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Supervised by Anna de Mier Vinué

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I would like to thank my project supervisor Anna de Mier for introducing me to the subject, helping me throughout the project and also for her patience and expertise. I also would like to thank my college friends Ernest Sorinas and Filip Cano for supporting and listening to me whenever I needed.

Abstract

Matroids arise from the abstract notion of dependency. Matroids can be studied from different points of view. From linear algebra we know matrices, which can be seen as matroids, however matroids generalise the concept of dependency. In matroids there also is combinatorics, graph theory and geometry. This project wants to be an introduction to matroids throughout two of its families: positroids and lattice path matroids. Starting with basic definitions and examples the project reaches deeper and more interesting relations between these two families.

Keywords

Matroid, Positroid, Lattice path matroid, Grassmann necklace, Le-Diagram, Decorated permutation, Excluded minor.

Contents

Introduction

The aim of this project is to introduce the general concept of matroids with more emphasis on one family of matroids, the positroids.

Gordon and McNulty's book (2012) ([7], *Matroids: A Geometric Introduction*) defines matroid theory as an active area of mathematics that uses ideas from abstract and linear algebra, geometry, combinatorics and graph theory. We will see there are many different families of matroids that all have in common an abstract notion of dependence. Trying to abstract the notion of dependence is what lead Whitney to the introduction of matroids in 1935 in [14]. We will give the formal definition of matroid in Chapter 1. Informally speaking, a matroid consists of a finite set E and a collection $\mathcal{I}$ of its subsets that somehow behaves like independent sets in a vector space. In particular, given a $k \times n$ matrix one can construct a matroid with $E = \{1, 2, \dots, n\}$ and taking $\mathcal{I}$ as the subsets of E that index column sets that are linearly independent.

Positroids are matroids representable over $\mathbb{R}$ by a $k \times n$ totally nonnegative matrix, that is a matrix with all its maximal minors being nonnegative. Positroids arose in the study of cell decompositions of the Grassmanian, but we will only be interested in their matroidal and combinatorial properties. Lusztig in [8] already defined a variety named $G_{\geq 0}$ over arbitrary connected algebraic reductive group G that *has a natural partition into "zones"* and conjectures that its totally nonnegative part has a cellular decomposition. More recently Postnikov in [13] extended the positroids notion in the following way. Let the *Grassmannian* $Gr_{kn}(\mathbb{R})$ be the set of k-dimensional subspaces in $\mathbb{R}^n$. For any element $V \in Gr_{kn}$ represented by a $k \times n$ matrix A we obtain the matroid M_V. Then the *totally nonnegative Grassmannian* Gr_{kn}^{tnn} are the elements of the Grassmannian that have a matrix representation with all its maximal minors nonnegative. For the Grassmannian we can obtain a partition into *Schubert cells* Ω_λ where λ is a partition of n whose diagram fits in a $k \times n$ rectangle (see [13] for the concrete definition). There is a finer subdivision of $Gr_{kn}(\mathbb{R})$ into *matroid strata* $S_M := \{V \in Gr_{kn}(\mathbb{R}) | M_V = M\}$ where M is a matroid. With these definitions Postnikov obtains a cell decomposition of Gr_{kn}^{tnn} where the cells correspond to positroids. Postnikov also obtains different objects with interesting combinatorial properties that are in bijection with positroids such as decorated permutations, Grassmann necklaces and ⅃-diagrams. Understanding these objects and their relationships is one of the main goals of this project.

We also study a specific family of positroids named lattice path matroids (LPM). Lattice path matroids are also a family of transversal matroids, and they are formally defined in Section 3.1; the name comes from the fact that each of them is determined by the region bounded by two lattice paths going from $(0, 0)$ to (m, r). This family of matroids has been studied from an enumerative and structural perspective by J. Bonin, A. de Mier, and M. Noy.

The project is structured as follows. In the second chapter we introduce the notion of matroid and define them in different (equivalent) ways. Then examples of matroid families are given. The third chapter refers to positroids, objects that are in bijection with them and some properties. The last chapter is mainly about studying lattice path matroids from the positroid perspective.

Chapter 1

Matroids

In this chapter we will introduce different but equivalent ways to define matroids and operations we can do with them. To familiarize with the concept we will study briefly different families of matroids.

As reference books, for general graph theory we suggest Diestel's [6] and for matroids Oxley's [12] and Gordon and McNulty's [7]. Also, we assume the reader to have the knowledge on discrete mathematics corresponding to a bachelor's degree on Mathematics.

1.1 Definitions

Definition 1. A *matroid* $M = (E, \mathcal{I})$ consists of a set E and a collection $\mathcal{I}$ of subsets of E such that the following properties hold:

I.1) $\mathcal{I} \neq \emptyset$.

I.2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.

I.3) If $I, J \in \mathcal{I}$ with $|I| < |J|$, then there exsists $x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$.

The set E is called the *ground set* and the elements of $\mathcal{I}$ are the *independent* sets.

In Section 1.2 we will study several families of matroids with interesting properties. Let us now introduce representable matroids to help the reader understand basic concepts of matroids with some linear algebra intuition.

Definition 2. Given a matrix A over a field $\mathbb{F}$ with m columns it defines a matroid with ground set $[m] = \{1, \ldots, m\}$. The independent sets of the matroid are the indices of independent sets of columns as vectors. This matroid is called $M(A)$. Matroids that can be constructed this way are called *representable* matroids.

Example 1. Let A be the following matrix over $\mathbb{R}$ with columns enumerated from 1 to 4:

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

Then $E = \{1,2,3,4\}$ and $\mathcal{I}(M) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}\}$ form a representable matroid $M = (E, \mathcal{I})$.

There are many other different ways to define a matroid. Some of them are by bases, by ranks, by circuits, by closure or by flats. Let $M = (E, \mathcal{I})$ be a matroid from now on. Let us define these elements and their properties. The proofs of the properties are not hard and they can be found in [12] .

Definition 3. A *circuit* is a minimal dependent set. The set of circuits of M will be denoted by C.

In circuits the following properties hold:

C.1) $\emptyset \notin C$.

C.2) If $C_1, C_2 \in C$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

C.3) If $C_1, C_2 \in C, C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in C$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Notation. As we did in property C.3) above, in this section and when talking about sets we will be using e to denote $\{e\}$, the set containing the element e.

Having $\mathcal{I}$, the collection C is determined. In the same way, from C we get $\mathcal{I}$ as all those subsets not containing any member from C. Moreover, given a set E and a collection C of subsets of E such that C.1, C.2, C.3 hold, then there is a matroid with ground set E whose circuits are the elements of C.

In Example 1 the circuits are $C = \{\{1\}, \{2,4\}\}$.

Definition 4. An element e such that $\{e\}$ is a circuit is called a *loop*. If $\{e, f\}$ is a circuit, the elements e, f are called *parallel*.

Definition 5. A *basis* or *base* is a maximal independent set. Given a matroid, its set of bases will be denoted by $\mathcal{B}$.

In bases the following properties hold:

B.1) $\mathcal{B} \neq \emptyset$.

B.2) If $B_1, B_2 \in \mathcal{B}$, $x \in B_1 - B_2$ then there exists $y \in B_2 - B_1$ with $(B_1 - x) \cup \{y\} \in \mathcal{B}$.

B.3) If $B_1, B_2 \in \mathcal{B}$, then $|B_1| = |B_2|$.

From $\mathcal{I}$ we can get $\mathcal{B}$ by selecting all the maximal indepentent sets, and we can get $\mathcal{I}$ from $\mathcal{B}$ by getting all the subsets of each base. Moreover, given a set E and a collection $\mathcal{B}$ of bases of E such that B.1, B.2 and B.3 hold, then there is a matroid with ground set E whose bases are the elements of $\mathcal{B}$.
In Example example 1 the bases are $\mathcal{B} = \{\{2,3\},\{3,4\}\}$.

Definition 6. An *isthmus* is an element of the ground set $E(M)$ that is in all the bases of the matroid M.

In Example 1 we have an isthmus that is 3.

Definition 7. Let $X \subseteq E$. We define $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$ and it is easy to check that the pair $(X, \mathcal{I}|X)$ is a matroid. This matroid is called the *restriction of* M *to* X and denoted by $M|X$.

Definition 8. For $X \subseteq E$ we define the *rank of* X, $rank(X)$ or simply $r(X)$ as the size of a basis of $M|X$.

The rank function has the following properties:

R.1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

R.2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

R.3) If $X, Y \subseteq E$, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Let $\mathcal{I}$ be the collection of subsets $X \in E$ such that $r(X) = |X|$. Then $(E, \mathcal{I})$ is a matroid with rank function r.

In matroids arising from linear algebra the rank can be seen as the dimension. The rank is the maximum cardinality of the subsets of the subset X that are independent.

Definition 9. Given $X \subseteq E$ the *closure* cl of X is defined by $cl(X) = \{x \in E : r(X \cup x) = r(X)\}$.

The closure operator has the following properties:

Cl.1) If $X \subseteq E$, then $X \subseteq cl(X)$.

Cl.2) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.

Cl.3) If $X \subseteq E$, then $cl(cl(X)) = cl(X)$.

Cl.4) If $X \subseteq E, x \in E, y \in cl(X \cup x) - cl(X)$, then $x \in cl(X \cup y)$.

In Example 1 the closure of $\{2\}$ is $cl\{2\} = \{1, 2, 4\}$.

Observation 1. Let $X \subseteq E$ and $x \in E$. Note that if $X \in \mathcal{I}$ and $X \cup x \notin \mathcal{I}$, then $x \in cl(X)$.

We can also know $\mathcal{I}$ from the closure by $\mathcal{I} = \{X \in E : x \notin cl(X - x)\ \forall x \in X\}$.

In matroids arising from linear algebra the closure can be seen as the span of a vector set.

Definition 10. A subset $X \subseteq E$ is a *flat* if $X = cl(X)$. The set of flats is called $\mathcal{F}$.

In flats the following properties hold:

F.1) $E \in \mathcal{F}$.

F.2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

F.3) If $F \in \mathcal{F}$ and $\{F_1, F_2, \dots, F_k\}$ is the set of minimal members of $\mathcal{F}$ that properly contain $\mathcal{F}$, then the sets $F_1 - F, F_2 - F, \dots, F_k - F$ partition $E - F$.

In Example 1 the set $\{1, 2, 4\}$ is a flat.

In matroids arising from linear algebra the flats can be seen as the subspaces of different dimensions over E.

Definition 11. Given a matroid $M = (E, \mathcal{I})$, the *truncation to rank k* of M is $T_k(M) = (E, \mathcal{I}')$ where a set I is in $\mathcal{I}'$ if and only if it is in $\mathcal{I}$ and $rank(I) \leq k$.

1.2 Families of matroids and operations

Let M be a matroid with ground set E and independent sets $\mathcal{I}$. We will now define several operations that is easy to check that are closed over matroids. After this, we will see some important families of matroids and see how these operations behave.

1.2.1 Operations

Definition 12. Let $M = (E, \mathcal{B})$ be a matroid. We define the *dual* of M as the matroid $M^* = (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$.

Definition 13. The *deletion* of an element $e \in E$ from M is the matroid denoted by $M \backslash e$ or $M - e$ and defined as $M \backslash e = (E \backslash e, \mathcal{I}')$ where $\mathcal{I}' = \{I \subseteq E \backslash e : I \in \mathcal{I}\}$.

In fact, using Definition 7, the deletion of e is $M \backslash e = M|(E - e)$.

Definition 14. The *contraction* of an element $e \in E$ from M is $M/e = (M^* \backslash e)^*$.

Definition 15. The *direct sum* of two matroids $M_1 = (E_1, B_1)$, $M_2 = (E_2, B_2)$ is

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2))\}.$$

1.2.2 Uniform matroids

Definition 16. A *uniform matroid* $U_{m,n}$ with $0 \leq m \leq n$ has as ground set $E = \{\text{an } n\text{−element set}\}$ and $\mathcal{I} = \{I \in E : |I| \leq m\}$.

A uniform matroid $U_{m,n}$ has rank m. Its circuits are the subsets of $m + 1$ elements.

When deleting an element e we get another matroid M' over $E \backslash e$. The independent sets are the same as in M except those containing e. Actually $M' = U_{min(m,n-1),n-1}$.

The dual of $U_{m,n}$ is $U_{n-m,n}$. When contracting an element e we get $U_{m,n}/e = U_{m-1,n-1}$ so uniform matroids are closed under duality, deletion and contraction.

It is easy to check that the direct sum of two uniform matroids does not have to be a uniform matroid, so uniform matroids are not closed under direct sum.

1.2.3 Graphic matroids

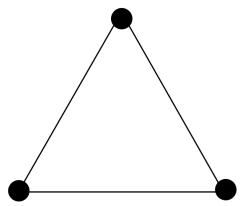
Definition 17. If G is a graph with edge set E and forests $\mathcal{I}$, then $M(G) = (E, \mathcal{I})$ is a matroid. Matroids that can be constructed this way are called *graphic* matroids.

A graphic matroid $M(G)$ has as independent sets the forests of G and as circuits the cycles. Its rank is the number of edges of a spanning forest.

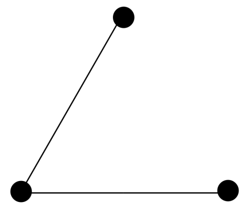
If the graph G is planar and G^* is its geometric dual, the dual of the graphic matroid $M(G)^*$ is $M(G^*)$ and then $M(G)^*$ is a graphic matroid. In fact, the inverse is also true and thus we have that the dual of a graphic matroid $M(H)$ is graphic if and only if H is planar.

When deleting an element e we get $M(G)\backslash e = M(G\backslash e)$, which is graphic. When contracting an element e we get $M(G)/e = M(G/e)$, which is also graphic. Here $G\backslash e$ and G/e stand for the usual deletion and contraction in a graph.

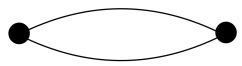
Example 2. A simple graphic matroid and the three operations defined above.



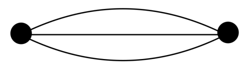
(a) Graphic matroid.



(b) Deletion of one edge.



(c) Contraction of one edge.



(d) Dual of the graphic matroid.

1.2.4 Transversal matroids

A family of subsets of a finite set E is a collection $\mathcal{A} = \{A_1, \dots, A_m\}$ of E.

Definition 18. A *partial transversal* of $\mathcal{A}$ is a subset $\{e_1, \dots, e_n\}$ of different elements of E such that $e_i \in A_i$ for $1 \leq i \leq n$.

Definition 19. It was proved by Edmonds and Fulkerson (also in [12], Theorem 1.6.2) that the collection of partial transversals of a family $\mathcal{A}$ on E are the independent sets of a matroid, $M[\mathcal{A}]$. A *transversal* matroid is any such matroid.

