On the Regularity and Stability of the Dual-Phase-lag Equation

Abstract

In this paper we consider the following linear partial different equation which is usually seen as an approximation to the dual-phase-lag heat equation proposed by Tzou.

$$\dot{T} + \tau_q \ddot{T} + \frac{\tau_q^2}{2}\dddot{T} = \kappa \triangle T + \kappa \tau_T \triangle \dot{T} + \kappa \frac{\tau_T^2}{2} \triangle \ddot{T}$$

on a bounded domain Ω in R^n with smooth boundary. We obtain analyticity for the associated C_0-semigroup. Moreover, we also obtain exponential stability of the solutions by spectrum analysis and Hurwitz criterion under one of the following conditions:
(i). $\frac{\tau_T}{\tau_q} > 2 - \sqrt{3}$; (ii). $2 - \sqrt{3} \geq \frac{\tau_T}{\tau_q} > \frac{\sqrt{(1+\kappa\tau_T\lambda_1)^2 + (\kappa\tau_T\lambda_1)^2 + (\kappa\tau_T\lambda_1)^3} - (1+\kappa\tau_T\lambda_1)}{\kappa\tau_T\lambda_1(1+\kappa\tau_T\lambda_1)}$,
where λ_1 is the smallest eigenvalue of the negative Laplacian on Ω with Dirichlet boundary condition.

Key words: dual-phase-lag heat equation, analyticity, exponential stability.
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1 Introduction

It is well-known that Fourier's heat equation theory implies that the thermal disturbances at some point will be felt instantly anywhere for every distant. This leads to the paradox of infinite speed of propagation. Most known alternative theory is the Maxwell-Cattaneo Law which proposes a hyperbolic damped equation for the heat conduction. We recall other models proposed by Lord and Shulman [8], Green and Lindsay [4] and Green and Naghdi [5, 6].

In 1995, Tzou [15] proposed a modification of the Fourier constitutive equation. He suggested a theory of thermal flux with delay. The basic constitutive equation is

$$q(x, t + \tau_q) = -\kappa \nabla T(x, t + \tau_T), \ \ \kappa > 0. \tag{1.1}$$

Where T is the temperature, q is the heat flux vector and τ_T and τ_q are two delay parameters. By Taylor approximations of different orders to the delay equations, we obtain the following two dual-phase-lag equations:

$$\dot{T} + \tau_q \ddot{T} + \frac{\tau_q^2}{2}\dddot{T} = \kappa \triangle T + \kappa \tau_T \triangle \dot{T}, \tag{1.2}$$

$$\dot{T} + \tau_q \ddot{T} + \frac{\tau_q^2}{2}\dddot{T} = \kappa \triangle T + \kappa \tau_T \triangle \dot{T} + \kappa \frac{\tau_T^2}{2} \triangle \ddot{T}. \tag{1.3}$$

Equation (1.2) has been well investigated. Quintanilla [11] proved the exponential stability when $\tau_q < 2\tau_T$ and the instability when $\tau_q > 2\tau_T$. Quintanilla, Borgmeyer and Racke [2] proved that the decay is not exponential in the critical case $\tau_q = 2\tau_T$. Later on, Liu, Quintanilla and Wang [9] clarified that the critical case actually is polynomially stable. Moreover, when the delay τ_T is assumed to depend on the material point, they obtained exponential stability again even if the non-negative $2\tau_T(x) - \tau_q$ is only locally positive on a proper subdomain in one-dimensional space. Some extension of these results in a thermoelastic context can be found in [7, 14]. In recent years, utilization of Dual-Phase-Lag model to simulate heat transfer in micro- or nano-structures has been considered by researchers. (see [3] and the reference therein).

On the other hand, few results have been obtained for equation (1.3). The best we know is that it is of parabolic type, and the point spectrum is far away from the imaginary axis when $(2 - \sqrt{3})\tau_q < \tau_T$, which hinted the exponential stability [12]. When $(2 - \sqrt{3})\tau_q = \tau_T$, there exist cases such that the solution of the equation is undamped and periodic [2]. This problem was also studied in [13], but the exponential decay of solution was obtained only when $\frac{\tau_T}{\tau_q} > 1$. Numerical simulation of this case was shown in [1].

