G is metabelian (that is, $G'' = \{1\}$) or G contains a copy of $\mathbb{Z}^2$.*

Proof. Note that $G' \leq PLF'(\mathbb{R})$. Applying the previous theorem, either G' is abelian, and thus $G'' = \{1\}$, or G' contains a copy of $\mathbb{Z}^2$. In the last case, since $G' \leq G$, also G contains a copy of $\mathbb{Z}^2$. $\square$

Corollary 4.49. *F does not contain non-abelian free subgroups.*

Proof. Free groups are not metabelian and neither contain copies of $\mathbb{Z}^2$. □

The absence of free groups does not share any light on the amenability of F. Despite this, if it is shown that F is amenable, it will be a torsion free, finitely presented counter-example to the Day-Neumann conjecture.

4.2.5 Wreath products and $\mathbb{Z} \wr \mathbb{Z}$

Thompson's group contains many copies of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Let us shortly introduce this product in general, and then look at this particular example and how it is embedded in F.

A wreath product is an special case of semidirect product: one of the factors is a direct sum of a group with itself, indexed by the other factor, and this other factor acts on the sum by permuting the coordinates in the sum.

Definition 4.50. Let H be a group and let G be a group acting on a set M. A (permutational) **wreath product** $H \wr_M G$ is the semi-direct product $\bigoplus_{m\in M} H \rtimes G$, where G acts on $\bigoplus_{m\in M} H$ by the permutations of the indices coming from its action on M.

Example 4.51. The lamplighter group, $\mathbb{Z}_2 \wr \mathbb{Z}$ is a semidirect product $\bigoplus_{-\infty}^{\infty} \mathbb{Z}_2 \rtimes \mathbb{Z}$. One could think of this group as having infinitely many lamps, and a lamplighter who can turn them on and off. Every element of the group corresponds to a state of the set of lamps (some on, some off), and a position for the lamplighter. There can only be finitely many lamps on at the same time (since we work with a direct sum).

This group admits a finite presentation

$$\langle a, t : a^2, [t^{-i}at^i, t^{-j}at^j] \text{ for any } i, j \in \mathbb{Z}\rangle.$$

The generator a corresponds to the lamplighter, starting at position 0, and the generator t corresponds to moving the lamplighter one step ahead. In this way, the element $t^{-i}at^i$ moves the lamplighter to position i, changes the state of the lamp in this position, and then takes the lamplighter back to its initial position. The relations mean that each lamp can only be in two states (on and off), and that it does not matter the order in which we change the state of the lamps.

This group is amenable, as it admits a short exact sequence

$$1 \to \bigoplus_{-\infty}^{\infty} \mathbb{Z}_2 \to \mathbb{Z}_2 \wr \mathbb{Z} \to \mathbb{Z} \to 1,$$

and both $\mathbb{Z}$, $\bigoplus_{-\infty}^{\infty} \mathbb{Z}_2$ are abelian and thus amenable.

Thompson's group contains many copies of a similar group: $\mathbb{Z} \wr \mathbb{Z}$, with presentation

$$\langle a, t : [t^i a t^{-i}, t^j a t^{-j}] \text{ for any } i, j \in \mathbb{Z} \rangle.$$

It works just like the lamplighter group, except for the fact that now the lamps can have countably many states, indexed by $\mathbb{Z}$, and $t^i a t^{-1}$ adds one to the state of the lamp in position i. We will denote $a_i = t^i a t^{-i}$.

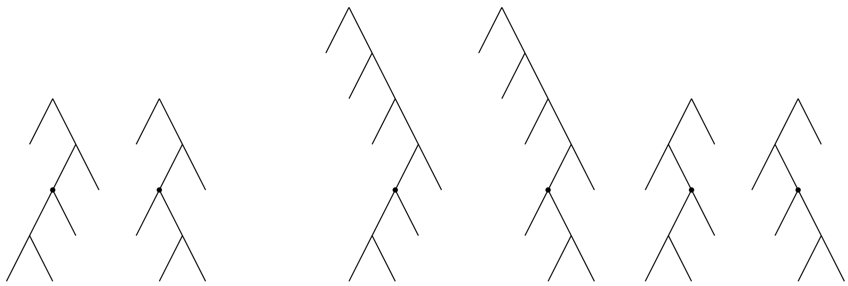
The idea to find these group inside F is the following: choose an element only having carets hanging from the same leaf in both trees. Then, conjugating by x_0 will move those same carets to the next leaf in both trees. So each leaf behaves like a lamp, and x_0 works as the lamplighter.

In this sense, we have as many subgroups of this type as things we can hang on a node; this also means that these subgroups are really small in F.

Let us formalize this idea. The element $x_1^2 x_2^{-1} x_1^{-1}$ only has carets hanging from the first right node.

Consider the subgroup $G = \langle x_0, x_1^2 x_2^{-1} x_1^{-1} \rangle \leq F$, and the map $\Phi \colon \mathbb{Z} \wr \mathbb{Z} \to G$ sending

$$t \to x_0, \quad a \to x_1^2 x_2^{-1} x_1^{-1}.$$



(a) $\Phi(a) = x_1^2 x_2^{-1} x_1^{-1}$ **(b)** $\Phi(a_2)$ **(c)** $\Phi(a_{-1})$

Theorem 4.52. *$\Phi \colon \mathbb{Z} \wr \mathbb{Z} \to G$ is a group isomorphism.*

Proof. The relations are satisfied: every pair $\Phi(a_i), \Phi(a_j)$ commute because, as $x_1 x_2^2$ hangs from a particular leaf, these elements have disjoint support. It is also surjective, since $x_0, x_1 x_2^2$ generate G.

To check injectivity, one can see that, since $\mathbb{Z} \wr \mathbb{Z}$ is a semidirect product, there is a normal form

$$a_{i_1}^{r_1} \dots a_{i_n}^{r_n} t^m$$

where the i_j, r_k, m are integers and $i_1 < \cdots < i_n$.

Each a_i maps to the corresponding pair of trees in the ith position, and then t^m moves the root to the position m, so the only way an element maps to the identity is if all exponents $r_1 = \cdots = r_n = m = 0$, i.e., if the element is the identity. □

Using the amenability of these subgroups, we will try to see how their Følner sets behave into F.

4.3 Other Thompson groups

Richard Thompson introduced another two groups T, V, with similar definitions to F. No proofs are included, but the curious reader can find more information in [8].

Thompson's group T is defined as the group of piecewise linear homeomorphisms of the circunference S^1, that are orientation preserving, send dyadics to dyadics, and have as slope a power of 2 on each piece.

We can look at S^1 as the interval $[0, 1]$ with the endpoints identified. With this idea, every element of F induces another element in T, so T contains F as a subgroup. Nevertheless, T contains more elements apart from those from F, because now we are not restricted to having $f(0) = 0$, or $f(1) = 1$, but just $f(0) = f(1)$. An element in $T \setminus F$ could be the following:

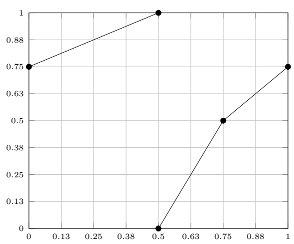


Figure 4.17: An element in $T \setminus F$.

In fact, the group T admits a finite presentation using generators x_0, x_1 from F and this new element. There is another version of tree diagrams for the elements of F: now, the leftmost tree in the left tree may not go to the leftmost leaf in the right tree, so one must mark this image of the leftmost leaf in the right tree.

