Scheme of pairs of matrices with vanishing commutator

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Abstract

This Bachelor thesis revolves around the scheme of pairs of matrices with vanishing commutator. It is conjectured to be reduced, irreducible, Cohen-Macaulay and normal. Some known results are overviewed and some new results are presented. Among them, generic reducedness and regularity in codimension one, some results for the jet schemes over our scheme of interest, results on simple and semi-simple quiver representations, small computations on the Bernstein-Sato polynomials and results on related schemes including the solution to a small open problem.

Keywords: commuting variety, jet schemes, quiver representations, log-canonical threshold.

Contents

Disclaimer:
I would like to remark that to simply understand the posed question required a lot of prior work. In fact, most of the time was actually invested in just understanding it, since I had to learn all the background knowledge by myself. The main reason is that I did not course neither the Algebraic Geometry elective course in the bachelor's degree nor the Algebraic Geometry and Commutative Algebra lectures from the master's degree.

Even though there is a lot of work and results that I studied (including a lot topics that at the end have not been used), presenting it all here probably would simply bother the reader. In this sense, I have tried to present the new results that have been obtained in the main body of the text, being concise with the general knowledge, and including in an appendix a basic compendium of the definitions and statements that are used in the main text.

1 Introduction

We will use the convention of defining a variety over an algebraically closed field as a finite type, reduced and irreducible (integral) scheme over it.

Sometimes in the definition of a scheme not necessarily reduced we just mention the associated set of closed points. Hopefully, the intended scheme structure will always be clear.

The purpose of this thesis is to present some results that we have obtained surrounding a long studied affine scheme. For $n > 0$, consider the scheme (associated to the following set with the natural scheme structure, which we also name X_n):

$$X_n = \{(A, B) \in \mathrm{Mat}(n, \mathbb{C})^{\times 2} \mid [A, B] = 0\}$$

where $[A, B] = AB - BA$, and we consider $\mathrm{Mat}(n, \mathbb{C})^{\times 2}$ as an affine $2n^2$-dimensional space, where A and B are generic matrices. Throughout the text, we refer to this scheme by the *commuting scheme* or X_n. Its reduced associated scheme is usually referred to as the *commuting variety.*

Conjecture 1.1. *X_n is reduced, irreducible, Cohen-Macaulay and normal for all $n > 0$.*

Presumably[1], it is a long standing conjecture (reduced and Cohen-Macaulay (cf. [Kad18], [Ngo14], [Knu03], [Bud18]), normal (cf. [Pop08], [Pre03])). In addition, it is thought to have rational singularities[2]. This conjecture is actually a specific case of a bigger one:

[1] It is cited as being posed by M.Artin and M.Hochster in 1982 ([Kad18], [Ngo14], [Knu03]), but none of the references cites those two authors directly and we have not been able to find a direct source that supports it.

[2] The statement of rational singularities is not a published conjecture or open problem, but it would fit in the behaviour of a more general family of schemes that are closely related to it, studied in [Bud18]

Conjecture 1.2. *Let $\mathfrak{g}$ be a reductive Lie algebra, then the associated scheme to*

$$\mathcal{C}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

is reduced, irreducible, Cohen-Macaulay and normal.

Conjecture 1.1 is a particular case of it considering $\mathfrak{g} = \mathfrak{gl}_n$. Even though we know of the existence of this wider conjecture, we will only focus on the specific case of X_n.

A long known important result relating to X_n, first proven by Motzkin and Taussky [MT55] (as well as a bit later by Gerstenhaber [Ger61]), is the following theorem:

Theorem 1.3. *X_n is irreducible and of dimension $n^2 + n$ for all $n \geqslant 1$.*

We will reproduce a proof of this fact extracted from [Gur92], since his methods relate to some results over the jet schemes (which, as we discuss further on, are of our interest).

This result is a concrete case of a theorem later proven:

Theorem 1.4. *(Richardson [Ric79]) Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field F of characteristic zero and let*

$$\mathcal{C}^{red}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

be the reduced scheme of pairs of commuting elements. Then $\mathcal{C}^{red}(\mathfrak{g})$ is irreducible.

Even though the following statement does not apply to our case, it is, nevertheless, somehow motivating for the conjecture being about Cohen-Macaulay and not Gorenstein:

Theorem 1.5. *(Corollary 9.3.18 [Vas94]) Let $\mathfrak{g}$ be a semi-simple Lie algebra over an algebraically closed field F of characteristic zero and let*

$$\mathcal{C}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

be the scheme of pairs of commuting elements. Then $\mathcal{C}(\mathfrak{g})$ is not Gorenstein.

On the other hand, for small n, the following is known:

Proposition 1.6. *(see [Hre94], [Hre06b]) X_n is reduced, irreducible and Cohen-Macaulay but not Gorenstein for $n < 5$.*

Even though it is a well known result, we present a proof by ourselves of the case $n = 2$ and we check, using Macaulay2 [GS], the cases $n = 3$ and $n = 4$, as well as we try to implement the ideas in [Hre06a] to the case $n = 5$. Furthermore, although it is, in a sense, a trivial result, we introduce a small improvement in performance reducing the number of variable of the polynomial ring we are working on:

Proposition 1.7. *$\mathcal{O}_{X_n}$ is Cohen-Macaulay (respectively reduced) iff, for any $1 \leqslant i, j \leqslant n$, the quotient $\mathcal{O}_{X_n}/(a_{i,i}, b_{j,j})$ is Cohen-Macaulay (respectively reduced). Where $(a_{i,i}, b_{j,j})$ is the ideal (sheaf) generated by the (i,i)-th entry of the matrix A and the (j,j)-th entry of the matrix B.*

On another direction, one of the main new results that we present here is the following:

Theorem 1.8. *X_n is generically reduced for all $n \geqslant 1$.*

In particular, this implies the following proposition:

Proposition 1.9. *If X_n is Cohen-Macaulay, then it is reduced.*

This implication was known previously (cf. [Hre94]), but, naturally, the argumentation was different (see Problem 2.7.1 [Vas98]).

This result can actually be improved by the following theorem:

Theorem 1.10. *([Pop08])*

Given $\mathfrak{g}$ a connected non-commutative reductive lie algebra over an algebraically close field F of characteristic 0, let

$$\mathcal{C}^{red}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

be the reduced scheme of pairs of commuting elements. Then $\operatorname{codim}_{\mathfrak{g}\times\mathfrak{g}}(\mathcal{C}^{red}(\mathfrak{g}))^{sing} \geqslant 2$, *where $(\mathcal{C}^{red}(\mathfrak{g}))^{sing}$ stands for the singular locus of $\mathcal{C}^{red}(\mathfrak{g})$.*

This implies:

Proposition 1.11. *If X_n is Cohen-Macaulay, then it is reduced and normal.*

Even though this proposition comes from results that were already known, its implications to X_n for $n < 5$ do not seem to be recorded in the literature. In any case, we have:

Proposition 1.12. *X_n is reduced, irreducible, Cohen-Macaulay and normal for $n < 5$.*

On the other hand, relating to the singularities of a variety, there is the following results by Mustață [Mus01] on jet schemes over complete intersection varieties:

Theorem 1.13. *([Mus01]) Let X be a locally complete intersection variety. The following are equivalent for $m \geqslant 1$:*

(i) X_m *is irreducible,*

(ii) $\dim \pi_m^{-1}(X_{sing}) < (\dim X)(m + 1)$,

(iii) X_m *is a locally complete intersection variety of dimension* $\leqslant (\dim X)(m + 1)$.

Theorem 1.14. *([Mus01]) Let X be a locally complete intersection variety. The following are also equivalent:*

(a) The conditions (i)-(iii) are fulfilled for all m,

(b) X has rational singularities,

(c) X has canonical singularities.

These, in conjunction with another result by Crawley-Boevey [CB01] on simple quiver representations and reducedness (which we will describe later on) where applied by N. Budur [Bud18] on a set of schemes related to X_n to establish reducedness and rational singularities:

Theorem 1.15. *([Bud18]) Let $g \geqslant 2$. The scheme*

$$X = \{(x_1, y_1, \ldots, x_g, y_g) \in \mathrm{Mat}(n, \mathbb{C})^{\times 2g} \mid [x_1, y_1] + \cdots + [x_g, y_g] = 0\}$$

is a variety with rational singularities for all $n \geqslant 1$.

These schemes relate to ours through the fact that they can all be constructed as the zero loci of a specific moment map on the representations of the doubles of the quivers with a single vertex and g loops (in his case, for $g \geqslant 2$, and in our case, $g = 1$). Since his results arouse from the study of the semi-simple and simple representations of the quivers and some results relating to them, we studied these in our case, leading to the following proposition:

Proposition 1.16. *If L is the loop quiver (single vertex, single loop) and $\overline{L}$ its double (in this case, obtained adjoining another loop), then $X_n^{red} \subset Rep(\overline{L}, n)$ (where X_n^{red} is the associated reduced scheme) and*

(i) The only simple representations intersecting X_n occur for $n = 1$.

(ii) The semi-simple representations in X_n are pairs of simultaneously diagonalisable matrices.

Furthermore, there was another result by Mustaţă:

Theorem 1.17. *([Mus02]) If X is a smooth variety and $Y \subset X$ is a closed sub-scheme, then the log canonical threshold of the pair (X, Y) is given by*

$$\mathrm{lct}(X, Y) = \dim X - \sup_{m \geqslant 0} \frac{\dim Y^{(m)}}{m + 1}$$

where $Y^{(m)}$ represents the m-th jet scheme over Y.

All these results motivated the study of the jet schemes over our scheme, because on the one hand we have these promising results for complete intersection varieties and, in a more general sense, they relate to the singularities of the scheme. On the other hand, we thought that imposing a bound on the log-canonical threshold or imposing some conditions on the jet schemes themselves could imply reducedness. We have explored it for the case of the log-canonical threshold and found counterexamples to some statements of this kind. For the conditions on the jet schemes for them to imply reducedness, we have not explored it enough for us to obtain any result.

About the jet schemes over our schemes, the main results known prior to our work were:

Theorem 1.18. *([Sv09]) For $n \leqslant 3$ and for all $m \geqslant 0$, the m-th jet scheme over X_n is irreducible and of dimension $(n^2+n)(m+1)$.*

Theorem 1.19. *([Sv09]) For all $m > 0$ exists an integer $N(m)$ such that for all $n \geqslant N(m)$ the m-th jet scheme over X_n is reducible.*

Even though it is not mentioned in that paper, the following corollary is immediate:

Corollary 1.19.1. *For all $m > 0$ exists an integer $N(m)$ such that for all $n \geqslant N(m)$ the m-th jet scheme over X_n is not equidimensional and of dimension $> (n^2+n)(m+1)$.*

Joining these results with the results by Mustață from Theorem 1.17, we obtain the following proposition:

Proposition 1.20. *For $n \leqslant 3$, $\mathrm{lct}(\mathrm{Mat}(n,\mathbb{C})^{\times 2}, X_n) = n^2 - n = \mathrm{codim} X_n$.*

Proposition 1.21. *For $n \geqslant 30$, $\mathrm{lct}(\mathrm{Mat}(n,\mathbb{C})^{\times 2}, X_n) < n^2 - n = \mathrm{codim} X_n$.*

The main results by Sethuraman and Šivic comes from the existence of an specific irreducible open set of dimension $(n^2+n)(m+1)$ on the m-th jet scheme over X_n. We prove the existence of another open set of such dimension.

We have mainly worked on $m = 1$, i.e., the first jet scheme. Using similar techniques, we have proven the belonging of certain matrices to the closure of the open set from Sethuraman and Šivic, which has led to the following results:

Proposition 1.22. *The first jet scheme over X_4 is irreducible of dimension* $2(4^2+4) = (m+1)(n^2+n)$

Proposition 1.23. *The first jet scheme over X_5 has dimension* $2(5^2+5) = (m+1)(n^2+n)$

These results on the jet schemes have implications on another open problem (see [Sv09]), that deals with the dimension of the $k[A_1, \ldots, A_m]$, the algebra generated by m square $n \times n$ commuting matrices over a field k. The question is whether it is bound by n or not. The answer is positive for $m = 2$ and negative for $m \geqslant 4$ (cf. [Sv09]).

Specifically, Sethuraman and Šivic introduce a relation between the jet schemes over X_n with algebras generated by three commuting matrices:

Proposition 1.24. *([Sv09]) Given F an algebraically closed field and $k \geqslant 0$ an integer, if J_{k+1} is the nilpotent Jordan block of dimension $k+1$, C is a* *block diagonal matrix in* $\mathrm{Mat}(n(k+1), F)$ *consisting of* n *copies of* J_{k+1} *along the diagonal upto addition of scalars and* A, B *two matrices commuting with* C*, then if* $X_n^{(k)}$ *is irreducible*

$$\dim F[A, B, C] \leqslant n(k+1)$$

In particular, if we combine this proposition with the results that we obtain on the jet schemes, we obtain the following new result:

Corollary 1.24.1. *Given* F *and algebraically closed field, if* J_2 *is the nilpotent Jordan block of dimension* 2*,* C *is a block diagonal matrix in* $\mathrm{Mat}(8, F)$ *consisting of* 4 *copies of* J_2 *along the diagonal upto addition of scalars and* A, B *two matrices commuting with* C*:*

$$\dim F[A, B, C] \leqslant 8$$

This is a direct implication of our result on the first jet scheme on X_4 and another result by [Sv09]. We will present the whole reasoning in a later section.

Furthermore, to study the singularities of our scheme, we have attempted at the computation of the Bernstein-Sato polynomial associeted to it, as well as the multiplier ideal of the pair $(\mathrm{Mat}(n, \mathbb{C})^{\times 2}, X_n)$. We have not spent much time on this and, consequently, the results are rather minor and coming from simple computations.

Finally, we have worked with some related schemes, which has lead to the solution of an small open problem:

Theorem 1.25. *Given* F *a field, the scheme associated to*

$$X = \{(a, b) \in \mathrm{Mat}(n, F) \mid \mathrm{diag}([a, b]) = 0\}$$

where $\mathrm{diag}(M)$ *applied to a matrix* M *is the projection onto the diagonal elements, (i.e.,* $M = (m_{i,j})_{1 \leqslant i,j \leqslant n} \mapsto \mathrm{diag}(M) = (m_{i,i})_{1 \leqslant i \leqslant n}$*), is a complete intersection integral normal scheme over* F*.*

This actually fits in a bigger result that we have proven concerning schemes of pairs of matrices with some entries of their commutator vanishing. For ease of lecture, we will present it in the corresponding section.

2 Some known results

We will start by proving some relevant known results related to some aspects of the problem. All the results presented here concern the scheme

$$X_n = \{(A, B) \in \mathrm{Mat}(n, \mathbb{C})^{\times 2} \mid [A, B] = 0\}$$

or, in some cases, more general results are presented, e.g. with the analogue construction over an arbitrary algebraically closed field F.

2.1 Generators of the defining ideal

Proposition 2.1. *If we name the defining ideal of X_n as I_n and the (r,s)-th entry of $[A,B]$ as $f_{r,s}$, we get:*

$$f_{i,j} = \begin{cases} \sum\limits_{\substack{k=0 \\ k\neq i}}^{n} (a_{i,k}b_{k,i} - a_{k,i}b_{i,k}) & \text{if } i = j \\ \sum\limits_{\substack{k=0 \\ k\neq i}}^{n} (a_{i,k}b_{k,j} - a_{k,j}b_{i,k}) + x_{i,j}(y_{j,j} - y_{i,i}) - y_{i,j}(x_{j,j} - x_{i,i}) & \text{if } i \neq j \end{cases}$$

and $\{f_{i,j}\}_{i\neq j} \cup \{f_{i,i}\}_{i\neq k}$ is a generating set of I_n for any k which has a minimal number of generators.

Proof. First, we want to remark that $\sum\limits_{i=0}^{n} f_{i,i} = \mathrm{Tr}([A,B]) = 0$, which is a direct consequence of the properties of the trace.

Therefore, $\{f_{i,j}\}_{i\neq j} \cup \{f_{i,i}\}_{i\neq k}$ for any k is a generating set of I_n.

Finally, we notice that each of the monomials appearing in the $f_{i,j}$ for $i \neq j$ appear only in that polynomial of the set $\{f_{i,j}\}_{i\neq j} \cup \{f_{i,i}\}_{i\neq k}$ for any k. On the other hand, if we consider the set $\{f_{i,j}\}_{i\neq j} \cup \{f_{i,i}\}_{i\neq k}$ for a specific k, we get that the monomial $a_{i,k}b_{k,i}$ only appears in the polynomial $f_{i,i}$. This observation, together with the fact that all these polynomials are of degree 2, we get that this set has the minimal number of defining elements for the ideal. □

2.2 Irreducibility and dimension of X_n

Theorem 2.2. *(Motzkin-Taussky [MT55]) Given F and algebraically closed field, the following scheme:*

$$X_n = \{(A,B) \in \mathrm{Mat}_{n\times n}(F) \times \mathrm{Mat}_{n\times n}(F) | [A,B] = 0\}$$

is irreducible and of dimension $n^2 + n$.

We will give a short proof following the one that R.M.Guralnick gives in [Gur92].

First of all recall the following definition:

Definition 2.3. A matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is called *regular* or *non-derogatory* if it fulfils any of the following equivalent conditions:

(i) All its eigenspaces have dimension one.

(ii) The geometric multiplicity of each eigenvalue is one.

(iii) Its minimal polynomial is of degree n.

(iv) $A^0, A^1, \ldots, A^{n-1}$ are linearly independent matrices.

(v) All matrices commuting with A are a polynomial evaluated on A (i.e. $\{f(A) \mid \deg f < n\} = C(A) = \{B \in \mathrm{Mat}(n,F) \mid [A,B] = 0\}$).

Let us notice that the condition (iii) implies that regularity is an open condition.

In this proof we will make use of the following lemma:

Lemma 2.4. *For every matrix A, there exists a regular matrix R that commutes with it.*

Proof. Consider A in Jordan canonical form: $A = \mathrm{diag}(J_1, \ldots, J_r)$ where the eigenvalue a_i corresponds to the i-th block. The matrix is regular if and only if all the a_i's are different. For given $b_1, \ldots, b_r$, take the matrix $R = \mathrm{diag}((b_1 - a_1)I + J_1, \ldots, (b_r - a_r)I + J_r)$ (where I denotes the identity matrix of the appropriate size). This matrix is in Jordan canonical form and the i-th block has b_i as the associated eigenvalue. A and R commute if and only if each block commutes, i.e. if J_i and $(b_i - a_i)I + J_i$ commute, which is trivial. For every set of b_i's where all of them are different, R is regular. $\square$

Proof of Theorem 2.2. Consider a commuting pair of matrices (A, B). If we take a regular matrix R commuting with A, we have that $(A, B + xR)$ is a commuting pair for all $x \in F$. Since the regularity condition is open (and the set $\{B + xR\}_{x \in F}$ is irreducible, of dimension 1 intersecting non-emptily with the set of regular matrices), we have that $B + xR$ is a regular matrix for all but finitely many values of x. Therefore, the closure of the set of pairs of commuting matrices (A, B) where B is regular, is dense in the commuting variety.

Finally, consider the following morphism:

$$\begin{array}{cccc} \phi: & P_{n-1} \times \mathrm{Mat}_{n\times n}(F) & \to & C \\ & (f, B) & \mapsto & (f(B), B) \end{array}$$

Where P_{n-1} is the set of polynomials of degree at most $n-1$. The domain is irreducible and, therefore, the image is irreducible. Since the image contains the dense set where the second component is regular, we have that the image is dense. Consequently, the commuting variety is irreducible.

On the other hand, since the regularity of a matrix B is equivalent to $B^0, B^1, \ldots, B^{n-1}$ being linearly independent. We have that the restriction of ϕ to the $P_{n-1} \times R_n(F)$, where $R_n(F)$ is the set of regular $n \times n$ matrices, is injective. Since regularity is open, we get that $\dim R_n(F) = n^2$, and $\dim P_{n-1} = n$. Thus $\dim C = n^2 + n$. $\square$

We have stated the proof here because, even though it is well known, it has applications to some proofs concerning the dimension of the jet schemes.

We should notice, as we mentioned in the introduction, that it is a concrete case of a more general result that we announce without proof:

Theorem 2.5. *(Richardson [Ric79]) Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field F of characteristic zero and let*

$$\mathcal{C}^{red}(\mathfrak{g}) = \{(a, b) \in \mathfrak{g} \mid [a, b] = 0\}$$

be the reduced scheme of pairs of commuting elements. Then $\mathcal{C}^{red}(\mathfrak{g})$ is irreducible.

2.3 Proof of conjecture for $n < 5$

This section move around the following theorem:

Theorem 2.6. *X_n is reduced, Cohen-Macaulay and normal for $n < 5$.*

This is a well known result (cf. [MZS11], [Hre94]), but we present a new proof for the case $n = 2$ for reducedness, and in all three cases we use algebraic computation tools (such as Macaulay2 [GS]) to check the results, applying some of the methods by Hreinsdóttir [Hre94][Hre06a].

2.3.1 Reducedness $n = 2$

The specific result we will prove here is the following:

Proposition 2.7. *X_2 is a reduced scheme.*

First of all, let's make some observations that will help us during the proof.

Lemma 2.8. *If R is a ring, and $a \in R$ is not a zero-divisor, then R is a domain (respectively reduced) if and only if R_a is a domain (respectively reduced).*

Proof for the domain case. ($\Rightarrow$): Assume that $\exists \frac{x}{a^n}, \frac{y}{a^m} \in R_a$ (with $x, y \in R$) such that $\frac{x}{a^n}\frac{y}{a^m} = 0$. This is equivalent to $\exists l \geqslant 0$ such that $a^l xy = 0$ in R. If R is a domain, we have that $xy = 0$ and, therefore, that either $x = 0$ or $y = 0$, which implies $\frac{x}{a^n} = 0$ or $\frac{y}{a^m} = 0$.

($\Leftarrow$): Assume $\exists x, y \in R$ such that $xy = 0$, then we have that $\frac{x}{1}\frac{y}{1} = 0$ in R_a. If R_a is a domain, without lost of generality, we can assume that $\frac{x}{1} = 0$ in R_a, which is equivalent to $\exists l \geqslant 0$ such that $a^l x = 0$. Since a is not a zero-divisor, we get that $x = 0$. □

Proof for the reduced case. ($\Rightarrow$): Assume that $\exists \frac{x}{a^n} \in R_a$ (with $x \in R$) and $\exists m \geqslant 1$ such that $\left(\frac{x}{a^n}\right)^m = 0$. This is equivalent to $\exists l \geqslant 0$ such that $a^l x^m = 0$ in R, which implies, if R a reduced ring and since a is not a zero-divisor, that $x = 0 \Rightarrow \frac{x}{a^n} = 0$.

($\Leftarrow$): Assume $\exists x \in R$ and $\exists n \geqslant 1$ such that $x^n = 0$, then we have that $\left(\frac{x}{1}\right)^n = 0$ in R_a. If R_a is reduced, we get that $\frac{x}{1} = 0$ in R_a, which is equivalent to $\exists l \geqslant 0$ such that $a^l x = 0$. Since a is not a zero-divisor, we get that $x = 0$. □

Remark: This implies that if we have an element $a \in R$ and an ideal such that $(I : (a)) = I$, I is prime (resp. radical) iff it is prime (resp. radical) in R_a (thanks to the localisation at a multiplicative set S being an exact functor from R-modules to $S^{-1}R$-modules).

Lemma 2.9. *Given a ring R, it is a domain (respectively reduced) iff the polynomial ring $R[X]$ is a domain (respectively reduced).*

Proof. The implication to the left is trivial.

For the implication to the right, assume that $R[X]$ is not a domain and take $f, g \in R[X] \backslash \{0\}$ such that $fg = 0$. Now, there exist $\tilde{f} = a_0 + a_1 X + \cdots +$

$a_nX^n, \tilde{g} = b_0 + b_1X + \cdots + b_mX^m$ such that $a_0 \neq 0, b_0 \neq 0$ and $f = \tilde{f}X^r, g = \tilde{g}X^s$. Therefore, $fg = 0 \Leftrightarrow \tilde{f}\tilde{g} = a_0b_0 + (\cdots)X = 0 \Rightarrow a_0b_0 = 0$. Consequently, R is not a domain.