Remember a *bipartite graph* B is a graph whose vertices can be divided into two disjoint sets U, V such that no edge has both endpoints in the same set. Let U be $\{e_1, \dots, e_n\}$ and V be $\mathcal{A}$.

Definition 20. A *matching* is a subset of edges of a bipartite graph such that there is not any pair of edges that share a vertex. We can see a matching as a partial transversal.

Example 3. All uniform matroids $U_{m,n}$ are transversal. It is easy to check by defining $U = [n]$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$ with $A_1 = \cdots = A_m = \{1, 2, \ldots, n\}$ because all matchings will have size at most m and all subsets U_i of U with $|U_i| \leq m$ will have matchings so they will be independents.

It is not hard to see that transversal matroids are closed under deletion. However, they are not closed under contraction, as we will now see.

Given the transversal matroid with $U = [7]$ and $\mathcal{A} = \{A_1, A_2, A_3\}$ where $A_1 = \{1, 2, 7\}, A_2 = \{3, 4, 7\}$ and $A_3 = \{5, 6, 7\}$ can be represented as the following graphic matroid.

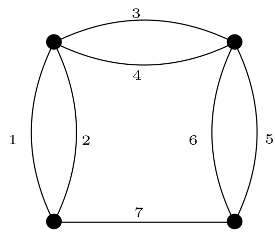


Figure 1.2: Graphic representation of a transversal matroid.

Contracting the edge 7 we get the following matroid

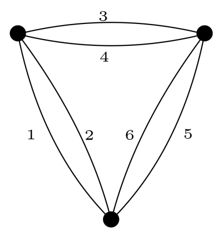


Figure 1.3: Graph after contracting edge 7.

which is not transversal.

To see it is not transversal assume it is and let us get a contradiction.

If it is transversal there is a family $\mathcal{A}$ of subsets of $\{1, 2, 3, 4, 5, 6\}$ such that all its partial transversals are independent sets. As $\{1, 2\}$ is a circuit there has to be a unique member of the family $\mathcal{A}$ meeting 1 and 2. Call this member A_1 and contains 1 and 2. Equivalently for 3 or 4 and for 5 or 6 we have the member A_2 containing 3 and 4 and the member A_3 containing 5 and 6. As we have that $\{1, 3\}$, $\{3, 5\}$ and $\{5, 1\}$ are independent they have to be partial transversals which implies that the three sets are different. If the sets are different any other partial transversal is an independent set of the matroid. Since $\{1, 3, 5\}$ is a partial transversal but is not an independent set of the matroid we have a contradiction.

Finally we can say that transversal matroids are not closed under duality because $M/e = (M^*\backslash e)^*$ and we have seen that they are not closed under contraction.

1.2.5 Representable matroids

Remember Definition 2 of representable matroids. In this section we study properties of this family of matroids.

We defined representable matroids by independent sets (subsets of columns linearly independent). The circuits are the minimal subsets linearly dependent and the rank is the rank of the matrix.

Since representable matroids are usually represented as a matrix, let us recall some elementary transformations over matrices to state some more properties of representable matroids. These are what we will call elementary (row) transformations:

- Row swap: Swap the position of two rows in a matrix.
- Row scalar multiplication: Multiply the row of a matrix by a non-zero scalar.
- Row addition: Add a multiple of a row to another row.

These operations can be made left-multiplicating by some elementary matrices.

We can define elementary column transformations analogously, and they are obtained right-multiplicating by the same elementary matrices. However, column transformations will surely affect the matroid because we are altering the positions and labels of the ground set elements.

If A' is a matrix obtained from A using elementary row operations then $M(A) = M(A')$. If we are allowed to relabel the elements of the ground set, then the same is true using column swaps.

Let A be a matrix of size $r \times n$ and rank r. Without loss of generality let us assume that $A = [I_{r,r}|D]$ for some matrix D of dimensions $r \times n-r$. We can assume this because from linear algebra we know that A can be converted into a matrix of this form using elementary row operation and column permutations. We also know that elementary transformations do not change the resulting matroid. We define A^* as $A^* = [-D^T|I_{(n-r)\times(n-r)}]$. Then, one checks that $M(A)^* = M(A^*)$.

If e is not an isthmus then $M(A)\backslash e = M(A\backslash e)$, where $A\backslash e$ is the result of deleting column e from matrix A.

In order to get $M(A)/e$ first let us transform A using elementary transformations in order to have just one value different from zero in the column represented by e. If the position of this value is i, j then by deleting the row i and the column j we get A/e. One checks $M(A)/e = M(A/e)$.

The direct sum of two representable matroids $M(A) \oplus M(B)$ is $M(A \oplus B)$ where $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Example 4. All transversal matroids are representable over $\mathbb{R}$. Let us assume that the transversal matroid has $E = \{e_1, \ldots, e_n\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$. The matrix A associated will have m rows and n columns and $a_{i,j}$ will be 0 if $e_j \notin A_i$ and $x_{i,j}$ if $e_j \in A_i$ where all $x_{i,j}$ are algebraically independent over $\mathbb{R}$.

Example 5. Not all transversal matroids are representable over an arbitrary field $\mathbb{F}$. For example $U_{2,4}$ -recall from Example 3 that all uniform matroids are transversal- is not representable over $\mathbb{F}_2$. This is because we would need to have at least four different non-zero vectors 2×1 and there are just three such vectors: $(1, 0), (0, 1)$ and $(1, 1)$.

1.3 Geometric representation

To represent geometrically a matroid it helps to see many properties of this matroid. A geometric respresentation of a matroid consists of dots (representing the ground set) in a space. The affine dependencies between the dots in the geometrical representation define the independent sets of the matroid.

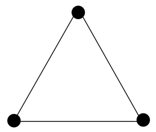
These are the rules to create the geometrical representation of a matroid:

- If k elements of the ground set form a circuit then they are represented by k different points in a $(k-2)$-dimensional space where any subset of $k-1$ points are independent.
- If k elements of the ground set are an independent set then they are represented by k independent points in a $(k-1)$-dimensional space.

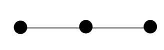
Small graphic matroids can be easily represented. Each edge (ground set element) is represented by a dot. If two edges are a cycle (a circuit in the matroid) they are represented by two points in the same place. If they form a tree -they are independent- they are represented by two points in a line. If three edges form a circuit they are represented by three points in a line. If they form a tree -they are independents- then they are represented by three points in a plane.

It may be easier to understand the procedure with some examples:

Example 6. A simple example of a graphic matroid and its geometric representation. Since the three edges of the triangle form a cycle, there will be three dots in a line.

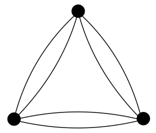


(a) Graphic matroid.

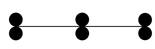


(b) Geometric representation of the graphic matroid.

Example 7. An example of graphic matroid with parallel edges and its geometric representation. Each pair of edges in parallel are represented by two dots in the same place.

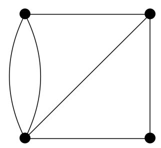


(a) Graphic matroid.

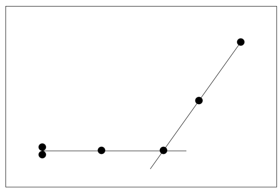


(b) Geometric representation of the graphic matroid.

Example 8. An example of a graphic matroid with four vertices and its geometric representation. As in Example 6, each triangle (cycle) is represented by three dots in a line. As they share an edge the two lines have a common dot.

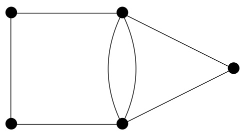


(a) Graphic matroid.

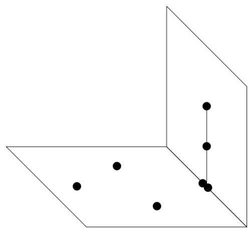


(b) Geometric representation of the graphic matroid.

Example 9. An example of a graphic matroid with five edges forming a cycle. Since it has five edges forming a cycle the dots are in a 3-dimensional space.



(a) Graphic matroid.



(b) Geometric representation of the graphic matroid.

Chapter 2

Positroids

Now we will study in more depth a class of representable matroids called positroids. We will focus on their fundamental properties and interesting objects with which we can find bijections.

Definition 21. Let A be a $k \times n$ matrix over $\mathbb{R}$ with rank k and all its maximal minors nonnegative. We call such matrix A a *totally nonnegative* matrix. A *positroid* is a representable matroid $M(A)$ over $\mathbb{R}$ with A being a totally nonnegative matrix.

In a representable matroid $M(A)$ where A is a $k \times n$ matrix, we will denote the determinant of the matrix formed by columns $\{i_1, \ldots, i_k\}$ of A as $\Delta_{i_1,\ldots,i_k}$.

In a positroid, as in any representable matroid, its independent sets are the subsets of linearly independent columns. The circuits are the minimal linearly dependent subsets and the rank is the rank of the matrix. Since the direct sum works as in the representable matroids, it is easy to check that the direct sum of two positroids is also a positroid.

In this chapter the ground set of a positroid will be identified same as before we can label the columns from 1 to n.

It is worth noting the importance of the ordering of the ground set. Reordering the elements of the ground set is equivalent to permuting columns in the matrix, and that may change signs in determinants.

Example 10. Let M be the matroid with ground set $E = \{1, 2, 3, 4\}$ and independent sets $\{1, 2\}$, $\{3, 4\}$, $\{1, 4\}$ and $\{2, 3\}$. The matrix A defined below yields this matroid, thus $M = M(A)$ is a representable matroid, but A is not totally nonnegative, and in fact one cannot find a totally nonnegative matrix that yields M. Therefore, M is a representable matroid but not a positroid.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

.

However, if we swap elements 2 and 3 in the ground set the resulting matroid does have a totally nonnegative representation. After this operation the independent sets are $\{1,3\}$, $\{2,4\}$, $\{1,4\}$ and $\{2,3\}$, and it is not hard to check that the permuted matrix B is totally nonnegative.

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

.

2.1 Bijections with positroids

In this section we will introduce several combinatorial objects that are in bijection with positroids, all of them defined as Postnikov did in [13].

These objects, defined in 2.1.1, are Grassmann necklaces, ⅃-Diagrams, ⅃-Graphs and decorated permutations. We will see how we can define a Grassmann necklace from any matroid, in particular from a positroid. We will show a bijection between Grassmann necklaces and decorated permutations, construct the ⅃-Diagram[1] and the ⅃-Graph associated to a Grassmann necklace. We will see how to get a totally nonnegative matrix from a ⅃-Graph to close the circle.

2.1.1 Definitions

Definition 22. (See [13], Definition 16.1) A *Grassmann necklace* is a sequence $\mathcal{G} = (G_1, \ldots, G_n)$ of subsets $G_r \subseteq [n]$ such that $|G_1| = \cdots = |G_n|$ and:

- If $i \in G_i$ then $G_{i+1} = \left(G_i \backslash \{i\}\right) \cup \{j\}$ for some $j \in [n]$.
- If $i \notin G_i$ then $G_{i+1} = G_i$.

The indices are taken modulo n.

[1] the ⅃ symbol is pronounced as "Le"

Observation 2. Notice that if $i \in G_i$ it may also happen $G_{i+1} = G_i$ as G_{i+1} can be $(G_i \backslash \{i\}) \cup \{i\}$.

Example 11. Before we show how we obtain a Grassmann necklace from a matroid and study the relation between positroids and these objects let us show an example of a Grassmann necklace to help the reader get familiar with them.

$$G_1 = \{1, 3, 5\} = G_6 \backslash \{6\} \cup \{5\}$$

$$1 \in G_1 \to G_2 = \{2, 3, 5\} = G_1 \backslash \{1\} \cup \{2\}$$

$$2 \in G_2 \to G_3 = \{3, 4, 5\} = G_2 \backslash \{2\} \cup \{4\}$$

$$3 \in G_3 \to G_4 = \{4, 5, 6\} = G_3 \backslash \{3\} \cup \{6\}$$

$$4 \in G_4 \to G_5 = \{5, 6, 1\} = G_4 \backslash \{4\} \cup \{1\}$$

$$5 \in G_5 \to G_6 = \{6, 1, 3\} = G_5 \backslash \{5\} \cup \{3\}$$

Definition 23. In the set $\{1, \ldots, n\}$ we define the *order* $a <_a$ to be the order $a <_a a + 1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a - 1$. We usually refer to the ground set as the ordered set $E = \{1, \ldots, n\}$ with the natural order $<_1$.

Definition 24. ([13], Definition 13.3) A *decorated permutation* $\pi^{:} = (\pi, col)$ is a permutation $\pi \in \mathfrak{S}_n$ together with a coloring function col from the set of fixed points $\{i | \pi(i) = i\}$ to $\{1, -1\}$.

We can think of a decorated permutation as a permutation where fixed points are colored with two different colors.

Now we will see a bijection between decorated permutations and Grassmann necklaces. In the next sections we will see a bijection between Grassmann necklaces and positroids. As a result of this we will have a bijection between decorated permutations and positroids.

First we will see how to obtain a decorated permutation $\pi^{:} = (\pi, col)$ from a Grassmann necklace $\mathcal{G} = (G_1, \ldots, G_n)$. An intuitive idea would be to set $\pi(i) = j$ when $G_{i+1} = (G_i \backslash \{i\}) \cup \{j\})$. We still have to deal with the coloration for fixed points, and we do this following these rules (see [11], Definition 15):

- If $G_{i+1} = (G_i \backslash \{i\}) \cup \{j\})$ and $i \neq j$ then $\pi(i) = j$.
- If $G_{i+1} = G_i$ and $i \notin G_i$ then $\pi(i) = i$ and $col(i) = 1$.
- If $G_{i+1} = G_i$ and $i \in G_i$ then $\pi(i) = i$ and $col(i) = -1$.

To obtain a Grassmann necklace from a decorated permutation we construct the following sets for $i \in [n]$:

$$G_i = \{j \in [n] | j <_i \pi^{-1}(j) \text{ or } (\pi(j) = j \text{ and } col(j) = -1)\}.$$

It is not hard to check that this is indeed a Grassmann necklace.

Example 12. For the Grassmann necklace in Example 11 the corresponding decorated permutation π: is $\pi = 246135$. Note that no coloration function is needed since there are no fixed points.