In this paper, we will investigate the regularity and stability of initial and boundary value problem

$$\begin{cases} \dot{T}+\tau_1\ddot{T}+\dfrac{\tau_1^2}{2}\dddot{T}=\kappa\triangle T+\kappa\tau_2\triangle\dot{T}+\kappa\dfrac{\tau_2^2}{2}\triangle\ddot{T}, \quad in \quad \Omega\times(0,\infty), & (1.4)\\ T(x,0)=T^0(x),\dot{T}(x,0)=\dot{T}^0(x),\ddot{T}(x,0)=\ddot{T}^0(x),\dddot{T}(x,0)=\dddot{T}^0(x) \quad in \quad \Omega, & (1.5)\\ T(\cdot,t)|_{\partial\Omega}=0, \quad for \quad t\in[0,\infty), & (1.6)\end{cases}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$.

Since an energy dissipative norm for system (1.3) is not easy to find, we first show analyticity for its associated C_0-semigroup in section 2. Hence, the semigroup possesses the spectrum determined growth property. Then, in section 3 and 4, we prove exponential stability by spectrum analysis and Hurwitz criterion under one of the following conditions:

(i). $\dfrac{\tau_T}{\tau_q}>2-\sqrt{3}$;

(ii). $2-\sqrt{3}\geq\dfrac{\tau_T}{\tau_q}>\dfrac{\sqrt{(1+\kappa\tau_T\lambda_1)^2+(\kappa\tau_T\lambda_1)^2+(\kappa\tau_T\lambda_1)^3}-(1+\kappa\tau_T\lambda_1)}{\kappa\tau_T\lambda_1(1+\kappa\tau_T\lambda_1)}$, where λ_1 is the smallest eigenvalue of the negative Laplacian on Ω with Dirichlet boundary condition.

Remark 1.1. *The contributions of this paper are the following. The observation of the parabolic behavior of system (1.3) is proved without any restriction on the two delay parameters. Moreover, the lower bound of* $\dfrac{\tau_T}{\tau_q}$ *for exponential stability is improved to the critical value* $2-\sqrt{3}$. *We would like to point out that Condition (ii) above is not satisfied if* τ_T *is small.*

Remark 1.2. *We note that the involvements of the high order terms in the lags and then on the dynamics are consequence of the handling of systems in which different energy carriers are involved. The interested reader can find a suitable discussion in ([16] p.376).*

2 Analyticity

Let

$$\mathcal{H}:=H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega),$$

equipped with the inner product

$$\langle Z,W\rangle_{\mathcal{H}}=\langle\nabla Z_1,\nabla W_1\rangle+\langle\nabla Z_2,\nabla W_2\rangle+\frac{\tau_1^2}{2}\langle Z_3,W_3\rangle.$$

Denoting $Z:=(Z_1,Z_2,Z_3)^T=(T,\dot{T},\ddot{T})^T$, we then convert system (1.4)-(1.6) to a first-order evolution equation on Hilbert space $\mathcal{H}$,

$$\begin{cases}\dfrac{dZ}{dt}=\mathcal{A}Z, & (2.1)\\ Z(0)=Z_0=(T^0,\dot{T}^0,\ddot{T}^0)^T, & (2.2)\end{cases}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}Z=\begin{pmatrix} Z_2\\ Z_3\\ \dfrac{2}{\tau_1^2}\big(-Z_2-\tau_1Z_3+\kappa\triangle Z_1+\kappa\tau_2\triangle Z_2+\kappa\dfrac{\tau_2^2}{2}\triangle Z_3\big)\end{pmatrix} \quad (2.3)$$

and

$$D(\mathcal{A}) = \{Z = (Z_1, Z_2, Z_3)^T \in \mathcal{H} | Z_1, Z_2 \in H^2(\Omega), Z_3 \in H_0^1(\Omega) \cap H^2(\Omega)\}, \tag{2.4}$$

