Physics Engineering
Bachelor’s Thesis
Gravitational waves and new gravitational astronomy

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Abstract

This project is aimed to provide the reader a guide to introduce oneself to the field of gravitational waves (GWs): it gives a global vision of the essential subjects concerning them to understand how GWs were predicted, the physics underlying them, the basis of the numerical methods employed and the detection events and consequences derived from the gravitational astronomy. We specially focus on the binary black hole coalescence problem, since it is the simplest, best understood source of gravitational waves for now.
1 Introduction

Gravitational waves (GWs) are ripples in spacetime rather than a disturbance superimposed on it [1]. They carry energy at the speed of light [2] and can do work being absorbed only very weakly [3] and so can travel cosmological distances without dispersion [4]. As far as we know, everything belonging to the Universe is affected by gravity, which means that detecting and studying gravitational waves—which is known as the emerging gravitational astronomy—may reveal us the Universe in its entirety, informing us from events never before witnessed [4].

The existence of GWs was predicted by Albert Einstein in 1916, who found them by linearizing the Einstein equations for the case of nearly flat spacetime [1]. A hundred of years later, the first detection of GWs was due to the LIGO interferometers [5], founded by Kip Thorne, Ronald Drever and Rainer Weiss. It was the most energetic event ever detected, apart from the Big Bang [6].

From the beginning of the human kind until 2016, any observation of the sky was based in electromagnetic waves; in some cases, neutrinos were the best we could aspire to detect too. This implies that gravitational astronomy does really open up a new era of the astronomy.

A historical review

Notwithstanding that GWs are a well known theoretical solution to Einstein’s equations since 1916, the existence and detection of this kind of cosmological information has been controversial along the history [5] until its recent discovery. Similarly to the role of charges in the case of electromagnetic waves, GWs are generated by any object with mass and no null acceleration. Since any gravitational wave produced on Earth is too small to be detected—in fact, they would produce displacements even lower than the Planck length

\[ l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.61629(38) \cdot 10^{-35} \text{m}, \]

which is theoretically the smaller measurable length–, the first thing coming to mind is that we need to search for events concerning extremely massive astronomical objects accelerating so that they produce GWs strong enough for the human means. Therefore, we irremediably derive to the sphere of astronomy and cosmology: binary systems composed by compact objects—ie. white dwarfs (WDs), neutron stars (NSs) and black holes (BHs)—, specially pairs of stellar mass NSs, BHs or NS-BH pairs, are the best candidates to generate GWs detectable for the forthcoming GW-antennae [7].

If the coalescence of, for example, a pair of black holes did not disturb the spacetime fabric too greatly, we could employ a linearized version of Einstein’s equations with the Minkowski metric plus a small perturbation [8]—as it is explained in Section 4.1—to solve the problem with precision. But this is not the case concerning, for instance, the merger of a BH-BH pair: the spacetime is distorted in such a non-linear manner that it forces us to solve the exact version of the Einstein equations instead [9]. Linearizing the equations makes no sense in the case of strong-field. Additionally, when the field equations are not linear and spacetime is non-flat and itself evolving, the definition of what constitutes a wave is vague. However, there is no reason to conclude that GWs would do not exist in that situation; one could intuitively think that they exist too, but for many years there was no certainty of that [9].
Another inconvenient came from the difficulty to estimate the energy output from the gravitational waves’ sources; it often depends on the details of the model employed, about which little was known [8]—even nowadays there are many potential sources of GWs still vaguely described by theoretical physics [10]. At many times, it was impossible to carry out algebraic computations [8]. This added to the nonexistence of numerical methods capable of solving the exact Einstein equations for the interesting cases, thus meaning that it was impossible to know the waveform of the GWs.

In summary, the main reasons to doubt about the existence and the possibility to detect GWs were the next: a) The proper physical definition of these waves was uncertain. b) Even if produced by compact astronomical objects, the perturbations are so small that the noise isolation must be extremely large to confirm the origin of the received signal. c) If they were actually detected, there was no way to theoretically describe complete waveform models to compare with the measurements.

Four decades of debate followed the Einstein prediction until some physicist concluded that, if General Relativity was right, gravitational waves must exist [11]. In 1952, the first steps to get stable, accurate, numerical solutions of the problem of compact binary systems were took: Fourès-Bruhat demonstrated that Einstein’s vacuum equations are locally at least, well-posed [12]. The solutions of the equations exists and they are non chaotic, which means that it is possible to solve them numerically without any approximation up to numerical error. Given the difficulty of Einstein’s equations, these seemingly reasonable expectations are far from obvious.