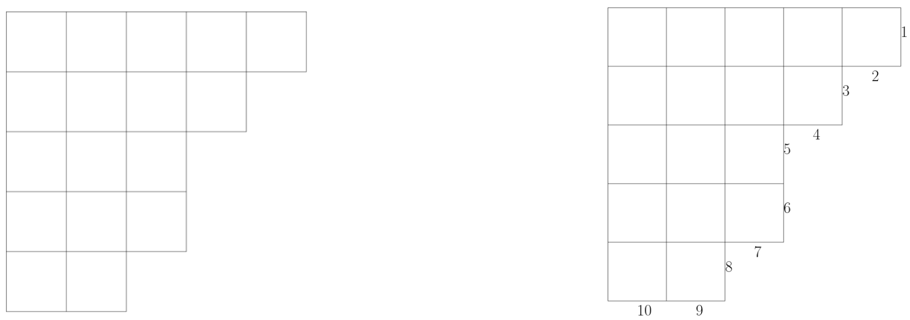
Definition 25. A *Young diagram* is a finite collection of cells arranged in left-justified rows. The number of cells in each row is not greater than in the row above. Listing the number of cells in each row we obtain a partition λ of the total number of cells of the diagram.

Definition 26. Given $n, m, k \in \mathbb{N}$ with $n = m + k$ and a partition λ that fits inside the rectangle $(n-k)^k$, the *boundary path* is the set of exterior edges of the corresponding Young diagram of λ from the top right corner to the down left corner.

In other words, the boundary path are the East and South external edges of the Young Diagram. Label these edges $1, \ldots, n$ from the top right to the bottom left corner. We define $I(\lambda)$ as the set of labels of the vertical edges.

Indices of the cells: The index for a cell is (i, j) where i is the label of the exterior vertical edge in the cell's row and j is the label of the horizontal exterior edge in the cell's column.

Example 13. Let λ be $\{5, 4, 3, 3, 2\}$. Let us construct the Young diagram associated to λ and label it to find $I(\lambda)$. See the result in Figure 2.1 below.



(a) Young diagram associated to λ.

(b) Young diagram labelled. Here, $I(\lambda) = \{1, 3, 5, 6, 8\}$.

Figure 2.1: Labelling of $\lambda = \{5, 4, 3, 3, 2\}$.

Definition 27. A ⅃-*Diagram* ("Le-Diagram") of shape λ is a Young diagram of shape λ where each box is either empty or filled with a cross (+) following the rule which we will reference as the ⅃-Diagram rule, ⅃-Diagram property or simply ⅃-property ([11], Definition 10).

⅃-*property*: For any three cells indexed $(i, j), (i', j), (i, j')$ where $i' < i$ and $j' > j$, if cells on positions (i', j) and (i, j') are filled, then the cell on (i, j) is also filled.

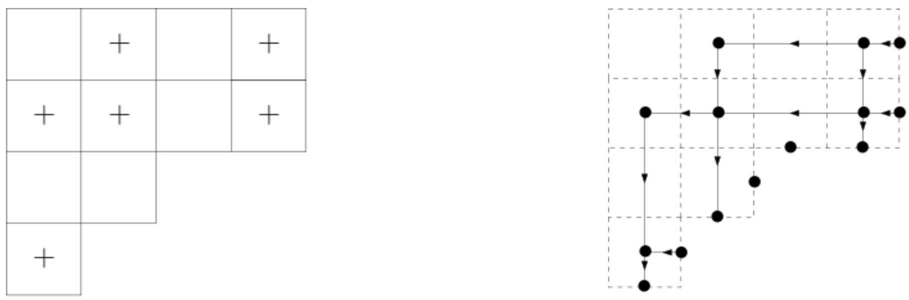
In other words, the ⅃-property says that a cell must be filled if both these conditions hold:

- It has a filled cell to the left in the same row
- It has a filled cell above in the same column.

Definition 28. (See [13], 6.3) The ⅃-*Graph* corresponding to a ⅃-*Diagram* is the directed graph obtained following these rules:

- Each filled box is a vertex.
- Also, put a vertex in the middle of each step of the boundary path.
- From each filled box draw an edge going down and another one going right until they reach a vertex of the boundary.
- All horizontal edges are oriented to the left and vertical ones downwards.

Example 14. Construction of the ⅃-graph associated to a ⅃-diagram.



(a) ⅃-Diagram.

(b) ⅃-Graph.

Figure 2.2: Example of ⅃-diagram and its corresponding ⅃-graph.

Definition 29. A *strictly northwest* cell of (i, j) is any cell (k, l) such that $i < k$ and $j > l$.

Definition 30. A *weakly northwest* cell of (i, j) is any cell (k, l) such that $i \leq k$ and $j \geq l$.

Observation 3. Let c be a cell in a ⅃-Diagram. It is clear by the ⅃-property, that if c has some strictly northwest + cell then the nearest strictly northwest + cell is unique. In this case, we will call this cell the *cover* of c.

This property is true for weakly northwest + cells as well, but now the nearest weakly northwest cell of c could be c itself. We will call this cell the *weak cover* of c.

2.1.2 From a matroid to a Grassmann necklace

Given a matroid $M = ([n], \mathcal{B})$ let G_j be the the lexicographically minimal basis of M with respect to the order $<_j$ in $[n]$ and $\mathcal{G}(M) := (G_1, G_2, ..., G_n)$.

Proposition 1. *([13], Lemma 16.3) For a matroid $M = ([n], \mathcal{B})$, the sequence $\mathcal{G}(M)$ is a Grassmann necklace.*

Note that if $i \notin G_i$ then $G_i = G_{i+1}$ because G_i will be the minimal basis in the order $<_{i+1}$ as well. If $i \in G_i$ and using the properties of a basis (B.3 has an important role) it can be seen that $G_{i+1} = (G_i \backslash \{i\}) \cup \{j\}$ and so it is indeed a Grassmann necklace.

Example 15. Given the representable matroid $M(A)$ with ground set $E = [4]$ and

$$A = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The Grassmann necklace $\mathcal{G}(M)$ is (G_1, G_2, G_3, G_4) with $G_1 = \{1, 3\}$, $G_2 = \{2, 3\}$, $G_3 = \{3, 4\}$, $G_4 = \{4, 1\}$.

Since a positroid is a matroid as well, its Grassmann necklace can be obtained in the same way. From now on we will focus again on positroids and the objects they are in bijection with, but it is worth noting that we can obtain a Grassmann necklace from any matroid.

Theorem 1. *([2], Theorem 4.3) Let $G = (G_1, G_2, \dots, G_n)$ be a Grassmann necklace with $|G_i| = d$, $G_i \subseteq [n]$. Then*

$$\mathcal{B}(G) := \left\{ B \in \binom{[n]}{d} \,\middle|\, B \geq_j G_j \text{ for all } j \in [n] \right\}$$

is the collection of bases of a rank d positroid $\mathcal{M}(G) := ([n], \mathcal{B}(G))$.

Moreover, for any positroid M we have

$$\mathcal{M}(\mathcal{G}(M)) = M.$$

We will see an interesting application of this theorem in 3.1.4.

2.1.3 From a ⅃-Diagram to a Grassmann necklace and back

We will now explain Suho Oh's algorithm to get a Grassmann necklace from a ⅃-diagram, which is described in [1].

- Let G_1 be $I(\lambda)$. Remember from Definition 26 this is the set of labels of the boundary path vertical edges.
- Make a path of cells from the top right to the bottom left cells following the boundary path and labelling them from $\overline{2}$ to $\overline{n}$ (see Figure 2.3).
- For each cell $\overline{i}$ labelled this way make a sequence following the next rules:
  - Start in the nearest weakly Northwest cell filled from $\overline{i}$ (which could be $\overline{i}$ itself).
  - Go to the nearest strictly Northwest cell filled and repeat this step until there are no more strictly Northwest cells filled.
- Let $G_i = (G_1 \setminus \{\text{rows involved in the path}\}) \cup \{\text{columns involved in the path}\}$

Observation 4. Since $G_1 = I(\lambda)$ is the set of rows, notice that $G_i = \{\text{rows } \textbf{not} \text{ used in the sequence}\} \cup \{\text{columns used in the sequence}\}$.

Observation 5. We stated that the cardinality of all sets in a Grassmann necklace must be the same. We still have to prove that the sets obtained by this algorithm are a Grassmann necklace, but for now notice that for these sets the cardinal property holds: $|G_i| = |G_j|$ for all $i, j \in \{1, \dots, n\}$.

This happens because the steps are taken to the nearest strictly Northwest filled cell, so for each step of the sequence we remove one element from G_1 (the row we are now using) and we add one element (the column we are now using). Therefore, no matter how many steps the sequence has, the cardinality will always be $|G_1|$.

Example 16. Let us find the Grassmann necklace associated to the ⅃-Diagram in Example 14. First we label the boundary path of the diagram and make the path of cells following this boundary path to label the boundary cells.

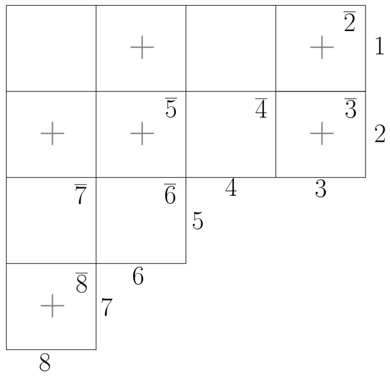


Figure 2.3: ⅃-Diagram from Example 14 with boundary path labels and boundary cells labels.

We can see in Figure 2.3 that the labels of the boundary path vertical edges (the labels of the rows) are $I(\lambda) = \{1, 2, 5, 7\}$.

We define $G_1 = I(\lambda) = \{1, 2, 5, 7\}$.

Now for G_2 we look at $\overline{2}$. Since there is a $+$ sign in this cell, it is its own weak cover, so we start the sequence at $\overline{2}$. In fact, we end the sequence there as well, because there are no $+$ cells strictly NW of $\overline{2}$ so we have

$$\text{Rows involved in this sequence } = \{1\}.$$

$$\text{Columns involved in this sequence } = \{3\}.$$

Then $G_2 = (G_1 \backslash \{1\}) \cup \{3\} = \{2, 3, 5, 7\}$.

For G_3 we look at $\overline{3}$. Again, since this cell is filled we start the sequence at this same cell.

Next step is the cell $(1, 6)$ because it is the cover of $\overline{3}$. The sequence ends there, so this time:

$$\text{Rows involved in this sequence } = \{1, 2\}.$$

$$\text{Columns involved in this sequence } = \{3, 6\}.$$

therefore $G_3 = (G_1 \backslash \{1, 2\}) \cup \{3, 6\} = \{3, 5, 6, 7\}$. Using the same procedure for $\overline{4}, \ldots, \overline{n}$ we get

$$G_4 = \{1, 5, 6, 7\} \quad G_5 = \{1, 5, 6, 7\} \quad G_6 = \{1, 5, 6, 7\} \quad G_7 = \{1, 5, 7, 8\} \quad G_8 = \{1, 2, 5, 8\}$$

It is left for the reader to check that this sets define a Grassmann necklace. The general case will be stated and proved in Theorem 2.

Now we will prove that the set $\{G_1, \dots, G_n\}$ is indeed a Grassmann necklace. This is stated without proof in [1], and in the original paper [11] the algorithm is defined implicitly inside a proof. Thus, we believe it is worth proving this result here so that the reader can follow it with the tools we have at this time. However, in order for the proof not to use any deeper results on matroid theory, it is not short and uses similar arguments several times. Let us start with a proposition that will be used in the proof.

Proposition 2. *Given a* ⅃*-Diagram, let* $+_u$ *and* $+_d$ *be two + cells in the same column without any + cell between them. Let* $+_u$ *be above* $+_d$ *and consider the NW sequences starting in these cells,* s_u *and* s_d *respectively.*

Then one of the following holds:

(a) The sequences differ by two rows and no columns.

More concretely, s_d *involves all rows in* s_u *except for one, and* s_u *involves all rows in* s_d *except for one (the row of* $+_d$*).*

(b) The sequences differ by one row and one column.

More concretely, s_d *involves all rows in* s_u *and one more (the row of* $+_d$*), and involves all columns in* s_u *and one more.*

Proof. Case 1: There are no + cells strictly NW of $+_u$.

In this case and if there is not a + cell strictly NW of $+_d$, then both sequences are trivial and differ by two rows and no columns, and thus *a)* holds.

If there is one + cell strictly NW of $+_d$, by the ⅃-property it must be in the same row as $+_u$. If there is a cover of $+_u$ call it $+_u^*$.

Hence, we are in the following configuration:

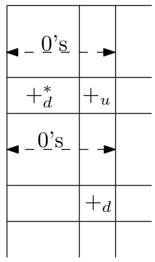


Figure 2.4: Case 1 configuration.

Therefore, since $+_u$ has no strictly NW + cells, the $+_d$ sequence ends after only one step. In this case, clearly *b)* holds.

Case 2: There is a + cell strictly NW of $+_u$.

First note that in this case there is a (unique) cover of $+_u$. Call it $+_u^*$.

Since $+_u^*$ is strictly NW of $+_d$ as well, we know that there is a unique cover of $+_d$, let it be $+_d^*$. Now there are two different options for $+_d^*$: either it is also $+_u^*$, or it is in the same row as $+_u$. Otherwise, as before, there would be a + cell between $+_u$ and $+_d$.

If $+_d^*$ is the same $+_u^*$, then the sequences are exactly the same from that cell on, and so the full sequences differ only in two rows and in no columns. Therefore *a)* holds.

Suppose then that $+_d^*$ is in the same row as $+_u$. Note that its column must be between the $+_u^*$ column (included) and $+_u$ column (not included), because of the $\lrcorner$-property. This can be seen in Figure 2.5a, where the grey zone represents the only possible cells for $+_d^*$ assuming it is not $+_u^*$.

Then if $+_d^*$ is not in the same column as $+_u^*$, the next step from $+_d^*$ will be $+_u^*$ and the sequences will be identical from this cell on (see Figure 2.5b). In this case, the sequence starting at $+_d$ involves a new row (row of $+_d$) and a new column (column of $+_d^*$), and so *b)* holds.

(a) Only possible places for u_d^* other than $+_u^*$. (b) If $+_u^*$ and $+_d^*$ are not in the same column.

Figure 2.5

But in the only case left, in which the cover $+_d$ is in the same row as $+_u$ and in the same column as $+_u^*$ (see Figure 2.6), then the configuration between $+_u^*$ and $+_d^*$ is the same as the one we had between $+_u$ and $+_d$, but we are strictly closer to the NW edge of the ⅃-Diagram. Repeating the process we could end in this situation once again for the nearest cells of $+_u^*$ and $+_d^*$, but since the ⅃-Diagram is finite, we cannot have this situation an infinite number of times: we will eventually come to Case 1 or another situation of Case 2 that we have already solved. Therefore, the proof is complete.

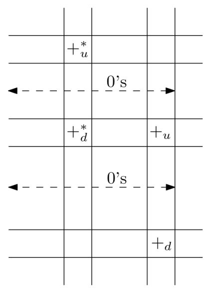


Figure 2.6: $+_d^*$ and $+_u^*$ in the same configuration as $+_d$ and $+_u$.