and $\tau_1 = \tau_q, \quad \tau_2 = \tau_T$. Since

$$\begin{aligned} Re\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} &= Re\bigg(\langle \nabla Z_2, \nabla Z_1\rangle + \langle \nabla Z_3, \nabla Z_2\rangle - \langle Z_2, Z_3\rangle - \tau_1 \|Z_3\|^2 \\ &\quad -\kappa\langle \nabla Z_1, \nabla Z_3\rangle - \kappa\tau_2\langle \nabla Z_2, \nabla Z_3\rangle - \kappa\frac{\tau_2^2}{2}\|\nabla Z_3\|^2 \bigg), \end{aligned} \tag{2.5}$$

the dissipativeness of $\mathcal{A}$ is not clear under this inner product. Therefore, we consider its translation $\mathcal{A}_1 = \mathcal{A} - \omega_0 I$ for $\omega_0 > 0$ big enough.

Theorem 2.1. *$\mathcal{A}_1$ is the infinitesimal generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$.*

Proof. Since

$$\begin{aligned} Re\langle \mathcal{A}_1 Z, Z\rangle_{\mathcal{H}} &= Re\bigg(\langle \nabla Z_2, \nabla Z_1\rangle - \omega_0\|\nabla Z_1\|^2 + \langle \nabla Z_3, \nabla Z_2\rangle - \omega_0\|\nabla Z_2\|^2 - \langle Z_2, Z_3\rangle - \tau_1\|Z_3\|^2 \\ &\quad -\kappa\langle \nabla Z_1, \nabla Z_3\rangle - \kappa\tau_2\langle \nabla Z_2, \nabla Z_3\rangle - \kappa\frac{\tau_2^2}{2}\|\nabla Z_3\|^2 - \omega_0\frac{\tau_1^2}{2}\|Z_3\|^2 \bigg). \end{aligned} \tag{2.6}$$

We can choose ω_0 big enough such that the right-hand side of (2.6) is less than $-C(\|\nabla Z_1\|^2 + \|\nabla Z_2\|^2 + \|\nabla Z_3\|^2) \le 0,\ C > 0$. Thus, $\mathcal{A}_1$ is dissipative. It is easily to show that $D(\mathcal{A}_1)$ is dense in $\mathcal{H}$. Suppose that $0 \in \sigma(\mathcal{A}_1)$. Then there exist a sequence $\lambda_n \to 0$ and a sequence $Z_n \in D(\mathcal{A}_1)$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$(\lambda_n I - \mathcal{A}_1)Z_n \to 0 \quad \text{in} \quad \mathcal{H}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$Re\langle \mathcal{A}_1 Z_n, Z_n\rangle_{\mathcal{H}} \ge C(\|\nabla Z_{n,1}\|^2 + \|\nabla Z_{n,2}\|^2 + \|\nabla Z_{n,3}\|^2) \to 0. \tag{2.8}$$

We conclude that $\|Z_n\|_{\mathcal{H}} \to 0$. This is a contradiction. Thus, $0 \in \rho(\mathcal{A}_1)$. By the modified Lumer-Philips Theorem [10], we prove that $\mathcal{A}_1$ generates a C_0-semigroup of contractions on $\mathcal{H}$. $\square$

We recall the following result on analytic semigroup which will be applied to prove Theorem 2.3.

Theorem 2.2. *[10] Let $S(t) = e^{At}$ be a C_0-semigroup of contraction in Hilbert space H. Suppose that*

$$\rho(A) \supseteq \{i\beta | \beta \in \mathbb{R}\} \equiv i\mathbb{R}. \tag{2.9}$$

Then, $S(t)$ is analytic if and only if

$$\overline{\lim_{|\beta|\to\infty}} \|\beta(i\beta I - A)^{-1}\|_H < \infty. \tag{2.10}$$