Moreover, technological advances that took place between the 40’s and the 50’s made feasible the first attempt to measure effects of the hypothetical GWs [11]. It was the major contribution to science performed by Joseph Weber, at the University of Maryland in 1966. He invented and constructed a device called Weber bar, which is a resonant-mass antenna with eletromechanical transducers consisting of an aluminium cylinder vibrating at a resonance frequency $\nu = 1660Hz$. The piezoelectric sensors inside it were capable of detecting a change of $10^{-16}$ meters in the cylinder’ length. It was designed to be set in motion by the GWs that Weber himself predicted. His positive results around 1968 and 1987 were widely discredited because of criticism to the data analysis and the incomplete definitions of the strength vibration associated to GWs [13].

Despite his failure to detect GWs, this both theorist and experimental physicist started the field of GW detection, not only by constructing the first gravitational-wave antenna but also by influencing his Ph.D. thesis student, Robert Forward, to construct the first laser-interferometry antenna. The idea was separately proposed by Felix Pirani in 1956, Gertsenshtein and Pusttovid in 1962, Weber in 1964, and Rainer Weiss, who came across a proposal by Pirani for detecting GWs while he was preparing for his course on relativity at the Massachusetts Institute of Technology. Pirani had suggested using a pulsed laser to analyse the displacements of neighboring particles as a GW went through them. This led to the genesis of LIGO when Weiss proposed to use a Michelson interferometer for making phase measurements instead of using the timing of short light pulses [11]. Ronald Drever, Kip Thorne, and many others made decisive contributions to refine this thought into what LIGO is today [14].
Meanwhile, the field of Numerical Relativity (NR) had been going through many difficulties from the very beginning in the fifties [9]. The complete development of the numerical method that enabled the first 3D simulations of binary black hole system mergers ended the 1998 [9]. The method we are talking about is known as the BSSNOK approach and it is in wide use today. As any NR method, it works by solving the highly nonlinear Einstein equations with no algebraic approximations [9]. Waveform data banks have been made up by means of numerical relativity algorithms; they strongly facilitated the labour of identifying GWs’ signals in the currently working detector laboratories.

The first experimental data backing up the hypothesis of compact objects emitting GWs was due to the first binary system of neutron stars ever detected, B1913+16 (Hulse & Taylor 1975). In 2004, Joel M. Weisberg and Joseph H. Taylor released a paper about 30 years of observation and analysis of this system [15]. Numerous relativistic phenomena measurements were carried out to demonstrate that the strange shift on the system periastron was in perfect agreement with the Einstein’s prediction of the pair of neutron stars emitting gravitational waves [15] –see Fig.1.1.

![Figure 1.1: Orbital decay of PSR B1913+16. The data points indicate the observed change in the epoch of periastron with date while the parabola illustrates the theoretically expected change in epoch for a system emitting gravitational radiation, according to General Relativity. Image source:[15]](image)

Weleve years later, Kip Thorne, David Reitze, Gabriela González and Rainer Weiss announced the success of the LIGO Collaboration and the Virgo Collaboration to make the first direct observation of gravitational waves from a BH-BH merger using the LIGO devices (U.S.A) [16]. Very soon, the Advanced Virgo interferometer, located in Italy, also demonstrated to be effective at detecting GWs [17]. Thus, a currently-operative world network of GW-antennae was born and it is on the way to increase its number of detectors today with projects such as KAGRA [18] (Japan) and IndIGO (India) [19]. Both gravitational and electromagnetic radiation from a NS-NS pair coalescence was detected for the first time on August the 17th, 2017, an event which initiated a branch of astronomy known as multimessenger astronomy [20].
A summary of the content

The first chapter of this project contains a deduction of gravitational waves as solutions to the Einstein equations in the case of linearized gravity approximation. The two possible polarization states are also deduced therein. In order to deduce the mathematical solution of GWs first found by Albert Einstein, a brief summary of the ideas and equations of General Relativity is done at Section 3. For a better and more extended explanation of these mathematical background, please, see [21] or [1],[8],[22],[23]. The algebraic computations that are required to develop the deduction are taken from reference [8], Ray D’Inverno’s *Introducing Einstein’s Relativity* (1998). An expansion of them is performed along Section 4. References [1],[22],[23] may be useful to the reader for a more complete understanding of the whole chapter.