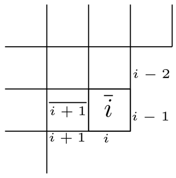
□

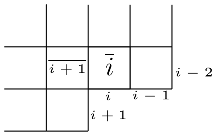
Theorem 2. *The sets obtained with Suho Oh's algorithm are indeed a Grassmann necklace.*

Proof. First of all, recall from Observation 4 that we can think of G_i for $i > 1$ as

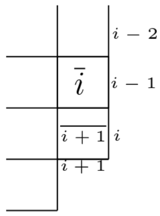
$$G_i = \{\text{rows not involved in the sequence starting from } \bar{i}\} \cup \{\text{columns involved in the sequence starting from } \bar{i}\}$$

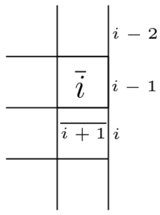
We will study two types of boundary cells differently: cells $\bar{i}$ such that $\overline{i+1}$ is in the same row and cells $\bar{i}$ such that $\overline{i+1}$ is in the same column. Examples can be seen in Figures 2.7 and 2.8 below.





(a) First type.





(b) Second type.

Figure 2.7: Examples of both types of cells.

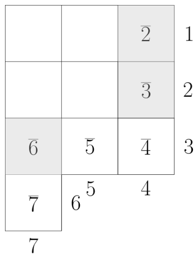


Figure 2.8: White boundary cells: First type. Grey boundary cells: Second type.

<u>First type:</u> $\bar{i}$ and $\overline{i+1}$ in the same row.

Note that in this kind of cells, the boundary edge number i corresponds to the column where $\bar{i}$ stands. In other words, $\bar{i}$ is right above i.

We have to check the Grassmann necklace condition in both possibilities, if $i \in G_i$ and if $i \notin G_i$.

- $i \in G_i$. It is obvious that $i \notin G_{i+1}$ because starting from $\overline{i+1}$ we would have to go East to get to the i column. It remains to see that we can create G_{i+1} adding only one element in $[n]$.

  Since i represents a column, $i \in G_i$ means that the column i is involved in the sequence starting from $\bar{i}$. The only possible step of the sequence that involves i is the first one, therefore the nearest weakly NW $+$ cell of $\bar{i}$ is in the same column i. Call this cell $+^*_i$. Note that rows between the row of $\bar{i}$ and the row of $+^*_i$ must be all zeroes by the ⅃-property. Figure 2.9a represents this situation.

  Therefore, the weak cover of $\overline{i+1}$ (which we will call $+^*_{i+1}$) must be in the row of $+^*_i$ or above, since all others are filled with zeroes.

  In particular, if $+^*_{i+1}$ is not in the same row as $+^*_i$ then it must be the nearest strictly NW $+$ cell of $+^*_i$. This configuration can be seen in Figure 2.9b. In this case sequences are identical from this step on, and the only thing that changed in the $\overline{i+1}$ sequence is that it does not involve column i anymore (which we had already stated) and it does not involve the row of $+_i^*$, because it jumps directly to its cover. Then $G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{row of } +_i^*\}$.

(a) Rows between $\bar{i}$ and $+_i^*$ all zeroes.

(b) When $+_{i+1}^*$ and $+_i$ are not in the same row.

Figure 2.9

Assume now that $+_{i+1}^*$ and $+_i^*$ are in the same row. Call $+_i^{**}$ the cover of $+_i^*$. Note that if $+_i^{**}$ is not in the same column as $+_{i+1}^*$ then $+_i^{**}$ is the cover $+_{i+1}^*$. In this case, which we can see in Figure 2.10a below, the sequences are identical from this step on, and $G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{column of } +_{i+1}^*\}$.

The remaining case is $+_{i+1}^*$ being in the same row as $+_i^*$ and in the same column as $+_i^{**}$, as shown in Figure 2.10b below.

(a) $+_i^{**}$ and $+_{i+1}^*$ not in the same column.

(b) $+_{i+1}^*$ and $+_i^{**}$ in the same column.

Figure 2.10: When $+_{i+1}^*$ and $+_i$ are in the same row.

In this case, $+_i^{**}$ and $+_{i+1}^*$ are in the same configuration as $+_u$ and $+_d$ in Proposition 2, so we can state that the sequences starting from this points differ in the row of $+_{i+1}^*$ and another row or column.

If we consider the original sequences starting at $\overline{i}$ and $\overline{i+1}$, the row of $+^*_{i+1}$ is involved in both and the rows and columns involved in the sequence from $\overline{i+1}$ are the same as the ones from $\overline{i}$ except it does not contain the column i anymore and it now contains a new column (if *b)* holds) or lacks a row (if *a)* holds), thus $G_{i+1} = (G_i \backslash \{i\}) \cup \{j\}$ for some j in $[n]$.

- $i \notin G_i$. In this case, the column i is not involved in the sequence starting from $\overline{i}$. That means that the weak cover of $\overline{i}$ is also a weak cover of $\overline{i+1}$. Therefore, the sequences are exactly the same, and thus $G_{i+1} = G_i$.

Second type. $\overline{i}$ and $\overline{i+1}$ in the same column.

Note that in this kind of cells the configuration is different. Instead of having the boundary edge labelled i under $\overline{i}$ in a horizontal edge, we now have the label i in the row below as a vertical edge, meaning that it represents a row in the diagram. See Figure 2.11 for an example.

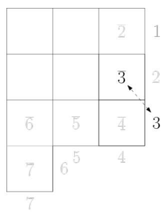


Figure 2.11: Example of label in second type of cells.

This time we do not need to check both possibilities of the Grassmann necklace because only one situation is possible. Since i labels the row below the cell $\overline{i}$, it cannot be involved in a NW sequence starting from $\overline{i}$, because it is south of $\overline{i}$. Therefore, in this type of cells we will always have $i \in G_i$.

Let $+^*_i$ and $+^*_{i+1}$ be, again, the weak covers of $\overline{i}$ and $\overline{i+1}$, respectively.

First we consider the case in which $+^*_i$ **is not in the same column as** $\overline{i}$. Note that the weakly NW cells of $\overline{i}$ are the weakly NW cells of $\overline{i+1}$ except for the row of $\overline{i+1}$. Therefore, either $+^*_{i+1} = +^*_i$ or $+^*_{i+1}$ is in the same row as $\overline{i+1}$, the row i.

If $+^*_{i+1} = +^*_i$, then both sequences are identical and so, again, $G_{i+1} = (G_i \backslash \{i\}) \cup \{i\}$.

If $+^*_{i+1}$ is in the same row of $\overline{i+1}$, the row label i, then it has to be in some column between the column of $+^*_i$ and the column of $\overline{i+1}$, both included, as shown in Figure 2.12.

If $+^*_{i+1}$ is not in the same column as $+^*_i$ then the next step from $+^*_{i+1}$ is $+^*_i$ and sequences coincide from this cell on. Therefore we have

$$G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{the column of } +^*_{i+1}\}$$

If, instead, $+^*_{i+1}$ is in the same column as $+^*_i$ (see Figure 2.13) we have, once again, the same composition as in Proposition 2.

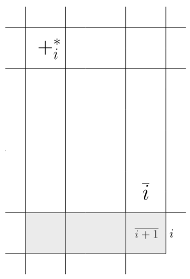


Figure 2.12: Possible locations for $+^*_{i+1}$.

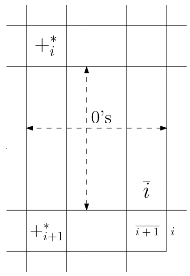


Figure 2.13: Configuration between $+^*_{i+1}$ and $+^*_i$ as required in Proposition 2.

So, using the Proposition:

- If the property that holds is *a)*, then

$$G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{the row in } s_u \text{ that is not in } s_d\}.$$

- If the property that holds is *b)*, then

$$G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{the column that is in } s_d \text{ and not in } s_u\}.$$

Now we consider the case in which $+^*_i$ **is in the same column as** $\bar{i}$ (and $\overline{i+1}$). In this case, and by the ⅃-property, either $+^*_{i+1}$ is $+^*_i$ or $+^*_{i+1}$ is $\overline{i+1}$ itself.

If $+^*_{i+1} = +^*_i$ then the sequences are identical and thus $G_{i+1} = G_i = (G_i \backslash \{i\}) \cup \{i\}$

On the other hand, if $+^*_{i+1} = \overline{i+1}$ then, as we can see in Figure 2.14, we have the configuration required in the Proposition 2.

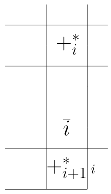


Figure 2.14: Configuration between $+^*_{i+1}$ and $+^*_i$ as required in Proposition 2.

Now,

- If the property that holds is *a)*, then

$$G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{the row in } s_u \text{ that is not in } s_d\}.$$

- If the property that holds is *b)*, then

$$G_{i+1} = (G_i \backslash \{i\}) \cup \{\text{the column that is in } s_d \text{ and not in } s_u\}.$$

□

Now we have a way to obtain a Grassmann necklace from a ⅃-Diagram, and we would like to have a way back as well.

We will show Agarwala and Freyer's algorithm from [1] to get the ⅃-diagram from a Grassmann necklace. The idea is to recover the sequences made in the previous algorithm by doing $G_1 \backslash G_i$ to obtain the rows used and $G_i \backslash G_1$ to obtain the columns used. The two subsets created this way have to be ordered one increasingly and one decreasingly because when the path is made in each step the row is smaller and the column greater than in the previous step. With this intuitive ideas in mind it is easy to see that the following algorithm is the inverse of Suho Oh's, for a detailed proof see [1].

The algorithm is the following:

- From G_1 we get the partition λ
- For each i, $2 \leq i \leq n$:
  - Write $G_1 \backslash G_i = \{a_1 > a_2 \cdots > a_n\}$ and $G_i \backslash G_1 = \{b_1 < b_2 \cdots < b_n\}$.
  - Fill the cells indexed by $(a_1, b_1), \ldots, (a_n, b_n)$.
- The cells that are not filled are left empty.

2.1.4 From a Ⅎ-Diagram to a positroid

To close the circle we would like to find a way to obtain a positroid from a Ⅎ-Diagram. We need to create then a totally nonnegative matrix which we will call A.

We will show Postnikov's construction which can be found in ([13], Section 4). Although we use the strictly necessary notions for our purpose, the reader can find in [13] more general notions and definitions as well.

We will actually get A from a Ⅎ-Graph, which we can easily obtain from a Ⅎ-Diagram as explained in Definition 28 and shown in Example 14. In order to work with these objects we need to define some concepts.

Let G be a Ⅎ-Graph associated to a Ⅎ-Diagram with partition λ that fits in a $k \times (n-k)$ rectangle. Remember each edge of the boundary path in the Ⅎ-Diagram has a vertex in the Ⅎ-Graph. Since these edges are labelled (recall Example 13) we have a labelling for the exterior vertices of the Ⅎ-Graph as well. Among these vertices we have two types: sources and sinks.

Definition 31. In this situation, the *source set* is $I = I(\lambda) \subseteq [n]$ (the set of exterior vertices corresponding to a vertical edge). The *sink set* is $\overline{I} = [n] \backslash I$ (the set of vertices corresponding to a horizontal edge).

Note this definition is the natural definition of source and sink for a directed graph: a source is a vertex with no incoming edges and a sink is a vertex with no outgoing edges.

Since the diagram fits in a $k \times (n-k)$ rectangle it would make sense to search a suitable matrix with these same dimensions. Instead we will construct a $k \times n$ matrix, but indeed there will be k special columns. We label the rows of this matrix by the labels of I and the columns of the matrix by the natural labelling $1, \ldots, n$.

Example 17. Figure 2.15 shows the labelling of the exterior vertices for the Ⅎ-Graph constructed in Example 14. In this case, the source set is $I = \{1, 2, 5, 7\}$ and the sink set is $\overline{I} = \{3, 4, 6, 8\}$.

Therefore, A will be a matrix with 4 rows labelled $1, 2, 5, 7$ and 8 columns labelled from 1 to 8.

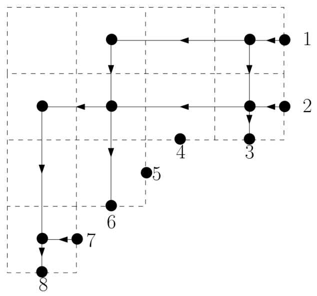


Figure 2.15: Ⅎ-Graph with labelled exterior vertices.

It only remains to fill this matrix. We do this following these rules ([13], Definition 4.6):

- Taking only the columns of A corresponding to the labels of the source set I, we must get the identity $k \times k$ matrix. Therefore, for $i, j \in I$ we define

$$a_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

- For $i \in I$, $j \in \overline{I}$, let p_{ij} be the number of different directed paths from the source i to the sink j. This number is of course nonnegative, and it is finite as well because there are no loops in a Ⅎ-Graph. We define $a_{ij} = (-1)^s p_{ij}$ where s is the number of sources strictly between the source i and the sink j.

These sign choices may look strange at first sight. Again, they are taken in order to fix determinants after column operations. Although we will not prove that this matrix is totally nonnegative, let us prove a particular case of maximal minor to get some intuition about this sign change. We will check that a maximal minor in which $k-1$ columns are sources is nonnegative.

Proposition 3. *Let A be a matrix constructed from a Ⅎ-Diagram following the rules above, and let I be the source set and $\overline{I}$ the sink set. Then, for any $j \in \overline{I}$ we have $\Delta_{(I \setminus \{i\}) \cup \{j\}}(A) = p_{ij}$. In particular, $\Delta_{(I \setminus \{i\}) \cup \{j\}}(A) \geq 0$.*

Proof. Note that $k-1$ columns of the determinant are identity matrix columns and they are ordered as in the identity matrix. However, identity column i is missing and instead we have column j, which is not necessarily in the i-th position. If we swap column j with its adjacent column until we place it on the

i-th position we need exactly s swaps, where s is the number of sources strictly between the source i and the sink j.

Therefore, the new determinant will be $(-1)^s$ times the original one. But once we have the j column in the i-th position we can use minor expansion for all the identity columns and, since the 1's will be in the diagonal there will be no more sign changes. Therefore, $\Delta_{(I\setminus\{i\})\cup\{j\}}(A) = (-1)^{2s} p_{ij} = p_{ij}$.

□

Example 18. (continuation of Examples 14 and 16). The matrix A obtained by this algorithm from the ⅃-Graph is

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

.

Note that looking only at the columns of $I = \{1, 2, 5, 7\}$, A_I is the identity matrix.

It is clear that some properties of the ⅃-Graph are reflected in this matrix. For example, since the source 5 is not connected to any vertex, $p_{5j} = 0$ for all $j \in \overline{I}$ because we cannot go to any sink starting from 5. Therefore, all entries in row 5 are zero except for $a_{5,5} = 1$.