The group T is not isomorphic to F, because it is not torsion free, as it contains the element $f(t) = t + \frac{1}{2}$.

The importance of group T lays on the fact that it is historically the first example of a finitely presented infinite simple group. It is known that T is not amenable, because it contains a non-abelian free subgroup, so this does not provide any information on the amenability of F. The proof for this can be checked in [5][Corollary 4.4].

Thompson's group V contains both T and F. It is defined analogously, as the group of orientation preserving, right continuous bijections of S^1 mapping dyadics to dyadics and having as slopes powers of 2 on each piece. Again, it is clear the elements in T, F have these properties, but now we are dropping continuity, so we can have elements like the following:

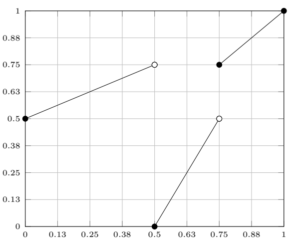


Figure 4.18: An element in $V \setminus T$.

As $T \subset V$, we know that V is neither amenable. Just like T, this group is also finitely presented and simple, and it has its own tree diagrams. Now, because of the lack of continuity, one needs to mark the target leaf where each leaf in the left tree goes, without any restrictions.

CHAPTER

5

Følner sets in F

Now that we know how to work with the group F, and its most important properties in relation to amenability, let us see why checking whether F is amenable can be difficult. We have seen that many of the most common tools for checking amenability fail when applied to F. Here, we will introduce another one: the fact that F is not elementary amenable. Then, we will dedicate the rest of the chapter to the study of Følner sets in F. Finding these sets is a really hard task, due to a theorem from Moore, but we will still try to get bounds on the Følner constant introduced in Chapter 3. We will introduce and prove the best bound to date, obtained by Belk and Brown, and see why it is hard to get better bounds. After this, we compute the ratios for other families, but we anticipate that our results are negative, and far from improving the constant found by Belk and Brown.

5.1 What we know

Recall that F has no free non-abelian subgroups, as shown in Corollary 4.49, and it has exponential growth, as proved in 4.33 so we cannot apply the criteria in Theorem 2.29. Furthermore, the idea to compute the Følner constant of $BS(2,3)$ in Chapter 3 does not work, because now, F is hopfian. We will prove now that F is not elementary amenable.

Recall the definition of elementary amenable groups in Chapter 2 There is another way to introduce this class inductively, using ordinals: let EG_0 be the class of all finite and abelian groups. Let α be an ordinal such that we have constructed all classes EG_β for $\beta < \alpha$. If α is a limit ordinal, we define

$$EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta$$

and if α is a succesor, we set EG_α to be the class of all groups that can be obtained from $EG_{\alpha-1}$ by taking (exactly) one time either a direct union or an extension of a group from $EG_{\alpha-1}$, by either a finite or an abelian group.

Then, one can see that

$$EG = \bigcup\{EG_\alpha : \alpha \text{ is an ordinal.}\}$$

This definition allows us to use transfinite induction. We will take advantage of this to prove the following:

Proposition 5.1. *F is not elementary amenable.*

Proof. Clearly F is not finite nor abelian, so $F \notin EG_0$. Moreover, F is finitely generated, so it cannot be expressed as a nontrivial direct union (this is because every generating set of F contains a finite generating set). Hence, the only way that F can appear in some EG_α is as a nontrivial group extension.

Assume that F does not belong to EG_β for $\beta < \alpha$, and suppose there is a short exact sequence

$$N \to F \to Q$$

where $N, Q \in EG_\beta$ for some $\beta < \alpha$.

Then, since F' is simple, it must be $F' \subset N$, and so N also contains copies of F. But now, since EG_β is closed under taking subgroups, we get $F \in EG_\beta$, which is a contradiction. $\square$

Thus, showing that F is amenable would give an example of a group in $AG \setminus EG$, and proving that F is not, an example of a group in $NF \setminus AG$.

In order to study F, we will search for some Følner sets, and compute limits on the boundary ratios, $\frac{|\partial A_n|}{|A_n|}$, in order to get upper bounds for the Følner constant in F.

5.2 Følner sets in F

As introduced in Chapter 1, Følner sets are families that "do not change too much" through the action of the generators. We can try finding Følner sets in F, but a theorem by Moore states that this is not going to be an easy task. It states that, if we actually find a Følner sequence in F, then the size of the sets must grow insanely quickly.

Let us state the theorem formally. It is remarkable that Moore does not use the inner boundary, but the symmetric difference boundary, defined as

$$|\partial A| = \sum_{x \in X} |xA \triangle A|.$$

As we mentioned in the first chapter, this is equivalent to the inner boundary, so it does not make that much of a difference. Define the function $\exp_p(n)$ recursively as

$$\begin{array}{lcl} \exp_0(n) & = & n, \\ \exp_{p+1}(n) & = & 2^{\exp_p(n)}. \end{array}$$

That is, $\exp_n(0)$ is an iterated exponential.

Theorem 5.2 (Moore [19])**.** *For any finite symmetric generating set $X \subset F$, there is a constant $C > 1$ such that if $A \subseteq F$ satisfies $\frac{|\partial A|}{|A|} \leq C^{-n}$, then A contains at least $\exp_n(0)$ elements.*

So, in a Følner sequence, the sets must grow as fast as an iterated exponential. This makes a hard task coming up with these kind of sets, and they also have attached a really expensive computational cost.

The proof of this theorem is long, and out of the scope of this project, but a reference is included for the curious reader. Although it treats elements as pairs of binary sequences, these are just another way to express tree diagrams.

We deduce from this theorem that all the families that we will study in this chapter will not be actual Følner sequences (as they do not grow sufficiently fast). Nevertheless, computing the ratio between the size of the boundary and the set will give us upper bounds for the Følner constant of F, as introduced in Chapter 2, for different sets of generators.

Notation. In this chapter we will call *Følner sequence* to any sequence of sets that we use to compute bounds on the Følner constants of F. These are not actual Følner sequences (as the limit of the boundary ratio does not go to zero), but it is just a matter of terminology.

First of all, it is remarkable that, since we have a lower bound from the growth in F, we can use Proposition 3.3 and Theorem 4.33 to deduce the following:

Proposition 5.3. *The Følner constant of F for the generating set $X = \{x_0, x_1\}$ satisfies*

$$Føl_X F \leq 1 - \frac{2}{3 + \sqrt{5}} \approx 0.618034.$$

Note that this bound uses the inner boundary, and that we get it using as Følner sequence the balls $B_X(n)$.

5.2.1 Belk-Brown sets

To this day, the best existing bound for the Følner constant of F is the one obtained by Belk and Brown. They also prove that it is hard to get a better bound, in the sense that sets achieving it must be a bit strange. It is now a conjecture whether this better bound actually exists (note that proving that it does not would immediately prove that F is not amenable.) Let us take a look at the sets used by Belk and Brown, and formalize these results. Remark that they also use a different boundary in their computations, namely the Cheeger boundary,

$$\partial A = \{(a, a', g) \in E : a \in A, a' \notin A\},$$

the set of edges joining a set and their complement.

Once we have done the original proof by Belk and Brown, we will adapt their constant for the inner boundary, in order to be able to compare it with the bound in Proposition 5.3.