Chapter 2 is about GWs as a physical phenomenon. In Section 5, an original classification of the different GW sources is performed. The main references employed to develop it are [24], [25] and [4]. In that section, we introduce the so called Compact Binary Systems (CBSs). The only gravitational waves that have been detected to date came from merging CBSs, so Section 6 is an extended explanation of the main subjects concerning this type of GWs. We make a review of various methods required to investigate the different phases of a CBS coalescence: Post-Newtonian approximation method, Numerical Relativity and the Black hole perturbation method are appropriate to respectively study the inspiral, the plunge and merger, and post merger stages of coalescing CBSs.

Subsection 6.2, which is about Numerical Relativity and, more specifically, about the binary black hole merger problem, is specially highlighted because of the historically important role of numerical methods in the development of the necessary data banks that enabled the identification of GWs among the signals received by the GW-antennae. The main reference used to write this subsection is [9], R. A. Eisenstein’s *Numerical relativity and the Discovery of Gravitational Waves* (2019).

Section 7 is a review of some subjects concerning the direct detection of GWs, such as the currently operative GW detectors, the future LISA interferometer, the physical parameters related to a CBS coalescence, and the most relevant detected events themselves. In subsection 7.2, we present the first noticed solar–mass IGW events: an individual, short description of them is exposed and their corresponding spectrograms are also included. Multi-messenger astronomy is introduced in this subsection. Reference [5] turns out to be an excellent summary of the methodology used to identify GWs, while [26] is of compulsory reading for those who want to get a more detailed explanation about the methodology and the features of the detected events, as well as to see a data comparison of those events detected along LIGO’s first and second observing runs.

In the last two sections of this thesis a summary of the consequences of the results obtained up to date by gravitational astronomy is made. Also a perspective of the future expectations of this field is described.
Chapter 1

2 Notation and tensors

Since the majority of equations and calculations appearing in the theoretical part of this project are not only tensorial but extensive, the notation employed is somehow condensed so that a pleasant reading is achievable. Here we expose some items about notation that are considered not to be obvious. They need to be taken into account in the next sections so that the maths are properly understood.

- **Covariant components and contravariant components**: if \( x \) is a \( k \)-covariant, \( l \)-contravariant tensor, then its components are:

\[
 x^{i_1...i_k}_{j_1...j_l}
\]

In this project, \( i_1...i_k \) and \( j_1...j_l \) will always represent the contravariant and covariant indexes and are represented by letters belonging to the Latin alphabet.

- **Continuity of the tensors**: All tensors appearing in this project are assumed to belong at least to \( C^2 \) continuity class.

- **Partial derivatives** are applied to every single component of a tensor \( z \) whatever is the order of the tensor.

\[
 z^{i_1...i_k}_{j_1...j_l} \equiv \frac{\partial z^{i_1...i_k}}{\partial x^a} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b}
\]

Second order derivatives are specified by:

\[
 z^{i_1...i_k}_{j_1...j_l,ab} \equiv \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b} = \frac{\partial^2 z^{i_1...i_k}}{\partial x^a \partial x^b}
\]

- **Einstein’s summation convention**: if an index appears twice in a single term –once as covariant and once as contravariant–, it implies the summation of that term over all the values of the index.

- **Contraction of an index using Minkowski’s metric**: The Minkowski metric \( \eta_{ab} \) downs one contravariant index of any tensor to which the metric is applied if the index of the arbitrary tensor \( z \) coincides with one of the metric.

\[
 \eta_{ab} z^{ac} = z^c_b
\]
If taking the contravariant form of the Minkowski metric $\eta^{ab}$, which is its own inverse –i.e. $\eta^{ab}\eta_{bc} = \delta^a_c$ – it rises one covariant index of any tensor to which the metric is applied if the index of the arbitrary tensor $z$ coincides with one of the metric.

$$\eta^{ab} z_{ac} = z^b_c.$$

This definition extends to the case of derivatives:

$$\eta^{ab} z_{c,a} = z^c_{b}; \quad \eta^{ab} z_{c,a} = z^b_c.$$

- The **covariant derivative** of a tensor field $X$ along the parametrized curve $c$ is expressed as: $\nabla_c X$, where $\dot{c}(\tau)$ is its derivative with respect to the parameter $\tau$.

## 3 Mathematical foundations

Spacetime curvature is mathematically determined by Einstein field equations shown below:

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = K T_{ab}; \quad (3.1)$$

where $G_{ab}$ is the Einstein tensor; $R_{ab}$ and $R$ –the Ricci’s tensor and the scalar of curvature, respectively– which are both obtained by contractions of the Riemann tensor $R^l_{\ jkr}$; $g_{ab}$ is the metric and $T_{ab}$ is the stress-energy-momentum tensor. These equations enable us to compute solutions to the spacetime curvature –left-hand equations– as a consequence of the presence of mass and energy –right-hand equations– at almost any region of the observable universe.