In a similar way, since the sink 4 is not connected to any other vertex then the column 4 is filled with zeroes because $p_{i4} = 0$ for all $i \in I$.

It is easy to check with a simple code that this matrix is indeed totally nonnegative and thus it defines a positroid.

The fact that this algorithm produces a totally nonnegative matrix is proved in ([13], Theorem 4.8), the main idea is to study how do determinants change under arbitrary permutations of the columns that are not the identity matrix columns.

2.2 Operations over positroids

In this section we will see that positroids are closed under the following operations:

- Cyclic shifts of the ground set.
- Duality.
- Contraction.
- Deletion.

2.2.1 Cyclic shifts of the ground set

Let $M = M(A)$ be a positroid with A being a $k \times n$ totally nonnegative matrix.

Recall that being representable is a property that does no depend of the labelling of the ground set, since permuting columns in a matrix yields isomorphic matroids.

However, we need to be careful working with positroids. The restriction of A being totally nonnegative involves determinants, which change signs with column swaps. Therefore, in general positroids are not closed under permutations of the ground set. We will now study a particular kind of permutation that preserves the positroid condition: cyclic shifts.

Definition 32. Given $a, b \in [n]$ the *cyclic interval* is

- $[a, b] = \{a, a+1, \dots, b\}$ if $a \leq b$.
- $[a, b] = \{a, a+1, \dots, n, 1, \dots, b\}$ if $a > b$.

For $d \in [n]$, the *cyclic shift* of the (ordered) ground set $E = \{1, \dots, n\}$ is $E' = \{d, d+1, \dots, n, 1, \dots, d-1\}$, which can be seen as E ordered with $<_d$.

Proposition 4. *([2], Lemma 3.3) If M is a positroid on $E = \{1, \dots, n\}$ then M is also a positroid on $E' = \{d, d+1, \dots, n, 1, \dots, d-1\}$.*

Proof. Consider the simplest cyclic shift. Let $d = 2$ and $E' = \{2, 3, \dots, n, 1\}$. Note that any cyclic shift is a composition of this cyclic shift. Therefore, we will only prove the proposition for this case and the general case will follow.

Let A be a totally nonnegative $k \times n$ matrix such that $M = M(A)$, and let $v_1, \ldots, v_n$ be its column vectors. That means $A = (v_1, \ldots, v_n)$ with $v_i \in \mathbb{R}^k$.

We want to find a suitable A' such that $M(A')$ is also a positroid (in this case $M(A') = M$ because the independent sets will be the same). Intuition leads us to think that we should do something similar to a cyclic shift on the matrix columns, maybe $(v_2, v_3, \ldots, v_n, v_1)$. This is nearly correct, although a detail is missing. Since we are working with determinants, we need to be careful with the column swaps, as they will change their sign. To obtain $(v_2, v_3, \ldots, v_n, v_1)$ from $(v_1, \ldots, v_n)$ we used exactly $n-1$ swaps, therefore we consider $A' = (v_2, \ldots, v_n, (-1)^{k-1} v_1)$. It is clear now using basic linear algebra that we have found the matrix we were looking for: for any subset $I \subseteq E$ and its cyclic shift $I' \subseteq E'$ we have $\Delta_I(A) = \Delta_{I'}(A')$.

□

2.2.2 Duality

Proposition 5. *If M is a positroid on $[n]$, then M^* is also a positroid.*

Proof. This proof follows the one on the paper [2] but with a little more detail. First we consider A to be a full rank totally nonnegative matrix $d \times n$ and $M = M(A)$ the corresponding positroid. We can reduce the matrix to row-echelon form by multiplying rows by constants, adding different rows and multiplying rows by -1 when necessary. Thus, we can assume without loss of generality that A is in row-echelon form.

Then A has the identity matrix in columns $i_1, \ldots, i_d$ where $i_1 < \cdots < i_d$. We will label the rows by $i_1, \ldots, i_d$ from top to bottom. This way each row will be named by the place where it has a 1 of a column that forms the identity. The columns will be labelled from 1 to n. We call $J := \{i_1, \ldots, i_d\}$ and $J^c := [n] - \{i_1, \ldots, i_d\}$. Note that $|J^c| = n - d$.

Let us construct $A' = (a'_{ij})$ such that $M(A') = M^*$. The matrix A' will be a $(n-d) \times n$ matrix. First we label the rows with the elements of J^c and place the identity in the columns J^c.

In order for the dual to be a positroid, given an arbitrary subset $I \subset [n]$ with $n - d$ elements we need to check that $\Delta_I(A') \geq 0$. To do this we will see that this maximal minor can be calculated as a maximal minor from A, and the proof will be complete because A is a positroid. Specifically we will see that $\Delta_I(A) = \Delta_{I^c}(A')$.

Some of the entries of A are fixed. Those corresponding to columns J are identity matrix columns. The others, instead of calling them a_{ij}, for convenience we will name them $(-1)^s a_{ij}$ where s is the number of elements in J strictly between i and j.

We will find suitable signs so that $a'_{ij} = \pm a_{ji}$ makes $\Delta_I(A) = \Delta_{I^c}(A')$ true.

Let us try to calculate $\Delta_I(A)$. Note that for any $i \in I$ either $i \in I \cap J$ or $i \in I \cap J^c$. Columns $I \cap J$ in A_I are identity matrix columns and we can use minor expansion to reduce the determinant by the cofactors 1's. The resulting determinant may be multiplied by a -1 but it will have the remaining columns $(I \cap J^c)$ without the rows $I \cap J$.

Now for $\Delta_{I^c}(A')$ the same reasoning tells us that, with maybe a -1, the determinant comes down to columns $I^c \cap J$ without the rows $I^c \cap J^c$. These columns that remain, $I^c \cap J$, are rows in A and, in fact, they are the rows that remain in $\Delta_I(A)$ after this procedure. In the same way, rows that remained calculating $\Delta_{I^c}(A')$ are the columns that remained in $\Delta_I(A)$. Therefore our definition of A' being $a'_{ij} = \pm a_{ji}$ makes sense, and it would remain to see which sign should we choose to make it work.

This sign choice is exactly the same we made in 2.1.4 with sources and sinks in a ⅃-Graph. For $i \in J^c$ (sources) and $j \in J$ (sinks) let s' be the number of elements of J^c strictly between i and j. We define now $a'_{ij} := (-1)^{s'} a_{ji}$. The reasoning behind this sign choice is similar to the one in 2.1.4: s is the number of column swaps we need in the determinant in order to place the identity columns in their respective natural positions, where the expansion does not change the sign.

□

Example 19. Let $M = M(A)$ be a positroid with $n = 6, k = 3$ and $J = \{2, 4, 5\}$. Writing A in row-echelon form with the signs as we defined in the proof we get:

$$A = \begin{array}{c} \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & a_{2,3} & 0 & 0 & a_{2,6} \\ 0 & 0 & 0 & 1 & 0 & -a_{4,6} \\ 0 & 0 & 0 & 0 & 1 & a_{5,6} \end{bmatrix} \end{array}$$

. Values that do not have a name are fixed because of the row-echelon form conditions. For the ones that can take different values, let us check the sign we assign to them: for $a_{2,3}$ there are no elements from J strictly between 2 and 3 so the sign will be $(-1)^0 = 1$, for $a_{2,6}$ there are two elements (2 and 4) so the sign is $(-1)^2 = 1$, for $a_{4,6}$ there is just one (5) so $(-1)^1 = -1$ and for $a_{5,6}$ again $(-1)^0 = 1$.

Now we construct $A' = (a'_{ij})$. Since $|J| = 3, n = 6$, A' will have $6 - 3 = 3$ rows. The labels for the rows will be $J^c = \{1, 3, 6\}$ and the columns 1, 3 and 6 will be the identity matrix. Finally, the other entries are $a'_{ij} := (-1)^{s'} a_{ji}$, where s' is the numbers of elements in $\{i + 1, \dots, j - 1\} \cap J^c$. This creates the following matrix

$$A' = \begin{array}{c} \\ 1 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,3} & 1 & 0 & 0 & 0 \\ 0 & -a_{2,6} & 0 & a_{4,6} & a_{5,6} & 1 \end{bmatrix} \end{array}$$

.

Signs for this entries can be checked as above but considering the set J^c instead of J.

2.2.3 Contraction and deletion

Lemma 1. *If M is a positroid on $[n]$, then $M/\{1\}$ is also a positroid.*

Proof. Given $M = M(A)$ positroid we can assume A is a $k \times n$ matrix of rank k in reduced row-echelon form. As the matrix is in row echelon form to contract $\{1\}$ works the same way that works in general for representable matroids. If $\{1\}$ is dependent it implies it is a null vector and then $M/\{1\}$ is the positroid that arises from the matrix A without its first column. If $\{1\}$ is independent $M/\{1\} = M(A')$ is the positroid that arises from the matrix A without its first row and column. Observe that as A is in row-echelon form $\{1\}$ is $e_1 \in R^k$ and that any maximal minor of A' has the same value than the maximal minor of A in the same columns plus the column $\{1\}$. □

Proposition 6. *Given a subset $S = \{s_1, \dots, s_j\} \subseteq [n]$, if M is a positroid then M/S is also a positroid.*

Proof. Note that if we have two subsets $S_1, S_2 \subseteq [n]$ where $S_1 \cap S_2 = \emptyset$ then $M/(S_1 \cup S_2) = (M/S_1)/S_2$ so $M/S = ((M(s_1)/s_2)/\dots)/s_j$ and it is enought to prove that M/S is a positroid for $S = \{s_i\}$. As we already have proved that positroids are closed under cyclic shifts we can assume $\{s_i\} = \{1\}$. Now using the previous lemma we are done. □

As positroids are closed under duality and contraction they are also closed under deletion.

2.2.4 Operations in terms of decorated permutations

Now that we know that positroids are closed under these operations and that positroids are in bijection with decorated permutations and Grassmann necklaces, a natural question arises. What relations are there between the decorated permutations and the Grassmann necklaces from the positroid before and after these operations? In other words, how do cyclic shifts, duality, contraction and deletion operate over decorated permutation and Grassmann necklaces?

In papers [10] and [9] Suho Oh studies matroid operations via decorated permutations and Grassmann necklaces and obtains the results we are looking for. Here we will only give the clues and the key results, for further explanation the reader should go to the original papers.

For duality we have the following result:

Theorem 3. *([10], Corollary 13) Let M be a positroid indexed by $\pi^{:} = (\pi, col)$. Let $M^{'}$ be the dual matroid of M. Then $M^{'}$ is indexed by $(\pi^{-1}, -col)$.*

For contraction, first Oh ([9]) finds how the Grassmann necklace corresponding to $M/\{j\}$ looks like with respect to the one corresponding to M, and then he studies the cases to determine the decorated permutation obtained from that Grassmann necklace to avoid computing all the sets of the Grassmann necklace in order to get to the permutation. In fact, instead of $M/\{j\}$ he actually uses $M' = \{I \in M | j \in I\}$ (here the positroid is defined by its bases) and states that the decorated permutation of M' will be the same as the one from $M/\{j\}$ except for the color of j. Let $\{K_1, \ldots, K_n\}$ be the Grassmann necklace associated to M'. Oh identifies them in the following way:

Proposition 7. *([9], Proposition 7) If $j \in G_a$, we have $K_a = G_a$. If not, then we have $K_a = (G_a \backslash \{max_a(G_a \backslash G_j)\}) \cup \{j\}$.*

With this result the decorated permutation is determined as well, but with a deeper study on the possibilities of the differences between K_a and K_{a+1} he determines how does it look like.

Proposition 8. *([9], Theorem 8) Let M be a positroid indexed by $\pi^{:} = (\pi, col)$. Let $M^{'}$ be $M/\{j\}$ for $j \in [n]$ with decorated permutation $\mu^{:} = (\mu, col^{'})$. Then we can obtain $\mu^{:}$ directly from $\pi^{:}$ in the following way:*

If $\pi(j) = j$ and $col(j) = -1$, then $\mu^{:} = (\pi, col^{'})$ where $col^{'}(j) = 1$ and $col^{'}(i) = col(i)$ for all $i \neq j$.

If $\pi(j) = j$ and $col(j) = +1$, then $\mu = 123 \dots n$ and $col(i) = 1$ for all $i \in [n]$.

If $\pi(j) \neq j$, then $\mu^{:}$ is obtained with the next algorithm.

1. *Initial states: $\mu \leftarrow \pi$, $col' \leftarrow col$, $\mu(j) \leftarrow j$, $col'(j) = 1$, $a \leftarrow j + 1$ and $q \leftarrow \pi(j)$.*

2. *While $\pi(a) \neq j$ do:*

   - *If $q = a$ or $q <_{a+1} \pi(a) <_{a+1} j$ then*
     - *Set $\mu(a) \leftarrow q$ and set $q \leftarrow \pi(a)$.*
     - *If $\mu(a) = a$ then set $col'(a) = 1$.*
   - *$a \leftarrow a + 1$ (modulo n).*

3. *Set $\mu(a) \leftarrow q$.*

2.2.5 Minors of Positroids

We now know that positroids are closed under deletion and contraction, so if we have a matroid and by doing these opeartions we get a different matroid wich is not a positroid we will know the original one was not a poisitroid either.

Definition 33. A *minor* of a matroid M is another matroid obtained from M by a series of deletion and contraction operations.

Definition 34. Given $\mathcal{M}$ a collection of matroids that are closed under contraction and deletion, a matroid N is an *excluded minor* for $\mathcal{M}$ if N is not in $\mathcal{M}$ but all its minors are.

Hence, a matroid M is in $\mathcal{M}$ if M does not contain any of the excluded minors of $\mathcal{M}$.

If a minor of a matroid M is not a positroid, then we know M is neither. That leads to the concept of excluded minors. An excluded minor of a positroid is any matroid that is not a positroid. Observe that positroids can be characterized by excluded minors.

In ([10], Section 4) Suho Oh studies the excluded minors of positroids and concludes with the next theorem.

Theorem 4. *([10], Theorem 16) Let $\mathcal{L}$ be $\{\{12, 34, 13, 23, 14\}, \{12, 34, 14, 23, 24\}, \{12, 34, 14, 23\}\}$. These are the independent sets of matroids of rank 2 over $E = \{1, 2, 3, 4\}$ that are not positroids.*

Moreover, a matroid is a positroid if and only if it has no minors among $\mathcal{L}$.

Chapter 3

Two families of positroids

In this chapter we are going to study the class of lattice path matroids and more briefly we are going to introduce the class of series-parallel graphs.