Analogously, assume that $R[X]$ is not reduced and take $f \in R[X]\backslash\{0\}$ such that $\exists m > 0$ such that $f^m = 0$. Now, there exists $\tilde{f} = a_0 + a_1X + \cdots + a_nX^n$ such that $a_0 \neq 0$ and $f = \tilde{f}X^r$. Therefore, $f^m = 0 \Leftrightarrow \tilde{f}^m = a_0^m + (\cdots)X = 0 \Rightarrow a_0^m = 0$. Consequently, R is not reduced. $\square$

Proof of Proposition 2.7. Given F an algebraically closed field, consider the following matrices over $F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant 2}]$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

Then we evaluate their commutator:

$$[A,B] = \begin{pmatrix} a_{1,2}b_{2,1} - a_{2,1}b_{1,2} & a_{1,2}(b_{2,2} - b_{1,1}) - b_{1,2}(a_{2,2} - a_{1,1}) \\ -a_{2,1}(b_{2,2} - b_{1,1}) + b_{2,1}(a_{2,2} - a_{1,1}) & -a_{1,2}b_{2,1} + a_{2,1}b_{1,2} \end{pmatrix}$$

The ideal defining X_2 is generated by the entries of that commutator and, hence, the following:

$$I_2 = \begin{pmatrix} a_{1,2}b_{2,1} - a_{2,1}b_{1,2} \\ a_{1,2}(b_{2,2} - b_{1,1}) - b_{1,2}(a_{2,2} - a_{1,1}) \\ a_{2,1}(b_{2,2} - b_{1,1}) - b_{2,1}(a_{2,2} - a_{1,1}) \end{pmatrix}$$

Now, consider the following F-algebra automorphism:

$$\begin{array}{cccc} \phi : & F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant 2}] & \to & F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant 2}] \\ & a_{i,j} & \mapsto & \begin{cases} a_{i,j} & \text{if } (i,j) \neq (2,2) \\ a_{2,2} + a_{1,1} & \text{if } (i,j) = (2,2) \end{cases} \\ & b_{i,j} & \mapsto & \begin{cases} b_{i,j} & \text{if } (i,j) \neq (2,2) \\ b_{2,2} + b_{1,1} & \text{if } (i,j) = (2,2) \end{cases} \end{array}$$

Our ideal is prime (respectively radical) if and only if its image is so. And its image is:

$$J = \begin{pmatrix} a_{1,2}b_{2,1} - a_{2,1}b_{1,2} \\ a_{1,2}b_{2,2} - b_{1,2}a_{2,2} \\ a_{2,1}b_{2,2} - b_{2,1}a_{2,2} \end{pmatrix}$$

Which, by Lemma 2.9, is prime or radical if the ideal generated by the same elements over $F[\{a_{i,j}, b_{i,j}\}_{\substack{1\leqslant i,j\leqslant 2 \\ (i,j)\neq(1,1)}}]$ is so.

Renaming $a_1 := a_{1,2}, a_2 := a_{2,1}, a_3 := a_{2,2}, b_1 := b_{1,2}, b_2 := b_{2,1}, b_3 := b_{2,2}$ and calling $F[a_1, a_2, a_3, b_1, b_2, b_3] = R$. And we get the following ideal:

$$J' = \begin{pmatrix} f_1 := a_1b_2 - b_1a_2 \\ f_2 := a_1b_3 - b_1a_3 \\ f_3 := a_2b_3 - b_2a_3 \end{pmatrix}$$

In the next step we need to prove the following lemma:

Lemma 2.10.
$$(I : (a_1)) = I$$

Proof. Take $f \in R$ such that $a_1 f \in J'$, which means that $\exists g_1, g_2, g_3 \in R$ such that $a_1 f = g_1 f_1 + g_2 f_2 + g_3 f_3$.

We can assume that a_1 does not appear in any of the g_i's (i.e. all the monomials that contain a_1 to a non-zero power have a zero coefficient).

In this situation, $a_1 f = a_1(g_1 b_2 + g_2 b_3) + b_1(g_1 a_2 + g_2 a_3) + g_3 f_3$, $f = g_1 b_2 + g_2 b_3$ and $b_1(g_1 a_2 + g_2 a_3) + g_3 f_3 = 0$. Since f_3 is an irreducible polynomial and a polynomial ring over a UFD is a UFD and a field is a UFD, we get that the ideal generated by f_3 is prime. Thus, since $b_1 \notin (f_3)$ because all non-zero elements in the ideal (f_3) have at least degree 2, we have that $(g_1 a_2 + g_2 a_3) \in (f_3)$. That is, $\exists \tilde{g}_3$ such that $(g_1 a_2 + g_2 a_3) + \tilde{g}_3 f_3 = (g_1 + \tilde{g}_3 b_3) a_2 + (g_2 - \tilde{g}_3 b_2) a_3 = 0$. Consequently, $a_3 \mid (g_1 + \tilde{g}_3 b_3)$, $a_2 \mid (g_2 - \tilde{g}_3 b_2)$ and there exists $h \in R$ such that $h a_3 = (g_1 + \tilde{g}_3 b_3)$ and $h a_2 = -(g_2 - \tilde{g}_3 b_2)$. That is, $f = g_1 b_2 + g_2 b_3 = h(a_3 b_2 - a_2 b_3) + \tilde{g}_3(b_3 b_2 - b_3 b_2) = h f_3 \in (f_3) \subset I$. $\square$

Therefore, by Lemma 2.8, it is equivalent to check the primality over R as over R_{a_1}, where the ideal is:

$$J'' = \begin{pmatrix} b_2 - a_1^{-1} b_1 a_2 \\ b_3 - a_1^{-1} b_1 a_3 \\ a_2 b_3 - b_2 a_3 \end{pmatrix}$$

Taking the following algebra automorphism:

$$\begin{array}{cccc} \psi : & F[a_1, a_2, a_3, b_1, b_2, b_3] & \to & F[a_1, a_2, a_3, b_1, b_2, b_3] \\ & b_2 & \mapsto & b_2 + a_1^{-1} b_1 a_2 \\ & b_3 & \mapsto & b_3 + a_1^{-1} b_1 a_3 \end{array}$$

The image ideal is

$$J''' = \begin{pmatrix} b_2 \\ b_3 \\ a_2(b_3 + a_1^{-1} b_1 a_3) - (b_2 + a_1^{-1} b_1 a_2) a_3 = a_2 b_3 - b_2 a_2 \in (b_2, b_3) \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix},$$

which is clearly prime. $\square$

2.3.2 $n \in \{3, 4\}$

For this two cases, at the moment of writing, there are no proof that does not use algebraic computation tools. However, the tool that we have access to, and that we want to use to check these results, can only do computations over finite type $\mathbb{Q}$-algebras. Consequently, our question is whether given $I \subset \mathbb{Q}[X_1, \ldots, X_n]$ a radical ideal, i.e. an ideal such that it is its own radical, its extension into $\mathbb{C}[X_1, \ldots, X_n]$ is a radical ideal or not. The answer is positive. In fact, there is a more general result:

Proposition 2.11. *Given a perfect field k, A a finite type k-algebra and K a k field extension, then, if A is a reduced algebra, $B = A \otimes K$ is also a reduced algebra.*

It can be seen that this implies the result we want to prove:

Obviously, $\mathbb{Q}$ is of characteristic 0 and, therefore, perfect.

If we take $A = \mathbb{Q}[X_1, \ldots, X_n]/I$, we get a finite type $\mathbb{Q}$-algebra, and A is reduced if and only if I is radical (as a reminder, a *reduced algebra* is an algebra such that its *nilradical* is 0).

Furthermore, $\mathbb{C}$ is a field extension of $\mathbb{Q}$, and, if $J = I\mathbb{C}[X_1, \ldots, X_n]$ is the extension ideal of $I \subset \mathbb{Q}[X_1, \ldots, X_n]$, then $B = A \otimes \mathbb{C} = (\mathbb{Q}[X_1, \ldots, X_n]/I) \otimes \mathbb{C} \cong \mathbb{C}[X_1, \ldots, X_n]/J$. Therefore, since B is reduced if and only if J is a radical ideal, we get the implication we want.

We are going to piece the proof in different steps, based on the proof of the same statement in [hR].

First of all, we will actually prove the following statement:

Proposition 2.12. *Given a perfect field k, A a finite type k-algebra and K a k field extension, then, if A is a domain, $B = A \otimes K$ is a reduced algebra.*

That is, we have reduced to the case of A a domain. It is actually not a lost of generality:

Lemma 2.13. *Proposition 2.12 implies Proposition 2.11.*

First of all, let us recall the following proposition:

Proposition 2.14. *If R is a Noetherian reduced ring with the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then $Q(R) \cong \bigoplus_{i=1}^{n} Q(R/\mathfrak{p}_i)$. Where, for a ring S we write $Q(S)$ for the total ring of fractions.*

Proof of Lemma 2.13. Since A is Noetherian (since it is a finite type algebra over a field) and reduced, from the proposition 2.14 in the appendix, we have that it injects into a direct sum of finitely many fields. Next, we have that there is a natural algebra isomorphism $(M \oplus M') \otimes N \cong (M \otimes N) \oplus (M' \otimes N)$ for M, M', N algebras over a certain ring. Since any field extension K over k is a flat module over the base field (since it is faithfully flat, what can be checked by noticing that the induced map $f* : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is surjective since it maps the single point at the domain to the single point at the image), which means that tensoring with it preserves injections. Finally, we have that a subring of (or a ring that injects in) a direct sum of reduced rings is reduced. Therefore, if it enough to prove the result for A a field, and, in particular, a domain. $\square$

Lemma 2.15. *Let K be a finite separable extension over k. Let A be a domain and a k-algebra. Then $B = A \otimes K$ is reduced (though not necessarily a domain).*

Proof. Since K is finite and separable, the Primitive Element Theorem applies and we have that $K = k(\alpha) = k[X]/f$, where $\alpha \in K$ and $f \in k[X]$. $A \otimes K = A \otimes k[X]/f \cong A[X]/f \subset (\mathrm{Frac}(A))[X]/f$. If K is a separable extension, f has no multiple roots and, therefore, $(\mathrm{Frac}(A))[X]/f$ is reduced. Finally, since a ring that injects into a reduced ring, is reduced, B is reduced. □

Corollary 2.15.1. *Let K be an algebraic separable extension over k. Let A be a domain and a k-algebra. Then $B = A \otimes K$ is reduced.*

Proof. If there exists $x = \sum_{i=1}^{m} a_i \otimes k_i \in B$ such that $x^n = 0$ for a certain $n > 0$, we can consider $x \in A \otimes k(k_1, \otimes, k_m)$. Since $k(k_1, \otimes, k_m)$ is a finite separable extension of k, we get that $x = 0$. □

Lemma 2.16. *Let K be a purely transcendental extension over k. Let A be a domain and a k-algebra. Then $B = A \otimes K$ is reduced.*

Proof. First, take a transcendence base of the extension, $\{X_i\}_{i \in \mathcal{I}}$, then $K = k(\{X_i\}_{i \in \mathcal{I}})$. Now we have that $B = A \otimes K = A \otimes k(\{X_i\}_{i \in \mathcal{I}})$. Since $k(\{X_i\}_{i \in \mathcal{I}}) = S^{-1}k[\{X_i\}_{i \in \mathcal{I}}]$ with S the set of all non-zero elements and, since A is a k-algebra, we have that $A \otimes k[\{X_i\}_{i \in \mathcal{I}}] \cong A[\{X_i\}_{i \in \mathcal{I}}]$ and $A \otimes S^{-1}k[\{X_i\}_{i \in \mathcal{I}}] \cong S^{-1}A[\{X_i\}_{i \in \mathcal{I}}]$. Since A is reduced, any localisation is reduced, so B is reduced. □

Corollary 2.16.1. *Let K be a separably generated extension over k. Let A be a domain and a k-algebra. Then $B = A \otimes K$ is reduced.*

Proof. It is a combination of the previous lemma and the corollary to the one before last lemma. □

Corollary 2.16.2. *Let K be a separable extension over k. Let A be a domain and a k-algebra. Then $B = A \otimes K$ is reduced.*

Proof. If there exists $x = \sum_{i=1}^{m} a_i \otimes k_i \in B$ such that $x^n = 0$ for a certain $n > 0$, we can consider $x \in A \otimes k(k_1, \otimes, k_m)$. Since $k(k_1, \otimes, k_m)$ is a finitely generated over k, we get that it is separably generated and, therefore, reduced, which implies that $x = 0$. □

Finally, we recall that any field extension over a perfect field is separable, which proves the Proposition 2.11.

In this way, if we check that our ideals of interest are radical over $\mathbb{Q}$, we know that they are radical over $\mathbb{C}$. And this can be done with a simple code such as the following (for Macaulay2 [GS]):

```
n = 3;
R = QQ[a_(1,1) .. a_(n,n), b_(1,1) .. b_(n,n)];
A = transpose genericMatrix(R, a_(1,1), n, n);
B = transpose genericMatrix(R, b_(1,1), n, n);
I = ideal(A*B-B*A);
time print(I == radical I);
```

A small improvement through Proposition 2.18 can be immediately implemented:

```
n = 3;
R = QQ[a_(1,1) .. a_(n-1,n),a_(n,1) .. a_(n,n-1),
       b_(1,1) .. b_(n-1,n),b_(n,1) .. b_(n,n-1)];
A = {};
B = {};
for i from 1 to n do (
    c = {};
    d = {};
    for j from 1 to n do (
        if (i,j)!=(n,n) then (
            c = append(c, a_(i,j));
            d = append(d, b_(i,j));
        )
        else (
            c = append(c, 0);
            d = append(d, 0);
        );
    );
    A = append(A,c);
    B = append(B,d);
);
A = matrix A;
B = matrix B;
M = A*B-B*A;
I = ideal()
for i from 0 to n-1 do
    for j from 0 to n-1 do
        if (i,j)!=(n-1,n-1) then I = I + ideal(M_(i,j));
time print(I == radical I);
```

However, it is worth noticing that this is quite computation intensive (in fact, we have only been able to use it for $n = 3$) and, actually, was not the method used by [Hre94]. She proved that it is Cohen-Macaulay, which implies reducedness and normality. In the next subsection we present a sketch of her method and we apply it.

2.3.3 Cohen-Macauly for $n < 5$

To check Cohen-Macaulayness, all the results to the moment use the following proposition:

Proposition 2.17. *(see [Hre94]) Let $S = F[x_1, \dots, x_n]$ be a polynomial ring over a field F, let $I \subset S$ be a homogeneous ideal, and let $d = \dim S/I$. Then S/I is Cohen-Macaulay if and only if $\deg S/I = \deg S/(I, f_1, \dots, f_d)$ for some (and hence all) system of parameters $\{f_1, \dots, f_d\}$.*

Here, the degree of a polynomial ring modulo a homogeneous ideal, makes reference to its multiplicity, that is, the leading coefficient of the Hilber polynomial multiplied by $e!$, where e is the degree of the Hilber polynomial.

Therefore, it is enough to compute two Groebner basis to determine Cohen-Macaulayness.

In her first paper [Hre94], Hreinsdóttir uses random numbers to generate a system of parameters, while in [Hre06b] she uses "extensive guessing to find a regular sequence that can be checked by a computer".

The main point of her early work was finding an efficient monomial ordering (see [Hre94], [Hre06a]) such that the computation of the Groebner basis becomes more efficient.

She also found Groebner basis with 11 elements for the case 3×3 and 51 elements for the case 4×4. It is still an open problem whether or not these are minimal.

Even though it is a new result (though elementary), we deemed it more appropriate to mention and prove it here:

Proposition 2.18. *$\mathcal{O}_{X_n}$ is Cohen-Macaulay (respectively reduced) iff, for any $1 \leqslant i, j \leqslant n$ the quotient $\mathcal{O}_{X_n}/(a_{i,i}, b_{j,j})$ is Cohen-Macaulay (respectively reduced). Where $(a_{i,i}, b_{j,j})$ is the ideal (sheaf) generated by the (i,i)-th entry of the matrix A and the (j,j)-th entry of the matrix B.*

Proof. It is immediate if we consider the following algebra automorphism:

$$\begin{array}{ccc} \mathbb{C}[\{a_{r,s}, b_{r,s}\}_{1 \leqslant r,s \leqslant n}] & \longrightarrow & \mathbb{C}[\{a_{r,s}, b_{r,s}\}_{1 \leqslant r,s \leqslant n}] \\ a_{r,r} & \mapsto & a_{r,r} + a_{i,i} \\ b_{s,s} & \mapsto & b_{s,s} + b_{j,j} \\ a_{r,s} & \mapsto & a_{r,s} \text{ if } r \neq s \\ b_{r,s} & \mapsto & b_{r,s} \text{ if } r \neq s \end{array}$$

And remember:

Lemma 2.19. *A ring R is Cohen-Macaulay (respectively reduced) if and only if $R[x]$ is Cohen-Macaulay (respectively reduced).*

The result follows. □

In what follows, we have adapted into Macaulay2 ([GS]) the code that she originally used with Macaulay ([BS]) to test Cohen-Macaulayness. The main point of her work was to introduce better monomial orderings, however, not to bother the reader with cumbersome code, we only introduce a functional though not optimal code:

```
n = 2;

R = QQ[a_(1,1) .. a_(n,n), b_(1,1) .. b_(n,n)]

A = transpose genericMatrix(R,a_(1,1),n,n);
```

```
B = transpose genericMatrix(R,b_(1,1),n,n);

I = ideal(A*B-B*A);

time degI = degree I;

lin = ideal();

for i from 1 to n^2+n do (
    p = 0;
    for j from 1 to n do (
        for k from 1 to n do (
            r1 = random QQ;
            r2 = random QQ;
            p = p + r1*a_(j,k) + r2*b_(j,k);
        );
    );
    lin = lin + ideal p;
);

J = I+lin;

time degLin = degree(J);

print (degLin == degI);
```

A word of caution should be taken since a negative answer by the program does not imply the refutation of Cohen-Macaulayness, since the ideal `lin` does not need to be a system of parameters.

We have used this code with the ideas on more efficient monomial orderings by Hreinsdóttir and the small improvement of Proposition 2.18 to check Cohen-Macaulayness for $n \leqslant 4$ and attempt the case $n = 5$.

2.4 Jet schemes

Recall that the m-th jet scheme over a scheme X over a field F is the set $X^{(m)}(F) = \mathrm{Hom}_F(\mathrm{Spec}(F[t]/t^{m+1}), X)$ with a natural scheme structure. Its construction and basic properties can be found in any introductory notes on jet schemes.

2.4.1 Defining equations

It is a well known result that the jet schemes over an affine scheme are again affine. Furthermore, there is the following result:

Theorem 2.20. *Given F a field and $X = \mathrm{Spec}(F[x_1, \dots, x_n]/I)$ an affine scheme over it, where $I = (f_1, \dots, f_r) \subset F[x_1, \dots, x_n]$ is an ideal, we have* *that the defining equations for the m-th jet scheme over the polynomial ring $F[\{x_{1,k},\dots,x_{n,k}\}_{0\leqslant k\leqslant m}]$ are:*

$$\begin{gathered}
f_1(\tilde{x}_1(t),\dots,\tilde{x}_n(t)) \cong 0 \mod t^{m+1}\\
\vdots\\
f_r(\tilde{x}_1(t),\dots,\tilde{x}_n(t)) \cong 0 \mod t^{m+1}
\end{gathered}$$

where $\tilde{x}_i(t) = x_{i,0} + x_{i,1}t + \cdots + x_{i,m}t^m$.

Equivalently, given the derivation D over that polynomial ring such that $D(x_{i,k}) = x_{i,k+1}$, the defining equations are:

$$\begin{aligned}
f_1(x_{1,0},\dots,x_{n,0}) &= 0\\
&\vdots\\
f_r(x_{1,0},\dots,x_{n,0}) &= 0\\
D(f_1(x_{1,0},\dots,x_{n,0})) &= 0\\
&\vdots\\
D(f_r(x_{1,0},\dots,x_{n,0})) &= 0\\
D^2(f_1(x_{1,0},\dots,x_{n,0})) &= 0\\
&\vdots\\
D^m(f_r(x_{1,0},\dots,x_{n,0})) &= 0
\end{aligned}$$

The proof of this result is quite immediate and can be found in any introductory material to jet schemes.

Given that, we have:

Proposition 2.21. *Over the ring $\mathbb{C}\left[\{a_{i,j,k}, b_{i,j,k}\}_{\substack{0\leqslant k\leqslant m\\ 1\leqslant i,j\leqslant n}}\right]$, we define the matrices $A_k = (a_{i,j,k})_{1\leqslant i,j\leqslant n}$, $B_k = (b_{i,j,k})_{1\leqslant i,j\leqslant n}$. In this situation, the elements generating the ideal that defines the m-th jet scheme, which we name $X_n^{(m)}$ are the entries of the following matrices:*

$$\begin{gathered}
[A_0, B_0]\\
[A_0, B_1] + [A_1, B_0]\\
[A_0, B_2] + [A_1, B_1] + [A_2, B_0]\\
\cdots\\
[A_0, B_m] + [A_1, B_{m-1}] + \cdots + [A_{m-1}, B_1] + [A_m, B_0]
\end{gathered}$$

Or, equivalently:

$$[A(t) = A_0 + A_1t + \cdots + A_mt^m, B(t) = B_0 + B_1t + \cdots + B_mt^m] \equiv 0 \quad (mod\ t^{m+1})$$

It is worth noticing that the group $GL_n(\mathbb{C})$ acts on the scheme by simultaneous conjugation on all the matrices $X_0,\dots,X_m,Y_0,\dots,Y_m$.

2.4.2 Distinguished open set

The following statement is weaker than the one proven in [Sv09] but it is enough to understand the whole reasoning:

Proposition 2.22. *Given a matrix $A(t) = A_0 + A_1 t + \cdots + A_m t^m$. A_0 is a regular (or non-derogatory) matrix if and only if the matrices that commute with $A(t)$ and with t, (mod t^{m+1}), can be described by $m+1$ polynomials of degree at most $n-1$.*

This implies the result that will be the base for our own ones:

Proposition 2.23. *The open set on $X_n^{(m)}$ where A_0 is non-derogatory, is irreducible and of dimension $(m+1)(n^2+n)$. We call it the distinguished open set of the scheme and denote it by $U_n^{(m)}$.*

This follows a similar proof as the proof of the irreducibility of X_n.

2.4.3 Irreducibility of the jet schemes for $n = 2$

Proposition 2.24. *The m-th jet scheme over X_2 is irreducible and of dimension $6(m+1)$ for all $m \geqslant 0$.*

We have not found an explicit proof of this result, but it is immediate from Proposition 2.23 and a simple induction argument.

Proof. The only possibility for a pair o matrices not to be in the distinguished open set is if A_0 is a scalar matrix. In this case, we have that if $m = 0$ it belongs to the closure of the distinguished open set. Assume that it happens for all $m < k$. The equations describing the closed subset where A_0 is a scalar matrix is described by the equations of the $(k-1)$-th jet, whose reduced subscheme we know to be irreducible. Therefore, since this closed irreducible set intersects with the open set where B_0 is regular, we get that it is included in the closure of this set. So now we know that the closure of the open set where A_0 is regular and the closure of the open set where B_0 is regular cover the whole reduced scheme. Since both of them are irreducible and the open sets intersect, we get that the closures are equal among them and to the whole reduced scheme, which is irreducible. □

2.4.4 Irreducibility of the reduced jet schemes for $n = 3$

Here we will present an outline of the proof found in the already mentioned paper by Sethuraman and Šiviv [Sv09].

Proposition 2.25. *The m-th jet scheme over X_3 is irreducible and of dimension $12(m+1)$ for all $m \geqslant 0$.*

The proof of this proposition is based on a series of lemmata which we will announce next.

Lemma 2.26. *If for all $n < N$, $X_n^{(m)}$ has been proven to be irreducible, then for any point $(A = A(t), B = B(t)) \in X_N^{(m)}$ such that A_0 or B_0 have two distinct eigenvalues, $(A, B) \in \overline{U}_N^{(m)}$, where $\overline{U}_N^{(m)}$ denotes the closure of the distinguished open set.*

Let us define $U_n'^{(m)}$ as the open set on $X_n^{(m)}$ where B_0 is non-derogatory.