The sixth element constituting Einstein’s equations is a constant whose value is taken from the approach of these equations to the Newton gravitation when the Newtonian limit is imposed, which, after comparing both theories, leads to the value: $K = \frac{8\pi G}{c^4}$.

The Riemann tensor, in terms of Christoffel symbols, is:

$$R^l_{\ jkr} = \Gamma^l_{\ jrs} - \Gamma^l_{\ rsj} + \Gamma^s_{\ jr} \Gamma^l_{\ ks} - \Gamma^s_{\ kr} \Gamma^l_{\ js}, \quad (3.2)$$

and the Christoffel symbols are:

$$\Gamma^a_{\ bc} = \frac{1}{2} g^{ad}(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}), \quad (3.3)$$

as we employ the Levi-Civita connection associated with the metric. From this we obtain an explicit expression of equation 3.1:

$$G_{ab} := (\delta^c_a \delta^d_b - \frac{1}{2} g_{ab} g^{cd})(\partial_d \Gamma^e_{\ cd} - \partial_d \Gamma^e_{\ cd} + \Gamma^i_{\ j} \Gamma^e_{\ ic} - \Gamma^i_{\ j} \Gamma^e_{\ ic}) = K T_{ab}. \quad (3.4)$$

Once the equations (1.1) are solved, the metric is obtained and the motion of matter and energy along spacetime is determined by the geodesics equation:

$$\nabla_\gamma \dot{\gamma} = 0 \iff \frac{d^2 \gamma^c}{d\tau^2} + \Gamma^c_{\ ab} \frac{d\gamma^a}{d\tau} \frac{d\gamma^b}{d\tau} = 0, \quad (3.5)$$

where $\gamma(\tau) = \gamma^a(\tau)$ is a parametrization of the curve.
4 Plane gravitational wave equations

Gravitational waves equations are a solution to the so-called linearized gravity: an approximation scheme in General Relativity in which the nonlinear contributions from the spacetime metric are ignored. This simplifies the study of many problems while still producing useful approximate results. Along this section many calculations are performed; note that some of them are written step-by-step meanwhile others—which are considered to be too extensive for the interest of this project— are reduced to just giving the final desired solution.

4.1 Linearized gravity

The ideal solution of gravitational waves is obtained from the assumptions and calculations that are exposed in this section. Given the Einstein’s equations:

\[ G^{ab} = \frac{8\pi G}{c^4} T^{ab}, \]  

(4.1)

- The first assumption to take into account is that the field is weak and, consequently, there exist frames of reference—the so-called approximately Lorentzian systems—at which the approach below is valid:

\[ g^{ab} = \eta^{ab} + h^{ab} + O(h^2) \approx \eta^{ab} + h^{ab}, \]  

(4.2)

with \( \eta^{ab} \) being the Minkowski metric corresponding to a non-curved spacetime and \( h_{ab} \) as a perturbation term. Eq. (4.2) means that the difference between the Minkowski metric and the hypothesized solution is so small that second order terms \( O(h^2) \) can be neglected. That is the main idea of the linearized gravity.

- spacetime is asymptotically flat, ie:

\[ \lim_{r \to \infty} h^{ab} = 0, \]  

(4.3)

where \( r \) is denoting a radial parameter.

Stated this, taking the Levi-Civita connection: the Christoffel symbols associated to the chosen metric are given by the next formula (obtained by straightforward calculations from eq. (3.3)):

\[ \Gamma^a_{bc} = \frac{1}{2}(h^a_{c,b} + h^a_{b,c} - h_{bc},^a) + O(h^2). \]  

(4.4)

The Riemann tensor then becomes:

\[ R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}) + O(h^2), \]  

(4.5)

and the Ricci tensor is:

\[ R_{ab} = g^{cd}R_{cadb} = \eta^{cd}R_{cadb} + h^{cd}R_{cadb} + O(h^2) = \]

\[ = \frac{1}{2}(\eta^{cd}h_{cb,ad} + \eta^{cd}h_{bd,ac} - \eta^{cd}h_{ac,cb} - \eta^{cd}h_{cb,ad}) + O(h^2) = \]
\[ = \frac{1}{2}(h_{b,ad}^d + h_{a,cb}^c - h_{ab} - \Box h_{ab}) + O(h^2) = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \Box h_{ab} - h_{ab}) + O(h^2), \quad (4.6) \]

where:
\[ h = \eta^{ab} h_{ab} = h_a^a, \quad (4.7) \]

and:
\[ \Box = \eta^{ab} \partial_a \partial_b = \partial^2 \partial_t^2 - \nabla^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \quad (4.8) \]