Theorem 5.4 (Belk, Brown [4]).

$$\inf\{\frac{|\partial A|}{|A|} : A \subset F \textit{ finite.}\} \leq \frac{1}{2}.$$

Proof. It is enough to find a sequence of sets, A_k, so that

$$\inf_k \frac{|\partial A_k|}{|A_k|} = \frac{1}{2}.$$

Given $n, k \in \mathbb{N}$, let $S_{n,k}$ be the set of positive elements whose pointed forest diagram has width at most n and whose trees have height at most k.

Claim.

$$\lim_k \lim_n \frac{|\partial S_{n,k}|}{|S_{n,k}|} = \frac{1}{2}.$$

First, we fix $k \in \mathbb{N}$, and compute the limit for n. After this, we compute the second limit in k.

Claim 1. For each $k \in \mathbb{N}$, if f is randomly (uniformly) chosen in $S_{n,k}$, then

$$\frac{|\partial S_{n,k}|}{|S_{n,k}|} - 2P[x_1^{-1} f \notin S_{n,k}] \xrightarrow{n\to\infty} 0.$$

The boundary is the sum over all edges, so

$$\frac{|\partial S_{n,k}|}{|S_{n,k}|} = P[x_0 f \ \notin S_{n,k}] + P[x_0^{-1} f \ \notin S_{n,k}] + P[x_1 f \ \notin S_{n,k}] + P[x_1^{-1} f \ \notin S_{n,k}].$$

For each of the generators, the number of incoming edges must be the same as the number of outgoing edges, because of the symmetry of the set. Thus,

$$\frac{|\partial S_{n,k}|}{|S_{n,k}|} = 2P[x_0^{-1}f \ \notin S_{n,k}] + 2P[x_1^{-1}f \ \notin S_{n,k}]. \tag{5.1}$$

Let us begin by computing the first term.

Note that the condition that $x_0^{-1}f \notin S_{n,k}$ is equivalent to the fact that the pointed tree of f is the leftmost tree. For each $n \in \mathbb{N}$, the number of trees in $S_{n,k}$ is upper bounded by n (all trees are trivial) and lower bounded by $\frac{n}{2^k}$ (all trees are balanced with height k), so as $n \to \infty$, the minimum number of trees goes to ∞, and thus the probability that the pointed tree of f is the leftmost one goes to zero. Hence,

$$\frac{|\partial S_{n,k}|}{|S_{n,k}|} - 2P[x_1^{-1}f \notin S_{n,k}] = 2P[x_0^{-1}f \notin S_{n,k}] \xrightarrow{n\to\infty} 0.$$

Let us now compute the second term in equation 5.1.

Claim 2. Let f be a random pointed forest, with n leaves and height at most k. Then,

$$\lim_n P[x_1^{-1}f \notin S_{n,k}] = p_k,$$

where p_k is the unique positive root of

$$t_{1,k}p_k + t_{2,k}p_k^2 + \cdots = 1,$$

where $t_{l,k}$ is the number of trees with l leaves and height at most k.

Let f_n be the number of binary (not pointed) forests with n leaves and height at most k. These forests can be constructed by joining a forest with less leaves and a tree with the missing leaves. Hence, there is a recurrence:

$$f_n = t_{1,k}f_{n-1} + t_{2,k}f_{n-2} + \cdots + t_{3,k}f_{n-3} + \dots$$

Note that $t_{n,k} \neq 0$ for $0 < n \leq 2^k$ and $t_{n,k} = 0$ for $n > 2^k$, so this sum is always finite (namely, it has $\min\{n, 2^k\}$ terms, which for n big enough is just 2^k). By the theory of linear recurrence equations (searching for solutions of the form q^k for some $q \in \mathbb{R}$), it can be seen that f_n is essentially of the form p_k^{-n}, where p_k is the positive root of the polynomial in the claim. In particular,

$$\lim_n \frac{f_{n-1}}{f_n} = p_k.$$

Now, let us define R_n to be the number of pointed binary forests with n leaves and height at most k. To construct a pointed forest, we just need to choose a place to put the arrow. Thus, we can construct a pointed forest by putting two binary forests side to side, and the arrow in the middle of them (for instance, in the first tree of the right forest). Hence, there is another recurrence

$$R_n = f_1 f_{n-1} + f_2 f_{n-2} + \dots f_n f_0$$

Let R_n^* be the number of pointed forests whose pointed tree is trivial. Using the same idea, and leaving one leaf for the trivial tree, we have

$$R_n^* = f_0 f_{n-1} + f_1 f_{n-2} + \dots + f_{n-1} f_0.$$

Then, the probability that the current tree is trivial corresponds to

$$\frac{R_n^*}{R_n} = \frac{f_0 f_{n-1} + f_1 f_{n-2} + \dots + f_{n-1} f_0}{f_1 f_{n-1} + f_2 f_{n-2} + \dots f_n f_0}.$$

With some computations, one can see that this ratio approaces p_k as $n \to \infty$, and so we finally deduce

$$\lim_n \frac{|\partial S_{n,k}|}{|S_{n,k}|} = 2p_k$$

for each $k \in \mathbb{N}$.

It is left to see how these roots behave when $k \to \infty$. Note that if we define the polynomial

$$t_k(x) = t_{1,k} x + t_{2,k} x^2 + \dots$$

then by definition of p_k, it is $t_k(p_k) = 1$.

If we have two trees of height at most $k-1$, we can put one of them on the left leaf of a caret, and the other one on the right leaf. This will give us a new tree with height at most k. Moreover, the number of leaves in this new tree will be the sum of the leaves in each of the previous trees. In fact, any tree with height at most k can be constructed this way, except for the trivial tree. This gives a recurrence that can be expressed using the generating polynomials, as

$$t_k(x) = t_{k-1}(x)^2 + x$$

where x come from adding the trivial tree.

If we iterate $t \to t^2 + c$, we see that $c = p_k$ if and only if we arrive at 1 after $k+1$ iterations (as it is $t_k(p_k) = 1$).

For instance, to get to 1 in one iteration, we have the golden ratio, $c = \frac{-1+\sqrt{5}}{2}$, which satisfies $c^2 + c = 1$, so $c = p_1$.

For p_2, we look for a number such that $t_2(p_2) = t_1(p_2)^2 + p_2 = (p_2^2 + p_2)^2 + p_2 = 1$, and so p_2 is the positive root of the equation

$$p_2^4 + 2p_2^3 + p_2^2 + p_2 = 1.$$

numerically, $p_2 \approx 0.557$.

As k grows, the parabola, $y^2 = x + c$ gets arbitrarily close to the line $y = x$. These become tangent at $c = \frac{1}{4}$, so $\lim_k p_k = \frac{1}{4}$, and thus

$$\lim_k \lim_n \frac{|\partial S_{n,k}|}{|S_{n,k}|} = 2\frac{1}{4} = \frac{1}{2}.$$

And this concludes the proof. □

Recall that this value is in terms of the Cheeger boundary. Nevertheless, notice that we can compute the ratio for the inner boundary from this one: we do not care about the boundary with x_0, because it goes to zero. For x_1, we only need to take away from this ratio those elements which go out when multiplied by both x_1 and x_1^{-1} (because we are counting them twice).

This corresponds to elements for which the current tree of f is trivial (for x_1^{-1}) and those for which either the current tree or the next tree has height $k-1$ (for x_1).

Note that the probability that the current tree has height $k-1$ is the same that the next tree has height $k-1$ (the limit cases are very few, and thus tend to zero), and the sum is precisely $\frac{1}{4}$, as we computed. Thus, the probability that the next tree has height $k-1$ must be $\frac{1}{8}$.

