On the Galois embedding problem associated to the universal central extension of the alternating group of degree 6

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Abstract

We obtain an explicit expression for the obstruction to the solvability of the Galois embedding problem associated to the universal central extension of the alternating group of degree 6.

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1 Introduction

Let $L|K$ be a Galois extension with Galois group the alternating group $A_6$. Let $6A_6$ be the nontrivial 6-fold cover of $A_6$. We consider the Galois embedding problem

$$6A_6 \rightarrow A_6 \simeq Gal(L|K).$$

(1)

A solution to (1) is a field $\tilde{L}$, containing $L$, which is a Galois extension of $K$ such that $Gal(\tilde{L}|K) \simeq 6A_6$ and the diagram

$$\begin{array}{ccc}
Gal(\tilde{L}|K) & \rightarrow & Gal(L|K) \\
\downarrow & & \downarrow \\
6A_6 & \rightarrow & A_6
\end{array}$$

commutes, where the epimorphism $Gal(\tilde{L}|K) \rightarrow Gal(L|K)$ is given by restriction. Our aim is to obtain an explicit expression for the obstruction to the solvability of (1).

It is known that, for $n \neq 6, 7$, the nontrivial double cover $2A_n$ is the universal central extension of $A_n$, i.e. we have a commutative diagram

$$\begin{array}{ccc}
1 & \rightarrow & \{\pm 1\} \\
\downarrow & & \downarrow \\
1 & \rightarrow & H
\end{array} \quad \begin{array}{ccc}
2A_n & \rightarrow & A_n \\
\rightarrow & & \rightarrow \\
A_n & \rightarrow & A_n
\end{array} \rightarrow 1$$

for any central extension $1 \rightarrow H \rightarrow \tilde{A}_n \rightarrow A_n \rightarrow 1$.

For $n = 6, 7$, the universal central extension of $A_n$ is

$$6A_n = \langle g_1, \ldots, g_{n-2}, z | g_i^2 = g_i = (g_{i-1}g_i)^3 = (g_jg_k)^2 = z^2, (g_tg_k)^2 = z, z^6 = [z, g_t] = 1, 1 \leq i \leq 5, 1 \leq j \leq k - 1, k \leq 5, (j, k) \neq (1, 4), 1 \leq t \leq 5 \rangle.$$

Given a finite Galois extension $L|K$ with Galois group $G$ and an epimorphism of groups $\tilde{G} \rightarrow G$ with abelian kernel $A$, the obstruction to the solvability for the Galois embedding problem

$$\tilde{G} \rightarrow G \simeq Gal(L|K)$$

is given by $\inf \varepsilon \in H^2(G_K, A)$, where $G_K$ is the absolute Galois group of $K$, $\varepsilon \in H^2(G, A)$ represents $\tilde{G}$ and $\inf : H^2(G, A) \rightarrow H^2(G_K, A)$ is the induced morphism between cohomology groups.

In the case of the Galois embedding problem given by the nontrivial double cover of the alternating group $A_n, n \geq 4$, over a field $K$ of characteristic different from 2, Serre [4] obtained a formula expressing the obstruction to the solvability in terms of the Hasse-Witt of a trace form and allowing its explicit computation. An explicit formula for the solutions to this embedding problem was obtained by Crespo in [1]. These results were generalized in [3] and [2]. An explicit determination of the obstruction to the solvability of embedding problems with a cyclic kernel of order $> 2$ in terms of generalized Clifford algebras was obtained by Vela in [6] together with an explicit construction of their solutions.
2 Preliminaries

The group $6A_6$ is isomorphic to the subgroup of $SL(6, \mathbb{Q}(\omega))$ generated by the two matrices

$$M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\omega$ denotes a primitive third root of unity, and the epimorphism $6A_6 \to A_6$ can be given by

$$M_1 \mapsto \sigma := (2,3)(4,5) \quad M_2 \mapsto \tau := (1,4,3,2)(5,6)$$

For $K$ a field containing $\mathbb{Q}(\omega)$, we obtain a representation

$$\rho : A_6 \to PGL(6, K) = Aut(M_{6 \times 6}(K))$$

and a commutative diagram

$$\begin{array}{ccc} 6A_6 & \to & A_6 \\ \downarrow & & \downarrow \rho \\ SL(6, K) & \to & PGL(6, K). \end{array}$$

In this situation, we may apply a result of Fröhlich and obtain the following statement.