Thus, the Ricci scalar is:
\[ R = g^{ab} R_{ab} = \eta^{ab} R_{ab} + h^{ab} R_{ab} + O(h^2) = \]
\[ = \frac{1}{2}(\eta^{ab} h_{a,bc}^c + \eta^{ab} h_{b,ac}^c - \Box \eta^{ab} h_{ab} - \eta^{ab} h_{ab}) + O(h^2) = \]
\[ = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \Box h_{ab} + O(h^2), \quad (4.9) \]

leading to the Einstein’s tensor:
\[ G_{ab} = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \Box h_{ab} - h_{ab} - \eta_{ab} h_{cd}^c + \eta_{ab} \Box h) + O(h^2). \quad (4.10) \]

At this point the following new variables are set up in order to simplify Einstein’s equations:
\[ \psi_{ab} \equiv h_{ab} - \frac{1}{2} \eta_{ab} h, \quad (4.11) \]

which is known as the ‘trace reversed’ perturbation because it satisfies: \( \psi^a_a = -h \) [27].

Employing the equality:
\[ h_{ab} = \eta_{bc} h_{a}^c = \eta_{bd} h_{a}^d = \eta_{b}^c h_{a,ac} = \eta_{b}^c h_{a,ac}, \quad (4.12) \]

it follows:
\[ R_{ab} = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \Box h_{ab} - h_{ab}) + O(h^2) = \]
\[ = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \frac{1}{2} \eta_{a,b} h_{b,ac} - \frac{1}{2} \eta_{a,b} h_{a,ac} - \Box h_{ab}) + O(h^2) = \]
\[ = \frac{1}{2}(h_{a,bc}^c + h_{b,ac}^c - \frac{1}{2} \eta_{a,b} h_{b,ac} - \frac{1}{2} \eta_{a,b} h_{a,ac} - \Box h_{ab}) + O(h^2). \quad (4.13) \]

On the other hand, using that:
\[ \eta^{ab} h_{a,ab} = \Box h, \quad (4.14) \]

the curvature scalar becomes:
\[ R = h_{a,ab}^{ab} - \Box h + O(h^2) = \]
\[ = h_{a,ab}^{ab} - \frac{1}{2} \Box h - \frac{1}{2} \Box h + O(h^2) = \]
and finally we obtain a shorter formula for the Einstein’s tensor from eq. (3.1), (4.13) and (4.15):

\[ G_{ab} = \frac{1}{2} (\psi_{c,a,bc} + \psi_{c,b,ac} - \Box h_{ab}) - \frac{1}{2} (2\psi_{cd,\eta} g_{ab} - \Box g_{ab}) = \]
\[ = \frac{1}{2} (\psi^c_{a,bc} + \psi^c_{b,ac} - \Box h_{ab} - \frac{\Box \eta_{ab}}{2} - \frac{2}{2} \psi_{cd,\eta_{ab}} + O(h^2)). \]

\[ \Rightarrow G_{ab} = \frac{1}{2} (\psi^c_{a,bc} + \psi^c_{b,ac} - \Box \psi_{ab} - \psi_{cd,\eta_{ab}}) + O(h^2). \] (4.16)

When dropping up the terms \( O(h^2) \), eq. (4.16) are called ‘linearized Einstein’s equations’ (LEE).

In addition, under the assumption that we are in the void, so that \( T^{ab} = 0 \), it is easy to see that LEE give rise to plane-wave equations in the variable \( \psi \):

\[ T^{ab} = 0 \Rightarrow G_{ab} = 0 \Rightarrow \frac{1}{2} (-\Box \psi_{ab}) = 0 \]
\[ \Rightarrow \Box \psi_{ab} = 0 \] (4.17)

if the condition bellow is imposed:

\[ \psi_{a,b} = 0, \] (4.18)

which in terms of the perturbation is:

\[ h^a_{b,a} - \frac{1}{2} h^b = 0. \] (4.19)

Eq. (4.18) –commonly called ‘Einsein’s, de Donder, Hilbert or Fock gauge’– plus eq. (4.17) –the LEE resulting after the imposition of the gauge– constitute the system to be solved.