On the other hand, the intersection we are looking for corresponds to elements for which the current tree is trivial and the next tree has height $k-1$. These are independent events (for n big enough), so the probability of the intersection is the product. Hence, the ratio of elements leaving by both x_1 and x_1^{-1} is $\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$.

We conclude that the bound from Belk and Brown in terms of the inner boundary is precisely $\frac{1}{2} - \frac{1}{32} = \frac{15}{32} \approx 0.46875$.

Let us now take a look on why it is so hard to get a better bound than this:

Corollary 5.5. *If T is a finite set of binary trees, which is close under taking subtrees, and $S_{n,T}$ is the set of poistive elements whose forest diagram has width at most n and trees in T, then*

$$\frac{|\partial S_{n,T}|}{|S_{n,T}|} > \frac{1}{2}.$$

Proof. Let a_i be the number of trees in T with i leaves, and

$$a(x) = a_1 x + a_2 x^2 + \dots$$

The same argument used before (under the hypothesis that T is closed under subtrees) shows that

$$\frac{|\partial S_{n,T}|}{|S_{n,T}|} = 2p,$$

where p is the positive root of the equation $a(x) = 1$. If k is the maximum height of a tree in T, then $a_i \leq t_{i,k}$ (as $T \subseteq S_{n,k}$), and so $a(x) \leq t_k(x)$ for all $x \geq 0$. All the coefficients of $a(x), t_k(x)$ are non-negative, so these two polynomials are increasing in $(0, \infty)$, and since $a(x) \leq t_k(x)$,it must be $p_k \leq p$ (i.e., t_k arrives to 1 before a does). Hence, $p > 1/4$. □

Note that we can only apply this result if we express our set in terms of forest diagrams. Anyway, most of the simple families that we can think of using forest or tree diagrams are closed by subtrees. Computations for families which are not might get really messy, as they might have really strange shapes.

Our approach in order to avoid this, is to define Følner sets in terms of the presentation. We have not checked whether our sequences are closed by subtrees, though, as there is not an easy transition from their expression to forest diagrams.

5.2.2 Følner sets from the presentation of F

The fact that Belk and Brown obtain such a great bound using a simple family of tree diagram invites to check whether other simple sets also give good bounds. In our case, we will look at simple sets coming from the presentation of F Remark that we can restrict ourselves to sets of positive elements, as translating a set does not change the size of the boundary.

5.2.2.1 Basic case

The first family that comes to mind when looking at the presentation is

$$A_n = \{x_0^{a_0} \dots x_n^{a_n} : 0 \leq a_1, \dots, a_n \leq n\}.$$

This works as a Følner sequence in $\mathbb{Z}^n$, for instance. We can think of this set as an $n+1 \times n+1$ box, where each point $(a_0, \dots, a_n)$ represents the element $x_0^{a_0} \dots x_n^{a_n}$. Even though our finite presentation of F has generators x_0, x_1, any pair $\{x_0, x_i\}$, with $i > 0$, is another finite generating system for F, since we can find x_1 as

$$x_1 = x_0 x_2 x_0^{-1} = \dots = x_0^{i-1} x_i x_0^{-(i-1)}.$$

Our work allows to compute the boundary ratio for any generating set of this form.

To compute thi ratio, just like in the proof above, we need to count how many elements escape A_n when multiplied by $x_0^{\pm 1}$ or $x_i^{\pm 1}$, i.e., their correspondent countributions to $\partial A_n|$.

Just like before, it is enough to compute the ratios for x_0 and x_i, as these ratios will be the same for its inverses because of the symmetry of the sets.

Let us begin with the case for x_0.

Proposition 5.6. *Let $|\partial_{x_0} A_n|$ be the number of elements in A_n which escape when multiplied by x_0, i.e., elements $x \in A_n$ such that $x_0 x \notin A_n$. Then,*

$$\lim_n \frac{|\partial_{x_0} A_n|}{|A_n|} = 0.$$

Proof. If $x_0^{a_0} \dots x_n^{a_n} \in A_n$, but $x_0 x_0^{a_0} \dots x_n^{a_n} \notin A_n$, then it must be $a_0 = n$. Hence,

$$\frac{|\partial_{x_0} A_n|}{|A_n|} = \frac{(n+1)^n}{(n+1)^{n+1}} = \frac{1}{n+1} \to 0.$$

□

Thus, almost all elements stay in A_n when multiplied by x_0 or x_0^{-1}.

The study with x_i, for $i > 0$ is not that easy, as now the subindex of this element will change as we find the normal form. Note that when we multiply x_i by an element, the subindex i will increase as it goes through the word, until it finally finds the normal form of the product.

There are two ways in which our element can leave A_n when multiplied by x_i: if this subindex increases until it becomes larger than n, or if this subindex stops in some x_k, before x_n, and the exponent a_k was already n. In the former case, we will say that the element leaves from the right, and otherwise that it leaves from the top. (this makes sense when understanding A_n as a box).

For instance, $x_0 x_1 x_2 \in A_2$, but $x_1 x_0 x_1 x_2 = x_0 x_1 x_2 x_3 \notin A_2$ is an example of an element leaving from the right. On the other hand, $x_0 x_2^2 \in A_2$ but $x_1 x_0 x_2^2 = x_0 x_2^3 \notin A_2$ represents the case of leaving from the top.

We will consider these two cases differently, as this helps the counting.

Let us start with the top case. We start counting how many elements leave from the top while stopping in the last possible place:

Proposition 5.7. *Let $a_{i,n}$ to be the number of elements in A_n that leave from the top in place x_n when multiplied by x_i. Then,*

$$a_{i,n} = \begin{cases} 0 & \textit{if } i > n, \\ 1 & \textit{if } i = n, \\ \sum_{k=i-1}^{n} a_{k,n-1} & \textit{if } i < n. \end{cases}$$

Proof. Consider an element $x_0^{a_0} \dots x_n^{a_n} \in A_n$, and $x_i \in F$. There are three possible cases:

1. If $i > n$, then
$$x_i x_0^{a_0} \dots x_n^{a_n} = x_0^{a_0} \dots x_n^{a_n} x_{i+a_0+\dots+a_n}$$
always leaves from the right. Hence, $a_{i,n} = 0$.

2. If $i = n$, then the only way that the element leaves from the top in x_n is if the subindex does not grow at all and $a_n = n$, i.e.,
$$\begin{cases} a_n = n, \\ a_0 = a_1 = \dots = a_{n-1} = 0. \end{cases}$$
Thus, $a_{n,n} = 1$.

3. If $i < n$, then it must be
$$i + a_0 + \dots + a_{n-1} = n,$$
so a_0 can take values from 0 to $n - i$. We consider each possible value of a_0 separately:

   If $a_0 = 0$, we have $x_i x_1^{a_1} \dots x_n^{a_n}$. Note that only a_n can be equal to n, since $\sum_{j=0}^{n-1} a_j = n - i$. Moreover, in order for the product to leave A_n, it must be $a_n = n$ maximal. Hence, the number of elements leaving here is the same as the number of elements leaving in the case $x_{i-1} x_0^{a_1} \dots x_{n-1}^{a_{n-2}} x_{n-1}^{n-1}$, and this is precisely $a_{i-1,n-1}$.

