EIGENVALUES WITH RESPECT TO A WEIGHT FOR DISCRETE ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. Any function $K\colon V\times V\longrightarrow\mathbb{R}$ is called *a kernel on* V and we denote by $\mathcal{C}(V\times V)$ the space of kernels. If we label the elements of V, then each kernel can be identify with the matrix of order $|V|\cdot|V|$ given by $\mathsf{K}=\big(\mathsf{K}(\mathsf{x},\mathsf{y})\big)_{\mathsf{x},\mathsf{y}\in\mathsf{V}}$. The symmetric kernels such that $K(x,y)\le 0$ for $x\neq y$ play a fundamental role in many areas of applied mathematics. They can be identified with symmetric Z-matrices of order $|V|$. Moreover, if for $t\in\mathbb{R}$ we consider $K_t=K+tI$, the matrix identified with K_t is positive definite for t large enough and hence it is an M-matrix.

In the framework of Discrete Mathematics it is usual to introduce the kernels as the main operators on a network. In this work we study boundary value problems for Schrödinger operators and we consider eigenvalue problems for boundary value problems with respect to a weight. As far as we know this is the first time that this type of problems is considered in the discrete setting.

1. Main Results

We state the Mercer Theorem for general BVP where the eigenvalues and eigenfunctions are computed with respect to an arbitrary weight. In this case, the function obtained from a Mercer-type theorem does not necessarily coincide with the group inverse of the associated operator, but with another generalized inverse. Then, the expression of the group inverse is obtained from the addition of suitable projectors.

Proposition 1.1. *The Green operator is singular and self-adjoint on $\mathcal{C}(F)$; that is,*

$$\int_F g\mathcal{G}_q(f)=\int_F f\mathcal{G}_q(g),\quad \textit{for any } f,g\in\mathcal{C}(F)$$

and when $q\neq q_\omega$ then

$$L_q^{-1}(x,y)=G_q(x,y)+\lambda_q^{-1}||\omega||_{\sigma_q}^{-2}\omega(x)\omega(y),\quad x\in V,\ y\in F$$

and hence, $\mathcal{L}_q^{-1}$ is self-adjoint on $\mathcal{C}(F)$. Moreover, given $\sigma\in\Omega_F$ we have that

$$G_q^{\sigma}(x,y)=G_q(x,y)-\tau_\sigma(x)\omega(y),\quad x\in V,\ y\in F$$

and hence, $\mathcal{G}_q^{\sigma}$ is self–adjoint on $\mathcal{C}(F)$ iff $\sigma=\sigma_q$; that is, iff $\tau_\sigma=0$ or equivalently iff $\mathcal{G}_q^{\sigma}=\mathcal{G}_q$. In addition, given $\widehat{\sigma}\in\Omega_F$ we have that

$$G_{q_\omega}^{\sigma,\widehat{\sigma}}(x,y)=G_{q_\omega}(x,y)+\mathcal{E}_{q_\omega}(\tau_\sigma,\tau_{\widehat{\sigma}})\omega(x)\omega(y)-\tau_\sigma(x)\omega(y)-\omega(x)\tau_{\widehat{\sigma}}(y),\quad x\in V,\ y\in F$$

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and hence, $\mathcal{G}^{\sigma,\widehat{\sigma}}_{q_\omega}$ is self–adjoint on $\mathcal{C}(F)$ iff $\widehat{\sigma} = \sigma$. Moreover,

$$G_{q_\omega}(x,y) = G^{\sigma,\sigma}_{q_\omega}(x,y) + \mathcal{E}_{q_\omega}(\zeta_\sigma,\zeta_\sigma)\omega(x)\omega(y) - \zeta_\sigma(x)\omega(y) - \omega(x)\zeta_\sigma(y), \quad x \in V, \;\; y \in F.$$

Now, we consider eigenvalue problems for boundary value problems with respect to a weight. As far as we know this is the first time that this type of problems is considered in the discrete setting.

A real number $\mu \in \mathbb{R}$ is named *eigenvalue of the BVP* (**??**)*, with respect to* $\sigma \in \Omega_{_F}$, if there exists a non-null function $v \in \mathcal{C}(\bar{F})$ such that

$$(1) \qquad \mathcal{L}_q(v) = \mu\sigma v \;\text{ on }\; F, \qquad \frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}} + qv = 0 \;\text{ on }\; F_{_N} \quad \text{ and } \quad v = 0 \;\text{ on }\; F_{_D}.$$

Equivalently, $\mu \in \mathbb{R}$ is an eigenvalue of (**??**) iff $\mathcal{V}^H_{q-\mu\sigma}$ is a non trivial subspace; that is, the problem is not regular.

If μ is an eigenvalue we say that v is an *associated eigenfunction* if v satisfies the equalities in (**??**). In addition, $\mathcal{V}(\mu) \subset \mathcal{V}_q$ denotes the subspace of eigenfunctions associated with μ. Observe that $v \in \mathcal{C}(\bar{F})$ is an eigenfunction associated with μ iff $\mathcal{L}_q\big(\gamma(v\chi_{_F})\big) = \mu\sigma v$ on F.

The self-adjointness of the BVP leads to the following results.