Additionally, when taking the trace of \( \psi \), we obtain:

\[ \eta^{ab} \Box \psi_{ab} = (\eta_{ab} \psi) = \Box (h_{ab} - \frac{1}{2} \eta_{ab} h) = \]
\[ = \Box (\eta^{ab} h_{ab} - \frac{1}{2} \eta^{ab} \eta_{ab} h) = \Box (h_a^a - \frac{1}{2} 4 h) = \Box (h - 2 h) = -\Box h = 0, \]
\[ \Box h = 0. \] (4.20)

From the definition of \( \psi_{ab} \) –eq. (4.11)– with \( \psi_{ab} \) fulfilling the wave-equation –eq. (4.17)– and using the previous result –eq. (4.20):

\[ \Box \psi_{ab} = \Box h_{ab} - \Box ((\frac{1}{2} \eta_{ab} h) = \Box h_{ab} - \frac{1}{2} \eta_{ab} \Box h = \Box h_{ab} + 0 = 0, \]
\[ \Box h_{ab} = 0; \] (4.21)

that is: the perturbation of the metric itself fulfills the plane-wave equation too.

\[ \frac{\partial^2 h_{ab}}{\partial t^2} - \nabla^2 h_{ab} = 0. \] (4.22)
4.2 Harmonic gauge transformations

Let us now make a parenthesis to demonstrate that there really exists a change of coordinates which leads to the gauge imposed in the previous section at eq. (4.18).

It can be proved that Einstein’s equations are invariant under an infinitesimal change of coordinates with a small parameter \( \epsilon \) of the form:

\[
x^a \rightarrow x'^a = x^a + \epsilon \xi^a
\]

\[
\Rightarrow x^a = x'^a - \epsilon \xi^a.
\]  

From the tensorial transformation of the metric, neglecting the terms of order \( O(\epsilon h) \) and \( O(\epsilon^2) \):

\[
g'_{ab}(x') = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}(x) = \frac{\partial(x^c - \epsilon \xi^c)}{\partial x'^a} \frac{\partial(x^d - \epsilon \xi^d)}{\partial x'^b} (\eta_{cd} + h_{cd}) =
\]

\[
= (\delta^c_a - \epsilon \frac{\partial \xi^c}{\partial x'^a})(\delta^d_b - \epsilon \frac{\partial \xi^d}{\partial x'^b})(\eta_{cd} + h_{cd}) =
\]

\[
= (\delta^c_a \delta^d_b - \epsilon(\delta^d_b \frac{\partial \xi^c}{\partial x'^a} + \delta^c_a \frac{\partial \xi^d}{\partial x'^b}))(\eta_{cd} + h_{cd}) \approx
\]

\[
\approx \eta_{ab} + h_{ab} - \epsilon(\delta^d_b \frac{\partial \xi^c}{\partial x'^a} + \delta^c_a \frac{\partial \xi^d}{\partial x'^b}) \approx \eta_{ab} + h_{ab} - \epsilon(\delta^d_b \frac{\partial \xi^c}{\partial x'^a} + \delta^c_a \frac{\partial \xi^d}{\partial x'^b}) =
\]

\[
= \eta_{ab} + h_{ab} - \epsilon(\delta^d_b \frac{\partial \xi^c}{\partial x'^a} + \delta^c_a \frac{\partial \xi^d}{\partial x'^b}) = \eta_{ab} + h_{ab}' = (4.25)
\]

the gauge transformation of the perturbation \( h_{ab} \) induced by the change 4.23 is:

\[
h_{ab} \rightarrow h'_{ab} = h_{ab} - \epsilon(\frac{\partial \xi_b}{\partial x'^a} + \frac{\partial \xi_a}{\partial x'^b}),
\]  

(4.26)

which transforms \( \psi_{ab} \) into:

\[
\psi_{ab} = h_{ab}(h'_{ab}) - \frac{1}{2} \eta_{ab} h(h', \xi),
\]  

(4.27)

where:

\[
h' = h'^a_a.
\]  

(4.28)

From eq. (4.26), we have:

\[
\psi_{ab} = h'_{ab} + \epsilon(\xi_{b:a} + \xi_{a:b}) - \frac{1}{2} \eta_{ab}(h'^{cd}(h'_{cd} + \epsilon(\xi_{c:d} + \xi_{d:c}))) =
\]

\[
= h'_{ab} + \epsilon(\xi_{b:a} + \xi_{a:b}) - \frac{1}{2} \eta_{ab}(h'^{ec} + \epsilon(\xi_{d:c} + \xi_{c:d})) =
\]
\[ h'_{ab} - \frac{1}{2} \eta_{ab} h' + e(\xi_{b,a} + \xi_{a,b}) - \frac{1}{2} \epsilon \eta_{ab} (\xi^d_{c} + \xi^c_{d}) = h'_{ab} - \frac{1}{2} \eta_{ab} h' + e(\xi_{b,a} + \xi_{a,b}) - \epsilon \eta_{ab} \xi^c_{c}. \]  