   For $a_0 = 1$, the element is of the form $x_0 x_1^{a_1} \dots x_n^{a_n}$, and thus
$$x_i x_0 x_{i+1} x_1^{a_1} \dots x_n^{a_n} = x_0 x_{i+1} x_1^{a_1} \dots x_n^{a_n}.$$

   In general, this argument shows that, if $a_0 = j$, then there are $a_{i+j-1,n-1}$ possible cases. Thus, summing over all possible values of a_0,
$$a_{i,n} = \sum_{k=i-1}^{n} a_{k,n-1}.$$

□

We wrote a small program in Python to compute the first terms of this recurrence:

$a_{i,n}$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	1	2	5	14	42	132	429
2	0	0	1	2	5	14	42	132	429
3	0	0	0	1	3	9	28	90	297
4	0	0	0	0	1	4	14	48	165
5	0	0	0	0	0	1	5	20	75
6	0	0	0	0	0	0	1	6	27

Note that row 1 seems to correspond to the *Catalan numbers*. Furthermore, some research on the rest of the rows seem to tell that these correspond to the convolutions of the Catalan numbers:

Definition 5.8. Given a sequence $(a_n)_n$, with generating polynomial $A(x)$, the **i-th convolution** of (a_n) is defined to be the sequence with generating polynomial $A^{i+1}(x)$.

We deduce from the definition the following recurrence formula, which looks exactly the same as the one for the $a_{i,n}$:

$$C_n^i = \begin{cases} 0 & \text{if } i > n, \\ 1 & \text{if } i = n, \\ \sum_{k=i-1}^{n} C_{n-1}^k & \text{if } i < n. \end{cases}$$

Catalan found a closed formula for these convolutions:

Theorem 5.9. *Let $i \geq 1$, and C_n^i the nth term of the ith Convolution of the catalan numbers. Then,*

$$C_n^i = \begin{cases} 0 & \textit{if } n < i, \\ \frac{i}{2n-i}\binom{2n-i}{n} & \textit{otherwise.} \end{cases}$$

This theorem is out of the scope of the project, but it will be useful for the asymptotic study of our sequences. The original proof can be found (in french) in [9].

Let us formally prove that, indeed, the $a_{i,n}$ are these convolutions:

Proposition 5.10. *The value $a_{i,n}$ corresponds to the nth term in the $(i-1)$th convolution of the Catalan numbers, i.e.*

$$a_{i,n} = C_n^{i-1}.$$

Proof. Since the recurrence is the same, it is enough to see that $a_{1,n}$ are the Catalan numbers. We do this by induction. First, recall that the Catalan numbers satisfy the recurrence:

$$C_n = \sum_{i=1}^{n} C_i \cdot C_{n-i}.$$

An element in A_n will leave in place x_n when multiplied by x_1 if and only if it n is the first position where it stops, and $a_n = n$. This is equivalent to the following equations:

$$\begin{cases} a_0 \neq 0, \\ a_0 + a_1 \neq 1, \\ \vdots \\ a_0 + \cdots + a_{n-2} \neq n-2, \\ a_0 + \cdots + a_{n-1} = n-1 \\ a_n = n. \end{cases} \tag{5.2}$$

For $n = 1$, this is just $a_0 = a_1 = 1$, so there is one unique case, and so $a_{1,1} = C_1$.

Now, assume $a_{1,j} = C_i$ for $j < k$. We claim that

$$a_{1,k} = \sum_{i=1}^{n} a_{1,i} \cdot a_{1,k-i}.$$

To see this, we consider different cases depending on the value of $a_0, \ldots, a_{k-1}$.

First, if $a_0 = 1$, then the constrains can be splitted separately into:

$$\begin{cases} a_0 = 1, \end{cases} \quad \begin{cases} a_1 \neq 0, \\ a_1 + a_2 \neq 1, \\ \vdots \\ a_1 + \cdots + a_{k-2} \neq k-3, \\ a_1 + \cdots + a_{k-1} = k-2. \\ a_k = k. \end{cases}$$

The condition on the left ($a_0 = 1$) corresponds to the case $n = 1$ in the conditions in equation 5.2, so this can happen in $a_{1,1} = 1$ ways. On the other hand, on the right, as all the a_j, $j = 1, \ldots, k-1$ take the same values, and a_k must be maximal, the number of possible cases is the same as if we took 1 out of each subindex j, i.e., if we had conditions

$$\begin{cases} a_0 \neq 0, \\ a_0 + a_1 \neq 1, \\ \vdots \\ a_0 + \cdots + a_{k-3} \neq k-3, \\ a_0 + \cdots + a_{k-2} = k-2. \\ a_{k-1} = k-1. \end{cases}$$

and this can be done in $a_{1,k-2}$ ways. Hence, the total number of possibilities here is $a_{1,1} \cdot a_{1,k-1}$.

We can keep doing this: again, if $a_0 \neq 1$, we can split the conditions in:

$$\begin{cases} a_0 \neq 0, \\ a_0 \neq 1, \\ a_0 + a_1 = 2. \end{cases} \quad \begin{cases} a_2 \neq 0, \\ a_2 + a_3 \neq 1, \\ \vdots \\ a_2 + \cdots + a_{k-2} \neq k-4, \\ a_2 + \cdots + a_{k-1} = k-3. \\ a_k = k. \end{cases}$$

Now, in the left, we can assume $a_0 > 0$ and reduce by 1 each independent term, so that the left part is equivalent to the case:

$$\begin{cases} a_0 \neq 0, \\ a_0 + a_1 = 1 \end{cases}$$

and this can happen in $a_{1,2}$ ways. On the other hand, taking two out from each subindex, the right part is equivalent to

$$\begin{cases} a_0 \neq 0, \\ a_0 + a_1 \ \neq 1, \\ \vdots \\ a_0 + \cdots + a_{k-4} \neq k-4, \\ a_0 + \cdots + a_{k-3} = k-3- \\ a_{k-2} = k-2. \end{cases}$$

and again, this happens in $a_{1,k-2}$ ways. Hence, this total case happens in $a_{1,2} \cdot a_{1,k-2}$.

The same process can be applied repeatedly, and these cases are all disjoint. At step i, one gets conditions:

$$\begin{cases} a_0 \neq 0, \\ a_0 \neq 1, \\ a_0 + a_1 \neq 2, \\ \vdots \\ a_0 + \cdots + a_{i-1} = i. \end{cases} \qquad \begin{cases} a_i \neq 0, \\ a_i + a_{i+1} \ \neq 1, \\ \vdots \\ a_i + \cdots + a_{k-2} \neq k-i-1, \\ a_i + \cdots + a_{k-1} = k-i. \\ a_k = k. \end{cases}$$

And, by the same argument, this contributes to $a_{1,k}$ in $a_{1,i} \cdot a_{1,k-i}$. Summing all of them, we get

$$a_{1,k} = \sum_{i=1}^{n} a_{1,i} \cdot a_{1,k-i}.$$

By induction hypothesis, $a_{1,j} = C_j$ for $j < k$, so,

$$a_{1,k} = \sum_{i=1}^{n} C_i \cdot C_{k-i} = C_k.$$

□

Now that we know the precise values of $a_{i,n}$, we are ready to compute the total number of elements leaving from the top:

Proposition 5.11. *Let $b_{i,n}$ to be the number of elements in A_n that leave by the top when multiplied by x_i. Then, for all $i \geq 1$,*

$$\lim_n b_{i,n} = 0.$$

Proof. Note that, as we saw in the proof, the conditions to stop at place k only depend on the values $a_0, \ldots, a_k$. Hence, we can free the last $n-k$ values of a_i, and on the first $k+1$ we have the same conditions as before. Thus, the number of elements in A_n that leave at place x_k is $a_{i,k}(n+1)^{n-k}$.