Lemma 2.27. *Let f be an automorphism of $X_n^{(m)}$ such that $f(U_n^{(m)}) = U_n^{(m)}$ or $f(U_n'^{(m)}) = U_n'^{(m)}$ or $f(U_n^{(m)} \cap U_n'^{(m)}) = U_n^{(m)} \cap U_n'^{(m)}$. Then $(A, B) \in \overline{U}_n^{(m)}$ iff $f(A, B) \in \overline{U}_n^{(m)}$.*

We can observe that $f : (A, B) \mapsto (A - \lambda I, B - \mu I)$ fulfils the hypothesis of this lemma for any λ, μ, therefore:

Corollary 2.27.1. *Let $(A, B) \in X_n^{(m)}$ such that A has a unique eigenvalue λ and B has a unique eigenvalue μ, then $(A, B) \in \overline{U}_n^{(m)}$ iff $(A - \lambda I, B - \mu I) \in \overline{U}_n^{(m)}$.*

As a consequence of this last corollary and of Lemma 2.26 and of the fact that the eigenvalues of A and A_0 and of B and B_0 are the same, when checking if (A, B) belongs to $\overline{U}_n^{(m)}$ we may assume that A_0 and B_0 are nilpotent.

Corollary 2.27.2. *Let $p(t)$ and $q(t)$ be polynomials in $F[t]$ of degree at most m, and assume that $q(0) = 0$. Then $(A(t), B(t)) \in \overline{U}_n^{(m)}$ iff the following occur:*

(i) $(B(t), A(t)) \in \overline{U}_n^{(m)}$

(ii) $(A(t) + p(t)I, B(t)) \in \overline{U}_n^{(m)}$

(iii) $(A(t), B(t) + p(t)I) \in \overline{U}_n^{(m)}$

(iv) $(A(t), B(t) + p(t)A(t)) \in \overline{U}_n^{(m)}$

(v) $(A(t)(1 + q(t)), B(t)) \in \overline{U}_n^{(m)}$

Corollary 2.27.3. *If $(A(t), B(t)) \in \overline{U}_n^{(m)}$ whenever A_0 or B_0 is non-zero, then $\overline{U}_n^{(m)} = X_n^{(m)}$*

All these lemmata are used to reduce the problem to only checking for when A_0 and B_0 are nilpotent and A_0 has two jordan blocks, one of order 2 and the other one of order 1. Furthermore, we can also consider A_0 in Jordan canonical form.

We will use some of these results to prove the irreducibility of $X_4^{(1)}$ and to compute the dimension of $X_5^{(1)}$.

2.4.5 Reducibility of the reduced jet schemes for big n

Theorem 2.28. *([Sv09]) For all $m > 0$ exists an integer $N(m)$ such that for all $n \geqslant N(m)$ the m-th jet scheme over X_n is reducible.*

We will reproduce the proof here since it has guided our attempts to obtain a better lower bound.

The basic idea is that if we find a proper closed subset of dimension at least $(m+1)(n^2+n)$ we get that the scheme is not irreducible. The proof will be a parafrasis of the one found in [Sv09].

Proof. First, let $n = 3a + b$ for $a, b \geqslant 0$ and write $n \times n$ matrices as 4×4 block matrices. Next, consider W, the closed set of matrices in the m-th jet scheme defined by:

$$A_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_0 = \begin{pmatrix} 0 & B_1^0 & B_2^0 & B_3^0 \\ 0 & 0 & B_1^0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B_4^0 & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} & A_{1,3}^{(1)} & A_{1,4}^{(1)} \\ A_{2,1}^{(1)} & A_{2,2}^{(1)} & A_{2,3}^{(1)} & A_{2,4}^{(1)} \\ 0 & A_{3,2}^{(1)} & A_{3,3}^{(1)} & A_{3,4}^{(1)} \\ A_{4,1}^{(1)} & A_{4,2}^{(1)} & A_{4,3}^{(1)} & A_{4,4}^{(1)} \end{pmatrix} \quad B_1 = \begin{pmatrix} B_{1,1}^{(1)} & B_{1,2}^{(1)} & B_{1,3}^{(1)} & B_{1,4}^{(1)} \\ B_{2,1}^{(1)} & B_{2,2}^{(1)} & B_{2,3}^{(1)} & B_{2,4}^{(1)} \\ 0 & B_{3,2}^{(1)} & B_{3,3}^{(1)} & B_{3,4}^{(1)} \\ B_{4,1}^{(1)} & B_{4,2}^{(1)} & B_{4,3}^{(1)} & B_{4,4}^{(1)} \end{pmatrix}$$

and $A_2, \dots, A_m, B_2, \dots, B_m$ are arbitrary as long as they fulfil the equations

$$\begin{aligned} [A_0, B_0] &= 0 \\ [A_0, B_1] + [A_1, B_0] &= 0 \\ [A_0, B_2] + [A_1, B_1] + [A_2, B_0] &= 0 \\ &\cdots \\ [A_0, B_m] + [A_1, B_{m-1}] + \cdots + [A_{m-1}, B_1] + [A_m, B_0] &= 0 \end{aligned}$$

The first equation is already fulfilled by construction and we observe that there are already some entries in the second one that already fulfil the equation, in particular, $(2,1), (3,1), (4,1), (3,2), (3,4)$. Furthermore, if we remember that the commutator of two matrices always has trace zero, we get that from the second block of equations we get at most $n^2 - 3a^2 - 2ab - 1 = 6a^2 + 4ab + b^2 - 1$. The other blocks give at most $n^2 - 1$ independent equations each.

In this way:

$$\begin{aligned} \dim W \geqslant & 2a^2 + 2ab + 2(8a^2 + 6ab + b^2) + 2(m-1)n^2 \\ & - (6a^2 + 4ab + b^2 - 1) - (m-1)(n^2 - 1) = \\ = & 12a^2 + 10ab + b^2 + (m-1)n^2 + m \end{aligned}$$

Considering the conjugation action by $GL_n(F)$ on the scheme, name V all the pairs $(A'(t), B'(t))$ where A'_0 is similar to $\lambda I + A_0$ for certain $\lambda \in F$. V contains $S = \{(\lambda I + GA(t)G^{-1}, \mu I + GB(t)G^{-1} | \lambda, \mu \in F, G \in GL_n(F), (A(t), B(t)) \in W\}$.

Hence:

$$\begin{aligned}\dim V \geqslant \dim S &= n^2 - \dim C(A_0) + \dim W + 2 \\ &\geqslant 18a^2 + 14ab + b^2 + m + 2 + (m-1)n^2\end{aligned}$$

Next, we can write $(m+1)(n^2+n) = 18a^2 + 12ab + 2b^2 + 6a + 2b + (m-1)n^2 + (m-1)(3a+b)$.

Since $\overline{V}$ (the closure of V) is a proper subvariety, if we determine that $\dim V \geqslant (m+1)(n^2+n)$, we get that the m-th jet scheme over X_n is reducible.

Consequently, if $(m+1)(n^2+n) \leqslant 18a^2 + 14ab + b^2 + m + 2 + (m-1)n^2$ we have that it is reducible.

This is equivalent to $b^2 + (m+1-2a)b + 3(m+1)a - m - 2 \leqslant 0$.

After some computations, the result is obtained. □

On the other hand even though it is not mentioned in that paper, the following corollary can be deduced from their proof and it brings information on the log-canonical threshold:

Corollary 2.28.1. *For all $m > 0$ exists an integer $N(m)$ such that for all $n \geqslant N(m)$ the m-th jet scheme over X_n is not equidimensional and of dimension $> (n^2+n)(m+1)$.*

Proof. In an obvious way, if we prove that there is a set with dimension strictly greater than $(m+1)(n^2+n)$, we get that it is not equidimensional, and this comes immediately by inspecting the proof of Theorem 2.28:

It is equivalent at finding a, b such that $b^2+(m+1-2a)b+3(m+1)a-m-2 < 0$ and checking that if there exist a, b fulfilling it such that $3a + b = n$, then for all $n' \geqslant n$ there exist a', b' fulfilling it. □

Remark. Actually, the result is stronger, that is, there is a lower bound $D(n, m)$ for the dimension of m-th jet scheme over X_n, such that, for all $n \geqslant N'(m)$, $\dim X_m^{(m)} \geqslant D(n, m) > (m+1)(n^2+n)$ and $D(n, m) - (m+1)(n^2+n) \overset{n\to\infty}{\longrightarrow} +\infty$.

3 New results concerning X_n

3.1 Generic reducedness

In this section we will prove the following theorem:

Theorem 3.1. *X_n is generically reduced for all $n \geqslant 1$.*

Remembering the defining polynomials of $I_n \subset \mathbb{C}[a_{1,1}, \ldots, a_{n,n}, b_{1,1}, \ldots, b_{n,n}]$ (the ideal associated to X_n) computed in Proposition 2.1, we observe the following:

Lemma 3.2. *Given a polynomial $g \in \mathbb{C}[a_{1,1}, \ldots, a_{n,n}, b_{1,1}, \ldots, b_{n,n}] \backslash \{0\}$, such that contains, with a non-zero coefficient, a monomial not divisible by any monomial of the type $a_{i,k}b_{k,j}$ or $a_{k,j}b_{i,k}$, then $g \notin \sqrt{I_n}$.*

Proof. It is an immediate consequence of all the monomials in the defining ideal only containing monomials of the form $a_{i,k}b_{k,j}$ or $a_{k,j}b_{i,k}$. □

The following corollaries can be immediately deduced:

Corollary 3.2.1. *Given a polynomial $g \in \mathbb{C}[a_{1,1}, \ldots, a_{n,n}] \backslash \{0\}$ or $g \in \mathbb{C}[b_{1,1}, \ldots, b_{n,n}] \backslash \{0\}$, then $g \notin \sqrt{I_n}$.*

Corollary 3.2.2. *Given a polynomial $g \in \mathbb{C}[a_{1,1}, \ldots, a_{n,n}, b_{1,1}, \ldots, b_{n,n}] \backslash \{0\}$ such that $\deg g \leqslant 1$, then $g \notin \sqrt{I_n}$.*

Combining Corollary 3.2.1 with Theorem 2.2 leads to the following:

Corollary 3.2.3. *Given a polynomial $g \in \mathbb{C}[a_{1,1}, ..., a_{n,n}] \backslash \{0\}$ and a polynomial $h \in \mathbb{C}[b_{1,1}, ..., b_{n,n}] \backslash \{0\}$, then $gh \notin \sqrt{I_n}$.*

Proof. It is an immediate consequence of the primality of the ideal $\sqrt{I_n}$ and Corollary 3.2.1. It can as well be deduced without a reference to Theorem 2.2, simply from the equations, but it does not provide much to do so. □

Now we will take a look at the Jacobian matrix of the ideal. We name the partial derivative of $f_{r,s}$ by $a_{i,j}$ (i.e. $\frac{\partial f_{r,s}}{\partial a_{i,j}}$) as $c^{r,s}_{i,j}$ and of $f_{r,s}$ by $b_{i,j}$ (i.e. $\frac{\partial f_{r,s}}{\partial b_{i,j}}$) as $d^{r,s}_{i,j}$.

Then we have:

$$c^{r,s}_{i,j} = \begin{cases} b_{r,r} - b_{s,s} & \text{if } (i,j) = (r,s) \\ b_{j,s} & \text{if } i = r, j \neq s \\ -b_{r,i} & \text{if } i \neq r, j = s \\ 0 & \textit{otherwise} \end{cases}$$

$$d^{r,s}_{i,j} = \begin{cases} -(a_{r,r} - a_{s,s}) & \text{if } (i,j) = (r,s) \\ -a_{j,s} & \text{if } i = r, j \neq s \\ a_{r,i} & \text{if } i \neq r, j = s \\ 0 & \textit{otherwise} \end{cases}$$

Definition 3.3. Given a monomial $g \in \mathbb{C}[b_{1,1}, ..., b_{n,n}] \backslash \{0\}$, we define

$$\deg_{\text{eq}} = \sum_{0 \leqslant i \leqslant n} \deg_{i,i} g$$

where $\deg_{i,j} g$ is defined as the degree of the polynomial respect to the variable $b_{i,j}$. This can obviously be extended to a general polynomial in that ring, assigning the maximal degree for all the monomials with non-zero coefficients.

Lemma 3.4.

$$\det\left(\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}}\right) \notin \sqrt{I_n}$$

Proof. First of all, take the matrix $\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}}$ with the lexicographical order in both pairs of indices, (i,j) and (r,s). In this matrix, we observe that outside the diagonal, $\deg_{\mathrm{eq}} c_{i,j}^{r,s} = 0$, and for the elements of the diagonal we have $\deg_{\mathrm{eq}} c_{i,j}^{i,j} = 1$.

Therefore, we have that the product of the elements of the diagonal (which is non-zero) has $\deg_{\mathrm{eq}} \prod\limits_{i\neq j} c_{i,j}^{i,j} = n$, which is strictly greater than for any other permutation in the determinant.

Thus, $\det\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}} \in \mathbb{C}[b_{1,1}, ..., b_{n,n}]\backslash\{0\}$, and, consequently by Corollary 3.2.1, $\det\left(\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}}\right) \notin \sqrt{I_n}$. □

Proof of Theorem 3.1. By Lemma 3.4 and given that X_n is irreducible of dimension n^2+n by Theorem 2.2, we have that $\operatorname{codim}\left(I_n + \left(\det\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}}\right)\right) > n^2 - n$. Which implies that X_n is generically reduced. □

There is actually a better result:

Proposition 3.5. *The dense open set containing all closed points where B has distinct eigenvalues is reduced.*

Proof. Let us consider B with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and in diagonal form. We see that if we evaluate $\det\left(\left(c_{i,j}^{r,s}\right)_{\substack{r\neq s\\ i\neq j}}\right) \prod\limits_{i\neq j}(\lambda_i - \lambda_j)$, so the Jacobian has rank $n^2 - n$ on those points. The rank of the Jacobian is not changed under an scheme isomorphism, therefore, considering the action by $\mathrm{GL}_n(\mathbb{C})$, we get the proposition. □

In particular, this implies generic reducedness as well.

Proposition 3.6. *The dense open set containing all closed points where B is non-derogatory is regular and, therefore, reduced.*

Proof. Let us consider A in Jordan Canonical form. If we name J_k the nilpotent Jordan block of size k, then there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ pairwise different elements and $a_1, \ldots, a_r > 0$ integers such that $a_1 + \cdots + a_r = n$, such that B is a block diagonal matrix of the form $B = \operatorname{diag}(\lambda_1 I_{a_1} + J_{a_1}, \ldots, \lambda_r I_{a_r} + J_{a_r}) = (b_{i,j})_{1\leqslant i,j\leqslant n}$.

In this situation:

$$c_{i,j}^{r,s} = \begin{cases} 1 & \text{if } i = r, s = j+1 \leqslant n \text{ and } b_{j,j} = b_{j+1,j+1} \\ -1 & \text{if } j = s, r = i-1 \geqslant 0 \text{ and } b_{i-1,i-1} = b_{i,i} \\ b_{j,j} - b_{i,i} & \text{if } (i,j) = (r,s) \text{ and } b_{j,j} \neq b_{i,i} \\ 0 & \text{otherwise} \end{cases}$$

First, we will prove that $\det\left(c_{i,j}^{r,s}\right)_{\substack{b_{r,r}\neq b_{s,s}\\ b_{i,i}\neq b_{j,j}}} \neq 0$, where the columns of the matrix are indexed by the (i,j) and the rows by (r,s), both with the same ordering.

We observe that the diagonal is $\prod_{\{(i,j)|b_{i,i}\neq b_{j,j}\}} (b_{j,j} - b_{i,i}) \neq 0$.

We will prove that all the other products in the determinant vanish.

Let us pick the column (i,j). If $j+1 \leqslant n$ and $b_{j,j} = b_{j+1,j+1}$, then $b_{i,i} \neq b_{j+1,j+1}$, so for the (i,j) column, we can get the entry of the $(i,j+1)$ row which has a value of 1. In this case, for the $(i,j+1)$ column we cannot get the diagonal element. If $i-1 \geqslant 0$ and $b_{i-1,i-1} = b_{i,i}$, then $b_{i-1,i-1} \neq b_{j,j}$ and for the (i,j) column we can get the entry of the $(i-1,j)$ entry that has a value of -1. In this case, for the $(i-1,j)$ column we cannot get the diagonal element.

A non-vanishing product would be equivalent to this process having a cycle, but either the i decreases or the j increases, so we can never have a cycle and all products, apart from the diagonal one, vanish, as we wanted to show.

Now, given (k,l) such that $b_{k,k} = b_{l,l}$, $l+1 \leqslant n$ and $b_{j,j} = b_{j+1,j+1}$, assume that all the columns with indexes in

$$\mathcal{S} = \{(i,j) \mid b_{i,i} \neq b_{j,j}\} \cup \{(i,j) \mid b_{i,i} = b_{j,j}, j+1 \leqslant n, b_{j,j} = b_{j+1,j+1} \text{ and } (i,j) < (k,l)\},$$

where the ordering is the lexicographic order, are linearly independent.

$c_{k,l}^{k,l+1} = 1$ and for all $(i,j) \in \mathcal{S}$, $c_{i,j}^{k,l+1} = 0$. This proves that the columns with indexes in $\mathcal{S} \cup \{(k,l)\}$ are linearly independent.

In this way, we have proven that the columns with indexes in

$$\mathcal{I} = \{(i,j) \mid b_{i,i} \neq b_{j,j}\} \cup \{(i,j) \mid b_{i,i} = b_{j,j}, j+1 \leqslant n, b_{j,j} = b_{j+1,j+1}\}$$

are linearly independent.

Since the cardinality of I is $n^2 - n$, we get that this closed point is reduced.

Through the action of$GL_n(\mathbb{C})$ we get that the open set that includes all closed points where B is non-derogatory is reduced. □

Corollary 3.6.1. *X_n is regular in codimension* 1.

Proof. The result will follow from the fact that the complementary of the set of closed points where A and B are non-derogatory has codimension at least 2.

We will do this working on reduced associated schemes, since the dimension does not change.

This subvariety is a finite union of sets of the following form:

B derogatory commuting with A, which is derogatory, has $0 \leqslant r \leqslant n$ different eigenvalues and has a concrete Jordan Canonical form. That is, there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that:

$$gAg^{-1} = \left(\begin{array}{ccccccc|cccccc}
\lambda_1 & \epsilon_{1,2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_1 & \epsilon_{2,3} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots & \lambda_1 & \epsilon_{a_1-1,a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_2 & \epsilon_{a_1+1,a_1+2} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda_2 & \epsilon_{a_1+2,a_1+3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \lambda_r & \epsilon_{n-1,n} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_r
\end{array}\right)$$

for arbitrary distinct $\lambda_1, \dots, \lambda_r$ specific $\epsilon_{i,j} \in \{0, 1\}$ that make it derogatory.

If there are strictly less than $n-1$ distinct eigenvalues, the set smaller than $n^2 - n - 2$, so we are done with it.

If we consider A in the Jordan Canonical form, the case of $n-1$ distinct eigenvalues $\lambda_2, \dots, \lambda_n$ gives us the following:

$$A = \left(\begin{array}{cc|cccc}
\lambda_2 & 0 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \lambda_3 & 0 & \cdots & 0 \\
0 & 0 & 0 & \lambda_4 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \lambda_n
\end{array}\right)$$

Which means that a generic $\widetilde{B}$ commuting with A has the form

$$\widetilde{B} = \left(\begin{array}{cc|cccc}
b_{1,1} & b_{1,2} & 0 & 0 & \cdots & 0 \\
b_{2,1} & b_{2,2} & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & b_{3,3} & 0 & \cdots & 0 \\
0 & 0 & 0 & b_{4,4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & b_{n,n}
\end{array}\right)$$

If B is derogatory and commuting with a matrix similar to A, the dimension of this set is $n^2 - n - 2$. Which proves the corollary. $\square$

Remark. When considering the associated reduced scheme, regular in codimension one was already known for the commuting variety of a reductive Lie algebra as we exposed in the introduction, Theorem 1.10 [Pop08]. When considering the associated reduced scheme of X_n, there is a simpler prove of this result.

Lemma 3.7. *X_n^{red}, the associated reduced scheme to X_n, fulfils R_1.*

Proof. As we saw in the proof of Theorem 2.2, there is a morphism

$$\begin{array}{rccc} \phi: & P_{n-1} \times R_n(\mathbb{C}) & \to & X_n^{red} \\ & (f,B) & \mapsto & (f(B),B) \end{array}$$

where $R_n(\mathbb{C})$ is the variety of non-derogatory square $n \times n$ matrices over $\mathbb{C}$ and P_{n-1} is the set of polynomials of degree at most $n-1$ over $\mathbb{C}$.

This is injective. So the points of the image, which are the ones where B is non-derogatory, are smooth.

We know that the complementary of the subvariety where A or B is non-derogatory has dimension at most $n^2 - n - 2$. Which proves the result. □

It is interesting to notice that Corollary 3.6.1 provides another proof of the reducedness of X_2:

Corollary 3.7.1 (Corollary to Corollary 3.6.1)**.** *X_2 is reduced.*

To prove this, let us remember Krull's height theorem:

Theorem 3.8 (Krull's height theorem)**.** *(see [AM94]) Let A be a Noetherian ring, $x_1, \ldots, x_r \in A$. Then any minimal prime $\mathfrak{p}$ belonging to $(x_1, \ldots, x_r)$ has height $\leqslant r$.*

Proof. It is immediate from the fact that the defining ideal has a minimal set of generators of size $n^2 + n - 1 = n^2 + 1 = 5$ so by Krull's height theorem, the result follows. □

Remark. Analogously, if we proved regularity in codimension n, the reducedness of X_{n+1} would follow. However, there is another remarkable result that stops us from following this way:

Proposition 3.9. *X_n is not regular in codimension 4.*

Proof. Take the closed points of the form (A,B) where A and B are both diagonalisable and they both have $n-1$ distinct eigenvalues. It is immediate to check that the Jacobian matrix has rank at most $n^2 - n - 2$, so these are all non-regular points and the codimension is 4. □

Remark. This bound is reached for X_2 and the reduced singular locus is exactly the one used in the proof, that is, in the case of $n = 2$, the pairs of scalar matrices (this can easily be checked using computational algebra programs such as Macaulay2 [GS]). For X_3, the singular locus has codimension 4 as well ([Hre06b]).

Remark. It is just speculation, but this result might hint to X_n for $n \geqslant 5$ behaving in a fundamentally different way than for $n \leqslant 4$.

Comment on the conjecture

A direct consequence of the Theorem 3.1 that we already announced in the introduction is the following proposition:

Proposition 3.10. *If X_n is Cohen-Macaulay, then it is reduced.*

This implication comes from Serre's criteria:

Theorem 3.11. *Given A a Noetherian commutative ring, Serre's conditions for it are:*

- R_k*:* $A_\mathfrak{p}$ *is a regular local ring for any prime ideal* $\mathfrak{p} \subset A$ *such that* $\mathrm{height}(\mathfrak{p}) \leqslant k$.
- S_k*:* $\mathrm{depth} A_\mathfrak{p} \geqslant \inf\{k, \mathrm{height}(\mathfrak{p})\}$ *for any prime* $\mathfrak{p}$.

Then:

- *A is reduced iff A satisfies R_0 and S_1.*
- *A is normal iff A satisfies R_1 and S_2.*
- *A is Cohen-Macauly iff A satisfies S_k for all $k \geqslant 0$*

Proof of Proposition 3.10. It is immediate from Theorem 3.11: R_0 is generic reducedness and Cohen-Macaulay is equivalent to S_k for all k. □

Proposition 3.10 was a previously known result (see Problem 2.7.1 [Vas98]), however, the argumentation was different.

R_1 was already known for the reduced associated scheme (Theorem 1.10), which already enabled us to assert the following proposition:

Proposition 3.12. *If X_n is Cohen-Macaulay, then it is reduced and normal.*

Proof. We have that, by Proposition 3.10, if X_n is Cohen-Macaulay, then it is reduced and, therefore by Theorem 1.10 satisfies Serre's condition R_1 and, consequently, it is normal. □

Remark. We have also proven Theorem 1.10 in our specific case (Lemma 3.7) but we have further proven a better result for X_n which implies it (Corollary 3.6.1).

3.2 Jet schemes

Let us think about the m-th jet scheme over X_n, that is $X_n^{(m)}$.