(4.29)

Defining:
\[ \psi'_{ab} = h'_{ab} - \frac{1}{2} \eta_{ab} h', \]  

(4.30)

it follows:
\[ \psi_{ab} = \psi'_{ab} + \epsilon (\xi_{b,a} + \xi_{a,b}) - \frac{1}{2} \epsilon \eta_{ab} (\xi^d_{c} + \xi^c_{d}) \]
\[ \Rightarrow \psi_{ab} \rightarrow \psi'_{ab} = \psi_{ab} - \epsilon (\xi_{b,a} + \xi_{a,b}) + \epsilon \eta_{ab} \xi^c_{c}. \]  

(4.31)

Then, when computing \( \psi'_{b,a} \) to obtain the Gauge defined at eq. (4.18):
\[ \psi_{b,a} = \partial_a (\psi_{b} - \epsilon (\xi_{b,a} + \xi_{a,b}) + \epsilon \eta^a_{b} \xi^c_{c}) = \psi_{b,a} - \epsilon (\xi_{b,a} + \xi_{a,b}) + \epsilon \eta^a_{b} \xi^c_{c} = \]
\[ \psi_{b,a} - \epsilon \xi^a_{b,a} - \epsilon \xi^c_{b,a} + \epsilon \xi^c_{a,b} = \psi_{b,a} - \epsilon \xi^a_{b,a} - \epsilon \xi^c_{b,a} + \epsilon \xi^c_{b,c} = \psi_{b,a} - \epsilon \xi^a_{b,a} \]
\[ \Rightarrow \psi_{b,a} \rightarrow \psi'_{b,a} = \psi_{b,a} - \epsilon \Box \xi_{b}. \]  

(4.32)

from which:
\[ \Box \xi_{b} = \psi_{b,a} \Rightarrow \psi'_{b,a} = 0. \]  

(4.33)

Then the required condition to obtain the LEE –eq. (4.18)– is naturally satisfied.

Please, note that additional infinitesimal changes of coordinates with \( \xi \) fulfilling the wave equation –eq. (4.34)– can be performed leaving the LEE unaltered:
\[ \Box \xi_{b} = 0. \]  

(4.34)

### 4.3 Criterion of propagation

Once the eq.(4.21) has been obtained, we could directly conclude that gravitational waves propagate at the velocity of light, \( c \). We could do this because, for any arbitrary event \( X = (x_0, x_1, x_2, x_3) \equiv (ct, x, y, z) \), time components are magnitude normalized with respect to spatial components in every equations shown before this point. This means that the d’Alembertian operator is a function of \( c \) as follows:
\[ \square = \frac{\partial^2}{\partial t^2} - c^2 \nabla^2. \]  

(4.35)

Thus, when applying \( \square \) to any function, the square root of the factor multiplying \( \nabla^2 \) –in this case: \( c^2 \)– is the velocity of the propagating wave:
\[ \sqrt{c^2} = c. \]  

(4.36)

Nevertheless, we have no reason to assume the existence of a reference system from which the resulting \( h_{ab} \), if not equal to 0, propagates. A better reasoning to see that a spacetime curvature does really propagate consists in analyzing the Riemann tensor, since its values represents the degree of spacetime curvature. Just because of eqs. 4.5 and 4.21 we obtain straightforwardly:
\[\Box R_{abcd} = 0;\]  
(4.37)

this means that, in linearized gravity, gravitational effects –curvature– propagate at the velocity of light independently from the metric value. Please note, however, that this criterion is just for the curvature propagation but it does not regard energy transport by itself, i.e. it is not a complete demonstration of the perturbation \( h_{ab} \) behaving as a real physical wave because we should demonstrate that it does transport energy in order to do so. There is a good, attainable demonstration of this at [1], S.Weinberg’s *Relativity and cosmology*, pages 255-260.

### 4.4 Polarization states

Next, we analyze the polarization states of GWs. First, we assume the following dependence:

\[ h_{ab} \equiv h_{ab}(t - x), \]  
(4.38)

which clearly represents a solution propagating in the \( x \)-direction and it requires that:

\[ h_{ab,2} = h_{ab,3} = 0. \]  
(4.39)

Then, after a very extensive analysis of the Riemann tensor components and the choice of the ‘transverse traceless’ gauge, which is defined by eqs. 4.18, 4.40 and 4.41:

\[ \psi_{0a} = 0, \]  
(4.40)

\[ h_\mu^\mu = 0 \Rightarrow \psi_{ab} = h_{ab}, \]  
(4.41)

it can be demonstrated that there exists a solution to the LEE such that:

\[
h = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & h_{22} & h_{23} \\
0 & 0 & h_{23} & -h_{22}
\end{bmatrix}
\]  
(4.42)