If we do this for all $k = i, \ldots, n$, since $a_{i,k} = 0$ if $k < i$, we get the following:

$$b_{i,n} = \sum_{k=0}^{n} a_{i,k}(n+1)^{n-k} = \sum_{k=i}^{n} a_{i,k}(n+1)^{n-k}.$$

And thus the ratio of elements that leave from the top is

$$\frac{\sum_{k=i}^{n} a_{i,k}(n+1)^{n-k}}{(n+1)^{n+1}} = \sum_{k=i}^{n} a_{i,k}(n+1)^{-(k+1)}.$$

Since the Catalan numbers (and its convolutions) grow approximately as 4^k (and in fact are upper bounded by 4^k), we get

$$\sum_{k=i}^{n} a_{i,k}(n+1)^{-(k+1)} \leq \frac{1}{n+1}\sum_{k=i}^{n}\left(\frac{4}{n+1}\right)^{-(k+1)}.$$

This is a geometric sum, so we can compute it to get

$$\frac{1}{n+1}\sum_{k=i}^{n}\left(\frac{4}{n+1}\right)^{-(k+1)} = \frac{1}{n+1}\frac{4((1+n)^{-i} + n(1+n)^{-i} - (1+n)^{-n})}{n}$$

And, if $i \geq 1$, this tends to 0 as $n \to \infty$, so the ratio of elements that leave from the top is precisely 0. $\square$

Let us now study proceed with the case of elements leaving by the right. When multiplying by x_i, if it does not stop before the end of the word, it will end up having index $i + a_0 + \cdots + a_n$. Hence, the elements leaving by the right are those whose sum of indices is bigger than $a_0 + \cdots + a_n \geq n+1-i$ which do not stop at any x_k, for $k < n$. We anticipate that this is the way most elements leave the set, because there are very few elements either stopping or having small sum of a_j. We first introduce a lemma for the computations:

Lemma 5.12. *Let*

$$B_n^k = \{x_0^{a_0} \dots x_n^{a_n} \in A_n : \sum_{i=0}^{n} a_i = k\}.$$

Then,

$$|B_n^k| = \binom{k+n}{n}.$$

Proof. Note that $|B_n^k|$ is precisely the number of non-negative integer solutions to the equation

$$\sum_{i=0}^{n} a_i = k.$$

To compute this number, consider a row of $(+)$ and $(-)$, where $(+)$ corresponds to adding 1 to a_i and $(-)$ corresponds to passing from a_i to a_{i+1}. Starting at $i = 0$, to get to a_n we need n steps, so there must be n symbols of type $(-)$. The total value of the sum is k, so there must be exactly k symbols $(+)$. Thus, it consists in writing a row of $k + n$ symbols, and choosing the n positions for the $(-)$ symbols. This can be done in $\binom{k+n}{n}$ ways. □

Now, let

$$\tilde{B}_{i,n} = \{x_0^{a_0} \dots x_n^{a_n} : \sum_{j=0}^{n} a_i \leq n - i\}.$$

Then, using Lemma 5.12,

$$|\tilde{B}_{i,n}| = \sum_{k=0}^{n-i} |B_n^k| = \sum_{k=0}^{n-i} \binom{k+n}{n} = \frac{1-i+n}{n+1}\binom{2n-i+1}{n}.$$

As a side note, remark the similarity with the convolution formula of the Catalan numbers in Theorem 5.9.

On the other hand, let us compute the cardinality of the set of elements which stop (independently of the sum of the a_i). This is just the same as we did for those elements leaving from the top, except now stopping at step k does not require that $a_k = n$, but instead, a_k can take any value. Hence, there are

$$\sum_{k=i}^{n} a_{i,k}(n+1)^{n-k+1}$$

elements stopping at some x_k when multiplied by x_i.

Let us put all things together: The number of elements leaving by the right is:

$$|\{x \in A_n : \sum a_i \geq n-i+1\}| - |\{x \in A_n : \sum a_i \geq n-i+1 \text{ and } x_i \text{ stops before } x_n\}|, \tag{5.3}$$

and we need to divide this by the size of $|A_n|$.

We can compute the first term by taking its complement in A_n,

$$\{x \in A_n : \sum a_i \geq n-i+1\}| = |A_n| - |\tilde{B}_{i,n}|.$$

We already computed these values, so the contribution of the first term to the boundary ratio is

$$\frac{(n+1)^{n+1} - \frac{1-i+n}{n+1}\binom{2n-i+1}{n}}{(n+1)^{n+1}} \xrightarrow{n\to\infty} 1.$$

This limit can be computed using Stirling's formula on the binomial.

For the second term in 5.3, we take the complement in the set of elements that stop, so that

$$|\{x \in A_n : \sum a_i \geq n-i+1 \text{ and } x_i \text{ stops before } x_n\}| =$$
$$|\{x \in A_n : x_i \text{ stops before } x_n\} - |\{x \in A_n : \sum a_i \leq n-i \text{ and } x_i \text{ stops before } x_n\}|$$

The first term here is precisely

$$\sum_{k=i}^{n} a_{i,k}(n+1)^{n-k+1}$$

and the second one can be upper bounded by $|\tilde{B}_{i,n}|$. Hence, the contribution is at most

$$\frac{-\sum_{k=i}^{n} a_{i,k}(n+1)^{n-k+1} + |\tilde{B}_{i,n}|}{(n+1)^{n+1}} \xrightarrow{n\to\infty} 0.$$

This limit can be done using the fact that $a_{i,k} \sim 4^k$ and Stirling's approximation once again.

In conclusion, summing up all these results the total contribution from x_i is

$$\lim_n \frac{|\partial_{x_i} A_n|}{|A_n|} = 1.$$

and so almost all elements leave when multiplied by x_i (and in particular, they mostly leave from the right.)

This concludes our study of the boundary ratio for this set. We proved the following:

Theorem 5.13. *Let $X_i = \{x_0, x_i\}$ be a generating system for F. Then,*

$$\frac{|\partial_{x_0} A_n|}{|A_n|} = 0 \quad \frac{|\partial_{x_i} A_n|}{|A_n|} = 1.$$

And so, if ∂A is the Cheeger boundary of $A \subset F$, it is

$$\lim_n \frac{|\partial_{X_i} A_n|}{|A_n|} = 2.$$

This does not tell us anything, since even finite sets have better boundary ratios. Nevertheless, this study gives some information on how to approach the problem: all the contribution to the boundary comes from x_i, and almost all elements leave from the right.

With this in mind, we can try introducing other threshold functions for the a_j, so that the number of elements such that $\sum a_j \geq n$ is small, and not so many elements leave from the right.

Note that, for any of these threshold functions f_n, the number of elements that leave with x_0 is approximately $\frac{1}{f(0)}$. Hence, in order to have no problem with x_0, we need to have $f_n(0) \xrightarrow{n} \infty$, so that $\frac{1}{f_n(0)} \xrightarrow{n} 0$. Moreover, it would be desirable that $f_n(0)$ tends to infinity slowly, because elements with large values of a_0 will likely escape with x_i, as it will probably be $\sum a_j \geq n$.