3.1 Lattice path matroids

Lattice path matroids are an interesting family of positroids with a very intuitive geometrical interpretation that have been studied from different approaches (see [5] and [4]). They were first introduced in [3] and we believe that their relationship with concepts like Grassmann necklaces, transversal matroids and excluded minors make the study of lattice path matroids a good way to put these concepts together.

We have seen that positroids are closed under minors and that transversal matroids are not. In this section we will see that lattice path matroids are both positroids and transversal matroids, and then naturally we will study their behaviour under minors. We will see that LPM are indeed closed under minors. We will also study relations between LPM and Grassmann necklaces.

Definition 35. A *lattice path* is a sequence $p_1, \dots, p_n$ of steps that are either North or East. We typically think that a lattice path forms a polygonal line in $\mathbb{Z}^2$ by drawing the steps starting at the origin, or at any other point of $\mathbb{Z}^2$.

Notation. If a North step joins the points (i, j) and $(i, j+1)$, we label this step with the integer $i+j+1$; similarly, an East step joining (i, j) and $(i+1, j)$ will also be labelled $i+j+1$ (see Figure 3.3). In what follows, whenever we speak of the set of North steps of a path we mean the set of labels of these steps.

Definition 36. Let $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$ be two lattice paths starting at origin and ending at (m, r), with $n = m + r$. Assume Q is never over P. Let $N_P = \{k_1, \dots, k_r\}$ and $N_Q = \{l_1, \dots, l_r\}$ be the labels of the North steps of paths P and Q, respectively. Let $\mathcal{A}$ be the intervals between these steps, $\mathcal{A} = \{[k_1, l_1], \dots, [k_r, l_r]\}$. As we have already stated, the partial transversals of $\mathcal{A}$ are the independent sets of a matroid $M[\mathcal{A}]$. A *lattice path matroid* is any such matroid.

Observation 6. Lattice path matroids have also a very intuitive geometrical approach. In the same conditions as in Definition 36, note that paths P and Q create a bounded region in $\mathbb{Z}^2$. For each lattice path L contained in this region (that is with Q never over L and L never over P), let N_L be its set of North steps. We claim that these sets $\{N_L\}_L$ are the independent sets of the matroid we defined in Definition 36. As bases are maximal independent sets, here they are the North steps from the maximal lattice paths (the ones from origin to (m, r)).

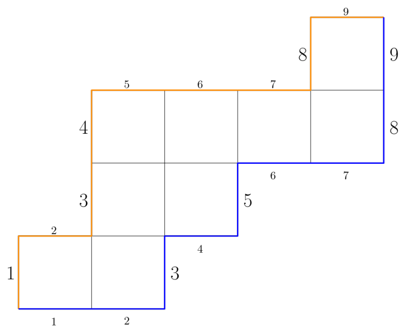
This claim is not hard to check, let us give the clues:

On one hand, given a lattice path in the region between P and Q its North steps must be a partial transversal over $\mathcal{A}$. This is because each North step of L will have a North step from P to its left and a North step from Q to its right, and so it will be in $[k_i, l_i]$ for exactly one i.

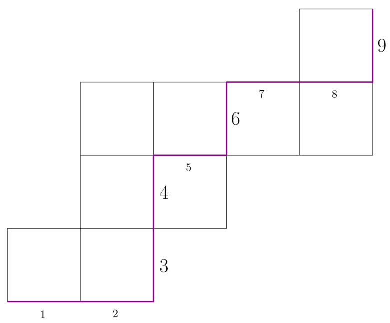
On the other hand, given a partial transversal $\{e_1, \dots, e_r\}$ over $\mathcal{A}$ there is a unique lattice path between P and Q having exactly these North steps. Labelling the vertical steps between k_i and l_i we use the one corresponding to e_i, and we add the horizontal steps needed for it to be a lattice path.

Observation 7. At this point we already see the importance of the labelling of North steps over East steps. This is why in most of the figures in this section the labels of the North steps will be bigger than the ones corresponding to the East or horizontal steps.

Example 20. Let P and Q be the lattice paths shown in Figure 3.1a in orange and blue, respectively.



(a) Example of a lattice path matroid.



(b) In purple, a lattice path between P and Q.

Figure 3.1

For each lattice path inside the region bounded by P and Q, its set of North steps is an independent set. For example, Figure 3.1b shows that $\{3, 4, 6, 9\}$ is an independent set in the lattice path matroid.

In this example, North steps of lattice paths P and Q are $N_P = \{1, 3, 4, 8\}$ and $N_Q = \{3, 5, 8, 9\}$, respectively. Now define $\mathcal{A} = \{[1, 3], [3, 5], [4, 8], [8, 9]\}$ as in Definition 36. It is clear that the set $\{3, 4, 6, 9\}$ is a partial transversal of $\mathcal{A}$.

Proposition 9. *([11], Lemma 21) A lattice path matroid is a positroid.*

The following proof can also be found in [11].

Proof. To see that a lattice path matroid is a positroid we will construct a matroid with all its maximal minors nonnegative. Recall from Example 4 that any transversal matroid is representable over $\mathbb{R}$. Therefore, we already know that a lattice path matroid is a representable matroid. The construction of the matrix will be similar to the one in Example 4 but the non-zero elements will be chosen in order for the matrix to be a Vandermonde matrix.

Let $N_P = \{k_1 < \cdots < k_r\}$ and $N_Q = \{l_1 < \cdots < l_r\}$ with $N_P \leq N_Q$ be the north steps of P and Q. Let V be a $r \times n$ Vandermonde matrix where:

$$v_{ij} = \begin{cases} x_i^{j-1} \text{ if } k_i \leq j \leq q_{k_i}, \\ 0 \text{ otherwise.} \end{cases}$$

It remains to assign values to $x_1, \dots, x_r$. Let $x_1 > 1$ and $x_{i+1} = x_i^{r^2}$ for all $i \in [r-1]$. The determinant of a maximal minor formed by r columns will be nonnegative if and only if the diagonal entries are non-zero. This will happen when those columns form a basis. Otherwise the determinant will be zero. The idea to check this is to transform this maximal minor into an upper diagonal matrix so that its determinant is the product of the diagonal elements. □

3.1.1 Grassmann necklaces and LPMs

Now we know that lattice path matroids are positroids, which are in bijection with Grassmann necklaces, and from previous sections we know how to obtain a Grassman necklace from a matroid. Some questions arise from these facts:

- Do Grassman necklaces obtained from LPM's have any specific characteristics?
- Given a Grassman necklace, how can we say if it can be obtained from a LPM?

In this section we will try to answer these questions which, to our knowledge, have not been investigated before.

Recall from 2.1.2 that a Grassmann necklace can be constructed with the minimal basis of order i for each $i \in E$. Also recall from Definition 36 that the bases in a LPM are obtained from maximal lattice paths inside the bounded region.

With this in mind, let us construct the Grassmann necklace associated to a LPM. Let M be a LPM. We will be using the same notations as in Definition 36. Label the North steps of each row from k_i to l_i as in Figure 3.2.

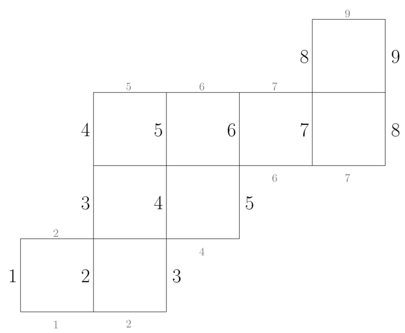


Figure 3.2

For $1 \leq i \leq n = m + r$:

1. Find the North step with label i in the lowest possible row. If there is no such step, search for $i+1$, $i+2$ and so on until there is a North step.

2. From there, make a lattice path to (m, r) prioritizing North steps over East steps, always inside the bounded region.

3. Like a continuation from this path, make a lattice path from $(0, 0)$ to the bottom of the step where we started in Step 1, again prioritizing North steps.

4. Joining these lattice paths we get a maximal lattice path. Label the steps of this path, and let G_i be its North steps.

We claim that $\{G_1, \ldots, G_n\}$ is a Grassmann necklace. Let us show an example first.

Example 21. For $i = 4$ in the LPM shown in Figure 3.2, we would create this path:

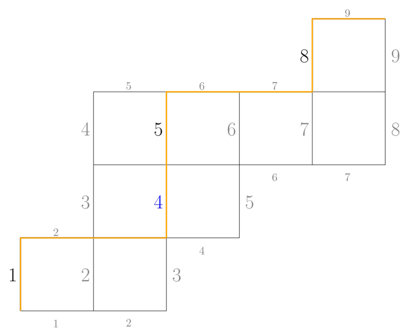


Figure 3.3: In blue, the starting step. In orange, the lattice path. $G_4 = \{1, 4, 5, 8\}$.

Proposition 10. *The sets $\{G_1, \ldots, G_n\}$ are indeed a Grassmann necklace.*

Proof. Let $i \in \{1, \ldots, n\}$.

First, if $i \notin G_i$ it means that in Step 1 we could not find a North step with label i. In that case we search for $i+1$, $i+2$ and so on until we find a North step and start the path there, so clearly $G_i = G_{i+1}$.

If, otherwise, $i \in G_i$ it means we started the i-th path to (m, r) -let it be P_i- at some North step labeled i. Two possibilities arise for $i+1$:

- If i and $i+1$ are in the same row, both paths P_i and P_{i+1} go North parallel until P_i cannot go North anymore, in which case it goes East and they coincide from that point on. The labels of these North steps are the same except for i, which is in P_i but not in P_{i+1}, and $j+1$ -where j is the label of the last North step that P_i could step on before joining P_{i+1}- which is in P_{i+1} but not in P_i.

From $(0,0)$ to the steps i and $i+1$ we have taken the North steps are the same. Therefore, $G_{i+1} = (G_i \backslash \{i\}) \cup \{j+1\}$.

- If i and $i+1$ are not in the same row, the paths to (m,r) will be identical. Then, from $(0,0)$ the only possible difference between the paths to i and $i+1$ is i, because if i is not the first North step of its row we will not use it in the path to $i+1$. Therefore, $G_{i+1} = (G_i \backslash i) \cup \{j\}$ where j is the label of the first North step in the row of i.

$\square$

With this tool we can now face the second question we asked: which Grassmann necklaces can be obtained this way from a LPM?

Lemma 2. *Let $G = \{G_1, \dots, G_n\}$ be a Grassmann necklace with $G_i = \{g_{i,1}, \dots, g_{i,r}\}$ and $n = m + r$. There exists a lattice path starting at the origin and ending at (m,r) and with set of North steps G_1. Call it P.*

For $1 \le i \le n$ order G_i (with the natural order $<_1$). For $1 \le s \le r$ let $q_s = \max_{1\le i\le n} g_{i,s}$ be the maximum among the s-th elements of each G_i. There exists a lattice path starting at the origin with set of North steps $\{q_1, \dots, q_r\}$. Call it Q.

Then, P and Q define a lattice path matroid M.

Proof. For convenience, in the proof we will refer to $\{g_{1,1}, \dots, g_{1,r}\}$ (the North steps of P) as $\{p_1, \dots, p_r\}$.

Existence of P and Q.

In order for P and Q to be lattice paths the only thing we need is the ordered sets $\{p_1, \dots, p_r\}$ and $\{q_1, \dots, q_r\}$ not to have repetitions. This is true for $\{p_1, \dots, p_r\}$. Now for $\{q_1, \dots, q_r\}$, remember $q_s = \max_{1\le i\le n} g_{i,s}$, so for any q_s there exists $m \in \{1, \dots, n\}$ such that $q_s = g_{m,s}$. Then, $g_{m,s+1} > g_{m,s}$ and

$$q_{s+1} = \max_{1\le i\le n} g_{i,s+1} \ge g_{m,s+1} > g_{m,s} = q_s$$

.

These paths define a LPM.

By construction, the label of the i-th North step of Q is the maximum among a set that includes the label of the i-th set of P. Therefore Q will never be above P so they define a LPM. $\square$

Observation 8. Although we proved this lemma in order to prove Proposition 11, it is worth noting that we have seen that each Grassmann necklace (and thus, each positroid) has a lattice path matroid canonically associated. We will not study this any further here, but it could be interesting to see if any special properties hold among positroids that share this canonically associated LPM, or among positroids modulo the relationship of having the same canonically associated LPM.

Proposition 11. *In the same conditions as in Lemma 2, if G is obtained from a lattice path matroid, it must be from M.*

Proof. First of all, if there is a LPM defined by some lattice paths P, Q (with Q never above P) that produces G, then the North steps of P have to be G_1, by construction. However, the North steps of Q are not so easy to determine.

Note that if we order each G_i then the s-th element will tell us which of the North steps we used in the s-th row of the LPM. Therefore, the set $\{g_{1,s}, \dots, g_{n,s}\}$ contains all the North steps used in the s-th row. The North step of Q in the s-th row has to be the greatest among these.

It remains to see that North steps of Q will always appear in some G_i. This is true by construction because if e is a North step of Q then it is the North step named e in the lowest possible row. $\square$

With this proposition, in order to see if a Grassmann necklace can be obtained from a LPM one can forge P and Q as in Proposition 11 and check if the result is a LPM that creates the necklace. Let us see an example to clarify this.

Example 22. Let G be the Grassmann necklace defined by $G_1 = \{1, 4\}$, $G_2 = \{2, 4\}$, $G_3 = \{3, 4\}$, $G_4 = \{4, 5\}$, $G_5 = \{1, 5\}$, and $G_6 = \{1, 6\}$.

We are looking for a LPM defined by two lattice paths P and Q with Q never above P. Following the proposition we set P to be the lattice path with north steps $\{1, 4\}$. For Q, we calculate $q_1 = \max\{1, 2, 3, 4, 1, 1\} = 4$ and $q_2 = \max\{4, 4, 4, 5, 5, 6\} = 6$. Then we set Q to be the lattice path with North steps 4 and 6 and we get the LPM shown in Figure 3.4.

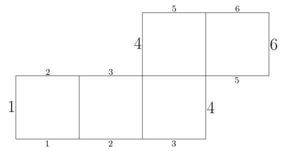


Figure 3.4

It is not hard to check that this LPM produces G using the method we defined at the beginning of the section.

3.1.2 From a LPM to a ⅃-Diagram

We know how to obtain a ⅃-Diagram from a Grassmann necklace and a how to obtain a Grassmann necklace from a LPM. Of course, composing these procedures we get a method to obtain a ⅃-Diagram from a LPM, but this composition does not look very clear at first sight. Here we will see how this method operates in a very simple way, which we find interesting and, to our knowledge, has not been studied from this perspective before.