Lemma 1.2. *Given $\mu, \widehat{\mu}$ two eigenvalues of the BVP* (**??**)*, then*

$$\mu\langle u,v\rangle_\sigma = \mathcal{E}^F_q(u,v) = \widehat{\mu}\langle u,v\rangle_\sigma \;\; \text{for any } u \in \mathcal{V}(\mu) \text{ and } v \in \mathcal{V}(\widehat{\mu}).$$

In particular, $\mathcal{E}^F_q(u,u) = \mu||u||^2_\sigma$ and when $\mu \neq \widehat{\mu}$, then $\mathcal{V}(\mu)$ and $\mathcal{V}(\widehat{\mu})$ are orthogonal each other with respect to σ.

Bearing in mind that the finite dimensionality of $\mathcal{V}_q$ implies that it is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\sigma$ and moreover that $\mathcal{L}_q$ is a *Hilbert-Schmidt operator*, we can prove that there exist an orthonormal basis of eigenfunctions. Although we could prove this result by using the Rayleigh quotient, see for instance [**?**] for a specific weight, we give here a direct proof based on the standard Spectral Theorem.

Theorem 1.3 (Hilbert-Schmidt Theorem)**.** *Given $\sigma \in \Omega_{_F}$, there exist a sequence $\mu_1 \le \cdots \le \mu_{|F|}$ and $\{v_j\}_{j=1}^{|F|} \in \mathcal{V}_{q_\omega}$ an orthonormal system, with respect to σ, such that*

(i) $\mathcal{L}_{q_\omega}(v_j) = \mu_j\sigma v_j$, $j = 1,\dots,|F|$.

(ii) $0 = \mu_1 < \mu_2$*; that is, $\mu_1 = 0$ is a simple eigenvalue and moreover* $v_1 = ||\omega||^{-1}_\sigma\omega$.

(iii) *If μ is an eigenvalue of the BVP* (**??**) *for $q = q_\omega$ with respect to σ, then $\mu = \mu_j$ for some $j = 1,\dots,|F|$.*

(iv) $\mathcal{L}_{q_\omega}(u) = \sigma\displaystyle\sum_{j=2}^{|F|}\mu_j\langle u,v_j\rangle_\sigma v_j$ *on F for any $u \in \mathcal{V}_{q_\omega}$.*

We call *sequence of eigenvalues* and *orthonormal basis of eigenfunctions*, with respect to σ, for the BVP (**??**) for $q = q_\omega$, to $\{\mu^\sigma_j\}_{j=1}^{|F|}$ and $\{v^\sigma_j\}_{j=1}^{|F|}$ given in the Hilbert-Schmidt Theorem. When σ is constant; that is $\sigma = \chi_{_F}$, then we drop the symbol σ in the above sequences.

Corollary 1.4. *Let $q \in \mathcal{Q}_\omega(F \cup F_{_N})$ such that $q = q_\omega + \lambda\sigma$ on F, where $\lambda > 0$ and $\sigma \in \Omega_{_F}$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then,*

(i) $\mathcal{L}_q(v_j^\sigma) = (\mu_j^\sigma + \lambda)\sigma v_j^\sigma$, $j = 1, \ldots, |F|$.

(ii) $\mathcal{L}_q(u) = \sigma \sum_{j=1}^{|F|} (\mu_j^\sigma + \lambda)\langle u, v_j^\sigma \rangle_\sigma v_j^\sigma$ *on F for any $u \in \mathcal{V}_q$.*

Finally we get the discrete version of Mercer Theorem

Lemma 1.5. *Let $\sigma \in \Omega_{_F}$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then, $\tau_\sigma = ||\omega||_\sigma^{-2}\mathcal{G}_{q_\omega}(\sigma\omega)$ is given by*

$$\tau_\sigma = \omega||\omega||_{_F}^{-4}\sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_{_F}^2 - ||\omega||_{_F}^{-2}\sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_{_F} v_j^\sigma$$

and hence, $\mathcal{E}_{q_\omega}(\tau_\sigma, \tau_\sigma) = ||\omega||_{_F}^{-4}\sum\limits_{j=2}^{|F|}(\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_{_F}^2$.

Corollary 1.6 (Mercer Theorem)**.** *Let $\sigma \in \Omega_{_F}$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then,*

$$G_{q_\omega}^{\sigma,\sigma}(x,y) = \sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}v_j^\sigma(x)v_j^\sigma(y), \qquad x \in V,\ y \in F$$

and hence,

$$\begin{aligned} G_{q_\omega}(x,y) = &\sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}v_j^\sigma(x)v_j^\sigma(y) \\ &-||\omega||_{_F}^{-2}\sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_{_F}\big(v_j^\sigma(x)\omega(y) + \omega(x)v_j^\sigma(y)\big) \\ &+||\omega||_{_F}^{-4}\omega(x)\omega(y)\sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_{_F}^2, \qquad x \in V,\ y \in F. \end{aligned}$$

In addition, if for any $\lambda > 0$ we consider $q = q_\omega + \lambda\sigma$, then $q \in \mathcal{Q}_\omega(F \cup F_{_N})$ and moreover

$$G_q(x,y) = \sum_{j=2}^{|F|}(\mu_j^\sigma + \lambda)^{-1}v_j^\sigma(x)v_j^\sigma(y) \quad \text{and} \quad L_q^{-1}(x,y) = \sum_{j=1}^{|F|}(\mu_j^\sigma + \lambda)^{-1}v_j^\sigma(x)v_j^\sigma(y),$$

$x \in V$, $y \in F$.

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