Even though we have tried to do this formally with different families of threshold functions, such as $\frac{n}{2^k}$, the computations get really messy. First of all, we could try to find a new sequence like $a_{i,k}$, and redo the analysis. This is hard to do in general, and even if we get a recurrence, it is hard to come up with a closed formula (in this case, we were lucky that we got the Catalan numbers) to study its asymptotics. Nevertheless, there are some simpler cases where an analysis like this can work.

5.2.2.2 **Threshold** $\log(n-x)$

We found a recurrence which works for any increasing function of the form $f(n-x)$ (and $f_n(0) < n$). It is hard getting a closed form for this recurrence, but we study it computationally. For simplicity, we do this for the case $f(x) = \log(x)$, but the same can be done for any other increasing function strictly dominated by n.

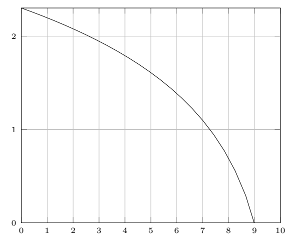


Figure 5.1: $\log(10-x)$.

Define

$$C_n = \{x_0^{a_0} \dots x_{n-1}^{a_{n-1}} : 0 \leq a_k \leq \log(n-k)\} \setminus \{1\}.$$

First of all, note that a_i takes values in $\{0, \dots, \lfloor \log(n-i) \rfloor\}$. Because of this, a general element in C_n is actually of the form $x_0^{a_0} \dots x_{n-3}^{a_{n-3}}$, as $a_{n-2} = a_{n-1} = a_n = 0$. In particular, $C_0 = C_1 = C_2 = \emptyset$, and $C_3 = \{x_0\}$. One could put inside the identity or take it away, as it just one more element. We choose to take it away, because it is helpful at some point during the computations. These set have size

$$|C_n| = \left(\prod_{k=0}^{n-3} (\lfloor \log(n-k) \rfloor + 1) \right) - 1 = \left(\prod_{k=3}^{n} (\lfloor \log(k) \rfloor + 1) \right) - 1 \sim n \log n.$$

The good thing with this recurrence is that

$$f(n-k) = f((n-1)-(k-1)) = f((n-2)-(k-2) = \dots$$

so the threshold in C_n for a_0 will be the threshold for a_1 in C_{n+1}, the one for a_2 in C_{n+2}, etc. With this, we can apply the argument in the recurrence for $a_{i,k}$ to reduce it to a sum of terms of the form $a_{j,k-1}$.

In any case, let us just begin with x_0.

Proposition 5.14. *Let $|\partial_{x_0} C_n|$ be the contribution of x_0 to $|\partial C_n|$.*

$$\lim_n \frac{|\partial_{x_0} C_n|}{|C_n|} = 0.$$

Proof. Again, an element leaves when multiplied by x_0 if and only if $a_0 = \lfloor \log n \rfloor$, so the ratio is

$$\frac{\prod_{k=1}^{n-3} \lfloor \log(n-k) \rfloor + 1}{-1 + \prod_{k=0}^{n-3} (\lfloor \log(n-k) \rfloor + 1)} = \frac{\prod_{k=3}^{n-1} \lfloor \log(k) \rfloor + 1}{-1 + \prod_{k=3}^{n} (\lfloor \log(k) \rfloor + 1)} \sim \frac{1}{\log(n)} \xrightarrow{n \to \infty} 0.$$

Hence,

$$\frac{|\partial_{x_0} C_n|}{|C_n|} \xrightarrow{n \to \infty} 0.$$

□

So just as we mentioned, the fact that $f_n(0) \xrightarrow{n} \infty$ solves any possible problem with x_0.

Let us see now what happens with the other generators, x_i.

Proposition 5.15. *Let $c_{i,n}$ be the number of elements in C_n which leave by the top in place x_{n-3} (last) when multiplied by x_i. Then,*

$$c_{i,n} = \begin{cases} 0 & \text{if } i > n-3, \\ 1 & \text{if } i = n-3, \\ \sum_{k=i-1}^{\lfloor log(n) \rfloor + i - 1} a_{k,n-1} & \text{if } i < n-3. \end{cases}$$

Proof. The proof is the same as Proposition 5.7. Only a couple of things change: the last place in the word is $n-3$, instead of n, and the values of a_0 go from 0 to $\lfloor \log(n) \rfloor$. $\square$

Again, we can look at the first values of these recurrence:

$c_{i,n}$	0	1	2	3	4	5	6	7	8
0	0	0	0	1	2	4	8	16	32
1	0	0	0	0	1	3	7	15	42
2	0	0	0	0	0	1	4	11	31
3	0	0	0	0	0	0	1	5	17
4	0	0	0	0	0	0	0	1	6

It is hard to extrapolate these sequences to a general case, though. For instace, even if the first row seems to correspond to powers of 2, one can see that $c_{1,9} = 96$. This is because $\log(7) < 2$, but $\log(8) > 2$, so there are "strange" changes whenever $\lfloor \log(n) \rfloor$ changes, i.e. in values of the form $\lfloor e^k \rfloor$ for some $k \in \mathbb{N}$.

Anyway, for any n, we can compute the contribution of these elements to the boundary ratio. Following the same argument as in the other case, this is:

$$\frac{\sum_{k=1}^{n} c_{i,k}(n+1)^{\lfloor \log(n-k) \rfloor}}{(\prod_{k=3}^{n} \lfloor \log(k) \rfloor) - 1}$$

And our only limitation for this computation is the memory size of the computer. Before getting overflow, we have got to $n = 423$, and the corresponding boundary ratio for x_1 is ≈ 0.0059. We might dare to say that the limit is zero, as also the sequence is decreasing. We cannot confirm this, though, beacuse of the slowness of the logarithm and the fact that there are these mentioned changes every long time.

For elements leaving by the right, we have a similar analysis:

Proposition 5.16. *Let $d_{i,n}$ be the number of elements that leave from the right in C_n when multiplied by x_i.*

$$d_{i,n} = \begin{cases} \sum_{k=0}^{\lfloor \log n \rfloor} d_{i-1+k,n-1} + \lfloor \log n \rfloor - (n-2-i) + 1 & \textit{if } \lfloor \log n \rfloor \geq n-2-i, \\ \sum_{k=0}^{\lfloor \log n \rfloor} d_{i-1+k,n-1} & \textit{otherwise.} \end{cases}$$

Proof. Again, the argument for the recurrence is the same: reduce the case

$$x_i \cdot x_0^{a_0} \dots x_{n-3}^{a_{n-3}}$$

to the case

$$x_{i+a_0-1} x_0^{a_1} \dots x_{n-4}^{a_{n-4}}.$$

The only time when we cannot do this is with elements of the form $x_0^{a_0}$. Note that these elements leave if and only if $i + a_0 \geq n - 2$. In particular, for this to happen, n must satisfy $\lfloor \log n \rfloor \geq n - 2 - i$, and the number of these elements leaving will be $\lfloor \log n \rfloor - (n - 2 - i) + 1$. □

Once again, we look at the first values of this recurrence:

$c_{i,n}$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	2	2	2	2	12
2	0	0	0	1	3	6	8	10	34
3	0	0	0	1	3	7	14	22	62
4	0	0	0	1	3	7	15	30	94

Again, as with the top case, we do not find any known sequence to compare it. We compute again the boundary ratio for these sets, and we get that, for x_1 and $n = 420$, the contribution is ≈ 0.82733.

We can not guarantee anything, but this seems to tend to one slowly, and in any case, the bounds that we get for this n are still bigger than Belk and Brown's.