Theorem 5. *Let M be a LPM defined by paths P and Q. The ⅃-Diagram obtained from M using the methods we defined in the previous sections is created in the following way (see Figure 3.5):*

Rotate the path diagram of M 180º as a figure and complete North-West of it as if it was a rectangle. In this way we obtain a Young Diagram with boundary path P. Now fill the cells in the region defined by the rotated paths P and Q, and leave the other cells empty.

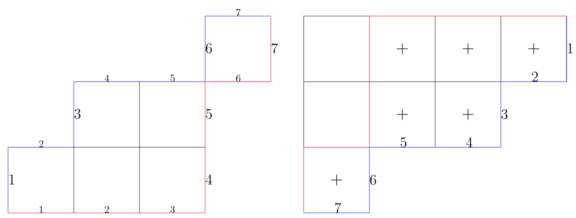


Figure 3.5: A LPM and its associated ⅃-Diagram. Blue: P and rotated P. Red: Q and rotated Q.

Proof. Let $G_1, \dots, G_n$ be the Grassmann necklace obtained from the LPM with $G_i = \{g_{i,1}, \dots, g_{i,r}\}$. We will see that, when obtaining the ⅃-Diagram from this necklace, the cells generated with $G_i \backslash G_1$ in increasing order and $G_1 \backslash G_i$ in decreasing order are the ones and the only ones that in the LPM have a right adjacent North step i (before the rotation).

In the path diagram, if a North step labelled i with a left adjacent cell appears k times then G_i has exactly k elements different from the ones of G_1. First let us see it for $k = 2$. Observe that in this case if i appears in the row $g_{1,j}$ it will also appear in the row $g_{1,j+1}$, one position to the left. Therefore there will be a $i + 1$ North step right above i. Let G_i be $\{i, i+1, g_{i,3}, \dots, g_{i,r}\}$.

- Case $i, i+1$ different to all $g_{1,*}$.

  With this condition $g_{i,3}$ will be a North step of G_1 because otherwise it would have an adjacent left cell. This left cell would have and adjacent left North step called $i+1$. We know $i+1$ is not from G_1 so again it would have an adjacent left cell with adjacent North step i with a cell to its left by the same reasoning. Therefore we would have $k = 3$. So in G_i we have two and only two elements different from the ones of G_1.

- Case i different to all $g_{1,*}$ and $i+1$ labelled with the same name as some $g_{1,*}$. In this case the North step $i+1$ will not be part of P so the label $i+1$ in a North step will have to appear in the row above. This means that we will have a North step labelled with $i+2$ and $g_{i,3} = i+2$. If $i+2$ is different to all $g_{1,*}$ we have the previous case and it is done, if it is labelled with the same name as some $g_{1,*}$ we go back to this case and the same argument applies.

- A similar argument works for i labelled with the same name as any $g_{1,*}$.

The proof for any $k \geq 2$ is analogous to this one.

The row labels of $G_1 \backslash G_i$ will be consecutive. They will be the ones where the left cells of the North i labelled steps are.

Once we have the subsets $G_i \backslash G_1$ and $G_1 \backslash G_i$ we order the second one in decreasing order to fill the Ⅎ-Diagram. Once this is done the cells to fill are the ones we wanted. For example the cell of the lowest row in the LPM will be the one with the highest value of $G_i \backslash G_1$ as column. It is because it is the label of the East step of G_1 that will label of the column in the Ⅎ-Diagram. The row labels will be the same by construction.

□

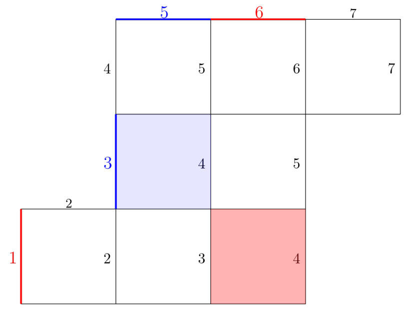


Figure 3.6: Example of how labels are changed.

3.1.3 Operations over lattice path matroids

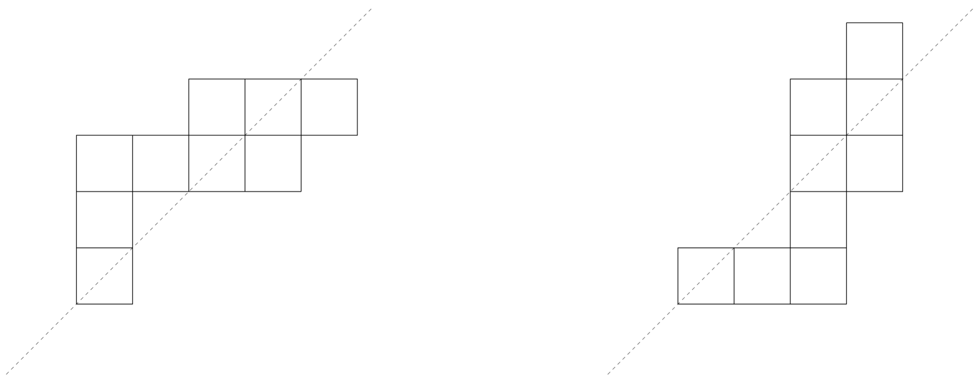
As we did for many different families of matroids, we will study wether lattice path matroids are closed or not under duality, deletion, contraction and direct sum. Formal proofs for the next properties are found in ([5], Theorem 3.1) and in ([3], Theorem 3.5 and Theorem 3.6).

Proposition 12. *Lattice path matroids are closed under duality and direct sums.*

To see lattice path matroids are closed under duality we want to see if chosiong the complementary of bases as bases make a lattice path matroid. That is easy to obtain if we reflect the diagram with respect to the line $x = y$.

To see lattice path matroids are close under direct sum if we sum two matroids $A \oplus B$ we need the independent sets to be the union of an independent set of A and one of B. That is easy to obtain if we draw B having as origin the ending of A.

The construction of the dual and the direct sum is illustrated in the next figures.



(a) Example of a lattice path matroid.

(b) The dual of the lattice path matroid.

Figure 3.7

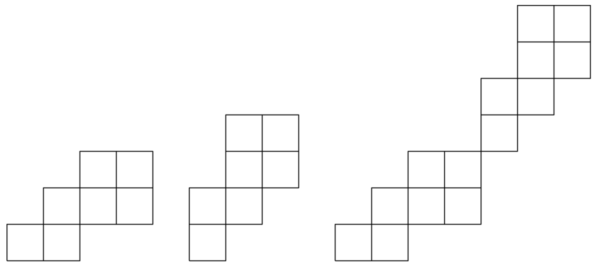


Figure 3.8: Example of two lattice path matroids and their direct sum.

In order to prove that lattice path matroids are closed under deletion, we will use the following lemma.

Lemma 3. *([3], Lemma 5.2) Let M be a lattice path matroid as defined in Definition 36 with transversal presentation $\{N_1 = [k_1, l_1], \ldots, N_r = [k_r, l_r]\}$ and let $\{b_1, \ldots, b_r\}$ be a basis of M. Assume $b_1 < \cdots < b_r$. Then b_i is in N_i for all $1 \leq i \leq r$.*

Proof. With the representation of the LPM this lemma is very intuitive. Remember that a basis is the set of North steps of a lattice path from $(0, 0)$ to (m, r). This lattice path will have one North step in each row i, whose label will be between the North step labels of paths P and Q in this row, which are k_i and l_i. □

Observation 9. Note that this lemma is equivalent to what we claim in Observation 6.

Proposition 13. *Lattice path matroids are closed under deletion and contraction.*

Proof. Again let M be a LPM with transversal presentation

$$\mathcal{A} = \{N_1 = [k_1, l_1], \ldots, N_r = [k_r, l_r]\}.$$

If we want $M \backslash x$ the intuitive idea would be to have $[m + r] - x$ as ground set and

$$\mathcal{A}' = \{N_1 - x = [k_1, l_1] - x, \ldots, N_r - x = [k_r, l_r] - x\}$$

as presentation. This is indeed a presentation of $M \backslash x$ but it does not show that $M \backslash x$ is a lattice path matroid. The only condition on $\mathcal{A}'$ for $M \backslash x$ to be a LPM is that the least (respectively, greatest) elements of each set $N_i - x$ have to be strictly bigger than the least (respectively, greatest) elements of the previous set $N_{i-1} - x$. There are only two ways in which this cannot happen:

1. If x is the least element of N_i and $x+1$ is the least element of N_{i+1}. In the LPM M this would mean two consecutive North steps in path P.

2. If x is the greatest element of N_i and $x-1$ is the greatest element of N_{i-1}. In the LPM M this would mean two consecutive North steps in path Q.

Let us study case 1 first. Let B be a base of $M\backslash x$ containing $x+1$. We claim that $B = \{b_1 < \cdots < b_r\}$ can be matched with $\mathcal{A}$ in a way that $x+1$ is not in N_{i+1}. Note that B is a basis of M as well (one that does not contain x), thus we can use Lemma 3 to say that we have a matching with $b_k \in N_k$ for all k. Now if $x+1$ was matched in N_{i+1} then the element matched in N_i would have to be x, which is a contradiction. Finally, with this we can say that replacing N_{i+1} for $N_{i+1} - (x+1)$ we still have a presentation of $M\backslash x$. Repeating this argument we can replace N_{i+2} for $N_{i+2} - (x+2)$ if necessary and so on. The argument for Case 2 is analogous to this one, and finally we get a presentation of $M\backslash x$ with the conditions we were looking for in order for it to be a lattice path matroid.

□

As lattice paths matroids are close under duality and deletion they are also closed under contraction.

3.1.4 Excluded minors of LPM

Remember from Definition 33 that a *minor* of a matroid M is another matroid obtained from M by a series of deletion and contraction operations. Also recall Definition 34 of excluded minors.

As lattice path matroids are closed under deletion and contraction they are closed under minors. It is natural to ask, then, are the excluded minors known? The answere is yes, they are known and studied in [4].

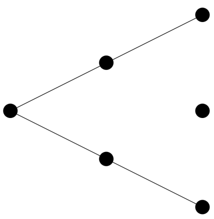
But why is this a natural and interesting question? Suppose we are given a matroid M and we use deletion and contraction to find a minor M' that is an excluded minor for LPMs. Then, since LPMs are closed under minors, we can state that M is not a LPM either. This is very similar to Graph Theory and excluded minors for planar graphs, where we can state that a finite graph is not planar if and only if it has K_5 or $K_{3,3}$ as minors.

The list of excluded minors of lattice path matroids we will show and its corresponding proof is a result from Bonin and can be found in [4], Theorem 3.1. We need to define some concepts in order to understand the list of excluded minors of lattice path matroids.

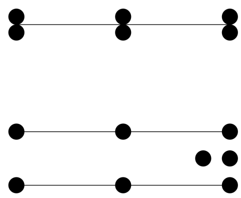
In this list, $P_n = T_n(U_{n-1,n} \oplus U_{n-1,n})$ is the truncation (recall Definition 11) to rank n of the direct sum of two n-circuits and P_n' is the truncation to rank n of two n-circuits with one common element. We will also use $+$ to denote adding a new point without increasing the rank and introducing the minimum possible number of dependencies (this operation is called *free extension*, see [12] for a more precise definition).

Finally, these are the excluded minors of lattice path matroids:

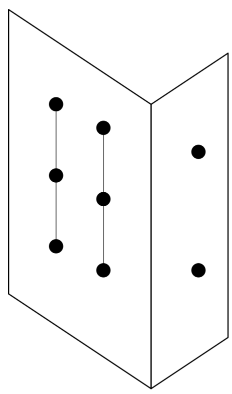
- $A_n = P_n' + x$ for $n \geq 3$. See Figure 3.9a.
- $B_{n,k} = T_n\left(U_{n-1,n} \oplus U_{n-1,n} \oplus U_{k-1,k}\right)$ and its dual $C_{n+k,k}$ for $n \geq k \geq 2$. See Figure 3.9b.
- $D_n = \left(P_{n-1} \oplus U_{1,1}\right) + x$ and its dual E_n for $n \geq 4$. See Figure 3.9c.



(a) A_3



(b) $B_{2,2}$ (top) and $B_{3,2}$ (bottom).

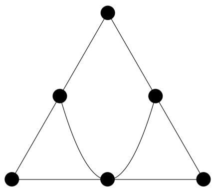


(c) D_4.

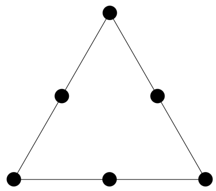
Figure 3.9: Particular cases of A_n, $B_{n,k}$ and D_n.

- The rank 3 wheel W_3. This is the graphic matroid of the complete graph K_4, which is also called wheel graph W_3. See Figure 3.10a.

- The rank 3 whirl W^3. See Figure 3.10b.



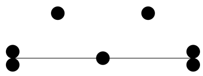
(a) W_3.



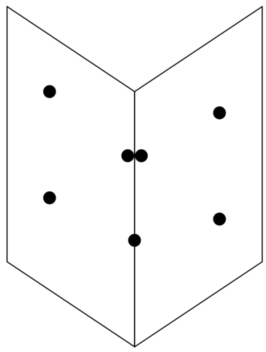
(b) W^3.

Figure 3.10: The rank 3 wheel and whirl.

- The matroid R_3 and its dual R_4. See Figure 3.11.



(a) R_3.



(b) R_4.

Figure 3.11: Matroids R_3 and R_4.

Recall from Proposition 9 that lattice path matroids are positroids, but not all positroids are lattice path matroids. With this in mind, a question arises: which of the excluded minors of LPM are positroids? To our knowledge this question has not been investigated before.

Theorem 6. *All excluded minors of lattice path matroids are positroids except for the rank* 3 *wheel* W_3.

Proof. Let us prove the theorem case by case:

- $A_n = P_n' + x$ for $n \geq 3$.

  We can work with A_n as the truncation to rank n of two n-circuits that have one and only one common element and we add there x.

To see if A_n is a positroid we will try to find a labelling of the elements such that we can do the next steps:

1. We label the elements and we obtain a Grassmann necklace $G = I(A_n)$.
2. We find the collection of bases

$$\mathcal{B}(I) := \left\{ B \in \binom{[n]}{d} \,\middle|\, B \geq_j G_j \text{ for all } j \in [n] \right\}.$$

3. This collection of bases by Theorem 1 from 2.1.2 form a positroid.
4. We want this bases to be the same as in A_n so we will know A_n is a positroid.

First we will label the elements of one circuit from $1, \dots, n$, and we set n to be the label of the element that is in both circuits. Then we label the rest of the elements of the other circuit from $n+1, \dots, 2n-1$. The element x will be labelled with $2n$.