**Proposition 2.1.** ([3], App. III) Let us denote by $B$ the twisted algebra of $A := M_{6 \times 6}(K)$ by the 1-cocycle $\rho : A_6 \to Aut(M_{6 \times 6}(K))$. The obstruction to the solvability of the embedding problem

$$6A_6 \to A_6 \simeq Gal(L|K)$$

is given by the class of the algebra $B$ in the Brauer group $Br(K)$ of the field $K$.

3 Main result

Let the field $L$ be given as the splitting field of a monic irreducible polynomial of degree 6 in $K[X]$.

**Theorem 3.1.** The twisted algebra $B$ of $M_{6 \times 6}(K)$ by the 1-cocycle $\rho$ is the subalgebra of $M_{6 \times 6}(L)$ whose elements $E = (a_{ij})_{1 \leq i,j \leq 6}$ satisfy

$$a_{ii} = \sum_{k=0}^{5} \lambda_k x_i^k, \quad \lambda_k \in K$$

$$a_{ij} = z_{ij} \sum_{k=0}^{5} \sum_{l=0}^{4} \mu_{kl} x_i^k x_j^l, \quad \mu_{kl} \in K, i \neq j,$$

where $x_i, 1 \leq i \leq 6$, denote the roots in $L$ of the polynomial $f$ realizing the group $A_6$ over $K$ and the elements $z_{ij}$ are given by

$$z_{ij} = \sum_{k=0}^{5} \sum_{l=0}^{4} \mu_{kl} x_i^k x_j^l, \quad \mu_{kl} \in K, i \neq j.$$
Proof. By Galois descent, the algebra $B$ is the $K$-subalgebra of $M_{6\times 6}(L) \simeq A \otimes_K L$ fixed by the action of the group $G = \text{Gal}(L|K)$, where $G$ acts on $A$ via the representation $\varphi$ given by

$$
\varphi : \quad A_6 \longrightarrow \text{Aut}(A)
$$

$$
s \mapsto \varphi(s) : \quad A \longrightarrow A
$$

$$
E \mapsto \tilde{s}E\tilde{s}^{-1},
$$

where $\tilde{s}$ is the preimage of $\rho(s)$ by the epimorphism $SL(6, K) \rightarrow PGL(6, K)$, and $G$ acts on $L$ by the Galois action (see [5] X §2).

If $E = (a_{ij})_{1 \leq i,j \leq 6}$ is a matrix in $M_{6\times 6}(L)$, we obtain

$$
\varphi(\sigma)(E) = M_1 E M_1^{-1} = \begin{pmatrix}
  a_{11} & a_{13} & a_{12} & a_{15} & a_{14} & a_{16} \\
  a_{31} & a_{33} & a_{32} & a_{35} & a_{34} & a_{36} \\
  a_{21} & a_{23} & a_{22} & a_{25} & a_{24} & a_{26} \\
  a_{51} & a_{53} & a_{52} & a_{55} & a_{54} & a_{56} \\
  a_{41} & a_{43} & a_{42} & a_{45} & a_{44} & a_{46} \\
  a_{61} & a_{63} & a_{62} & a_{65} & a_{64} & a_{66}
\end{pmatrix},
$$

and

$$
\rho(\tau)(E) = M_2 E M_2^{-1} = \begin{pmatrix}
  a_{22} & \omega^2 a_{23} & a_{24} & \omega a_{21} & a_{26} & a_{25} \\
  \omega a_{32} & a_{33} & \omega a_{34} & \omega^2 a_{31} & \omega a_{36} & \omega a_{35} \\
  a_{42} & \omega^2 a_{43} & a_{44} & \omega a_{41} & a_{46} & a_{45} \\
  \omega^2 a_{12} & \omega a_{13} & \omega^2 a_{14} & a_{11} & \omega^2 a_{16} & \omega^2 a_{15} \\
  a_{62} & \omega^2 a_{63} & a_{64} & \omega a_{61} & a_{66} & a_{65} \\
  a_{52} & \omega^2 a_{53} & a_{54} & \omega a_{51} & a_{56} & a_{55}
\end{pmatrix}.
$$