We know, as we explained in the section of known results (Proposition 2.23) and as Sethuraman and Šivic showed in [Sv09], that the open set where A_0 is regular, which we have been calling $U_n^{(m)}$ is irreducible of dimension $(m + 1)(n^2 + n)$.

Therefore, as they did with $X_3^{(m)}$, the main idea is to reduced the problem of irreducibility of $X_n^{(m)}$ to check the belonging of some $(A(t), B(t))$, with a concrete description, to the closure of $U_n^{(m)}$.

Thanks to Lemma 2.26 and Corollary 2.27.1, for the case $n = 4$ it is enough to check when A_0 is nilpotent (it can be reduced to A_0 and B_0 both nilpotent, but we do not make use of it). Despite the fact that we cannot apply those results to $n \geqslant 5$ yet, they point towards the fact that studying the belonging or not of the elements where A_0 is nilpotent entails some interest towards understanding the jet schemes.

Our main method consists in proving that the closed subvariety where A_0 is in a specific nilpotent Jordan Canonical form is irreducible. In this case, the set

$$S_{A_0} = \{(A'(t), B'(t)) \mid \exists g \in \mathrm{GL}_n(F), \lambda \in F \text{ such that } A'_0 = gA_0g^{-1} + \lambda I\}$$

is irreducible. Finally, we have, as can be seen in the proof of Theorem 2.2, that there is a non-derogatory matrix B_0 commuting with A_0. Taking $A(t) = A_0 + 0t + \cdots + 0t^m$ and $B(t) = B_0 + 0t + \cdots + 0t^m$, we have that this pair belongs to $U_n^{(m)}$ and, therefore, $S_{A_0} \cap U_n^{(m)} \neq \varnothing$. Which, by the irreducibility of S_{A_0}, implies $S_{A_0} \subset U_n^{(m)}$.

We also use similar methods to set bounds on the dimension of the jet schemes.

3.2.1 1st jet scheme

Since the first jet scheme only involves four matrices it is much easier to treat than the other jet schemes. As we have explained, we will try to prove the irreducibility of the closed subvarieties where A_0 is in Jordan Canonical Form, is nilpotent and has a specific Jordan structure. We have named each subsection with the partition of n that corresponds to the Jordan structure studied in that subsection.

We adopt the convention that $J_{(a_1,\ldots,a_r)}$ where $a_1 + \cdots + a_r = n$ is the Jordan Canonical nilpotent matrix with r blocks of size $a_1, \ldots, a_r$.

3.2.1.1 Case $(1, \ldots, 1)$

Proposition 3.13. *The reduced scheme associated to*

$$\{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = 0\}$$

is irreducible.

To prove this, we will first notice the following lemma:

Lemma 3.14. *As reduced schemes:*

$$\{(A(t), B(t)) \in X_n^{(m)} \mid A_0 = 0\} \simeq X_n^{(m-1)} \times \mathrm{Mat}(n, F)$$

Proof. This follows immediately from the defining equations of $X_n^{(m)}$. $\square$

Proof of Proposition 3.13. By Theorem 2.2 $X_n^{(0)} = X_n$ is irreducible and, therefore, $X_n^{(m-1)} \times \mathrm{Mat}(n, F)$ is as well. $\square$

Corollary 3.14.1 (Corollary of Proposition 3.13)**.** *For all $n \geqslant 1$*

$$\{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = 0\} \subset \overline{U}_n^{(1)}$$

3.2.1.2 Case $(2, 1, \overset{n-2}{\dots}, 1)$

Proposition 3.15. *The reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(2,1,\dots,1)}\}$$

is irreducible for $n = 4$ and given

$$S = \{(A(t), B(t)) \in X_n^{(1)} \mid \exists g \in \mathrm{GL}_n(F), \lambda \in F \ s.t. \ gA_0g^{-1} + \lambda F = J_{(2,1,\dots,1)}\}$$

$\dim S \leqslant 2(n^2 + n)$.

For the proof of this proposition we have used the computation engine Macaulay2 [GS]. However, it can only do computations over the base field $\mathbb{Q}$. Therefore we need to prove the following:

Lemma 3.16. *Given an ideal $I \subset \mathbb{Q}[x_1, \dots, x_n]$ then $(I : (x_1)) = I$ implies that, if $J = I\mathbb{C}[x_1, \dots, x_n]$, $(J : (x_1)) = J$.*

There are obviously more general results related to this one, but we do not need them.

Proof. First of all, let us notice that, considering the ideal $(It + x_1(1-t)) \subset \mathbb{Q}[x_1, \dots, x_n, t]$, then $(I : (x_1)) = \frac{1}{x_1}((It + x_1(1-t)) \cap \mathbb{Q}[x_1, \dots, x_n])$. This is the basis of the algorithm that computes the quotient ideal. Given a Gröbner basis of $(It+x_1(1-t))$ with a monomial order such that any monomial with t raised to a positive power is greater than any with t^0, then $(It+x_1(1-t)) \cap \mathbb{Q}[x_1, \dots, x_n]$ is generated by the elements of the Gröbner basis that do not contain t (whose monomials with t to a positive power have a zero coefficient).

Finally, we use the fact that Gröbner basis are preserved under scalar extension between fields, i.e. the inclusion of a Gröbner basis of $(It + x_1(1-t)) \subset \mathbb{Q}[x_1, \dots, x_n, t]$ with a certain monomial odering into $\mathbb{C}[x_1, \dots, x_n, t]$ is a Gröbner basis of $(It + x_1(1-t))\mathbb{C}[x_1, \dots, x_n, t]$ with the same monomial ordering on $\mathbb{C}[x_1, \dots, x_n, t]$.

This last statement comes from the fact that a field extension is a faithfully flat (and therefore flat) module over the base field (which can be seen since the morphism between the prime spectra is bijective and, in particular, surjective, a condition for faithful flatness). This, combined with the following theorem:

Theorem 3.17. *([BGS91]) A Gröbner basis over a polynomial ring $A[x_1, \ldots, x_n]$ is preserved under the scalar extension to $B[x_1, \ldots, x_n]$, where B is an A-algebra, if and only if B is a flat A-algebra.*

□

Proof of Proposition 3.15: irreducibility for $n = 4$. Even though the actual proof is only for $n = 4$, we work with general n, given that we will use it to prove a bound on the dimension for arbitrary n. It is equivalent to check for A_0 with the following shape:

$$A_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Now we compute the commutator of A_0 with a generic matrix $(b_{i,j})_{1 \leqslant i,j \leqslant n}$:

$$[A_0, (b_{i,j})] = \begin{pmatrix} b_{n,1} & \cdots & b_{n,n-1} & b_{n,n} - b_{1,1} \\ 0 & \cdots & 0 & -b_{2,1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -b_{n,1} \end{pmatrix}$$

This means that B_0 must have the following form for some values $b_{i,j}$:

$$B_0 = \left(\begin{array}{c|cccc|c} b_{1,1} & b_{1,2} & \cdots & b_{1,n-1} & & b_{1,n} \\ \hline 0 & b_{2,2} & \cdots & b_{2,n-1} & & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & b_{n-1,2} & \cdots & b_{n-1,n-1} & & b_{n-1,n} \\ \hline 0 & 0 & \cdots & 0 & & b_{1,1} \end{array}\right) = \left(\begin{array}{c|c|c} B_{1,1} & B_{1,2} & B_{1,3} \\ \hline 0 & B_{2,2} & B_{2,3} \\ \hline 0 & 0 & B_{1,1} \end{array}\right)$$

We see that it is convenient to work with block matrices.

Now we consider two generic matrices A_1 and B_1 as block matrices $(1 + (n-2) + 1) \times (1 + (n-2) + 1)$:

$$A_1 = \left(\begin{array}{c|c|c} A_{1,1} & A_{1,2} & A_{1,3} \\ \hline A_{2,1} & A_{2,2} & A_{2,3} \\ \hline A_{3,1} & A_{3,2} & A_{3,3} \end{array}\right) \qquad B_1 = \left(\begin{array}{c|c|c} B'_{1,1} & B'_{1,2} & B'_{1,3} \\ \hline B'_{2,1} & B'_{2,2} & B'_{2,3} \\ \hline B'_{3,1} & B'_{3,2} & B'_{3,3} \end{array}\right)$$

Now we compute the second matrix of equations ($[A_1, B_0] + [A_0, B_1]$):

$$
[A_1, B_0]+[A_0, B_1] =
$$

$$
= \left(\begin{array}{c|c|c}
\begin{array}{c}[A_{1,1}, B_{1,1}] \\ -B_{1,2}A_{2,1} - B_{1,3}A_{3,1}\end{array} & \begin{array}{c}A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ -B_{1,1}A_{1,2} - B_{1,2}A_{2,2} \\ -B_{1,3}A_{3,2}\end{array} & \begin{array}{c}A_{1,1}B_{1,3} + A_{1,2}B_{2,3} \\ +[A_{1,3}, B_{1,1}] - B_{1,2}A_{2,3} \\ -B_{1,3}A_{3,3}\end{array} \\
\hline
\begin{array}{c}A_{2,1}B_{1,1} - B_{2,2}A_{2,1} \\ -B_{2,3}A_{3,1}\end{array} & \begin{array}{c}A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \\ -B_{2,2}A_{2,2} - B_{2,3}A_{3,2}\end{array} & \begin{array}{c}A_{2,1}B_{1,3} + A_{2,2}B_{2,3} \\ +A_{2,3}B_{1,1} - B_{2,2}A_{2,3} \\ -B_{2,3}A_{3,3}\end{array} \\
\hline
[A_{3,1}, B_{1,1}] & \begin{array}{c}A_{3,1}B_{1,2} + A_{3,2}B_{2,2} \\ -B_{1,1}A_{3,2}\end{array} & \begin{array}{c}A_{3,1}B_{1,3} + A_{3,2}B_{2,3} \\ +[A_{3,3}, B_{1,1}]\end{array}
\end{array}\right)
$$

$$
+ \left(\begin{array}{c|c|c}
B'_{3,1} & B'_{3,2} & B'_{3,3} - B'_{1,1} \\
\hline
0 & 0 & -B'_{2,1} \\
\hline
0 & 0 & -B'_{3,1}
\end{array}\right)
$$

Where we can deduce that $B'_{3,1} = -[A_1, B_0]_{1,1}$, $B'_{3,2} = -[A_1, B_0]_{1,2}$, $B'_{2,1} = [A_1, B_0]_{2,3}$ and $B'_{3,3} = B'_{1,1} - [A_1, B_0]_{1,2}$. Where, given a (block) matrix M, $M_{i,j}$ represents the (i, j)-th entry (block).

Which leaves us with the following equations:

$$
\begin{cases}
[A_1, B_0]_{1,1} + [A_1, B_0]_{3,3} = 0 \\
[A_1, B_0]_{2,1} = 0 \\
[A_1, B_0]_{2,2} = 0 \\
[A_1, B_0]_{3,1} = 0 \\
[A_1, B_0]_{3,2} = 0
\end{cases}
$$

Now we notice that the trace of a commutator always vanishes, which means that if the third equation is fulfilled, then the first is. Furthermore, we observe that $B_{1,1}, A_{3,1} \in \mathbb{C}$, which means that they commute, and the fourth equation is always fulfilled.

Renaming $\tilde{B}_{2,2} := B_{1,1}\mathrm{I} - B_{2,2}$ we obtain that the defining equations are:

$$
\begin{cases}
\tilde{B}_{2,2}A_{2,1} + A_{3,1}B_{2,3} \\
A_{3,2}\tilde{B}_{2,2} + A_{3,1}B_{1,2} \\
\left[A_{2,2}, \tilde{B}_{2,2}\right] + A_{2,1}B_{1,2} - B_{2,3}A_{3,2}
\end{cases}
$$

If we rename again $A = A_{2,2}, B = \tilde{B}_{2,2}, \alpha = A_{2,1}, \beta = B_{2,3}, \alpha' = A_{3,2}, \beta' = B_{1,2}, a = A_{3,1}$:

$$
\begin{cases}
B\alpha + a\beta = 0 \\
\alpha' B + a\beta' = 0 \\
[A, B] + \alpha\beta' - \beta\alpha' = 0
\end{cases}
$$

Therefore, proving the irreducibility of the variety defined by this equations is equivalent at proving the primality of the ideal

$$
I = \begin{pmatrix} B\alpha + a\beta \\ \alpha' B + a\beta' \\ [A, B] + \alpha\beta' - \beta\alpha' \end{pmatrix} \subset \mathbb{C}[a, \{\alpha_i, \alpha'_i, \beta_i, \beta'_i\}_i, \{b_{i,j}, a_{i,j}\}_{i,j}]
$$

If $(I : a) = I$, then we can localise at a using Proposition 2.8, which leaves us with the following equations:

$$\begin{cases} \beta = -\frac{1}{a} B\alpha \\ \beta' = -\frac{1}{a}\alpha' B \\ [A, B] - \frac{1}{a}\alpha\alpha' B + \frac{1}{a} B\alpha\alpha' = \left[A - \frac{1}{a}\alpha\alpha', B\right] = 0 \\ a \neq 0 \end{cases}$$

Renaming $\widetilde{A} = A - \frac{1}{a}\alpha\alpha'$, we get that the resultant equations are the ones of the commuting variety:

$$[\widetilde{A}, B] = 0$$

which is irreducible.

Furthermore, looking carefully, it can be noticed that the dimension is $m^2 + 3m + 1$ (where $m = n - 2$).

The only thing that remains to be checked is that $(I : a) = I$. This can be done for $m = 2$ using Macaulay2 [GS] and a code such as the following:

```
n=4;
m = n-2;
R = QQ[a_(1,1) .. a_(m,m), b_(1,1) .. b_(m,m),
       al_1 .. al_m, alp_1 .. alp_m, be_1 .. be_m,
       bep_1 .. bep_m, c];

A = transpose genericMatrix(R, a_(1,1), m, m);
B = transpose genericMatrix(R, b_(1,1), m, m);
alpha = genericMatrix(R, al_1, m, 1);
alphap = genericMatrix(R, alp_1, 1, m);
beta = genericMatrix(R, be_1, m, 1);
betap = genericMatrix(R, bep_1, 1, m);

I = ideal(A*B-B*A + alpha*betap - beta*alphap,
          B*alpha + c*beta, alphap*B + c*betap);

time J = quotient(I, ideal(c));

time J == I
```

□

On the other hand, even though we do not have irreducibility results for higher n, we can obtain bounds on its dimension.

To prove this, we will need another known result:

Theorem 3.18. *(combination of [Hul81] and [Neu89]) Given an algebraically closed field F and two integers $n \geqslant r \geqslant 0$, the scheme defined by*

$$Z = \{(A, B) \in \mathrm{Mat}(n, F)^{\times 2} \mid \mathrm{rank}[A, B] \leqslant r\}$$

is irreducible for $r \neq 1$ and $r = 1, n = 2$, for $r = 1$ it has $n-1$ irreducible components and its dimension is

$$\dim Z = \begin{cases} n^2 + n & \text{if } r = 0 \\ n^2 + 2rn - r^2 & \text{if } r \geqslant 1 \end{cases}$$

We also need to prove the following lemma:

Lemma 3.19. *Given an algebraically closed field F, the dimension of the scheme*

$$\left\{(A, B, \alpha, \alpha'^T, \beta, \beta'^T) \in \mathrm{Mat}(n, F)^{\times 2}\,(F^m)^4 \middle| \beta\alpha' - \alpha\beta' = [A, B]\right\}$$

is at most $m^2 + 4m + 1$.

Proof. Using Theorem 3.18 we divide the associated reduced scheme in three subschemes:

If $\mathrm{rank}[A, B] = 2$, we get that $\dim \mathrm{Im}[A, B] = \dim \mathrm{Im}[A, B]^T = 2$ and and $\mathrm{Im}[A, B] = \langle \alpha, \beta \rangle$ (that is, the image of the automorphism described by $[A, B]$ is generated by α and β), and $\mathrm{Im}[A, B]^T = \langle \alpha'^T, \beta'^T \rangle$. Consider the projection from this scheme to $(A, B, \alpha, \alpha'^T)$. We will bound the dimension of the fibres. Now, consider a concrete $0 \neq \alpha \in \mathrm{Im}[A, B]$ and a concrete $0 \neq \alpha'^T \in \mathrm{Im}[A, B]^T$. Then, $\beta\alpha' - \alpha\beta' = \widetilde{\beta}\alpha' - \alpha\widetilde{\beta}'$ iff $(\widetilde{\beta} - \beta)\alpha' = \alpha(\widetilde{\beta}' - \beta)$ iff exists $\lambda \in F$ such that $\widetilde{\beta} = \beta + \lambda\alpha$ and $\widetilde{\beta}' = \beta' + \lambda\alpha'$. Consequently, the fibers have dimension at most 1, and we can bound the dimension of this subscheme by $m^2 + 4m + 1$.

If $\mathrm{rank}[A, B] = 1$. A rank one matrix is a matrix of the type $uv^T \neq 0$ for some vectors u, v, both non-zero. If we have $uv^T = u'v'^T$, then there is a constant $\lambda \in F \backslash 0$ such that $u' = \lambda u$ and $v = \lambda v'$. Therefore, the dimension can be bounded by $(m^2 + 2m - 1) + (2m + 1) = m^2 + 4m$.

If $\mathrm{rank}[A, B] = 0$, the dimension can be bounded by $(m^2 + m) + (2m + 1) = m^2 + 3m + 1$. $\square$

Corollary 3.19.1. *Given an algebraically closed field F, the dimension of the scheme*

$$\left\{(A, B, \alpha, \alpha'^T, \beta, \beta'^T) \in \mathrm{Mat}(n, F)^{\times 2}\,(F^m)^4 \middle| \beta\alpha' - \alpha\beta' = [A, B], \det B = 0\right\}$$

is at most $m^2 + 4m$.

Proof. A parallel proof to the one from Lemma 3.19. $\square$

Proof of Proposition 3.15: dimension bound. Given the equations expression that we reached in the previous proof:

$$\begin{cases} B\alpha + a\beta = 0 \\ \alpha' B + a\beta' = 0 \\ [A, B] + \alpha\beta' - \beta\alpha' = 0 \end{cases}$$

We can consider that $a = 0$, if $a \neq 0$ the elements belong to $\overline{U}_n^{(m)}$, which leaves us with the following equations:

$$\begin{cases} B\alpha = 0 \\ \alpha' B = 0 \\ [A, B] = \beta\alpha' - \alpha\beta' \end{cases}$$

If B is invertible: Let us take: $\tilde{B} = I, \tilde{\alpha} = -B^{-1}(\beta - \alpha), \tilde{\alpha}' = -(\beta' - \alpha')B^{-1}, \tilde{A} = \tilde{\alpha}\tilde{\alpha}'$

$(A, B, \alpha, \alpha', \beta, \beta', a = 0) \mapsto (A + x\tilde{A}, B + x\tilde{B}, \alpha + x\tilde{\alpha}, \alpha' + x\tilde{\alpha}', \beta, \beta', x^2)$

$$\begin{aligned} &\left[A + x\tilde{A}, B + x\tilde{B}\right] = \beta(\alpha' + x\tilde{\alpha}') - (\alpha + x\tilde{\alpha})\beta' \\ &(B + x\tilde{B})(\alpha + x\tilde{\alpha}) = x^2\beta \\ &(\alpha' + x\tilde{\alpha}')(B + x\tilde{B}) = x^2\beta' \end{aligned}$$

Consequently, this points belong to the closure of the points where $a \neq 0$.

Now we can consider $\det B = 0$, which leaves us with:

$$\begin{cases} B\alpha = 0 \\ \alpha' B = 0 \\ [A, B] = \beta\alpha' - \alpha\beta' \\ \det B = 0 \end{cases}$$

We name as $\widetilde{W}$ the associated scheme.

By Corollary 3.19.1, $\dim \widetilde{W} \leqslant m^2 + 4m$.

Therefore the dimension of the closed subscheme of the 1st jet scheme where A_0 is in the predefined Jordan Canonical Form and $a_{n,1}^{(1)} = 0$ (the $(n, 1)$-th entry of the matrix A_1, which we renamed a) and $\det\left((b_{i,j}^{(0)})_{1<i,j<n} - b_{1,1}^{(0)}\right) = 0$ (which is the determinant of the matrix that we named B), which we name W, can be bounded in the following manner:

$\dim W = (\dim C(A_0)) + (2m + 3) + (2) + \dim \tilde{W}$

Where $C(A_0)$ is the centraliser of A_0. The first summand comes from B_1, the second from A_1 (the elements that are free and do not appear in the generators of the defining ideal of $\tilde{W}$) and the third from B_0 (analogously the elements that do not appear in the generators of the defining ideal of $\tilde{W}$).

If now we consider the image of the following morphism, which we name V:

$$\begin{array}{rccc} \phi: & W \times \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C} & \to & V \\ & ((A(t), B(t)), g, \lambda) & \mapsto & (gA(t)g^{-1} + \lambda I, gB(t)g^{-1}) \end{array}$$

$\dim V = \dim W + n^2 + 1 - \dim C(A_0) = n^2 + 2m + 3 + 2 + \dim \tilde{W} \leqslant n^2 + 2m + 5 + m^2 + 4m = 2n^2 + 2n - 3 = 2(n^2 + n) - 3 < 2(n^2 + n)$. □

3.2.1.3 Case (even n) $(n/2, n/2)$

Proposition 3.20. *For even n, the reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(n/2,n/2)}\}$$

is irreducible if and only if $X^{(1)}_{n/2}$ *is irreducible.*

In particular, it is irreducible for $n \leqslant 6$ (and with a later result, for $n \leqslant 8$).

Proof. It is equivalent if we check that for A_0 with the following form:

$$A_0 = \left(\begin{array}{c|c} 0_{n/2} & \mathbb{I}_{n/2} \\ \hline 0_{n/2} & 0_{n/2} \end{array}\right)$$

Now we compute the commutator with a generic block matrix $(B_{i,j})$ (with the same block dimensions as A_0):

$$[A_0, (B_{i,j})] = \left(\begin{array}{c|c} B_{2,1} & B_{2,2} - B_{1,1} \\ \hline 0_{n/2} & -B_{2,1} \end{array}\right)$$

Equating this to zero, leads to B_0 having the following form:

$$B_0 = \left(\begin{array}{c|c} B_{1,1} & B_{1,2} \\ \hline 0_{n/2} & B_{1,1} \end{array}\right)$$

Next, if we compute $[A_1, B_0] + [A_0, B_1]$ for generic matrices A_1, B_1, we get the following:

$$[A_1, B_0] + [A_0, B_1] = \left(\begin{array}{c|c} \begin{array}{c}[A_{1,1}, B_{1,1}] \\ -B_{1,2}A_{2,1} + B'_{2,1}\end{array} & \begin{array}{c}A_{1,1}B_{1,2} + [A_{1,2}, B_{1,1}] \\ -B_{1,2}A_{2,2} + B'_{2,2} - B'_{1,1}\end{array} \\ \hline [A_{2,1}, B_{1,1}] & A_{2,1}B_{1,2} + [A_{2,2}, B_{1,1}] - B'_{2,1} \end{array}\right)$$

Hence, it is irreducible iff the following is irreducible:

$$\begin{cases} [A_{2,1}, B_{1,1}] = 0 \\ [A_{1,1} + A_{2,2}, B_{1,1}] + [A_{2,1}, B_{1,2}] = 0 \end{cases}$$

Which is irreducible iff the following is irreducible:

$$\begin{cases} [A'_0, B'_0] = 0 \\ [A'_1, B'_0] + [A'_0, B'_1] = 0 \end{cases}$$

□

3.2.1.4 Case $r \mid n$, $(n/r, \overset{r}{\ldots}, n/r)$

It can be realised that the result in the previous section fits into a more general one:

Proposition 3.21. *For* $r \mid n$*, the reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X^{(1)}_n \mid A_0 = J_{(n/r,\ldots,n/r)}\}$$

is irreducible if and only if $X^{(r-1)}_{n/r}$ *is irreducible.*

Proof. We will work again with block matrices and the proof is really similar to the previous one. It is equivalent to consider A_0 with the following shape:

$$A_0 = \begin{pmatrix} 0_{n/r} & \mathbb{I}_{n/r} & 0_{n/r} & \cdots & 0_{n/r} \\ 0_{n/r} & 0_{n/r} & \mathbb{I}_{n/r} & \cdots & 0_{n/r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n/r} & 0_{n/r} & 0_{n/r} & \cdots & \mathbb{I}_{n/r} \\ 0_{n/r} & 0_{n/r} & 0_{n/r} & \cdots & 0_{n/r} \end{pmatrix}$$

Now, we compute the commutator with a generic block matrix $(B_{i,j})$ (with the same block structure as A_0):

$$[A_0, (B_{i,j})] = \begin{pmatrix} B_{2,1} & B_{2,2} - B_{1,1} & B_{2,3} - B_{1,2} & \cdots & B_{2,r} - B_{1,r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{r,1} & B_{r,2} - B_{r-1,1} & B_{r,3} - B_{r-1,2} & \cdots & B_{r,r} - B_{r-1,r-1} \\ 0_{n/r} & -B_{r,1} & -B_{r,2} & \cdots & -B_{r,r-1} \end{pmatrix}$$

From where we can deduce B_0 has the following form:

$$B_0 = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,r-1} & B_{1,r} \\ 0_{n/r} & B_{1,1} & \cdots & B_{1,r-2} & B_{1,r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n/r} & 0_{n/r} & \cdots & B_{1,1} & B_{1,2} \\ 0_{n/r} & 0_{n/r} & \cdots & 0_{n/r} & B_{1,1} \end{pmatrix}$$

Now we take a look at the other set of equations and we name them: $[A_1, B_0] + [A_0, B_1] = M = (M_{i,j})$.