Theorem 2.3. *The semigroup $e^{\mathcal{A}_1 t}$ is analytic.*

Proof. We will use Theorem 2.2 to prove this result. It consists of the following two steps:
Step I: Assume that (2.9) is false, i.e., there is a $\lambda = i\beta \in \sigma(\mathcal{A}_1)$. Then there exist $\lambda_n (= i\beta_n) \to \lambda$ and normalized $Z_n = (Z_{1n}, Z_{2n}, Z_{3n})^T$ such that

$$\|(i\beta_n I - \mathcal{A}_1)Z_n\|_{\mathcal{H}} \to 0, \tag{2.11}$$

which implies

$$Re\langle \mathcal{A}_1 Z, Z\rangle_{\mathcal{H}} = o(1). \tag{2.12}$$

For convenience, we have omitted the subscript n hereafter. Thus, $\|Z\|_{\mathcal{H}}^2 = o(1)$, which is a contradiction to the assumption $\|Z\|_{\mathcal{H}}^2 = 1$. Therefore, $i\mathbb{R} \subseteq \rho(\mathcal{A}_1)$.
Step II: Assume that (2.10) is false. Then by the uniform boundedness theorem, there exist a sequence $\beta \to \infty$ and a unit sequence $Z = (Z_1, Z_2, Z_3)^T \in D(\mathcal{A}_1)$ such that

$$\|\frac{1}{\beta}(i\beta I - \mathcal{A}_1)Z\|_{\mathcal{H}} \to 0. \tag{2.13}$$

We rewrite (2.13) as

$$\begin{cases} \frac{1}{\beta}\big(i\beta\nabla Z_1 - \nabla(Z_2 - \omega_0 Z_1)\big) = o(1), & (2.14)\\ \frac{1}{\beta}\big(i\beta\nabla Z_2 - \nabla(Z_3 - \omega_0 Z_2)\big) = o(1), & (2.15)\\ \frac{1}{\beta}\big(i\beta Z_3 - \frac{2}{\tau_1^2}(-Z_2 - \tau_1 Z_3 + \kappa\Delta Z_1 + \kappa\tau_2\Delta Z_2 + \kappa\frac{\tau_2^2}{2}\Delta Z_3) + \omega_0 Z_3\big) = o(1). & (2.16) \end{cases}$$

Then, by dissipative $Re\frac{1}{\beta}\langle \mathcal{A}_1 Z, Z\rangle_{\mathcal{H}} = o(1)$, which implies that

$$\frac{1}{\beta}\|\nabla Z_1\|^2 = o(1), \quad \frac{1}{\beta}\|\nabla Z_2\|^2 = o(1), \quad \frac{1}{\beta}\|\nabla Z_3\|^2 = o(1). \tag{2.17}$$

From (2.14),

$$i\nabla Z_1 - \frac{1}{\beta}\nabla(Z_2 - \omega_0 Z_1) = o(1). \tag{2.18}$$

Then by (2.17), we obtain

$$\|\nabla Z_1\|^2 = o(1). \tag{2.19}$$

Similarly, it follows from (2.15) and (2.17) that

$$\|\nabla Z_2\|^2 = o(1). \tag{2.20}$$

Taking the inner product of (2.16) with Z_3 in $L^2(\Omega)$ gives

$$\begin{aligned} & i\|Z_3\|^2 + \frac{2}{\tau_1^2}\langle\frac{1}{\beta}Z_2, Z_3\rangle + \frac{2}{\tau_1}\frac{1}{\beta}\|Z_3\|^2 + \frac{2}{\tau_1^2}\kappa\langle\frac{1}{\beta}\nabla Z_1, \nabla Z_3\rangle + \frac{2\tau_2}{\tau_1^2}\kappa\langle\frac{1}{\beta}\nabla Z_2, \nabla Z_3\rangle \\ + \quad & \frac{\tau_2^2}{\tau_1^2}\kappa\frac{1}{\beta}\|\nabla Z_3\|^2 + \omega_0\frac{1}{\beta}\|Z_3\|^2 = o(1). \end{aligned} \tag{2.21}$$

In reference of (2.19)-(2.20) and (2.17), (2.21) can be simplified to

$$i\|Z_3\|^2 = o(1). \tag{2.22}$$

Therefore, we have arrived at $\|Z\|_{\mathcal{H}}^2 = o(1)$, which is a contradiction with assumption $\|Z\|_{\mathcal{H}}^2 = 1$. □

Since the operator $\mathcal{A}_1$ is the translation of the operator $\mathcal{A}$, we have