Hence, the equivalent conditions for a matrix $E = (a_{ij})_{1 \leq i,j \leq 6}$ to be fixed under $\sigma$ are
\[ a_{11} = a_{11}^\sigma, \quad a_{12} = a_{12}^\sigma, \quad a_{13} = a_{13}^\sigma, \quad a_{14} = a_{14}^\sigma, \quad a_{15} = a_{15}^\sigma, \quad a_{16} = a_{16}^\sigma, \quad (2) \]

\[ a_{21} = a_{21}^\sigma, \quad a_{22} = a_{22}^\sigma, \quad a_{23} = a_{23}^\sigma, \quad a_{24} = a_{24}^\sigma, \quad a_{25} = a_{25}^\sigma, \quad a_{26} = a_{26}^\sigma, \]

\[ a_{31} = a_{31}^\sigma, \quad a_{32} = a_{32}^\sigma, \quad a_{33} = a_{33}^\sigma, \quad a_{34} = a_{34}^\sigma, \quad a_{35} = a_{35}^\sigma, \quad a_{36} = a_{36}^\sigma, \]

\[ a_{41} = a_{41}^\sigma, \quad a_{42} = a_{42}^\sigma, \quad a_{43} = a_{43}^\sigma, \quad a_{44} = a_{44}^\sigma, \quad a_{45} = a_{45}^\sigma, \quad a_{46} = a_{46}^\sigma, \]

\[ a_{51} = a_{51}^\sigma, \quad a_{52} = a_{52}^\sigma, \quad a_{53} = a_{53}^\sigma, \quad a_{54} = a_{54}^\sigma, \quad a_{55} = a_{55}^\sigma, \quad a_{56} = a_{56}^\sigma, \]

\[ a_{61} = a_{61}^\sigma, \quad a_{62} = a_{62}^\sigma, \quad a_{63} = a_{63}^\sigma, \quad a_{64} = a_{64}^\sigma, \quad a_{65} = a_{65}^\sigma, \quad a_{66} = a_{66}^\sigma. \]

and those to be fixed under \( \tau \)

\[ a_{11} = a_{22}^\tau, \quad a_{12} = \omega^2 a_{23}^\tau, \quad a_{13} = a_{13}^\tau, \quad a_{14} = \omega a_{14}^\tau, \quad a_{15} = a_{15}^\tau, \quad a_{16} = a_{16}^\tau, \]

\[ a_{21} = \omega a_{21}^\tau, \quad a_{22} = a_{22}^\tau, \quad a_{23} = \omega^2 a_{23}^\tau, \quad a_{24} = \omega^2 a_{24}^\tau, \quad a_{25} = \omega a_{25}^\tau, \quad a_{26} = \omega^2 a_{26}^\tau, \]

\[ a_{31} = a_{31}^\tau, \quad a_{32} = \omega^2 a_{32}^\tau, \quad a_{33} = a_{33}^\tau, \quad a_{34} = \omega a_{34}^\tau, \quad a_{35} = a_{35}^\tau, \quad a_{36} = a_{36}^\tau, \]

\[ a_{41} = \omega^2 a_{41}^\tau, \quad a_{42} = \omega a_{42}^\tau, \quad a_{43} = \omega^2 a_{43}^\tau, \quad a_{44} = a_{44}^\tau, \quad a_{45} = \omega^2 a_{45}^\tau, \quad a_{46} = \omega^2 a_{46}^\tau, \]

\[ a_{51} = a_{51}^\tau, \quad a_{52} = \omega^2 a_{52}^\tau, \quad a_{53} = a_{53}^\tau, \quad a_{54} = \omega a_{54}^\tau, \quad a_{55} = a_{55}^\tau, \quad a_{56} = a_{56}^\tau, \]

\[ a_{61} = a_{61}^\tau, \quad a_{62} = \omega^2 a_{62}^\tau, \quad a_{63} = a_{63}^\tau, \quad a_{64} = \omega a_{64}^\tau, \quad a_{65} = a_{65}^\tau, \quad a_{66} = a_{66}^\tau. \]

Let us look at the conditions imposed on the elements in the diagonal.