It is easy to check that

$$M_{i,j} = \sum_{k=1}^{j} A_{i,k} B_{1,j-k+1} - \sum_{k=i}^{n} B_{1,k-i+1} A_{k,j}$$

Now we operate with the equations:

$$\sum_{l=1}^{t} M_{n-t+l,l} = \sum_{l=1}^{t}\left(\sum_{k=1}^{l} A_{n-t+l,k}B_{1,l-k+1} - \sum_{k=n-t+l}^{n} B_{1,k-n+t-l+1}A_{k,l}\right) =$$
$$= \sum_{l=1}^{t}\left(\sum_{k=1}^{l} A_{n-t+l,l-k+1}B_{1,k} - \sum_{k=1}^{t-l+1} B_{1,k}A_{k+n-t+l-1,l}\right) =$$
$$= \sum_{k=1}^{t}\left(\sum_{l=k}^{t} A_{n-t+l,l-k+1}B_{1,k} - B_{1,k}\sum_{l=1}^{t-k+1} A_{k+n-t+l-1,l}\right) =$$
$$= \sum_{k=1}^{t}\sum_{l=1}^{t-k+1}\left(A_{n-t+l+k-1,l}B_{1,k} - B_{1,k}A_{k+n-t+l-1,l}\right) =$$
$$= \sum_{k=1}^{t}\left[\left(\sum_{l=1}^{t-k+1} A_{n-t+l+k-1,l}\right), B_{1,k}\right]$$

where we have applied the corresponding index changes and the adequate formula for the swapping of the summations.

Now, renaming:

$$A_i := \sum_{l=1}^{i+1} A_{n-i+l-1,l} \qquad B_j := B_{1,j+1}$$

We obtain that the initial scheme is irreducible iff the one describe the following equations is:

$$\begin{gathered}
[A_0, B_0] = 0\\
[A_1, B_0] + [A_0, B_1] = 0\\
[A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0\\
\cdots\\
[A_{r-1}, B_0] + [A_{r-2}, B_1] + \cdots + [A_1, B_{r-2}] + [A_0, B_{r-1}] = 0
\end{gathered}$$

which are the same generators as the defining ideal for the r-th jet scheme for $n' = n/r$. □

3.2.1.5 Case $(n-1, 1)$

Proposition 3.22. *The reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(n-1,1)}\}$$

is irreducible for all $n > 1$

Proof. A_0 has the following form:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Now, let us compute its commutator with a generic matrix $(b_{i,j})$:

$$[A_0, (b_{i,j})] = \begin{pmatrix} b_{2,1} & b_{2,2} - b_{1,1} & b_{2,3} - b_{1,2} & \cdots & b_{2,n-1} - b_{1,n-2} & b_{2,n} \\ b_{3,1} & b_{3,2} - b_{2,1} & b_{3,3} - b_{2,2} & \cdots & b_{3,n-1} - b_{2,n-2} & b_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} - b_{n-2,1} & b_{n-1,3} - b_{n-2,2} & \cdots & b_{n-1,n-1} - b_{n-2,n-2} & b_{n-1,n} \\ 0 & -b_{n-1,1} & -b_{n-1,2} & \cdots & -b_{n-1,n-2} & 0 \\ 0 & -b_{n,1} & -b_{n,2} & \cdots & -b_{n,n-2} & 0 \end{pmatrix}$$

Which leads to:

$$B_0 = \left(\begin{array}{cccccc|c} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-2} & b_{1,n-1} & b_{1,n} \\ 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-3} & b_{1,n-2} & 0 \\ 0 & 0 & b_{1,1} & \cdots & b_{1,n-4} & b_{1,n-3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{1,1} & b_{1,2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{1,1} & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & b_{n,n-1} & b_{n,n} \end{array}\right) = \left(b_{i,j}^{(0)}\right)_{1 \leqslant i,j \leqslant n}$$

Which is equivalent to:

$$b_{i,j}^{(0)} = \begin{cases} 0 & \text{if } i = n, j < n-1 \text{ or } i < n, j < i \text{ or } 1 < i < n, j = n \\ b_{i,j} & \text{if } i = n, j \geqslant n-1 \text{ or } i = 1, j = n \\ b_{1,j-i+1} & \text{otherwise} \end{cases}$$

If we write now $M = [A_1, B_0] = (M_{i,j})$, we want to prove that $\forall k \in \{1, \ldots, n-2\}$:

$$\sum_{l=0}^{k-1} M_{n-k+l,l+1} = 0$$

Expanding each summand:

$$M_{n-k+l,l+1} = \sum_{h=1}^{l+1} a_{n-k+l,h} b_{1,l+2-h} - \sum_{h=n-k+l}^{n-1} a_{h,l+1} b_{1,h-n+k-l+1}$$

If we now put that into the summation that we want to prove that vanishes:

$$\begin{aligned}\sum_{l=0}^{k-1} M_{n-k+l,l+1} &= \sum_{l=0}^{k-1}\left(\sum_{h=1}^{l+1} a_{n-k+l,h}b_{1,l+2-h} - \sum_{h=n-k+l}^{n-1} a_{h,l+1}b_{1,h-n+k-l+1}\right)\\ &= \sum_{h=1}^{k}\left(\sum_{l=h-1}^{k-1} a_{n-k+l,l+2-h} - \sum_{l=0}^{k-h} a_{n-k+l+h-1,l+1}\right) b_{1,h} = 0\end{aligned}$$

Where we have used:

$$\begin{aligned}\sum_{h=1}^{l+1} a_{n-k+l,h}b_{1,l+2-h} &= \sum_{h=1}^{l+1} a_{n-k+l,l+2-h}b_{1,h}\\ \sum_{h=n-k+l}^{n-1} a_{h,l+1}b_{1,h-n+k-l+1} &= \sum_{h=1}^{k-l} a_{h+n-k+l-1,l+1}b_{1,h}\\ \sum_{l=0}^{k-1}\sum_{h=1}^{l+1} &= \sum_{h=1}^{k}\sum_{l=h-1}^{k-1}\\ \sum_{l=0}^{k-1}\sum_{h=1}^{k-l} &= \sum_{h=1}^{k}\sum_{l=0}^{k-h}\\ \sum_{l=h-1}^{k-1} a_{n-k+l,l+2-h} &= \sum_{l=0}^{k-h} a_{n-k+l+h-1,l+1}\end{aligned}$$

This implies that the scheme is irreducible iff the following ideal is prime.

$$I = \begin{pmatrix} f_{n,1} = a_{n,1}(b_{1,1} - b_{n,n}) - b_{n,n-1}a_{n-1,1} \\ f_{n-1,n} = a_{n-1,1}b_{1,n} + a_{n-1,n}(b_{n,n} - b_{1,1}) \\ f_{n,n} = a_{n,1}b_{1,n} - b_{n,n-1}a_{n-1,n} \end{pmatrix}$$

And I is prime in $\mathbb{C}[a_{n,1}, a_{n-1,1}, a_{n-1,n}, b_{1,n}, b_{n,n-1}, b_{1,1}, b_{n,n}]$ iff J is prime in $\mathbb{C}[a_{n,1}, a_{n-1,1}, a_{n-1,n}, b_{1,n}, b_{n,n-1}, \beta]$:

$$J = \begin{pmatrix} f_1 = a_{n,1}\beta + b_{n,n-1}a_{n-1,1} \\ f_2 = a_{n-1,1}b_{1,n} + a_{n-1,n}\beta \\ f_3 = a_{n,1}b_{1,n} - b_{n,n-1}a_{n-1,n} \end{pmatrix}$$

Now, if we rename $a_1 := a_{n,1}$, $a_2 := a_{n-1,n}$, $a_3 := a_{n-1,1}$, $b_1 := b_{n,n-1}$, $b_2 := b_{1,n}$, $b_3 := -\beta$. We get the following ideal:

$$J = \begin{pmatrix} -a_1b_3 + b_1a_3 \\ a_3b_2 - a_2b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Which are the same generators as the ones for the defining ideal of X_2, which is prime (Proposition 2.7). □

3.2.1.6 Case $(n-r,1,\overset{r}{\ldots},1)$

While studying the case $(n-2,1,1)$ we realised that that case and the case $(n-1,1)$ fit into a major result. Even though we have omitted the case $(n-2,1,1)$ for this reason, we have kept the case $(n-1,1)$ as a reference for the process of deduction of this proposition:

Proposition 3.23. *Given $r \geqslant 0$, the reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(n-r,1,\overset{r}{\ldots},1)}\}$$

is irreducible for all $n \geqslant r+2$ if and only if it is for some $n \geqslant r+2$.

Proof. A_0 has the following shape:

$$A_0 = \left(\begin{array}{ccccc|ccc}
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

Where the first block is of dimension $n-r$ and the second, of dimension r. Now we compute its commutator with a generic matrix $(b_{i,j})$:

$$[A_0, (b_{i,j})] = \left(\begin{array}{ccccc|ccc}
b_{2,1} & b_{2,2} - b_{1,1} & b_{2,3} - b_{1,2} & \cdots & b_{2,n-r} - b_{1,n-r-1} & b_{2,n-r+1} & \cdots & b_{2,n} \\
b_{3,1} & b_{3,2} - b_{2,1} & b_{3,3} - b_{2,2} & \cdots & b_{3,n-r} - b_{2,n-r-1} & b_{3,n-r+1} & \cdots & b_{3,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n-r,1} & b_{n-r,2} - b_{n-r-1,1} & b_{n-r,3} - b_{n-r-1,2} & \cdots & b_{n-r,n-r} - b_{n-r-1,n-r-1} & b_{n-r,n-r+1} & \cdots & b_{n-r,n} \\
0 & -b_{n-r,1} & -b_{n-r,2} & \cdots & -b_{n-r,n-r-1} & 0 & \cdots & 0 \\
\hline
0 & -b_{n-r+1,1} & -b_{n-r+1,2} & \cdots & -b_{n-r+1,n-r-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -b_{n,1} & -b_{n,2} & \cdots & -b_{n,n-r-1} & 0 & \cdots & 0
\end{array}\right)$$

Which tells us that B_0 has the following shape:

$$B_0 = \left(\begin{array}{cccccc|ccc}
b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-r-1} & b_{1,n-r} & b_{1,n-r+1} & \cdots & b_{1,n} \\
0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-r-2} & b_{1,n-r-1} & 0 & \cdots & 0 \\
0 & 0 & b_{1,1} & \cdots & b_{1,n-r-3} & b_{1,n-r-2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b_{1,1} & b_{1,2} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & b_{1,1} & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & b_{n-r+1,n-r} & b_{n-r+1,n-r+1} & \cdots & b_{n-r+1,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & b_{n,n-r} & b_{n,n-r+1} & \cdots & b_{n,n}
\end{array}\right) = \left(b_{i,j}^{(0)}\right)$$

Equivalently:

$$b^{(0)}_{i,j} = \begin{cases} 0 & \text{if } i > n-r, j < n-r \text{ or } i \leqslant n-r, j < i \text{ or } 1 < i < n-r, j > n-r \\ b_{i,j} & \text{if } i > n-r, j \geqslant n-r \text{ or } i = 1, j > n-r \\ b_{1,j-i+1} & \text{otherwise} \end{cases}$$

We want to prove that $\forall k \in \{1, \dots, n-r-1\}$:

$$\sum_{l=0}^{k-1} M_{n-k+l-r+1,l+1} = 0$$

$$M_{n-k+l-r+1,l+1} = \sum_{h=1}^{l+1} a_{n-k+l-r+1,h} b_{1,l+2-h} - \sum_{h=n-k+l-r+1}^{n-r} a_{h,l+1} b_{1,h-n+k-l+r}$$

$$\begin{aligned}\sum_{l=0}^{k-1} M_{n-k+l-r+1,l+1} &= \sum_{l=0}^{k-1} \left(\sum_{h=1}^{l+1} a_{n-k+l-r+1,h} b_{1,l+2-h} - \sum_{h=n-k+l-r+1}^{n-r} a_{h,l+1} b_{1,h-n+k-l+r} \right) \\ &= \sum_{h=1}^{k} \left(\sum_{l=h-1}^{k-1} a_{n-k+l-r+1,l+2-h} - \sum_{l=0}^{k-h} a_{n-k+l+h-r,l+1} \right) b_{1,h} = 0\end{aligned}$$

Where we have used:

$$\begin{aligned}\sum_{h=1}^{l+1} a_{n-k+l-r+1,h} b_{1,l+2-h} &= \sum_{h=1}^{l+1} a_{n-k+l-r+1,l+2-h} b_{1,h} \\ \sum_{h=n-k+l-r+1}^{n-1} a_{h,l+1} b_{1,h-n+k-l+r} &= \sum_{h=1}^{k-l} a_{h+n-k+l-r,l+1} b_{1,h} \\ \sum_{l=0}^{k-1} \sum_{h=1}^{l+1} &= \sum_{h=1}^{k} \sum_{l=h-1}^{k-1} \\ \sum_{l=0}^{k-1} \sum_{h=1}^{k-l} &= \sum_{h=1}^{k} \sum_{l=0}^{k-h} \\ \sum_{l=h-1}^{k-1} a_{n-k+l-r+1,l+2-h} &= \sum_{l=0}^{k-h} a_{n-k+l+h-r,l+1}\end{aligned}$$

Finally, we observe that, for $n \geqslant r+2$, the defining ideal is generated by:

$$\left\{a_{n-r+s,1}b_{1,n-r+t} + \sum_{h=1}^{r} a_{n-r+s,n-r+h}b_{n-r+h,n-r+t} - \sum_{h=0}^{r} b_{n-r+s,n-r+h}a_{n-r+h,n-r+t}\right\}_{\substack{1\leqslant s\leqslant r\\1\leqslant t\leqslant r}} \cup$$
$$\cup \left\{a_{n-r,1}b_{1,n-r+t} - \sum_{h=1}^{r} a_{n-r,n-r+h}b_{n-r+h,n-r+t} - b_{1,1}a_{n-r,n-r+t}\right\}_{1\leqslant t\leqslant r} \cup$$
$$\cup \left\{a_{n-r+s,1}b_{1,1} - \sum_{h=0}^{r} b_{n-r+s,n-r+h}a_{n-r+h,1}\right\}_{1\leqslant s\leqslant r}$$

Therefore, if it is prime for some $n \geqslant r+2$, it is prime for all $n \geqslant r+2$. □

Corollary 3.23.1. *The reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(n-2,1,1)}\}$$

is irreducible for all $n \geqslant 4$.

Proof. We have proved so for $n = 4$ and, therefore, for all $n \geqslant 4$. □

Corollary 3.23.2. *Given $r \geqslant 0$, the reduced scheme associated to*

$$Y = \{(A(t), B(t)) \in X_n^{(1)} \mid A_0 = J_{(n-r,1,\overset{r}{\dots},1)}\}$$

has the same codimension for all $n \geqslant r+2$.

Proof. It can easily be deduced from the proof of the proposition. □

Corollary 3.23.3. *Given $r \geqslant 0$, the reduced scheme associated to*

$$\widetilde{Y} = \{(A(t), B(t)) \in X_n^{(1)} \mid \exists g \in \mathrm{GL}_n(F), \lambda \in F \ s.t.\ gA_0g^{-1} + \lambda I = J_{(n-r,1,\overset{r}{\dots},1)}\}$$

has dimension at most $(m+1)(n^2+n)$.

Proof. This is a direct consequence of Corollary 3.23.2 and Proposition 3.15. □

3.2.1.7 Case odd n, $((n-1)/2, (n-1)/2, 1)$

Note: I think this result can be improved with not much effort.

Proposition 3.24. *Given an odd $n \geqslant 1$, the reduced scheme associated to*

$$S = \{(A(t), B(t)) \in X_n^{(1)} \mid \exists g \in \mathrm{GL}_n(F), \lambda \in F \ s.t.\ gA_0g^{-1} + \lambda F = J_{((n-1)/2,(n-1)/2,1)}\}$$

$\dim S \leqslant 2(n^2+n)$ *for $n = 5$.*

Proof. We work again with block matrices. This time, $((n-1)/2+1+(n-1)/2)\times((n-1)/2+1+(n-1)/2)$. A_0 takes the following shape (we name the closed subscheme defined by this as W):

$$A_0 = \left(\begin{array}{c|c|c} 0 & 0 & I \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

And the commutator with a generic matrix $(B_{i,j})$ is:

$$[A_0,(B_{i,j})] = \left(\begin{array}{c|c|c} B_{3,1} & B_{3,2} & B_{3,3}-B_{1,1} \\ \hline 0 & 0 & -B_{2,1} \\ \hline 0 & 0 & -B_{3,1} \end{array}\right)$$

So B_0 is of the following form:

$$B_0 = \left(\begin{array}{c|c|c} B_{1,1} & B_{1,2} & B_{1,3} \\ \hline 0 & B_{2,2} & B_{2,3} \\ \hline 0 & 0 & B_{1,1} \end{array}\right)$$

Which leads to the commutator of A_1 with B_0 to be:

$$[A_1,B_0] = \left(\begin{array}{c|c|c} \begin{matrix}-B_{1,2}A_{2,1}-B_{1,3}A_{3,1}\\ +[A_{1,1},B_{1,1}]\end{matrix} & * & * \\ \hline \begin{matrix}A_{2,1}(B_{1,1}-B_{2,2}I)\\ -B_{2,3}A_{3,1}\end{matrix} & A_{2,1}-B_{2,3}A_{3,2} & * \\ \hline [A_{3,1},B_{1,1}] & \begin{matrix}A_{3,1}B_{1,2}\\ -(B_{1,1}-B_{2,2}I)A_{3,2}\end{matrix} & \begin{matrix}A_{3,1}B_{1,3}+A_{3,2}B_{2,3}\\ +[A_{3,3},B_{1,1}]\end{matrix} \end{array}\right)$$

So the defining equations are:

$$\begin{cases} [A_{1,1}+A_{3,3},B_{1,1}]+[A_{3,1},B_{1,3}]+A_{3,2}B_{2,3}-B_{1,2}A_{2,1}=0 \\ [A_{3,1},B_{1,1}]=0 \\ A_{2,1}B_{1,1}-B_{2,3}A_{3,1}=0 \\ A_{3,1}B_{1,2}-B_{1,1}A_{3,2}=0 \end{cases}$$

Were we have used the fact that the trace of a commutator vanishes.

Renaming the variables we may obtain the following:

$$\begin{cases} [A_0,B_0]=0 \\ [A_1,B_0]+[A_0,B_1]=\beta\alpha'-\alpha\beta' \\ \alpha' B_0=\beta' A_0 \\ A_0\beta=B_0\alpha \end{cases}$$

Where $A_0,A_1,B_0,B_1\in\mathrm{Mat}_m(\mathbb{C})$ and $\alpha,\alpha'^T,\beta,\beta'^T\in\mathbb{A}^m_{\mathbb{C}}$, where $m=\frac{n-1}{2}$. We name the scheme defined so by $\widetilde{W}$.

For $n=5$, $m=2$, we will obtain a bound on the dimension of this scheme.

If we consider the first equation, we obtain that if we project onto (A_0, B_0) the dimension is at most $m^2 + m$. If we take a look at the second equation, since $m = 2$ we see that for any value of (A_0, A_1, B_0, B_1), the left hand side has always rank at most 2, so we will be able to find some $(\alpha, \alpha', \beta, \beta')$ fulfilling the equation. In this case, we see that given (α, α') both different that zero, if $\beta\alpha' - \alpha\beta' = \tilde{\beta}\alpha' - \alpha\tilde{\beta}'$, then there exists $c \in \mathbb{C}$ such that $\widetilde{beta} = \beta + c\alpha$ and $\widetilde{beta}' = \beta' + c\alpha'$. So in this case the dimensio. is at most $2m + 1$. If $\alpha' = 0$ and $\alpha \neq 0$, then $\beta' = \tilde{\beta}'$ and $\tilde{\beta} - \beta$ is free. So the dimension is at most $2m$. If both $\alpha = 0$ and $\alpha' = 0$, then $\tilde{\beta} - \beta$ and $\tilde{\beta}' - \beta'$ are free, so the dimension is at most $2m$. This implies that the scheme defined by the two first equations has dimension at most $3m^2 + 3m + 1$.

Consequently, $\dim W = (\dim C(A_0)) + (2m^2 + 2m + 1) + (1) + \dim \tilde{W}$. Where the first summand comes from B_1, the second from the variables of A_1 that do not appear in the generators of the ideal of $\tilde{W}$, and the third is the variable $B_{2,2}^{(}0)$.

Finally, if we name the subscheme of the jet scheme where A_0 is simlar to $\lambda I + \tilde{A}_0$ for some $\lambda \in \mathbb{C}$, where $\tilde{A}_0$ is the A_0 that we have defined previously.

$\dim V = n^2 - \dim C(A_0) + \dim W + 1 = n^2 + (2m^2 + 2m + 1) + 1 + \dim \tilde{W} + 1 \leqslant n^2 + 2m^2 + 2m + 3 + 3m^2 + 3m + 1 = n^2 + 5m^2 + 5m + 4 = 25 + 20 + 10 + 4 = 59 < 60 = 2(n^2 + n)$. ☐

3.2.2 Irreducibility of the 1st jet scheme for $n = 4$

Proposition 3.25. *The first jet scheme over X_4, $X_4^{(1)}$, is irreducible of dimension $2(4^2 + 4) = (m + 1)(n^2 + n)$.*

Proof. The possible Jordan Canonical Forms of a nilpotent 4×4 matrix are $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$, $\{1, 1, 1, 1\}$. The first one is non-derogatory and all the other ones have been checked to belong to $\overline{U}_n^{(m)}$ in the previous subsubsection. Therefore, the 1st jet scheme over X_4 is irreducible. ☐

3.2.3 Dimension of the 1st jet scheme for $n = 5$

Proposition 3.26. *The first jet scheme over X_5, $X_5^{(1)}$, has dimension $2(5^2 + 5) = (m + 1)(n^2 + n)$.*

Proof. Since the 1st jet scheme over X_4 is irreducible, we get that, to obtain the dimension of the 1st jet scheme over X_5 it is enough to check the dimension of the subscheme where A_0 has a single eigenvalue and is derogatory.

This set can be covered by a finite closed sets where A_0 has a single eigenvalue and has a concrete Jordan canonical form. In the previous section we have bounded the dimension of all this closed sets by $2(5^2 + 5) = (m + 1)(n^2 + n)$, except for the case $(3, 2)$, which we have checked with Macaulay2. The proposition follows. ☐

3.2.4 Attempt at improving the lower bound for reducibility and non-equidimensionality of the jet schemes

Remark. If we find a better bound on the dimension of 1st jet scheme, we immediately obtain a better bound for all jet schemes. This can be done through the closed subscheme where A_0 is a scalar matrix. Since:

$$\{(A(t), B(t)) \in X_n^{(m)} \mid A_0 = \lambda I\} \simeq X_n^{(m-1)} \times \mathrm{Mat}(n, \mathbb{C}) \times \mathbb{C},$$

so its dimension is $n^2 + 1 + \dim X_n^{(m-1)}$.