Theorem 2.4. *The Semigroup $e^{\mathcal{A}t}$ is analytic.*

3 Spectrum analysis

Since $e^{\mathcal{A}t}$ is analytic, its growth rate is

$$\omega = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathcal{A})\}.$$

Thus, for stability analysis, we need to study the spectrum of $\mathcal{A}$. Suppose $\mathcal{A}Z = \lambda Z$ with $Z \in D(\mathcal{A})$ and $Z \neq 0$, then

$$\begin{cases} Z_2 = \lambda Z_1, & (3.1) \\ Z_3 = \lambda Z_2, & (3.2) \\ \dfrac{2}{\tau_1^2}(-Z_2 - \tau_1 Z_3 + \kappa\Delta Z_1 + \kappa\tau_2\Delta Z_2 + \kappa\dfrac{\tau_2^2}{2}\Delta Z_3) = \lambda Z_3. & (3.3) \end{cases}$$

Substituting (3.1) and (3.2) into (3.3), we have

$$\kappa(1 + \tau_2\lambda + \frac{\tau_2^2}{2}\lambda^2)\Delta Z_1 - \lambda(1 + \tau_1\lambda + \frac{\tau_1^2}{2}\lambda^2)Z_1 = 0. \tag{3.4}$$

Theorem 3.1. $\sigma_r(\mathcal{A}) = \emptyset$.

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$ if and only if $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$, it suffices to show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. It is easy to check that $\mathcal{A}^*W = \lambda W$ for $W \in D(\mathcal{A}^*)$ and $W \neq 0$ is given by

$$\begin{cases} -\kappa W_3 = \lambda W_1, & (3.5) \\ \Delta W_1 + W_3 - \kappa\tau_2\Delta W_3 = \lambda\Delta W_2, & (3.6) \\ -\dfrac{2}{\tau_1^2}\Delta W_2 - \dfrac{2}{\tau_1}W_3 + \kappa\dfrac{\tau_2^2}{\tau_1^2}\Delta W_3 = \lambda W_3. & (3.7) \end{cases}$$

Eliminating W_2, W_3 in (3.5)-(3.7), we get

$$\kappa(1 + \tau_2\lambda + \frac{\tau_2^2}{2}\lambda^2)\Delta W_1 - \lambda(1 + \tau_1\lambda + \frac{\tau_1^2}{2}\lambda^2)W_1 = 0, \tag{3.8}$$

which is the same with (3.4). Hence, $\lambda \in \sigma_p(\mathcal{A}^*)$ if and only if $\lambda \in \sigma_p(\mathcal{A})$, and consequently $\sigma_r(\mathcal{A}) = \emptyset$. $\square$

For $\lambda = \lambda_\pm = \dfrac{-1}{\tau_2} \pm i\dfrac{1}{\tau_2}$, which are the roots of $1 + \tau_2\lambda + \dfrac{\tau_2^2}{2}\lambda^2 = 0$, equation (3.4) reduces to

$$(1 + \tau_1\lambda_\pm + \frac{\tau_1^2}{2}\lambda_\pm^2)Z_1 = 0. \tag{3.9}$$

Case I: If $\tau_1 \neq \tau_2$. As $1 + \tau_1\lambda_\pm + \dfrac{\tau_1^2}{2}\lambda_\pm^2 \neq 0$, then by (3.9), we have $Z_1 = 0$, which together with (3.1) and (3.2), we obtain $Z_2 = 0$ *and* $Z_3 = 0$, i.e., $Z = 0$. That is, $\ker(\mathcal{A} - \lambda_\pm I) = 0$. Let's show that $(\mathcal{A} - \lambda_\pm I)^{-1}$ is unbounded, then $\lambda_\pm \in \sigma_c(\mathcal{A})$. We will prove that there exists $\|F_n\|_{\mathcal{H}}^2 = 1$, such that

$$\|(\mathcal{A} - \lambda_\pm I)^{-1}F_n\|_{\mathcal{H}}^2 \to \infty. \tag{3.10}$$