With this labelling we obtain the Grassmann necklace $G = \{G_1, \dots, G_{2n}\}$:

$$\begin{cases} G_1 = \{1, \dots, n-1, n+1\} & \\ G_i = \{i, \dots, n+i-1\} & \text{for } 1 < i < n \\ G_n = \{n, \dots, 2n-2, 2n\} & \\ G_{n+1} = \{n+1, \dots, 2n\} & \\ G_{n+i} = \{n+i, \dots, 2n, 1, \dots, i-1\} & \text{for } 1 < i \leq n \end{cases}$$

From this Grassmann necklace we obtain the bases of a positroid $\mathcal{M}(G)$ with Theorem 1. We want to check that this positroid and A_n have the same bases, thus $\mathcal{M}(G) = A_n$ so A_n is a positroid. Instead of doing this, we will check that they have the same sets of n elements that are not bases. For A_n there are two such sets: $\{1, \dots, n\}$ and $\{n, \dots, 2n-1\}$. Now for $\mathcal{M}(G)$, from G_1 we see that $\{1, \dots, n\}$ is not a basis because

$$\{1, \dots, n\} \leq_1 \{1, \dots, n-1, n+1\}.$$

In the same way, from G_n we see that $\{n, \dots, 2n-1\}$ is not a basis because

$$\{n, \dots, 2n-1\} \leq_n \{n, \dots, 2n-2, 2n\}.$$

It is not hard to check that for any other set B of n elements in $\mathcal{M}(G)$ the following statement holds:

$$B \geq_j G_j \text{ for all } j \in [2n].$$

Therefore all other n-element sets are basis, so we are done.

- $B_{n,k} = T_n(U_{n-1,n} \oplus U_{n-1,n} \oplus U_{k-1,k})$ and its dual $C_{n+k,k}$ for $n \geq k \geq 2$.

  We will just see if $B_{n,k}$ is a positroid because we know that positroids are closed under duals and thus $C_{n+k,k}$ will be a positroid if and only if $B_{n,k}$ is a positroid.

  We will work with $B_{n,k}$ as two circuits of n elements and a circuit of k elements with no common element between them, all this truncated to rank n. We will follow the same steps that we used for A_n.

  First we will label the elements of one of circuits of n elements from $1, \dots, n$, and the elements of the other circuit from $n+1, \dots, 2n$. We wil label the elements of the circuit of k elements from $2n+1, \dots, 2n+k$.

  With this labelling we obtain the Grassmann necklace $G = \{G_1, \dots, G_{2n+k}\}$:

$$\begin{cases} G_1 = \{1, \dots, n-1, n+1\} & \\ G_i = \{i, \dots, n+i-1\} & \text{for } 1 < i < n+1 \\ G_{n+1} = \{n+1, \dots, 2n-1, 2n+1\} & \\ G_{n+i} = \{n+i, \dots, 2n, \dots, 2n+k-1, 1, \dots, i-k\} & \text{for } 1 < i \leq n+1 \\ G_{2n+1} = \{2n+1, \dots, 2n+k-1, 1, \dots, n-k+1\} & \\ G_{2n+i} = \{2n+i, \dots, 2n+k, 1, \dots, n-k+i\} & \text{for } 1 < i < k+1 \end{cases}$$

  From this Grassmann necklace we obtain the bases of a positroid $\mathcal{M}(G)$ with Theorem 1. As we did in the previous case we want to see that $B_{n,k}$ and the positroid we obtain have the same sets of n elements that are not bases. For $B_{n,k}$ there are three kinds of such sets: $\{1, \dots, n\}$, $\{n+1, \dots, 2n\}$ and $\{2n+1, \dots, 2n+k, x_1, \dots, x_{n-k}\}$ with x_i any other label not used for $i \in [n-k]$.

  Now for $\mathcal{M}(G)$, from G_1 we *miss* the set $\{1, \dots, n\}$. This means that $\{1, \dots, n\}$ is not a basis because

$$\{1, \dots, n\} \leq_1 \{1, \dots, n-1, n+1\}.$$

In the same way, from G_{n+1} we see that $\{n+1,\dots,2n\}$ is not a basis because

$$\{n+1,\dots,2n\} \le_{n+1} \{n+1,\dots,2n-1,2n+1\},$$

and from G_{2n+1} we see that $\{2n+1,\dots,2n+k,x_1,\dots,x_{n-k}\}$ with x_i any other label not used, is not a basis because

$$\{2n+1,\dots,2n+k,x_1,\dots,x_{n-k}\} \le_{2n+1} \{2n+1,\dots,2n+k-1,1,\dots,n-k+1\}$$

Finally we see that any other base that $\mathcal{M}(G)$ misses is from $B \ge_j G_{n+j}$ for $j \in \{k,\dots,n+1\}$, but these missing bases are particular cases of $\{2n+k,x_1,\dots,x_{n-k}\}$ so we are done.

- $D_n = (P_{n-1} \oplus U_{1,1}) \oplus x$ and its dual E_n for $n \ge 4$.

  Similar as in the previous case this time we will just see if E_n is a positroid because we know that positroids are closed under duals and so D_n will be a positroid if and only if E_n is a positroid.

  We know that D_n is made of two circuits of $n-1$ elements truncated in rank $n-1$ with 2 independent points in rank n. Then its dual E_n is made of two spaces of rank $n-1$ each of them with $n-1$ independent points and 2 more points that are equal and in the intersection of the two spaces. Now we will follow the same steps that we used for A_n.

  First we will label the two points of the intersection with 1 and 2. The points from one of the spaces of rank $n-1$ will be labelled with $3,\dots,n+1$ and the ones from the other space with $n+2,\dots,2n$. With this labelling we obtain the Grassmann necklace $G = \{G_1,\dots,G_{2n}\}$:

$$\begin{cases} G_1 = \{1,3,\dots,n,n+2\} \\ G_2 = \{2,3,\dots,n,n+2\} \\ G_i = \{i,\dots,n+1,\dots n+i-1\} & \text{for } 2 < i < n+2 \\ G_{n+2} = \{n+2,\dots,2n,3\} \\ G_{n+i} = \{n+i,\dots,2n,1,3\dots i\} & \text{for } 2 < i < n+1 \end{cases}$$

  From this Grassmann necklace we obtain the bases of a positroid $\mathcal{M}(G)$ with Theorem 1. As we did in the previous case we want to see that E_n and the positroid we obtain $\mathcal{M}(G)$ have the same sets of n elements that are not bases.

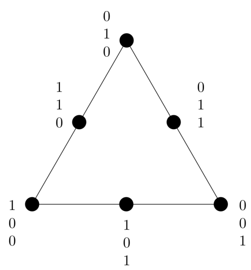
For E_n there are three kinds of such sets:

1. n elements all from one space of rank $n-1$.
2. n elements all from the other space of rank $n-1$.
3. Any set that contain 1 and 2.

The sets of n elements from the same space will be $\{1,3,\dots,n+1\}$, $\{2,3,\dots,n+1\}$, $\{1,n+2,\dots,2n\}$ and $\{1,n+2,\dots,2n\}$. Now for $\mathcal{M}(G)$, from G_1 we miss any set with both 1 and 2 so we have any set that contains 1, 2 and $n-2$ more elements. Clearly we also miss the set $\{1,3,\dots,n+1\}$. From G_2 we miss $\{2,3,\dots,n+1\}$, from G_i we do not miss any set and from G_{n+2} we miss $\{n+2,\dots,2n,1\}$ and $\{n+3,\dots,2n,2\}$. Finally from G_{n+i} we miss sets that contain both 1 and 2 but we already miss them from G_1. Finally we can say that these matroids have the same n-element dependent sets and therefore the same bases, so they must be the same. Since $\mathcal{M}(G)$ is a positroid, we conclude that E_n is a positroid and its dual D_n is a positroid as well.

- The rank 3 whirl W^3

  W^3 is the geometric representation of the representable matroid $M(A)$ where A is



$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 3.12

It is not hard to check using a simple algorithm that A is a nonnegative matroid, and thus $M(A)$ is a positroid.

- The rank 3 wheel W_3 is not a positroid.

  This matroid has ground set $E = \{1,2,3,4,5,6\}$. From this matroid, and using Proposition 1 from 2.1.2 we obtain a Grassmann necklace $G = \mathcal{I}(W_3)$ from its lexicographically minimal basis with order $<_j$. Also, if W_3 was a positroid, using Theorem 1 from 2.1.2 we know that the positroid produced by G, should be W_3 again: $\mathcal{M}(\mathcal{I}(W_3)) = W_3$.

The Grassmann necklace we get from W_3 depends on how we label the points of the geometric representation. We will see that the Grassman necklace $G = \{G_1, \dots, G_6\}$ we get from any possible labelling produces a positroid that is not W_3. Therefore, using Theorem 1 we will conclude that W_3 is not a positroid.

Note that each G_i will be either $G_i = \{i, i+1, i+2\}$ or $G_i = \{i, i+1, i+3\}$. It will be $i+2$ if $i+2$ is not in the same line that form $i, i+1$, otherwise it will be $i+3$. As we have four and only four lines in W_3 we want that when we do $\mathcal{M}(\mathcal{I}(W_3))$ we just get four 3-element sets that do not appear. That means we want to have four different G_i such that $G_i = \{i, i+1, i+2, i+3\}$. To achieve this we need consecutive labels modulo 6 in the four lines.

Arbitrarily we set the left bottom point to be labelled with 1. Then arbitrarily we can set the opposite point or the upper adjacent point to 2. First let us focus on the case with the opposite point as labelled with 2. Now we just have two options, to set the adjacent top point to 5 and the top to 6 or the other way. To have all lines with consecutive numbers we need the point in the middle of the bottom line to be 6 or 1 respectively, but that is not possible because we already used these labels.

In a similar way we can see that labelling the upper adjacent point of 1 with 2 we neither can get the four lines with consecutive numbers, so the wheel W_3 is not a positroid. We can see the different options we would be forced to get in the next figure. When we get to a cross we do not have any available label right for the point.

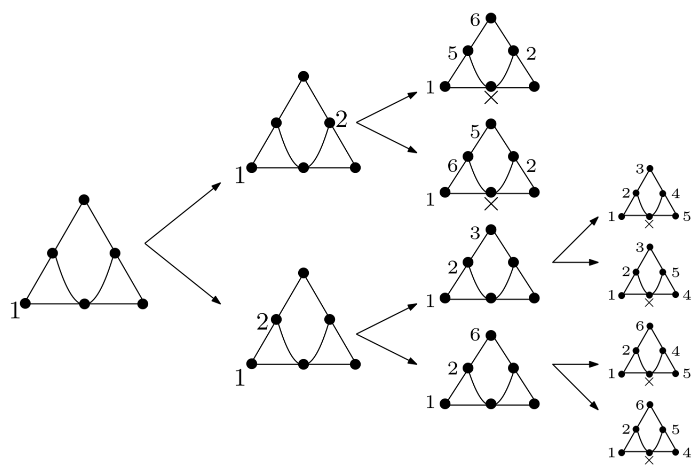
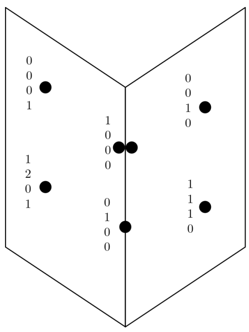


Figure 3.13: The possible results we can get without loss of generality.

- The matroid R_3 and its dual R_4:

  R_4 is the geometric representation of the matroid $M(B)$ where B is



$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Figure 3.14

  As with W_3, it is not hard to check that B is nonnegative and therefore $M(B)$ is not an excluded minor of positroids.

$\square$

3.2 Series-parallel graphs

Definition 37. A series-parallel graph is a graph formed recursively by the series and the parallel operation. The series operation consists in subidividing a step so it is replaced by two steps in series. The parallel operation consist in replacing a single step that connects two endpoints by two steps in parallel having the original step endpoints as common endpoints.

Proposition 14. *Let $M(G)$ be a graphic matroid and assume it is also a positroid. If G' is another graph obtained from G using series and parallel operations, then $M(G')$ is also a positroid.*

Proof. Clearly we only need to see it for one single operation and one can iterate to make any number of operations. Let us check both possible operations.

- **Parallel operation:**

  This one is easy to check as adding a parallel step e' to an existing one e consists on duplicating the column of e in its matrix. If the graphic matroid corresponds to a positroid then by duplicating the column of e it will still be a positroid as all determinants do not have e' will stay the same and the ones that have e' will be the same as substituting e' by e. If both e, e' are in the maximal minor, then the determinant will be zero as there will be two equal columns.

- **Series operation:**

  To subdivide an step e into two e, e' what we are doing is: each cycle that contained e is not a cycle anymore except if e' is added. The rest of the cycles must remain the same. To do that, suppose the graph is represented by a $d \times n$ matrix with all maximal minors nonnegative. We add a row and a column -to represent e'- to the matrix so it becomes a $(d+1) \times (n+1)$ matrix. The row we add is the last one with all entries zero except the ones corresponding e and e' that are a 1. The column that respresents e' is a $(d+1) \times 1$ vector all with zeros except the last number being a 1, $(0, \ldots, 0, 1)$. If we originally have

$$\begin{bmatrix} | & | & \ldots & | & \ldots & | \\ a_0 & a_1 & \ldots & e & \ldots & a_n \\ | & | & \ldots & | & \ldots & | \end{bmatrix},$$

  then after the series operation we get

$$\begin{bmatrix} | & | & \ldots & | & 0 & \ldots & | \\ a_0 & a_1 & \ldots & e & 0 & \ldots & a_n \\ | & | & \ldots & | & 0 & \ldots & | \\ 0 & 0 & \ldots & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

  This represents a series operations because given a group of columns which neither e nor e' are in the relation between those columns stay the same. If e was part of a cycle then without e' is not anymore. It is easy to check as the determinant now can be developed thorough the one of the column representing e becoming $\pm\Delta$(all the columns minus e and the las row), but if e was part of a cycle and e' is added to the group developing by e', $\pm\Delta$ (all the columns minus e' and the las row) $= \pm\Delta$(of a cycle). The problem here is that this may not be a positroid, it is just a way of representing the graphical matroid as a matrix. In order to still have a positroid after the series operation first we do a cyclic shift of the ground set till e is the last vector of the matrix. As positroids are closed under cyclic shifts we still have a positroid. Now if we add the column for e' and the last row we still have a positroid. Suppose we have a maximal minor, if the columns of e

nor e' are taken the determinant will be the same, so it is nonnegative, if e is taken then developing by the last row we get a minor we had before the operation, so it is nonnegative, and if e' is taken developing by the the columns we also get a minor of before the operation.

□

Now we know series-parallel graphs are positroids so they do not have W_3 as a minor. In fact, unions of series-parallel graphs are exactly the graphs that do not have W_3 as a minor (see the graph theory reference book [6] for more details). Since W_3 is the only excluded minor of lattice path matroids that is graphic, we can conclude that all series-parallel graphs are lattice path matroids as well.

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