Furthermore, the bound obtained in [Sv09] for $\dim X_n^{(m)}$, name it $B_S(n, m)$, is such that $B_S(n, m + 1) = B_S(n, m) + n^2 + 1$. Thus if we obtain a better bound for the first jet scheme, we obtain better bounds for all jet schemes.

Our attempts have consisted basically on trying to apply the same reasoning done in the paper of reference to a certain kind of matrix. Specifically, matrices with the following block form of size $(x + (y + z)) \times (y + x + z)$:

$$A_0 = \left(\begin{array}{c|c|c} 0 & I & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

The one that is used in the proof by Sethuraman and Šivic is of this type with $y = a$, $x = 2a$, $z = b$.

In that case, the condition that had to be fulfilled was:

$$b^2 + (k + 1 - 2a)b + 3(k + 1)a - k - 2 \leqslant 0$$

For the case $y = a$, $x = 3a$, $z = b$, applying the same reasoning we get the following condition:

$$b^2 + (k + 1 - 2a)b + 4(k + 1)a - k - 2 \leqslant 0$$

Therefore, since some a, b satisfying it satisfy the one for $y = a$, $x = 2a$, $z = b$, we get that this bound is not better than the previous one.

When we take $x = y = a$ and $z = b$, applying the same kind of reasoning as in the paper, we cannot obtain any bound neither for reducibility nor for non-equidimensionality.

Taking this into consideration, we studied the case when $y \leqslant x \leqslant 2y$. And we obtained the condition:

$$z^2 + (k + 1 - s(x - y))z + (k + 1)(x + y) - k - 2 \leqslant 0$$

from which, for a given n and a given z, the best bound is obtained for $x = 2y$, which is the situation in [Sv09].

It is not easy to do a general study of all the possible A_0 of this form. In general, it can be studied for each $m \geqslant 0$ for the interval $my \leqslant x \leqslant (m + 1)y$ using block matrices of $(m + 4$ blocks$) \times (m + 4$ blocks$)$.

It is speculation, but given the situation, one might think that the best bound that can be obtained with this kind of matrices and reasoning, is indeed achieved for $x = 2y$ and, therefore, to improve it, new kinds of arguments should be used.

3.3 Log-canonical threshold

The (global) log-canonical threshold of the pair of a scheme and an ambient variety is of interest since it relates to the type of singularities of that scheme.

For that goal, there is an interesting result by Mustață:

Theorem 3.27. *([Mus02]) If X is a smooth variety and $Y \subset X$ is a closed sub-scheme, then the log canonical threshold of the pair (X, Y) is given by*

$$\operatorname{lct}(X, Y) = \dim X - \sup_{m \geqslant 0} \frac{\dim Y^{(m)}}{m+1}$$

where $Y^{(m)}$ represents the m-th jet scheme over Y.

Observation. In particular, we have that $\operatorname{lct}(X, Y) \leqslant \operatorname{codim}_X Y$, and, in our case, we are interested in $X = \operatorname{Mat}(n, \mathbb{C})^{\times 2}$ and $Y = X_n$, so:

$$\operatorname{lct}(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, X_n) \leqslant n^2 - n$$

Proposition 3.28. *For $n \leqslant 3$*

$$\operatorname{lct}(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, X_n) = n^2 - n = \operatorname{codim}_{\operatorname{Mat}(n,\mathbb{C})^{\times 2}} X_n$$

Proposition 3.29. *For $n \geqslant 30$*

$$\operatorname{lct}(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, X_n) < n^2 - n = \operatorname{codim}_{\operatorname{Mat}(n,\mathbb{C})^{\times 2}} X_n$$

That is, the pair $(\operatorname{Mat}(n, \mathbb{C})^{\times 2}, (\operatorname{codim}_{\operatorname{Mat}(n,\mathbb{C})^{\times 2}} X_n) \cdot X_n)$ is not log-canonical.

Remark. There is actually a better bound that comes from the remark to Corollary 2.28.1. However, the interest of the proposition is to point at the general fact that the *log-canonical threshold* and the jet schemes behave essentially different for small n and for big n, hinting at a possibly different behaviour of the singularities for small and big n.

3.3.1 Relation between log-canonical threshold and reducedness

One of our interests on the log-canonical threshold was that bounds on it might relate to reducedness. First, we thought that $\operatorname{lct} \geqslant \frac{\operatorname{codim}}{2}$ or $\operatorname{lct} > \frac{\operatorname{codim}}{2}$ might imply reducedness. But this turned out to be false, even with the assumption of generically reduced and even when lct = codim:

Observation. Given F an algebraically closed field and the ideal $\mathfrak{a} = (x_1, \ldots, x_{n-1}, x_n^2) \subset F[x_1, \ldots, x_n]$, which is not radical,

$$\operatorname{lct}(\mathbb{A}^n_F, \mathfrak{a}) = n - 1/2 = -\frac{1}{2} + \operatorname{codim}_{\mathbb{A}^n_F} \operatorname{Spec}(F[x_1, \ldots, x_n]/\mathfrak{a}).$$

Furhtermore, if we take $\mathfrak{a} = (x_1, x_2) \cdot (x_1, \ldots, x_4, x_5^2) \subset F[x_1, \ldots, x_5]$, it is not radical but is generically reduced, and

$$\operatorname{lct}(F^5, \mathfrak{a}) = 2 = \operatorname{codim}_{F^5} \operatorname{Spec}(F[x_1, \ldots, x_5]/\mathfrak{a}).$$

Remark. The second counterexample fits into a more general set of ideals, $(x_1,\ldots,x_r)(x_1,\ldots,x_{(n-1)},x_n^2)\subset k[x_1,\ldots,x_n]$. A general formula for the lct of these ideals can be obtained, since they are monomial ideals, but it does not seem to be relevant.

However, the study of these conditions, sprouted the following open question:

Open problem 3.30. *If lct=codim, does this imply Serre's condition R_k for some $k\geqslant 0$ or reducedness to some codimension?*

It seems to be the case at least for hypersurfaces and R_0, i.e. generic reducedness.

In the ring $\mathbb{C}[x_1,x_2]$, the scheme X associated to the ideal $(x_1x_2,x_1^3)=(x_1)\cdot(x_2,x_1^2)$ has $\mathrm{lct}(\mathbb{A}^2_{\mathbb{C}},X)=1=\mathrm{codim}_{\mathbb{A}^2_{\mathbb{C}}}X$ but it does not fulfil R_1 and is not reduced to codimension 1.

Therefore, conditions on the lct cannot give us any new information about reducedness, reducedness to a certain codimension or any Serre condition R_k.

Even though the *log-canonical threshold* did not lead to any useful condition for reducedness, there is still another open question that might be more promising:

Open problem 3.31. *Is there any condition on the jet schemes that might imply reducedness of the base scheme?*

3.3.2 Bernstein-Sato Polynomials

It is known that the Bernstein-Sato polynomial of polynomial of a complex scheme not necessary reduced or irreducible (introduced by Nero Budur, Mircea Mustață, and Morihiko Saito [BMS06]) is closely related to the multiplier ideals of the pair of that scheme on an ambient smooth variety, as well as to its singularities. For these reasons, it is of interest to study it.

We already have some information arising from the following theorem and the knowledge that we have on the log-canonical threshold.

Theorem 3.32. *([BMS06]) Given Z a complex scheme embedded in a smooth affine scheme X, naming $b_f(s)$ the Bernstein-Sato polynomial of the ideal defining Z in X, then the log-canonical threshold of (X,Z) coincides with the smallest root α'_f of $b_f(-s)$ (in particular,$\alpha'_f>0$), and any jumping coefficients of (X,Z) in $[\alpha'_f,\alpha'_f+1)$ are roots of $b_f(-s)$.*

Some of the interest, even though not applicable to our case, comes as well from a criterion for ration singularities for reduced complete intersections:

Theorem 3.33. *([BMS06]) Assume Z is a reduced complete intersection of codimension r in X, a smooth affine scheme. Then Z has at most rational singularities if and only if $\mathrm{lct}(X,Z)=r$ and its multiplicity as a root of $b_f(s)$ (the Bernstein-Sato polynomial of the defining ideal of Z in X) is 1.*

Since already the attempts to compute the lct have not been really fruitful, we have not invested much time in trying to obtain results on the Bernstein-Sato polynomials. Appart from understanding them and the theoretical background around them, we have computed some small cases with the computational algebra system Macaulay2 ([GS]):

Proposition 3.34. *The Bernstein-Sato polynomial for the defining ideal of X_2 is $b_{X_2}(s) = (s+2)(s+3)$, and $b_{X_2}(s - \mathrm{codim} X_2) = s(s+1)$.*

The Bernstein-Sato polynomial for the defining ideal of

$$X_{2,2} = \{(A_1, B_1, A_2, B_2) \in \mathrm{Mat}(2, \mathbb{C})^{\times 4} \mid [A_1, B_1] + [A_2, B_2] = 0\}$$

is $b_{X_{2,2}}(s) = (s+3)(s+4)$, and $b_{X_{2,2}}(s - \mathrm{codim} X_{2,2}) = s(s+1)$.

The Bernstein-Sato polynomial for the defining ideal of

$$X_{3,2} = \{(A_1, B_1, \ldots, A_3, B_3) \in \mathrm{Mat}(2, \mathbb{C})^{\times 6} \mid [A_1, B_1] + \cdots + [A_3, B_3] = 0\}$$

is $b_{X_{2,3}}(s) = (s+3)(s+6)$, and $b_{X_{2,3}}(s - \mathrm{codim} X_{2,3}) = s(s+3)$.

It is worth noticing the following lemmata:

Lemma 3.35. *Given a ring $R = \mathbb{C}[X_1, \ldots, X_n]$ and an ideal $I \subset R$, consider the ideal $J = IR[Y] \subset R[Y]$ in the polynomial ring. Now, if we name their Bernstein-Sato polynomials as $b_I(s)$ and $b_J(s)$ respectively, then $b_I(s) = b_J(s)$.*

Proof. This follows from the definition of Bernstein-Sato polynomial. □

Lemma 3.36. *Given a ring $R = \mathbb{C}[X_1, \ldots, X_n]$, an ideal $I \subset R$ and an automophism ϕ on R, consider the ideal $J = \phi(I)$. Now, if we name their Bernstein-Sato polynomials as $b_I(s)$ and $b_J(s)$ respectively, then $b_I(s) = b_J(s)$.*

Proof. This follows from the following theorem:

Theorem 3.37. *([BMS06]) Given a ring $R = \mathbb{C}[X_1, \ldots, X_n]$ and $Z = \mathrm{Spec}(R/I) \subset X = \mathbb{A}^n_{\mathbb{C}}$ and $b_I(s)$ the Bernstein-Sato polynomial of the ideal $I \subset R$, the polynomial $b_I(s - \mathrm{codim}_X Z)$ only depends on Z.*

Since the codimension of Z in X does not change under the automorphism, the lemma follows. □

These two propositions enable us to perform the computations of the Bernstein-Sato polynomials over smaller rings and, therefore, to reduce the computational load.

3.4 Quiver representations

Definition 3.38. A *quiver* $\mathcal{Q}$ is a quadruple (V, E, s, t) of two sets V (the set of vertices) and E (the set of edges), and two maps $s, t : E \to V$ which correspond to the source and target of each edge.

That is, a quiver is a directed graph where loops and multple arrows between two given vertices are allowed.

Definition 3.39. Given $\mathcal{Q} = (I, E, s, t)$ a quiver and F a field, the *representations* of $\mathcal{Q}$ of dimension vector $\alpha \in \mathbb{N}^{|I|}$ are the elements of:

$$\operatorname{Rep}(\mathcal{Q}, \alpha) = \bigoplus_{a \in E} \operatorname{Mat}(\alpha_{s(a)} \times \alpha_{t(a)}, F)$$

Observe that the group

$$G(\alpha) = (\prod_{i \in I} \operatorname{GL}_{\alpha_i}(F))/F*$$

acts by conjugation on $\operatorname{Rep}(\mathcal{Q}, \alpha)$. $F*$ represents the multiplicative group of F.

Definition 3.40. A *morphism* between two representations of a quiver $\mathcal{Q} = (I, E, s, t)$, $x \in \operatorname{Rep}(\mathcal{Q}, \alpha)$,$y \in \operatorname{Rep}(\mathcal{Q}, \beta)$ is an element

$$f \in \bigoplus_{i \in I} \operatorname{Mat}(\alpha_i \times \beta_i, F)$$

such that the following diagram commutes:

$$\begin{tikzcd} x_{s(a)} \arrow[r, "x_a"] \arrow[d, "f_{s(a)}"'] & x_{t(a)} \arrow[d, "f_{t(a)}"] \\ y_{s(a)} \arrow[r, "y_a"'] & y_{t(a)} \end{tikzcd}$$

Observation. Direct sums and exact sequences of representations have the natural definitions.

Definition 3.41. A *quiver subrepresentation* of a representation $x \in \operatorname{Rep}(\mathcal{Q}, \alpha)$ is a representation $y \in \operatorname{Rep}(\mathcal{Q}, \beta)$ together with an in injective quiver representation morphism into x.

Definition 3.42. The *quiver algebra of a quiver* $\mathcal{Q}$ over a field F is its path algebra with coefficients in F and it is usually denoted as $F\mathcal{Q}$.

We notice that it can naturally be associated a $F\mathcal{Q}$-module to each quiver representation of the quiver $\mathcal{Q}$ over F.

Definition 3.43. A quiver representation is said to be *simple* if its quiver algebra is a simple algebra or, analogously, if it has no proper subrepresentation apart from the trivial one.

A quiver representation is said to be *semi-simple* if it is a direct sum of simple representations.

Definition 3.44. When given a semi-simple representation $x \in \mathrm{Rep}(\mathcal{Q}, \alpha)$, it has a decomposition in simple representations

$$x \simeq x_1^{\oplus e_1} \oplus \cdots \oplus x_r^{\oplus e_r}$$

where $x_1, \ldots, x_r$ are non-isomorphic simple representations. If $\beta^{(i)}$ is the dimension vector of x_i, we say that x has *representation type*

$$\tau = (e_1, \beta^{(1)}; \ldots; e_r, \beta^{(r)})$$

Given a semi-simple representation $x \in \mathrm{Rep}(\mathcal{Q}, \alpha)$ of type $\tau = (e_1, \beta^{(1)}; \ldots; e_r, \beta^{(r)})$, we name $e = (e_1, \ldots, e_r)$ and $G(e)$ is a conjugate of $G(\alpha)_x$.

Definition 3.45. Given a quiver $\mathcal{Q}$ we construct its *double* $\overline{\mathcal{Q}}$ by adjoining a reverse arrow $a*$ for each arrow a in $\mathcal{Q}$.

Then there is a $G(\alpha)$-equivariant map

$$\begin{array}{rccc} \mu_{\mathcal{Q},\alpha}: & \mathrm{Rep}(\overline{\mathcal{Q}}, \alpha) & \to & \mathrm{End}(\alpha) = \bigoplus_{i \in I} \mathrm{Mat}(\alpha_i, F) \\ & x & \mapsto & \sum_{a \in E} [x_a, x_{a*}] \end{array}$$

where $[x_a, x_{a*}] = x_a x_{a*} - x_{a*} x_a$ represents the commutator.

We denote its zero locus by

$$X(\mathcal{Q}, \alpha) = \mu_{\mathcal{Q},\alpha}^{-1}(0)$$

and we consider it as a closed subscheme of $\mathrm{Rep}(\overline{\mathcal{Q}}, \alpha)$. It can be noticed that it does not depend on the orientation of the arrows of $\mathcal{Q}$ (see Lemma 2.2 in [CBH98]).

Furthermore, given the action by conjugation of $G(\alpha)$, the affine quotient

$$M(\mathcal{Q}, \alpha) = X(\mathcal{Q}, \alpha) /\!\!/ G(\alpha)$$

parametrises the isomorphism classes of semi-simple representations in $X(\mathcal{Q}, \alpha)$ which are the closed orbits of $G(\alpha)$ in $X(\mathcal{Q}, \alpha)$.

For a given quiver $\mathcal{Q} = (I, E, s, t)$ define for any dimension vector α, $p_{\mathcal{Q}} = 1 - \langle \alpha, \alpha \rangle_{\mathcal{Q}}$, where we define $\langle \alpha, \beta \rangle_{\mathcal{Q}} = \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in E} \alpha_{s(a)} \beta_{t(a)}$. Furthermore, define $(\alpha, \beta)_{\mathcal{Q}} = \langle \alpha, \beta \rangle_{\mathcal{Q}} + \langle \beta, \alpha \rangle_{\mathcal{Q}}$.

If L_g is the quiver with a single vertex and g loops, we notice that $X_n = X(L_1, n)$ and the schemes studied by Budur in [Bud18] are $X(L_g, n)$ for $g \geqslant 2$.

Given this, Theorem 2.2 about the irreducibility of X_n fits into a bigger result:

Theorem 3.46. *(see [CB03]) The underlying reduced scheme of*

$$X(\mathcal{Q}, \alpha)$$

is irreducible but not always a normal variety.

In the paper by Budur [Bud18], semi-simple representations and simple representations contained in the scheme of interest were the key to control the jet schemes and to obtain the results on reducedness and ration singularities. In this sense, we thought that it might lead to something of interest if we understood the simple and semi-simple representations of X_n.

The main results that he used are the theorems that we already announced in the introduction by Mustață (Theorem 1.13 and Theorem 1.14) and:

Theorem 3.47. *([CB01], cf. [Bud18]) If $X(\mathcal{Q},\alpha)$ contains a simple representation from* $\mathrm{Rep}(\overline{\mathcal{Q}},\alpha)$ *then:*

(a) $X(\mathcal{Q},\alpha)$ is a reduced and irreducible complete intersection of dimension $\alpha\cdot\alpha-1+2p_{\mathcal{Q}}(\alpha)$,

(b) the general element of $X(\mathcal{Q},\alpha)$ is a simple representation,

(c) the dimension of $M(\mathcal{Q},\alpha)$ is $2p_{\mathcal{Q}}(\alpha)$,

(d) $p_{\mathcal{Q}}(\alpha)>0$ if and only if $M(\mathcal{Q},\alpha)$ contains an open dense subset of isomorphism classes of simple representations,

(e) the simple representations in $M(\mathcal{Q},\alpha)$ are smooth points.

In conjuction with Luna's étale slices theory. Specifically, one of the results of that theory that he uses and might be useful in our case is the following theorem:

Theorem 3.48. *([Bud18]) Let $x\in X(\mathcal{Q},\alpha)$ be a semi-simple representation of type τ. Then there exists a morphism*

$$f:S\to X(\mathcal{Q}_\tau,e)$$

from an étale slice S for $X(\mathcal{Q},\alpha)$ at x, sending x to 0, such that f is equivariant via the canonical isomorphism $G(\alpha)_x\simeq G(e)$, and the restriction of f is strongly étale from a $G(\alpha)_x$-saturated open neighbourhood of x onto a $G(e)$-saturated open neighbourhood of 0.

Where, given a semi-simple representation $x\in X(\mathcal{Q},\alpha)$ of type $\tau=(e_1,\beta^{(1)};\ldots;e_r,\beta^{(r)})$, we define $\mathcal{Q}_\tau$ as the quiver with r vertices whose double $\overline{\mathcal{Q}}_\tau$ has $2p_{\mathcal{Q}}(e_i)$ loops at vertex i and $-(\beta^{(i)},\beta^{(j)})$ arrows from i to j if $i\neq j$.

It might be interesting to introduce here the étale slices theory because it might entail as in [Bud18] part of the solution to our problem. However, they are not applicable to our case (yet) since they are only defined for varieties. However, we think that there might be an analogue with similar behaviour for more general schemes. We introduce some of the definitions in the Appendix.

All these results and their application in the case of [Bud18] motivated us to compute the simple and semi-simple representations in X_n:

Proposition 3.49. *Given the identification $X_n=X(L_1,n)\subset\mathrm{Rep}(\overline{L}_1,n)$:*

(i) The only simple representations intersecting X_n occur for $n = 1$.

(ii) The semisimple representations in X_n are pairs of simultaneously diagonalisable matrices.

Proof. Since $p_{L_1}(n) = 1$, we have that $\alpha \cdot \alpha - 1 + 2p_{\mathcal{Q}}(\alpha) = n^2 + 1$, but $\dim X_n = n^2 + n$. This proves that there are only simple representations for $n = 1$ by Theorem 3.47.

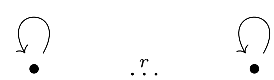
Another way of proving (i) is to observe that if two matrices commute over an algebraically closed field they are simultaneously triangularisable and, consequently, they have a common eigenvector, which produces a subrepresentation of dimension 1.

Therefore, since $(A_1 \oplus A_2, B_1 \oplus B_2)$ are a commuting pair of matrices iff (A_1, B_1) and (A_2, B_2) are, all semisimple representations are sums of representions of dimension 1. That is, all semisimple representations are the pairs of matrices simultaneously diagonalisable. □

Corollary 3.49.1. *All the semi-simple representation types that occur in X_n are of the form*

$$\tau = (e_1, 1; \ldots ; e_r, 1),$$

the quiver $\mathcal{Q}_\tau$ consists of r disconnected vertices with a single loop each:



and the associated scheme $X(\mathcal{Q}_\tau, e) = \prod_{i=1}^{r} X_{e_i}$, where X_{e_i} is the commuting scheme of dimension e_i.

Although we obtained all the semi-simple representations, it does not seem as simple as in Budur's case to apply these for our goal. In particular, because in his paper he made an strong use of the fact that the underlying variety was a complete intersection, which implied through previous results of Crawley-Boevey ([CB03], cf. [Bud18]) the reducedness and irreducibility of his scheme, which allowed for the application of Luna's étale slices machinery.

3.5 Some determinants of the Jacobian matrix

If $f_{i,j} = [A, B]_{i,j}$ is the (i, j)-th entry of the commutator of two generic matrices, we define $c^{r,s}_{i,j} = \frac{\partial f_{r,s}}{\partial a_{i,j}}$ and $C^{i,j}_{r,s} = \left(c^{k,l}_{t,u}\right)_{\substack{k\neq l,(k,l)\neq(i,j)\\ t\neq u,(t,u)\neq(r,s)}}$. Then:

Proposition 3.50.

$$\det C^{i,j}_{r,s} = \begin{cases} -\det C^{s,r}_{i,j} & \text{if } 2 \mid n \text{ or } (r-s) \equiv (j-i) \mod 2 \\ \det C^{s,r}_{i,j} & \text{if } 2 \nmid n \text{ and } (r-s) \not\equiv (j-i) \mod 2 \end{cases}$$

Proof. Notice that

$$c^{i,j}_{r,s} = -c^{s,r}_{j,i}$$

Which leads to the following identity:

$$\det C^{i,j}_{r,s} = -\mathrm{sgn}(\sigma_{r,s})\mathrm{sgn}(\sigma_{i,j}) \det C^{s,r}_{j,i} \tag{1}$$

where $\sigma_{i,j}$ corresponds to the permutation on $\{(r,s)|(r,s) \neq (i,j)\}$ that corresponds to sending the k-th element in in this set with the lexicographic order to the k-th element in the same set but considering the ordering where $(i,j) > (k,l)$ iff $j > l$ or $j = l$ and $i > k$.

This comes from the fact that applying the permutation $\sigma_{r,s}$ to the columns of $C^{i,j}_{r,s}$, afterwards applying to its rows the permutation $\sigma_{i,j}$ and, finally, transposing the matrix, gives the matrix $-C^{s,r}_{j,i}$.

Taking the set $\{(r,s)|(r,s) \neq (i,j)\}$ considering the ordering where $(i,j) > (k,l)$ iff $j > l$ or $j = l$ and $i > k$ and apply the order preserving bijection that sends (r,s) to (s,r), where the image lives in $\{(s,r)|(s,r) \neq (j,i)\}$ with lexicographic order. Now take the set $\{(r,s)|(r,s) \neq (i,j)\}$ with the lexicographical order and apply the order preserving bijection $(r,s) \mapsto (s,r)$ where the image set is $\{(s,r)|(s,r) \neq (j,i)\}$ with the ordering $(i,j) > (k,l)$ iff $j > l$ or $j = l$ and $i > k$. If we consider now $\sigma_{j,i}$, we notice that, through the order preserving bijections, it is equivalent to $\sigma^{-1}_{i,j}$, so it has the same sign as $\sigma_{i,j}$.