The element \( a_{11} \) is fixed by \( \sigma = (2,3)(4,5) \) and \( \sigma \tau \sigma \tau = (2,6,4,3,5) \). So, \( a_{11} \) belongs to the field \( L^{<\sigma,\sigma \tau \sigma \tau>} \) which clearly contains \( K(x_1) \). Now, the subgroup \( <\sigma,\sigma \tau \sigma \tau> \) of \( A_6 \) has order 60, so \( L^{<\sigma,\sigma \tau \sigma \tau>} = K(x_1) \). As \( a_{11} \in L^{<\sigma,\sigma \tau \sigma \tau>} = K(x_1) \), we may write \( a_{11} = \sum_{i=0}^{5} \lambda_i x_1^i \), for some \( \lambda_i \) in \( K \).

From the relations \( a_{44} = a_{11}^\tau, a_{33} = a_{44}, a_{22} = a_{33}, a_{55} = a_{44}, a_{66} = a_{55} \), we obtain

\[ a_{kk} = \sum_{i=0}^{5} \lambda_i x_1^i \quad k = 1, \ldots, 6. \]

We consider now the elements out of the diagonal.

The element \( a_{61} \) is fixed by the subgroup \( H \) of \( A_6 \) generated by \( \sigma \) and the permutation \( (2,4)(3,5) = \sigma \tau^2 \sigma \tau^3 \sigma \tau^3 \sigma \tau \). Let \( K_2 := L^H \); as \( H \) has order 4, the degree of \( K_2 \) over \( K \) is 90. The field \( K_2 \) contains clearly the field \( K_1 := K(x_1, x_6) \) which has order 30 over \( K \), so \( [K_2 : K_1] = 3 \). Let us look for a generator of the extension \( K_2/K_1 \). We have the following towers of fields

\[
A_4 \begin{pmatrix}
4 \\
3 \\
30 \\
1
\end{pmatrix}
\]

\[
\begin{array}{c}
L \\
K_2 = L^H \\
K_1 = K(x_1, x_6) \\
K
\end{array}
\]

The extension \( L/K_1 \) is the splitting field of the polynomial

\[
f_4(X) = \frac{f(X)}{(X-x_1)(X-x_5)} = (X-x_2)(X-x_3)(X-x_4)(X-x_5) \in K_1[X].
\]

By the Galois correspondence, the field \( K_2 \) is associated to the subgroup \( H_2 = <\sigma, (2,4)(3,5)> \) and the field \( K_1 \) to the alternating group in the four letters \( \{2,3,4,5\} \). Hence, the extension \( K_2/K_1 \) is the splitting field of the cubic resolvent of \( f_4 \). We may then write \( K_2 = K_1(y_1, y_2, y_3) \), where
\[ y_1 = (x_2 + x_3)(x_4 + x_5) \]
\[ y_2 = (x_2 + x_4)(x_3 + x_5) \]
\[ y_3 = (x_2 + x_5)(x_3 + x_4). \]

The Galois group of \( \text{Gal}(K_2/K_1) \) is generated by \( \gamma H_2 \), where \( \gamma = (2, 3, 4) = \sigma \tau \sigma^3 \sigma \tau^2 \tau \). We have

\[ a_1^\gamma = \omega a_{16}, \quad y_1^\gamma = y_3, \quad y_2^\gamma = y_1, \quad y_3^\gamma = y_2. \]

Hence the element \( z = y_1 + \omega y_2 + \omega^2 y_3 \) satisfies \( z^\gamma = \omega z \) and we have \( a_{16} = \alpha z \), with \( \alpha \in K(x_1, x_6) \). We obtain then the expression for \( a_{16} \) given in the statement. By using the relations (2) and (3), we obtain the formulas for all elements \( a_{ij}, 1 \leq i, j \leq 6, i \neq j \).

References