We choose $F_n = (0, 0, e_n)^T$, where λ_n is the eigenvalue of $-\Delta$ with Dirichlet boundary condition, $\lambda_n \to \infty$ as $n \to \infty$; e_n is the corresponding normalized eigenfunction. Let $Z_n = (Z_{n1}, Z_{n2}, Z_{n3}) = (\mathcal{A}-\lambda_\pm I)^{-1}F_n$, i.e.,

$$(\mathcal{A} - \lambda_\pm I)Z_n = F_n, \tag{3.11}$$

which is equivalent to

$$\begin{cases} Z_{n2} - \lambda_\pm Z_{n1} = 0, & (3.12) \\ Z_{n3} - \lambda_\pm Z_{n2} = 0, & (3.13) \\ -Z_{n2} - \tau_1 Z_{n3} + \kappa\Delta Z_{n1} + \kappa\tau_2\Delta Z_{n2} + \kappa\dfrac{\tau_2^2}{2}\Delta Z_{n3} - \dfrac{\tau_1^2}{2}\lambda_\pm Z_{n3} = \dfrac{\tau_1^2}{2}e_n. & (3.14) \end{cases}$$

Eliminating Z_{n2}, Z_{n3} in (3.12)-(3.14), we have

$$-\lambda_\pm(1+\tau_1\lambda_\pm+\frac{\tau_1^2}{2}\lambda_\pm^2)Z_{n1}=\frac{\tau_1^2}{2}e_n, \tag{3.15}$$

i.e., $Z_{n1}=c_1e_n$, for some constant $c_1\neq 0$. Now it is clear that $\|Z_{n1}\|_{H^1}\to\infty$, which concludes that $(\mathcal{A}-\lambda_\pm I)^{-1}$ is unbounded. Therefore, $\lambda_\pm\in\sigma_c(\mathcal{A})$.

For $\lambda\in\mathbb{C}\setminus(\sigma_p(\mathcal{A})\cup\{\lambda_\pm\})$, and for any $F=(f_1,f_2,f_3)\in\mathcal{H}$, we look for solution to

$$(\mathcal{A}-\lambda I)Z=F \tag{3.16}$$

in $D(\mathcal{A})$, i.e.,

$$\begin{cases} Z_2-\lambda Z_1=f_1, & (3.17)\\ Z_3-\lambda Z_2=f_2, & (3.18)\\ \frac{2}{\tau_1^2}(-Z_2-\tau_1Z_3+\kappa\Delta Z_1+\kappa\tau_2\Delta Z_2+\kappa\frac{\tau_2^2}{2}\Delta Z_3)-\lambda Z_3=f_3. & (3.19)\end{cases}$$

Again, (3.17)-(3.19) can be reduced to an elliptic partial differential equation

$$\begin{aligned}\kappa(1+\tau_2\lambda+\frac{\tau_2^2}{2}\lambda^2)\Delta Z_1-\lambda(1+\tau_1\lambda+\frac{\tau_1^2}{2}\lambda^2)Z_1=(1+\tau_1\lambda+\frac{\tau_1^2}{2}\lambda^2)f_1-\kappa(\tau_2+\frac{\tau_2^2}{2}\lambda)\Delta f_1\\ +(\tau_1+\frac{\tau_1^2}{2}\lambda)f_2-\kappa\frac{\tau_2^2}{2}\Delta f_2+f_3\end{aligned} \tag{3.20}$$

with boundary condition $Z_1|_{\partial\Omega}=0$, which is solvable by the standard elliptic theory. Thus, $\lambda\in\rho(\mathcal{A})$ and

$$\sigma(\mathcal{A})=\mathbb{C}\backslash\rho(\mathcal{A})=\sigma_p(\mathcal{A})\cup\{\lambda_\pm\}. \tag{3.21}$$

Case II: If $\tau_1=\tau_2$, then we also have $1+\tau_1\lambda_\pm+\frac{\tau_1^2}{2}\lambda_\pm^2=0$. Thus,

$$Z=(1,\lambda_\pm,\lambda_\pm^2)Z_1,\quad \forall Z_1\in H^2(\Omega)\cap H_0^1(\Omega). \tag{3.22}$$