Thus it is enough to compute $\mathrm{sgn}(\sigma_{r,s})$ for $r > s$.

The way we will go about this is by computing the cycle decomposition.

First we take a look at the 2-cycles. These occur when we have (i,j) mapping to (j,i) and (j,i) mapping to (i,j). And (i,j) maps to (j,i) iff $(i,j) > (r,s)$ or $(i,j) < (s,r)$ in lexicographical order. We call that number, N. So we are going to count all the possibilities:

If $(i,j) < (s,r)$ and $(j,i) < (s,r)$, there are $(s-1)s$ values of (i,j) that fulfil this. If $(i,j) > (s,r)$ and $(j,i) > (s,r)$, there are $(n-r+1)(r-n)$ values of (i,j). If $(i,j) > (r,s)$ and $(j,i) < (s,r)$, there are $(n-r)(s-1)$ pairs. Finally, the reversed case ($(i,j) < (r,s)$ and $(j,i) > (s,r)$) has the same number of pairs. We have counted every pair twice, so in total:

$$2N = s(s-1) + (n-r+1)(n-r) + 2(s-1)(n-r)$$

Then, we take a distinguished element, (s,r).

First, we will name the permutation σ to simplify the notation. Then, we will denote σ^k to indicate $\prod_{i=1}^{k} \sigma$, that is, σ composed k times with itself.

Now, for $k \leqslant n-r$, $\sigma^{2k}((s,r)) = (s, r+k)$, since, for $0 \leqslant k \leqslant n-r$ $\sigma^{2k-1}((s,r)) = (r+k, s) > (r,s)$.

$$\sigma^{2(n-r)}((s,r)) = (s,n)$$

So, $\sigma^{2(n-r)+1}((s,r)) = (1, s+1)$ and, for all $t \geqslant 0$ such that $(t+1, s+1) \leqslant (s,r)$, we have that $\sigma^{2(n-r)+2t+1}((s,r)) = (t+1, s+1)$.

Now, if $r = s+1$:

$$\sigma^{2(n-s-1)+2s-1}((s,s+1)) = (s,s+1)$$

Otherwise, $\sigma^{2(n-r)+2s-1}((s,r)) = (s,s+1) < (s,r)$ and $\sigma^{2(n-r)+2s}((s,r)) = (s{+}1,s)$, and for all $t \geqslant 0$ such that $(s{+}1{+}t, s{+}t) \leqslant (r,s)$, $\sigma^{2(n-r)+2s+t}((s,r)) = (s+1+t, s+t)$.

This leads to:

$$\sigma^{2(n-r)+2s+r-s-1}((s,r)) = (r,r-1) > (r,s)$$

and

$$\sigma^{2(n-r)+2s+r-s}((s,r)) = (r-1,r) < (r,s)$$

Hence, for all $t \leqslant n-r$, $\sigma^{2(n-r)+2s+r-s+2t}((s,r)) = (r-1,r+t)$ and

$$\sigma^{2(n-r)+2s+r-s+2(n-r)}((s,r)) = (r-1,n).$$

This implies that $\sigma^{2(n-r)+2s+r-s+2(n-r)+1}((s,r)) = (1,r)$, and for all $t \leqslant s$, $\sigma^{2(n-r)+2s+r-s+2(n-r)+1+2(t-1)}((s,r)) = (t,r)$. Leading finally to:

$$\sigma^{2(n-r)+2s+r-s+2(n-r)+1+2(s-1)}((s,r)) = (s,r)$$

So the cardinality m of its orbit is:

$$m = \begin{cases} 2n-3 & \text{if } r = s+1 \\ 4n+3(s-r)-1 & \text{otherwise} \end{cases}$$

Now, observe the elements of the form $(k, s{+}i{+}1)$ for $s < k < s{+}i{+}1 < r{-}1$. We have that $\sigma((k,s+i+1)) = (s+i+2,k)$. Given that $(s,r) < (s+i+2,k) < (r,s)$, $\sigma^2((k,s+i+1)) = (k+1,s+i+2)$.

For k such that $s < k < r-1$ and $t \leqslant n-r+1$, $\sigma^{2t}((k,r-1)) = (k,r+t-1)$. This can be seen because, for $1 \leqslant t \leqslant n-r+1$, $\sigma^{2t-1}((k,r-1)) = (r+t-1,k) > (r,s)$. Following this, one arrives at:

$$\sigma^{2(n-r+1)}((k,r-1)) = (k,n)$$

Next, $\sigma^{2(n-r+1)+1}((k,r{-}1)) = (1,k{+}1)$ and, for $0 \leqslant t \leqslant s$, $\sigma^{2(n-r+1)+1+2t}((k,r-1)) = (t+1,k+1)$. Which implies:

$$\sigma^{2(n-r+1)+2s+1}((k,r-1)) = (s+1,k+1) \tag{2}$$

If we now take $(s+1,s+i+1)$ for some $i \geqslant 0$ such that $s+i+1 < r-1$, we get that $\sigma^{2t}((s+1,s+i+1)) = (s+1+t,s+i+1+t)$ for t such that $s+i+1+t \leqslant r-1$, thus:

$$\sigma^{2(r-s-i-2)}((s+1,s+i+1)) = (r-i-1,r-1)$$

We already know that, $t \leqslant n{-}r{+}1$, $\sigma^{2t}((r{-}i{-}1,r{-}1)) = (r{-}i{-}1,r{+}t{-}1)$, hence applying the equation 2:

$$\sigma^{2(r-s-i-2)+2(n-r+1)+2s+1}((s+1,s+i+1)) = (s+1,r-i)$$

If $(s+1, r-i) = (s+1, s+i+1)$ we have finished, $2 \nmid r-s$, $i = \frac{r-s-1}{2}$ and the cardinality of the orbit is $l = 2n - r + s$.

If not, then if we now consider $(s+1, s+1+j) = (s+1, r-i)$, we get that $j = r - s - i - 1$, which still fulfils $s+1 < s+j+1 < r-1$, so we can apply the same reasoning and we obtain:

$$\sigma^{(2(r-s-i-2)+2(n-r+1)+2s+1)+(2(r-s-j-2)+2(n-r+1)+2s+1)}((s+1, s+i+1)) = (s+1, s+1+i)$$

Therefore, the cardinality of its orbit is $2(2n - r + s) = 2l$, which does not deppend on i.

So given $l = 2(n-1) - r - s$, the cardinality of the orbit of $(s+1, s+1+i)$ for $s+1 < s+i+1 < r-1$ is:

$$\begin{cases} l & \text{if } 2 \nmid r-s \text{ and } i = \frac{r-s-1}{2} \geqslant 1 \\ 2l & \text{otherwise} \end{cases}$$

Now, we check that we have obtained the orbit for each element. We observe that the orbits corresponding to 2-cycles, to (s, r) and to $(s+1, s+1+i)$ for $i \leqslant \lfloor \frac{r-s-1}{2} \rfloor$ are disjoint and the cardinality of the union is:

$$\begin{cases} 2N + m = n^2 - n - 1 & \text{if } r - s = 1 \\ 2N + m + (r-s-2)l = n^2 - n - 1 & \text{otherwise} \end{cases}$$

So in both cases it is equal to the total number of elements. Therefore, we have the decomposition in disjoint cycles and we can compute its sign:

$$\operatorname{sgn}\sigma_{r,s} = \begin{cases} (-1)^N (-1)^{m+1} (-1)^{\frac{r-s-2}{2}(2l+1)} & \text{if } r-s-2 \geqslant 2, 2 \mid (r-s) \\ (-1)^N (-1)^{m+1} (-1)^{\frac{r-s-2-1}{2}(2l+1)} (-1)^{l+1} & \text{if } r-s-2 \geqslant 2, 2 \nmid (r-s) \\ (-1)^N (-1)^{m+1} & \text{if } r-s-2 < 0 \Leftrightarrow r-s = 1 \end{cases}$$

which working out through it leads to:

$$\operatorname{sgn}\sigma_{r,s} = \begin{cases} (-1)^{\frac{n^2-n}{2}-1} & \text{if } 2 \mid (r-s) \\ (-1)^{\frac{n^2-n}{2}-n+1} & \text{if } 2 \nmid (r-s) \end{cases}$$

This, together with equation 1 gets us to the desired result. □

Corollary 3.50.1.

$$D_t = \det \left(c_{i,j}^{r,s}\right)_{\substack{r \neq s \ or \ r=s=t \\ i \neq j \ or \ i=j=t}} = 0$$

Proof. By the properties of the determinant, we get the following expression:

$$D_t = \sum_{\substack{j=1 \\ j\neq t}}^{n} b_{t,j} \left(\sum_{\substack{s=1 \\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{t,j}} b_{s,t} \det C_{t,s}^{t,j} + \sum_{\substack{r=1 \\ r\neq t}}^{n} (-1)^{\sigma_{r,t}^{t,j}} (-b_{t,r}) \det C_{r,t}^{t,j} \right) +$$

$$+ \sum_{\substack{i=1 \\ i\neq t}}^{n} (-b_{i,t}) \left(\sum_{\substack{s=1 \\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{i,t}} b_{s,t} \det C_{t,s}^{i,t} + \sum_{\substack{r=1 \\ r\neq t}}^{n} (-1)^{\sigma_{r,t}^{i,t}} (-b_{t,r}) \det C_{r,t}^{i,t} \right)$$

where $\sigma_{r,s}^{i,j}$ is the sign that corresponds to the associated minor.

$$\sigma_{r,s}^{i,j} = l_{i,j} + l_{r,s} + 1$$

where (noticing that $i \neq j$ and $r \neq s$)

$$l_{r,s} = \begin{cases} s + (r-1)(n-1) & \text{if } s < r \\ s - 1 + (r-1)(n-1) & \text{if } s > r \end{cases}$$

One thing that can be immediately noticed is that $\sigma_{r,s}^{i,j} = \sigma_{i,j}^{r,s}$.
We will divide the proof into two cases:

• $\boldsymbol{2 \mid n}$:

Notice that $\det C_{j,i}^{i,j} = -\det C_{j,i}^{i,j} = 0$.

$$\begin{aligned} D_t = &\sum_{\substack{i=1 \\ i\neq t}}^{n} \sum_{\substack{r=1 \\ r\neq t}}^{n} \left((-1)^{\sigma_{t,i}^{t,r}+1} + (-1)^{\sigma_{r,t}^{i,t}} \right) b_{i,t} b_{t,r} \det C_{r,t}^{i,t} \\ &+ \sum_{\substack{j=1 \\ j\neq t}}^{n} \sum_{\substack{r=1 \\ r\notin\{t,j\}}}^{n} (-1)^{\sigma_{r,t}^{t,j}+1} b_{t,j} b_{t,r} \det C_{r,t}^{t,j} \\ &+ \sum_{\substack{i=1 \\ i\neq t}}^{n} \sum_{\substack{s=1 \\ s\notin\{t,i\}}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} b_{i,t} b_{s,t} \det C_{t,s}^{i,t} \end{aligned}$$

First we will take charge of the first summation:

$$\sigma_{t,i}^{t,r} = l_{t,r} + l_{t,i} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i + r + 1 & \text{if } r > t, i > t \text{ or } r < t, i < t \\ i + r & \text{if } r > t, i < t \text{ or } r < t, i > t \end{cases}$$

$$\sigma_{r,t}^{i,t} = l_{i,t} + l_{r,t} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i + r + 1 & \text{if } r > t, i > t \text{ or } r < t, i < t \\ i + r & \text{if } r > t, i < t \text{ or } r < t, i > t \end{cases}$$

Therefore, we have that $\sigma_{t,i}^{t,r} \equiv \sigma_{r,t}^{i,t} \mod 2$. Which implies that $(-1)^{\sigma_{t,i}^{t,r}+1} + (-1)^{\sigma_{r,t}^{i,t}} = 0$ for all (i,r) such that $t \notin \{i,r\}$.

$$
\begin{aligned}
D_t &= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\notin\{t,i\}}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{i,t}b_{s,t}\det C_{t,s}^{i,t}\right)\\
&= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\neq t}}^{i-1} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right)\\
&+ \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=i+1\\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right)\\
&= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\neq t}}^{i-1} (-1)^{\sigma_{t,s}^{i,t}} \left(b_{t,s}b_{t,i}\det C_{s,t}^{t,i} + b_{s,t}b_{i,t}\det C_{t,i}^{s,t}\right)\\
&+ \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=i+1\\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right)\\
&= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=i+1\\ s\neq t}}^{n} \left((-1)^{\sigma_{t,i}^{s,t}} + (-1)^{\sigma_{t,s}^{i,t}+1}\right) \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right)
\end{aligned}
$$

$$
\sigma_{t,i}^{s,t} = l_{s,t} + l_{t,i} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i+s+1 & \text{if } s>t, i>t \text{ or } s<t, i<t \\ i+s & \text{if } s>t, i<t \text{ or } s<t, i>t \end{cases}
$$

$$
\sigma_{t,s}^{i,t} = l_{i,t} + l_{t,s} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i+s+1 & \text{if } s>t, i>t \text{ or } s<t, i<t \\ i+s & \text{if } s>t, i<t \text{ or } s<t, i>t \end{cases}
$$

which leads to $\sigma_{t,i}^{s,t} \equiv \sigma_{t,s}^{i,t} \mod 2$ and, therefore: $D_t = 0$ for $2 \mid n$.

- $\mathbf{2} \nmid \boldsymbol{n}$:

$$
\begin{aligned}
D_t &= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{r=1\\ r\neq t}}^{n} \left((-1)^{\sigma_{t,i}^{t,r}+1+i-r} + (-1)^{\sigma_{r,t}^{i,t}}\right) b_{i,t}b_{t,r}\det C_{r,t}^{i,t}\\
&+ \sum_{\substack{j=1\\ j\neq t}}^{n} \sum_{\substack{r=1\\ r\neq t}}^{n} (-1)^{\sigma_{r,t}^{t,j}+1} b_{t,j}b_{t,r}\det C_{r,t}^{t,j}\\
&+ \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} b_{i,t}b_{s,t}\det C_{t,s}^{i,t}
\end{aligned}
$$

Now we take a look at the first summation and observe that

$$\sigma_{t,i}^{t,r} = l_{t,r} + l_{t,i} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i+r+1 & \text{if } r > t, i > t \text{ or } r < t, i < t \\ i+r & \text{if } r > t, i < t \text{ or } r < t, i > t \end{cases}$$

$$\sigma_{i,t}^{r,t} = l_{i,t} + l_{r,t} + 1 \equiv_{(\text{mod } 2)} \begin{cases} 1 & \text{if } r > t, i > t \text{ or } r < t, i < t \\ 0 & \text{if } r > t, i < t \text{ or } r < t, i > t \end{cases}$$

Therefore, $(-1)^{\sigma_{t,i}^{t,r}+1+i-r} + (-1)^{\sigma_{r,t}^{i,t}} = 0$ for all (i,r) such that $t \notin \{i,r\}$.

Now, using some simple changes of variables:

$$\begin{aligned} D_t &= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\notin\{t,i\}}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{i,t}b_{s,t}\det C_{t,s}^{i,t}\right) \\ &= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=1\\ s\neq t}}^{i-1} (-1)^{\sigma_{t,s}^{i,t}+1+i+s+1} \left(b_{t,s}b_{t,i}\det C_{s,t}^{t,i} + b_{s,t}b_{i,t}\det C_{t,i}^{s,t}\right) \\ &+ \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=i+1\\ s\neq t}}^{n} (-1)^{\sigma_{t,s}^{i,t}+1} \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right) \\ &= \sum_{\substack{i=1\\ i\neq t}}^{n} \sum_{\substack{s=i+1\\ s\neq t}}^{n} \left((-1)^{\sigma_{s,t}^{t,i}+i+s} + (-1)^{\sigma_{t,s}^{i,t}+1}\right) \left(b_{t,s}b_{t,i}\det C_{i,t}^{t,s} + b_{s,t}b_{i,t}\det C_{t,s}^{i,t}\right) \end{aligned}$$

$$\sigma_{s,t}^{t,i} = l_{s,t} + l_{t,i} + 1 \equiv_{(\text{mod } 2)} \begin{cases} i+t+1 & \text{if } s > t, i > t \text{ or } s < t, i < t \\ i+t & \text{if } s > t, i < t \text{ or } s < t, i > t \end{cases}$$

$$\sigma_{t,s}^{i,t} = l_{i,t} + l_{t,s} + 1 \equiv_{(\text{mod } 2)} \begin{cases} t+s+1 & \text{if } s > t, i > t \text{ or } s < t, i < t \\ t+s & \text{if } s > t, i < t \text{ or } s < t, i > t \end{cases}$$

Hence,

$$(-1)^{\sigma_{s,t}^{t,i}+i+s} + (-1)^{\sigma_{t,s}^{i,t}+1} = 0$$

so $D_t = 0$ for all n. □

Note on this result

Once we know the underlying scheme is irreducible of dimension n^2+n it is immediate that all the minors strictly bigger than n^2-n must vanish. In particular, all the ones that solely contain $b_{i,j}$'s or $a_{i,j}$'s must be identically zero in the polynomial ring. Therefore, this last corollary brings nothing new. However, we consider that the relation between the minors of order n^2-n-1 could be useful to prove or disprove regularity in codimension 2 and 3.

4 New results concerning related schemes

4.1 Variety of pairs of matrices with zero diagonal

The main result of this section is:

Theorem 4.1. *Given F a field, the scheme associated to*

$$X = \{(A,B) \in \mathrm{Mat}(n,F)^{\times 2} \mid \mathrm{diag}([A,B]) = 0\}$$

where $\mathrm{diag}(M)$ *applied to a matrix M is the projection onto the diagonal elements, (i.e., $M = (m_{i,j})_{1 \leqslant i,j \leqslant n} \mapsto \mathrm{diag}(M) = (m_{i,i})_{1 \leqslant i \leqslant n}$), is a complete intersection normal integral scheme over F.*

Hsu-Wen Young in his PhD dissertation [You10] posed this as an open problem. He proved it to be a reduced complete intersection for general n and checked it to be irreducible for $n \leqslant 3$. Our proof is more elementary than his, only making use of elementary results. As a matter of fact, we proved it before noticing he had studied it before.

Our motivation was mainly to attempt a proof of Conjecture 1.1 by using the Lemma 2.8 and Lemma 2.9 or, at least, find a more efficient algorithm or, at least, some partial results.

His motivation was mainly as a counterpart to the *diagonal commutator scheme*, which is the scheme:

$$D_n = \{(A,B) \in \mathrm{Mat}(n,F)^{\times 2} \mid [A,B] = \mathrm{diag}([A,B])\}$$

that is, the pairs of matrices whose commutator is diagonal.

This scheme has some interesting properties and might hold the clue to solve the conjecture.

Theorem 4.2. *[You10] The scheme defined by*

$$D_n = \{(A,B) \in \mathrm{Mat}(n,F)^{\times 2} \mid [A,B] = \mathrm{diag}([A,B])\}$$

is a complete intersection scheme of dimension $n^2 + n$ with two components, one of which is the commuting variety (this holds for any algebraically closed field F). Furthermore, for characteristic zero it is reduced.

Proof of Theorem 4.1 (1st part: integral scheme). The ideal that we want to prove to be prime is $J = \left(f_i := \sum_{\substack{k=1 \\ k \neq i}}^{n} (a_{i,k} b_{k,i} - a_{k,i} b_{i,k}) \right)_{2 \leqslant i \leqslant n}$.

Since the elements of the diagonal of a commutator of two matrices are linearly dependent, $n-1$ generators are enough. Now consider the following ideals:

$J_j = (f_i)_{1 < i \leqslant j}$

Since each of the f_i's is irreducible, we get that $J_2 = (f_2)$ is prime.

We will argue by induction. Assume that all J_j have been proven to be prime for all $j < l$.

We will first prove that, in this case, $(J_l : (a_{l,1})) = J_l$:

Suppose that $a_{l,1}f \in J_l$ for some f. That means that $a_{l,1}f = \sum_{j=2}^{l-1} g_j f_j + g_l f_l$. We want to prove that $f \in I_l$. For that, we can assume that $a_{l,1}$ does not appear in any of the g_i's (i.e. such that all the monomials that contain $a_{l,1}$ raised to a non-zero power have a zero coefficient). Now we have:

$a_{l,1}f = \sum_{j=2}^{l-1} g_j f_j + a_{l,1}g_l b_{1,l} - a_{1,l}g_l b_{l,1} + g_l(f_l - (a_{l,1}b_{1,l} - a_{1,l}b_{l,1}))$

Consequently, $f = g_l b_{1,l}$ and $\sum_{j=2}^{l-1} g_j f_j - a_{1,l}g_l b_{l,1} + g_l(f_l - (a_{l,1}b_{1,l} - a_{1,l}b_{l,1})) = 0$.

If we write $g_i = \sum_{r=0}^{m} h_{i,r}a_{1,l}^r$, we get the following equations for $0 \leqslant r \leqslant m+1$ (considering $h_{i,r} = 0$ for all $r > m$ or $r < 0$):

$\sum_{j=2}^{l-1} h_{j,r}f_j - h_{l,r-1}b_{l,1} + h_{l,r}(f_l - (a_{l,1}b_{1,l} - a_{1,l}b_{l,1})) = 0$ Which is equivalent to:

$h_{l,r-1}b_{l,1} = \sum_{j=2}^{l-1} h_{j,r}f_j + h_{l,r}(f_l - (a_{l,1}b_{1,l} - a_{1,l}b_{l,1}))$

For the case $r = m+1$ we get that $h_{l,m}b_{l,1} = 0$, and so $h_{l,m} = 0$. Consequently, for the case $r = m$ we get that $h_{l,m-1}b_{l,1} \in J_{l-1}$. It can easily be deduced that $h_{l,m-i}b_{l,1}^i \in J_{l-1}$. Therefore, $g_l b_{l,1}^m \in J_{l-1}$. Given that I_{l-1} is prime by the induction hypothesis and that $b_{l,1} \notin J_{l-1}$ since all the non-zero elements in that ideal have at least degree 2 and $\deg b_{l,1} = 1$, we get that $g_l \in J_{l-1}$, and thus, $f = a_{l,1}g_l \in I_{l-1} \subset J_l$. Hence $(J_l : (a_{l,1})) = J_l$.

Now if we remember the propositions that allowed us to prove the primality for the variety of commuting 2×2 matrices, we can consider the ideal over the ring $R_{a_{l,1}}$ (the localisation of the ring R at the ideal $(a_{l,1})$). In this ring, the ideal takes the following shape:

$J_l = J_{l-1} + (b_{1,l} + a_{l,1}^{-1}(-a_{1,l}b_{l,1} + \sum_{\substack{k=1 \\ k\neq i}}^{n} (a_{i,k}b_{k,i} - a_{k,i}b_{i,k})))$

So now we can consider the F-algebra automorphism:

$$\begin{array}{cccc} \phi & R_{a_{l,1}} & \to & R_{a_{l,1}} \\ & b_{1,l} & \mapsto & b_{1,l} - a_{l,1}^{-1}(-a_{1,l}b_{l,1} + \sum_{\substack{k=1 \\ k\neq i}}^{n} (a_{i,k}b_{k,i} - a_{k,i}b_{i,k})) \end{array}$$

So the image ideal is $J_{l-1} + (b_{1,l})$. Since $b_{1,l}$ does not appear in the generators of the ideal J_{l-1}, we can apply the other proposition that says that if $I \subset R$ is an ideal, it is prime or radical if and only if $IR[X] \subset R[X]$ is so. Therefore, J_l is prime if and only if J_{l-1} is prime in $R/(b_{1,l})$, which it is by thhe induction hypothesis and the proposition.

Consequently, we have proven that all the ideals J_l are prime and, in particular, the ideal $J_n = J$ is prime, which is the ideal corresponding to the matrices whose commutator has zero diagonal. □

Even though it was already proven by Young, we provide another proof of complete intersection.

Proof of Theorem 4.1 (2nd part: complete intersection). We will obtain this proof through computation of the dimension of the scheme. To do so, we will use the Jacobian ideal.