We also tried $\sqrt{n-k}$, and one could do this again for any $(n-k)^{1/l}$, with $l \geq 2$, but we stay far from getting any good bound. These are all concave functions, and for which

$$\int_0^n f(x) >> n.$$

It would be desirable to try other functions of $n - x$ whose integral was asymptotically like n, but this is quite hard, since we always need $f_n(0) \to \infty$, and if we ask to have $0 \leq a_j \leq 1$ for $j = 0, \ldots, n$, we are already adding n to the integral.

5.2.3 Sets defined by breakpoints.

Just as we searched for Følner sets in the presentation, we could define other sets via the definition of F. We are not going to do the whole study of these sets, but see how the Corollary from Belk and Brown can be applied. Consider the set A_n of positive elements in F whose breakpoints are of the form $(\frac{k}{2^n}, y)$, for each $n \in \mathbb{N}$. If we think about this set as a family of tree diagrams, these elements corresponds to diagrams whose left tree has height at most n.

This family of tree diagrams is closed by subtrees, but we cannot directly apply the Corollary of the theorem from Belk and Brown, as that only works for forest diagrams. Let us see how the forest diagram of these elements look.

Consider a balanced tree of height k. Its corresponding pointed diagram is triangular:

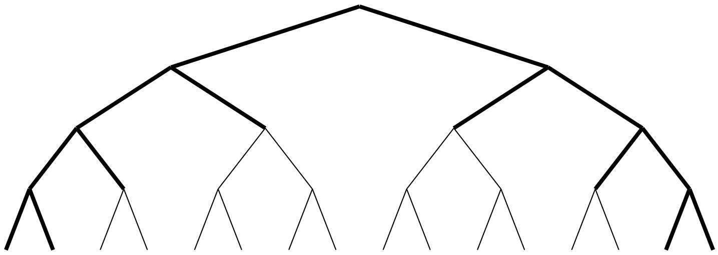


Figure 5.2: A balanced tree of height 4

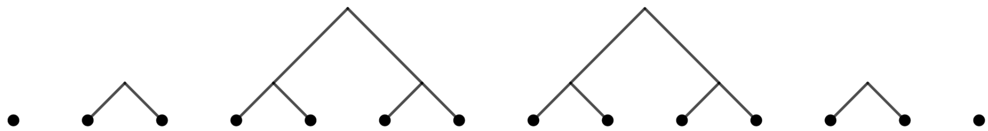


Figure 5.3: Pointed forest diagram of a balanced tree of height 4.

Any subtree in the original balanced tree corresponds to a subforest of the pointed forest diagram, so we can conclude that this family is closed by subtrees (in the forest sense), and thus Corollary 5.5 applies.

5.2.4 **Følner sets in wreath products.**

Amenable subgroups of F have their own Følner sequences. We could think that, as these sets have small boundary inside the subgroup, they might have small boundary in the whole F aswell. As we proved, F contains copies of $\mathbb{Z} \wr \mathbb{Z}$, and this is an amenable group. Let us see whether we can transfer its Følner sequence to a Følner sequence in F.

The monoid of positive words, $\langle a, t\rangle \; \leq \mathbb{Z} \wr \mathbb{Z}$ is free, so the group has exponential growth, and thus the balls do not work as Følner sets. We take advantage of the fact that there is an exact sequence

$$1 \to \bigoplus_{k \in \mathbb{Z}} \mathbb{Z} \to \mathbb{Z} \wr \mathbb{Z} \to \mathbb{Z}.$$

The group $\mathbb{Z}$ is abelian and thus it has subexponential growth, so we have Følner sets $\{t^k \colon -n \leq k \leq n\}$. The group $\bigoplus_{-\infty}^{\infty} \mathbb{Z}$ is not finitely generated, but one can see that the set

$$\{a_{i_1}^{r_1} \dots a_{i_l}^{r_l} : -n \leq i_1, \dots, i_l \leq n \text{ and } \sum |r_i| \leq n\}$$

is a Følner sequence (care that we need to use here the definition of a Følner sequence for a non-finitely generated group, so the condition must be satisfied for any finite subset of generators.)

Thus, we propose as Følner sequence:

$$A_n = \{a_{i_1}^{r_1} \dots a_{i_l}^{r_l} t^k : -n \leq k \leq n, -n \leq i_1, \dots, i_l \leq n \text{ and } \sum |r_i| \leq n\}.$$

Proposition 5.17. *A_n is a Følner sequence in $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. Multiplying by t, one gets

$$t \cdot a_{i_1}^{r_1} \dots a_{i_l}^{r_l} t^k = a_{i_1-1}^{r_1} \dots a_{i_l-1}^{r_l} t^{k+1}$$

so this only takes out those elements for which $k = n$ and $i_1 = -n$, and thus the ratio for t is

$$\frac{|\partial_t A_n|}{|A_n|} = \frac{1}{(2n+1)^2} \xrightarrow{n \to \infty} 0.$$

The symmetry of the sets implies that the ratio for t^{-1} is the same.

For a,

$$a \cdot a_{i_1}^{r_1} \dots a_{i_l}^{r_l} t^k = a_{i_1}^{r_1} \dots a_0^{i_0+1} \dots a_{i_l}^{r_l}$$

since the a_i commute. Hence, it only adds one to the sum of the $|r_i|$, and thus the only elements leaving are those for which $\sum |r_i| = n$. Thus, we confirm that these are Følner sets for $\mathbb{Z} \wr \mathbb{Z}$. □

Now, consider a subgroup pf F isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, with generators a and $t = x_0$.

The action by x_0 works just like the action by t in the previous proof, so this does not present any problem.

Nevertheless, when multiplying by any other x_i, generally we go out of the subgroup, as

$$x_i a_j = x_i x_0^{-j} a x_0^j$$

and this usually changes the structure of the element.

This is similar to what happens if we use the Følner sets for $\mathbb{Z} = \langle t \rangle$ in $\mathbb{Z}^2 = \langle t, u \rangle$. The action by t is fine, but multiplying any element by u will make the element leave the set, so the u contributes with 1 to the ratio, and so this is not a Følner set for $\mathbb{Z}^2$.

CHAPTER

6

Conclusions

The purposes presented in the introduction have been attained: we have introduced the property of amenability, along with its most important examples and results. Følner's theorem has been used to introduce a weaker form of amenability. We have studied the behavior of the Følner constant through subgroups and quotients, and computed it for some particular groups. Apart from this, we have introduced Thompson's group F, and shown how most of the existing criteria for amenability either do not apply, such as subexponential growth, or existence of free subgroups, or they are hard to check, such as Følner subsets. Finally, even though we completed all our computations, our method turned out unsuccessful in getting better bounds on the Følner constant of F. This work, along with Moore's theorem and Belk and Brown's, highlights how hard it can be to study any possible Følner sequence in F and obtain good results.

To conclude, it is remarkable how we treated a measure theoretical property from a purely combinatorial point of view. Most of the difficulty of the original work presented in Chapter 4 had more to do with enumerative combinatorics than actual group theory, and this was all done in order to search for a measure. Moreover, we have studied groups through their Cayley graph, in order to turn them into metric spaces and give them a topology. Apart from this, we found several interpretations, in terms of binary trees, of a group which was initially defined as a set of piecewise linear homeomorphisms of the interval. All these ideas make beautiful examples of how different fields of mathematics interlace, and feed back one another.

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