Then $\dim\ker(\mathcal{A}-\lambda_\pm I)=\infty$, i.e., $\lambda_\pm\in\sigma_c(\mathcal{A})$. Similar to Case I, we can prove that if $\lambda\notin\sigma_p(\mathcal{A})$ and $\lambda\neq\lambda_\pm$, then $\lambda\in\rho(\mathcal{A})$. Therefore,

$$\sigma(\mathcal{A})=\mathbb{C}\backslash\rho(\mathcal{A})=\sigma_p(\mathcal{A})\cup\{\lambda_\pm\}. \tag{3.23}$$

4 The exponential stability of the system

In [12], it was proved by Hurwitz criterion that all three roots of the characteristic polynomial

$$\beta^3+(\frac{2}{\tau_1}+\frac{\kappa\tau_2^2\lambda_n}{\tau_1^2})\beta^2+(\frac{2}{\tau_1^2}+\frac{2\kappa\tau_2\lambda_n}{\tau_1^2})\beta+\frac{2\kappa\lambda_n}{\tau_1^2}=0 \tag{4.1}$$

have negative real parts if and only if for all n

$$l_1=\frac{2}{\tau_1}+\frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}>0,\quad l_2=\frac{2}{\tau_1^2}+\frac{2\kappa\tau_2\lambda_n}{\tau_1^2}>0,\quad l_3=\frac{2\kappa\lambda_n}{\tau_1^2}>0\quad \textit{and}\quad l_1l_2>l_3, \tag{4.2}$$

where λ_n is the increasing sequence of eigenvalues of $-\Delta$ with Dirichlet boundary conditions. To make sure $l_1l_2>l_3$, $\frac{\tau_2}{\tau_1}$ should satisfies condition (i) or (ii) in the section of introduction.

Let $\beta = \lambda - \varepsilon$, $\forall \varepsilon > 0$ small enough, we have

$$(\lambda-\varepsilon)^3 + (\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2})(\lambda-\varepsilon)^2 + (\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2})(\lambda-\varepsilon) + \frac{2\kappa\lambda_n}{\tau_1^2} = 0, \tag{4.3}$$

i.e.,

$$\begin{aligned}\lambda^3 &+ [(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) - 3\varepsilon]\lambda^2 + [3\varepsilon^2 - 2\varepsilon(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) + (\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2})]\lambda \\ &+ \frac{2\kappa\lambda_n}{\tau_1^2} - \varepsilon^3 + \varepsilon^2(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) - \varepsilon(\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2}) = 0. \end{aligned}\tag{4.4}$$

Denote

$$\widehat{l_1} = (\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) - 3\varepsilon, \quad \widehat{l_2} = 3\varepsilon^2 - 2\varepsilon(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) + (\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2})$$

and

$$\widehat{l_3} = \frac{2\kappa\lambda_n}{\tau_1^2} - \varepsilon^3 + \varepsilon^2(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) - \varepsilon(\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2}).$$

For $\varepsilon > 0$ small enough, the following inequalities still hold.

$$\widehat{l_1} = l_1 - 3\varepsilon > 0, \quad \widehat{l_2} = l_2 + 3\varepsilon^2 - 2\varepsilon(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) > 0,$$

$$\widehat{l_3} = l_3 - \varepsilon^3 + \varepsilon^2(\frac{2}{\tau_1} + \frac{\kappa\tau_2^2\lambda_n}{\tau_1^2}) - \varepsilon(\frac{2}{\tau_1^2} + \frac{2\kappa\tau_2\lambda_n}{\tau_1^2}) > 0,$$

and

$$\widehat{l_1}\widehat{l_2} - \widehat{l_3} = l_1 l_2 - l_3 - 8\varepsilon^3 + 8\varepsilon^2 l_1 - 2\varepsilon l_2 - 2\varepsilon l_1^2 > 0,$$

under the condition (i) or (ii). Thus, $Re\lambda < 0$ which implies that $Re\beta < -\varepsilon$, for some $\varepsilon > 0$ small enough. The growth rate of the semigroup $e^{\mathcal{A}t}$ is

$$\omega = \max\{-\varepsilon, -\frac{1}{\tau_2}\}.$$

Therefore, we have proved that equation (1.3) is exponentially stable under condition either (i) or (ii).

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