Using the same convention as we did for Theorem 3.1, we name the partial derivative of $f_{r,r}$ by $a_{i,j}$ (i.e. $\frac{\partial f_{r,r}}{\partial a_{i,j}}$) as $c_{i,j}^{r,r}$ and of $f_{r,r}$ by $b_{i,j}$ (i.e. $\frac{\partial f_{r,s}}{\partial b_{i,j}}$) as $d_{i,j}^{r,r}$.

Then we have:

$$c_{i,j}^{r,r} = \begin{cases} b_{j,r} & \text{if } i = r, j \neq r \\ -b_{r,i} & \text{if } i \neq r, j = r \\ 0 & otherwise \end{cases}$$

$$d_{i,j}^{r,r} = \begin{cases} -a_{j,r} & \text{if } i = r, j \neq r \\ a_{r,i} & \text{if } i \neq r, j = r \\ 0 & otherwise \end{cases}$$

If we take submatrix $(c_{i,1}^{rr})_{1<r,i\leqslant n}$ of the Jacobian matrix, taking the same order in r and i we obtain a diagonal matrix with the monomials $b_{1,r}$ along the diagonal. Its determinant is $\prod_{r=2}^{n} b_{1,r} \notin J$, where J is the defining ideal of the scheme. Therefore, the codimension of the scheme is $n-1$, as we wanted to prove. □

Proof of Theorem 4.1 (3rd part: normal). Since it is a complete intersection it is Cohen-Macaulay and, therefore, if we prove that the singular locus has codimension at least 2, by Serre's criteria, it will imply normality.

We will proof smoothness to codimension 1 through the Jacobian of the ideal. We already saw the Jacobian matrix associated to the ideal in the previous part of the proof. Now take the following two matrices: $(c_{i,1}^{rr})_{1<r,i\leqslant n} = \operatorname{diag}(b_{1,r})_{1<r\leqslant n}$, $(c_{i,n}^{rr})_{1\leqslant r,i<n} = \operatorname{diag}(b_{n,r})_{1\leqslant r<n}$.

The determinant of the first is $\prod_{r=2}^{n} b_{1,r}$ and of the second $\prod_{r=1}^{n-1} b_{n,r}$. we will try to prove that the reduced associated scheme to $J + (\prod_{r=2}^{n} b_{1,r}, \prod_{r=1}^{n-1} b_{n,r})$ has codimension 2 in VD_n.

We will denote the reduced scheme associated to the ideal I as $V(I)$.

First of all, we will decompose $V(J + (\prod_{r=2}^{n} b_{1,r}))$ into irreducible varieties. we claim that

$$V(J + (\prod_{r=2}^{n} b_{1,r})) = \bigcup_{r=2}^{n} V(J + (b_{1,r}))$$

and the reduced schemes $V(J+(b_{1,r}))$ are irreducible (we will actually prove that the ideal $J+(b_{1,r})$ is prime).

The method of proof of the primality of $J+(b_{1,r})$ is the same as the one we used for the primality of J.

First we consider the following morphism:

$$\phi : F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant n}] \longrightarrow F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant n}\backslash\{b_{1,r}\}] \simeq F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant n}]/(b_{1,r})$$

Since ϕ is surjective and $\ker(\phi) \subset J+(b_{1,r})$, if we prove that the image of $J+(b_{1,r})$, $\overline{J} \subset F[\{a_{i,j}, b_{i,j}\}_{1\leqslant i,j\leqslant n}\backslash\{b_{1,r}\}]$ is prime, we have that $J+(b_{1,r})$ is prime.

To prove the primality of $\overline{J}$ we will use the same method that we used for J.

$$\begin{aligned}\overline{J} = &\left(g_{i,i} := \sum_{\substack{k=1\\k\neq i}}^{n} (a_{i,k}b_{k,i} - a_{k,i}b_{i,k}) \right)_{\substack{1\leqslant i\leqslant n-1\\ i\notin\{1,r\}}} \\ &+ \left(g_{1,1} := a_{1,r}b_{r,1} + \sum_{\substack{k=1\\k\notin\{1,r\}}}^{n} (a_{1,k}b_{k,1} - a_{k,1}b_{1,k}) \right) \\ &+ \left(g_{r,r} := -a_{1,r}b_{r,1} + \sum_{\substack{k=1\\k\notin\{1,r\}}}^{n} (a_{r,k}b_{k,r} - a_{k,r}b_{r,k}) \right)\end{aligned}$$

Now, we consider the ideals $\overline{J}_j = (g_{i,i})_{1\leqslant i\leqslant j}$ for $j \leqslant n-1$. Notice that $\overline{J}_{n-1} = \overline{J}$.

It is immediate that, if $n \geqslant 2$, then $J_1 = (g_1)$ is prime, since g_1 is irreducible for all $n \geqslant 2$.

Next, assume that J_j is prime for all $j < k$.

Since $b_{n,k}$ only appears in g_k, if we prove that $(\overline{J}_k : (b_{n,k})) = \overline{J}_k$, we will have that J_k is prime if and only if J_{k-1} is, which is true by the induction hypothesis.

The proof of $(\overline{J}_k : (b_{n,k})) = \overline{J}_k$ is identical as the one we used in the proof of primality of J, so we think it does not have any interest to reproduce it again.

Furthermore, $\dim V(J+(b_{1,r})) = \dim J - 1$, that is, $V(J+(b_{1,r})) \simeq V(\overline{J})$ is a complete intersection. We will prove this through the Jacobian matrix of $\overline{J}$.

We name the partial derivative of $g_{r,r}$ by $a_{i,j}$ (i.e. $\frac{\partial g_{r,r}}{\partial a_{i,j}}$) as $\overline{c}_{i,j}^{r,r}$ and of $g_{r,r}$ by $b_{i,j}$ (i.e. $\frac{\partial f_{r,s}}{\partial b_{i,j}}$) as $\overline{d}_{i,j}^{r,r}$.

If one takes $(\overline{c}_{i,n}^{rr})_{1\leqslant r,i<n} = \mathrm{diag}(b_{n,r})_{1\leqslant r<n}$, we see that the determinant is $\prod_{r=1}^{n-1} b_{n,r} \notin \sqrt{\overline{J}} = \overline{J}$. This implies that $\dim V(J+(b_{1,r})) = \dim J - 1$ and, furthermore

$$\dim\left(V(J+(b_{1,r}))\cap V\left(J+\left(\prod_{r=2}^{n} b_{1,r},\prod_{r=1}^{n-1} b_{n,r}\right)\right)\right)=\dim V(J)-2$$

Thus

$$\begin{aligned}\dim\bigcup_{r=2}^{n}&\left(V(J+(b_{1,r}))\cap V\left(J+\left(\prod_{r=2}^{n} b_{1,r},\prod_{r=1}^{n-1} b_{n,r}\right)\right)\right)\\&=\dim\left(\left(\bigcup_{r=2}^{n}V(J+(b_{1,r}))\right)\cap V\left(J+\left(\prod_{r=2}^{n} b_{1,r},\prod_{r=1}^{n-1} b_{n,r}\right)\right)\right)\\&\qquad=\dim V\left(J+\left(\prod_{r=2}^{n} b_{1,r},\prod_{r=1}^{n-1} b_{n,r}\right)\right)=\dim V(J)-2\end{aligned}$$

So we have proven smoothness to codimension 1, which, together with complete intersection and Serre's criteria, implies normality. □

4.2 Other schemes of pairs of matrices with vanishing of some entries of their commutator

In an analogous manner as we proved the previous result, there are some other schemes that can be proven to be reduced irreducible complete intersections.

Definition 4.3. Given $\mathcal{I}\subset\{(i,j)\}_{1\leqslant i,j\leqslant n}$, the scheme $X_{\mathcal{I}}$ associated to the following set:

$$\{(A,B)\in \mathrm{Mat}(n,F)\mid \forall (i,j)\in\mathcal{I},\ \ [A,B]_{i,j}=0\}$$

where $[A,B]_{i,j}$ is the (i,j)-th entry of the commutator $[A,B]$.

Theorem 4.4. *For any $\mathcal{J}\subset\{(i,j)\}_{1\leqslant i,j\leqslant n}$ such that $\mathcal{J}=\{(i,i)\}_{1\leqslant i\leqslant n}\cup\{(i,j_i)\}_{1\leqslant i\leqslant n}$ or $\mathcal{J}=\{(i,i)\}_{1\leqslant i\leqslant n}\cup\{(i_j,j)\}_{1\leqslant j\leqslant n}$ for a specific choice of $j_i\neq i$ or $i_j\neq j$, and for any $\mathcal{I}\subset\mathcal{J}$, then $X_{\mathcal{I}}$ is a reduced irreducible complete intersection scheme for any field F.*

Proof. The proof follows the same structure as the one of Theorem 4.1. It consists on finding a pair of monomials that only appear in one polynomial, see that we can localise at one of the four appearing variables and apply induction. □

Based on these cases and the work and computations that we have done surrounding these schemes, we conjecture the following:

Conjecture 4.5. *For any $\mathcal{I}\subset\{(i,j)\}_{1\leqslant i,j\leqslant n}$ and any field F, $X_{\mathcal{I}}$ is reduced.*

This can be checked to be true for $n \leqslant 2$.

However, not all of them are irreducible. For example, when $\mathcal{J} = \{(i, j)\}_{i \neq j}$ it is the diagonal commutator variety, which has two components (see [You10]). For specific cases we have checked computationally that there are others which are not irreducible.

5 Final remarks

After the results that we have obtained, some questions have been left open and might be fruitful if pursued in future research. Among them:

- Try to use or use the method of Hreindóttir with the slight improvement that we have introduced with a more powerful computer (the last results are from 2006, so proving or rejecting the conjecture for $n = 5$ might be in the reach of computation).
- Try to obtain bounds on the *log-canonical threshold* and more information on the jet schemes.
- Try to prove (or disprove) R_2 and R_3.
- Try to find conditions on the jet schemes for reducedness of the base scheme.
- Try to generalise the étale slices machinery to be able to apply it to our case in order to attempt to prove the conjecture.

6 Appendix

Most of the definitions and properties to be found in standard textbooks such as [Har06],[AM94], [BH98], [Sha13]. We have also used [Sta19]. When other sources are used, they are referenced specifically.

6.1 General algebra definitions and properties

We will assume all rings to be commutative and unital.

6.1.1 Serre criteria

In this subsection, we introduce Serre's criteria and the definitions needed to understand them. We have extensively used them in the main text as a way to relate diferent parts of the conjecture.

Definition 6.1. Let R be a ring and M and R-module. A sequence of elements $f_1, \ldots, f_r \in R$ is called an *M-regular sequence* if:

1. f_i is a non-zerodivisor in $M/(f_1, \ldots, f_{i-1})$ for each $i \in \{1, \ldots, r\}$, and

2. the module $M/M(f_1,\dots,f_r) \neq 0$.

If $I \subset R$ is an ideal and $f_1,\dots,f_r \in I$ is an M-regular sequence, we call it an *M-regular sequence in I*. If $M = R$ and $f_1,\dots,f_r$ is an M-regular sequence, we call it simply a *regular sequence (in I)*.

Definition 6.2. Let R be a ring and $I \subset R$ an ideal. Given M a finite R-module, the *I-depth* of M, denoted by $\mathrm{depth}_I M$, is defined as follows:

1. if $IM \neq M$, then $\mathrm{depth}_I M$ is the supremum in $\{0, 1, \dots, \infty\}$ of the lengths of M-regular sequences in I,

2. if $IM = M$, then $\mathrm{depth}_I M = \infty$.

If $(R, \mathfrak{m})$ is a local ring, we call $\mathrm{depth}_{\mathfrak{m}} M = \mathrm{depth} M$ the *depth* of M.

Definition 6.3. Let R be a ring and $I \subset R$ be an ideal. The *height* of I is the Krull dimension of R_I, the localisation of R at I.

Definition 6.4. Given A a Noetherian commutative ring and an integer $k \geqslant 0$, A is said to fulfil Serre's condition

- R_k if $A_{\mathfrak{p}}$ is a regular local ring for any prime ideal $\mathfrak{p} \subset A$ such that $\mathrm{height}(\mathfrak{p}) \leqslant k$,

- S_k if $\mathrm{depth} A_{\mathfrak{p}} \geqslant \inf\{k, \mathrm{height}(\mathfrak{p})\}$ for any prime $\mathfrak{p}$.

Theorem 6.5 (Theorem (Serre's criteria))**.** *Given A a Noetherian commutative ring, then*

- *A is reduced iff A satisfies R_0 and S_1,*

- *A is normal iff A satisfies R_1 and S_2,*

- *A is Cohen-Macauly iff A satisfies S_k for all $k \geqslant 0$*

6.1.2 Some types of rings

Different kinds of rings correspond to different kinds of singularities. In this sense, regular rings correspond to smooth schemes and complete intersection, Gorenstein and Cohen-Macaulay rings correspond, in some specific sense, to mild singularities. Specifically, Cohen-Macaulay and Gorenstein rings arise because of their nice duality theories. In what follows we present the definitions of the objects mentioned in the text.

Definition 6.6. Let R be a Noetherian local ring, we say that R is a *local Cohen-Macaulay ring* if $\mathrm{depth} R = \dim R$, where $\dim R$ is its Krull dimension.

For an arbitrary ring R, we say that it is a *Cohen-Macaulay ring* if it is Noetherian and its localisation to every prime ideal is a local Cohen-Macaulay ring.

Definition 6.7. ([BH98]) Let R be a Noetherian local ring, we say that R is a *local Gorenstein ring* if it has finite injective dimension over itself.

For an arbitrary ring R, we say that it is a *Gorenstein ring* if it is Noetherian and its localisation to every prime ideal is a local Gorenstein ring.

Definition 6.8. Let F be a field and S a finite type F-algebra.

1. We say that S is a *global complete intersection over* F if there exists a presentation $S = F[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim S = n - c$.

2. We say that S is a *local complete intersection over* F if there exists a covering of $\mathrm{Spec}(S) = \bigcup D(g_i)$ by principal open sets, such that each of the rings S_{g_i} is a global complete intersection over F.

Definition 6.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring, we say that R is a *regular local ring* if the minimal number of generators of $\mathfrak{m}$ is equal to the Krull dimension of R.

Let R be an arbitrary ring, we say that R is a *regular ring* if it is Noetherian and the localisation at each prime ideal is a regular local ring.

Proposition 6.10. *Local regular ring $\Rightarrow$ Local complete intersection $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay*

Definition 6.11. Let R be a ring, we say that R is a *reduced ring* if $\mathrm{nilrad} R = 0$, where nilrad is the nilradical of R.

Definition 6.12. Let R be a ring, we say that it is a *normal domain* if it is an integral domain and it is integrally closed in its field of fractions.

The main result used to prove Cohen-Macaulayness in [Hre94] makes use of the following definitions:

Definition 6.13. ([Sha13]) Let F be a field and $I \subset R = F[x_1, \ldots, x_n]$ a homogeneous ideal of the polynomial ring. Name $I^{(k)}$ the set of forms of degree k in I. Name $R^{(k)}$ the set of forms of degree k in R. Set $a_k(R/I) = \dim R^{(k)}/I^{(k)}$. Then, there exists a polynomial $P_{R/I}(T) \in \mathbb{Q}[T]$ such that $P_{R/I}(k) = a_k(R/I)$ for sufficiently large k. This polynomial is unique and we call it the *Hilbert polynomial* of the ring R/I.

If $e = \deg P_{R/I}(T)$, then there is an integer d such that the leading term of $P_{R/I}(T)$ is $\frac{d}{e!}T^e$. We call d the *multiplicity* or *degree* of R/I.

Definition 6.14. Given $(R, \mathfrak{m})$ a local ring of Krull dimension m, given a set of elements $\{x_1, \ldots, x_d\}$, we say that it is a *system of parameters* if any of the following equivalent conditions is fulfilled:

1. $\mathfrak{m}$ is a minimal prime over $(x_1, \ldots, x_d)$,

2. $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$,

3. $\exists r \geqslant 1$ such that $\mathfrak{m}^r \subset (x_1, \ldots, x_d)$,

4. $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary.

6.1.3 Lie algebras

Our problem can be seen as a part of a broader conjecture on some schemes over a certain type of Lie algebras. In that sense, the following definitions are meant to be a reference for the types of Lie algrebras mentioned in the main text. Our studied case corresponds to $\mathfrak{g} = \mathfrak{gl}_n$, which is reductive but not semisimple or abelian.

Definition 6.15. Let $\mathfrak{g}$ be a Lie algebra, we say that it is *simple* if it is a non-abelian Lie algebra whose single proper ideal is (0).

Let $\mathfrak{g}$ be a Lie algebra, we say that it is *semisimple* if it is a direct sum of simple Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra, we say that it is *abelina* if the Lie bracket vanishes for all pairs of elements (that is, $\forall x, y \in \mathfrak{g}$, $[x, y] = 0$).

Let $\mathfrak{g}$ be a Lie algebra, we say that it is *reductive* if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

6.2 Algebraic Geometry

It is maybe remarkable that during the main text we work with affine schemes of finite type over a field F, that is, if $I \subset F[x_1, \ldots, x_n]$ is an ideal, we work with schemes of the type $\mathrm{Spec}(F[x_1, \ldots, x_n]/I)$. In this case, reducedness is equivalent to the ideal I being radical and irreducibility to the radical of I being prime.

6.2.1 General scheme properties

In what follows, there are the scheme counterparts of many of the properties that we stated over rings in the previous section.

Definition 6.16. Let X be a scheme, we say that it is *Cohen-Macaulay* (resp. *Gorenstein*, resp. *regular*) if it is locally Noetherian and for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay (resp. *Gorenstein*, resp. *regular*).

Definition 6.17. Let X be a scheme of finite type over a field F, we say that it is a *local complete intersection* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that $\mathcal{O}_X(U)$ is a local complete intersection over F.

Let $X = \mathrm{Spec}(A)$ be an affine scheme of finite type over a field F, we say that it is a *(global) complete intersection* if A is a global complete intersection over F.

Definition 6.18. Let X be a scheme, we say that it is *reduced* (resp. *normal*) if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced (resp. a normal domain).

Lemma 6.19. *For every scheme X there is an associated reduced scheme X^{red} with the same topological space.*

Definition 6.20. Let X be a scheme, we say that it is *generically reduced* if for every $x \in X$, there exists an open neighbourhood $U \ni x$ such that $\mathcal{O}_X(U)$ fulfils Serre's condition R_0.

Given an integer $d \geqslant 0$, we say that X is *reduced to codimension d* if all the components of codimension at most d are reduced.

Lemma 6.21. *Generically reduced is equivalent to reduced to codimension 0.*

6.2.2 Singularities and invariants

When studying singularities one is usually interested in studying mild types of singularities such as rational singularities and one usually studies certain invariants. In our case, we got interested in three tightly related invariants: the jet schemes, the log-canonical threshold and the Bernstein-Sato polynomials. In what follows we introduce the definitions of the properties and objects that are used or mentioned in the main text in what refers to singularities and their study.

Definition 6.22. Let X be a scheme, we say that it has *rational singularities* if it is normal, of finite type over a field of characteristic 0 and there exists a regular scheme Y and a proper birational map

$$f : Y \to X$$

such that

$$R^i f_* \mathcal{O}_Y = 0 \quad \forall i > 0$$

where $R^i f_*$ stands for the i-th higher direct image of f_*.

Proposition 6.23. *Rational singularities implies Cohen-Macaulayness.*

Definition 6.24. Let F be a field and X an F-scheme. For $m \geqslant 0$, set theoretically, the *m-th jet scheme* over X is

$$X^{(m)} = \mathrm{Hom}(\mathrm{Spec}(F[t]/(t^{m+1})), X)$$

and the space of arcs,

$$X^{(\infty)} = \mathrm{Hom}(\mathrm{Spec}(F[[t]]), X)$$

There is a natural structure sheaf that makes it into a scheme.

Definition 6.25. ([BMS06]) Let Z be a complex algebraic scheme embedded in a smooth affine variety X. Let $f_1, \ldots, f_r$ be non-zero generators of the ideal of Z. Let $\mathcal{D}_X$ be the sheaf of linear differential operators on X. It acts naturally on $\mathcal{O}_X[\prod_i f_i^{-1}, s_1, \ldots, s_r] \prod_i f_i^{s_i}$, where the s_i are independent variables. Let us define a $\mathcal{D}_X$-linear action t_j on it by $t_j(s_i) = s_i + 1$ if $i = j$ and $t_j(s_i) = s_i$ otherwise. In particular, $t_j \prod_i f_i^{s_i} = f_j \prod_i f_i^{s_i}$, and the action of t_j is bijective. Let $s_{i,j} = s_i t_i^{-1} t_j$ and $s = \sum_i s_i$.

The *Bernstein-Sato polynomial* (also called the *b-function*) $b_f(s)$ of $f := (f_1, \ldots, f_r)$ is defined to be monic polynomial of the lowest degree in s satisfying the relation

$$b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_i f_i^{s_i}$$

where $P_k \in \mathcal{D}_X[\{s_{i,j}\}_{i,j}]$.

Definition 6.26. (cf. [Mus12]) Let X be a non-singular irreducible complex variety and $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero (coherent) ideal sheaf. Let $f : W \to X$ be a log resolution of $\mathfrak{a}$, and consider a divisor with simple normal crossings $\sum\limits_{i=1}^{N}$ on W such that if $\mathfrak{a}\mathcal{O}_W = \mathcal{O}_W(-D)$, then we may write

$$D = \sum_{i=1}^{N} a_i D_i \text{ and } F_{W/X} = \sum_{i=1}^{N} k_i E_i,$$

where $K_{W/X}$ is the relative canonical bundle.

Then the *log-canonical threshold* at a point $P \in X$, lct_P is

$$\mathrm{lct}_P(\mathfrak{a}) = \min_{\{i \mid P \in f(E_i)\}} \frac{k_i + 1}{a_i}.$$

The *(global) log-canonical threshold* is

$$\mathrm{lct}(\mathfrak{a}) = \inf_{\{P \in X\}} \mathrm{lct}_P(\mathfrak{a}).$$

Remark. The infimum in the definition of the (global) log-canonical threshold is actually a minimum and, therefore, it is a rational number.

6.2.3 Étale slices

One of the tools used in [Bud18] to study a similar problem to ours are the étale slices. Even though they are not directly applicable to our case, because we do not know our scheme to be a variety, it might be possible to construct an analogous machinery for more general schemes. Some basic definitions follow.

Definition 6.27. Given $X = \mathrm{Spec}(R)$ an affine scheme and G a group scheme acting on it, the *affine quotient* $X /\!\!/ G$ is $\mathrm{Spec}(R^G)$, where R^G denotes the fixed elements by the action of G.

The following two definitions belong more appropriately to the Commutative Algebra section but we deemed it better to mention them here.

Definition 6.28. A module N over a ring R is said to be *flat* if the functor $M \mapsto M \otimes_R N$ is an exact functor on the category of R-modules. If it is also a faithful functor, then we say that it is *faithfully flat.*

Definition 6.29. Let $f : A \to B$ be a ring morphism. We say that it is *G-unramified* if it is of finite presentation and $\Omega_{B/A} = 0$. Where $\Omega_{B/A} = 0$ is the module of differentials of B over A.

Definition 6.30. Let $f : X \to S$ be a morphism of schemes.

1. We say that it is *flat* if, for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
2. We say that it is *G-unramified* if, for every $x \in X$, there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
3. We say that it is *étale* if it is flat and G-unramified.

Definition 6.31. ([Bud18]) Let G be a reductive group acting on affine varieties X and Y, and let $f : X \to Y$ be a G-equivariant morphism.We say that f is *strongly étale* if

- $f/G : X /\!\!/ G \to Y /\!\!/ G$ is étale, and
- f, f/G and the quotient morphisms induce a G-isomorphism $X \simeq Y \times_{Y/G} (X /\!\!/ G)$.

Definition 6.32. ([Bud18]) Let G be a reductive group acting on an affine variety X. Let $x \in X$ be a point with closed orbit. An *étale slice* is a G_x-invariant locally closed affine subvariety S of X containing x such that the induced G-equivariant morphism

$$\psi : G \times_{G_x} S \to X$$

is strongly étale onto a G-saturated affine open subset U of X.

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