–see [8], Ray D’Inverno, pages 275-278–, being \( h_{22} \) and \( h_{23} \) two independent functions only depending on \( t - x \). Therefore, it turns out to be comfortable decomposing the tensor \( h \) into:

\[
h = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix} h_{22}(t - x) + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} h_{23}(t - x).
\]  
(4.43)

Now, taking our ‘whole’ metric:

\[
g = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 + h_{22} & 0 & 0 \\
0 & 0 & h_{23} & -1 - h_{22} \\
0 & 0 & h_{23} & -1 + h_{22}
\end{bmatrix}
\]  
(4.44)
we analyze separately the properties of a line element in the cases of having \( h_{22} = 0 \) and \( h_{23} = 0 \).

Let us start with the case \( h_{23} = 0 \). Looking at the metric components \( g_{ab} = \eta_{ab} + h_{ab} \), when \( h_{23} = 0 \), the line element is:

\[
 ds^2 = dt^2 - dx^2 - [1 - h_{22}(t - x)]dy^2 - [1 + h_{22}(t - x)]dz^2. \quad (4.45)
\]

Taking 2 particles placed on points belonging to the \( x \)-plane, one at \((y_0, z_0)\) and another one at \((y_0 + dy, z_0)\), then the proper distance between them—which is a coordinate independent quantity—is:

\[
 dl^2 = -[1 - h_{22}(t - x)]dy^2. \quad (4.46)
\]

This means that, if \( h_{22} \) is an oscillating function taking positive and negative values, when \( h_{22} > 0 \), \( dl \) becomes smaller and therefore the test particles move closer together. When \( h_{22} < 0 \), the consequences are the opposite and the test particles move further apart.

It is easy to imagine that, if an oscillatory plane gravitational wave propagating along the \( x \)-axis is incident on a ring of test particles placed in a plane parallel to the \( yz \)-plane, then the ring distorts into a pulsating ellipse whose major axis switches its orientation, alternating 2 positions: one parallel to the \( y \)-axis and the other one parallel to the \( z \)-axis, passing through all the intermediate states between both orientations. This solution is named \( + \) polarization and is denoted by: \( h_+ \).

![Figure 4.1: From left to right, evolution of a ring made of dot particles due to the effects of a \( h_+ \) GW passing through it perpendicularly to the plane of the ring. The phase difference between two circles/ellipses is \( \pi/4 \).](image)

In the case \( h_{22} = 0 \), the line element is:

\[
 ds^2 = dt^2 - dx^2 - dy^2 + 2h_{23}(t - x)dydz - dz^2. \quad (4.47)
\]

If a rotation of 45° is performed in the \((y,z)\) plane:

\[
 \begin{align*}
 y &\rightarrow y' = \frac{1}{\sqrt{2}}(y + z) \\
 z &\rightarrow z' = \frac{1}{\sqrt{2}}(-y + z)
\end{align*} \quad (4.48)
\]
\begin{align*}
\Rightarrow \left\{ \begin{array}{l}
dy = \frac{dy' - dz'}{\sqrt{2}}, \\
dz = \frac{dy' + dz'}{\sqrt{2}}, 
\end{array} \right. \tag{4.49}
\end{align*}

the line element becomes:

\begin{align*}
\Rightarrow \quad & dl^2 = dt^2 - dx^2 - \left[ \frac{dy'}{\sqrt{2}} - \frac{dz'}{\sqrt{2}} \right]^2 + 2h_{23}(t - x) \left[ \frac{dy'}{\sqrt{2}} + \frac{dz'}{\sqrt{2}} \right] - \left[ \frac{dy'}{\sqrt{2}} + \frac{dz'}{\sqrt{2}} \right]^2 = \\
= & dt^2 - dx^2 - \left[ \frac{dy'^2}{2} + \frac{dz'^2}{2} - dy'dz' \right] + 2h_{23}(t - x) \left[ \frac{dy'^2}{2} - \frac{dz'^2}{2} \right] - \left[ \frac{dy'^2}{2} + \frac{dz'^2}{2} + dy'dz' \right] \\
\Rightarrow \quad & dl^2 = dt^2 - dx^2 - \left[ 1 - h_{23}(t - x) \right] dy'^2 - \left[ 1 + h_{23}(t - x) \right] dz'^2. \tag{4.50}
\end{align*}

The result is exactly the same as in the previous case, but with the 2 orientations of the major axis rotated $45^\circ$ with respect to the $+$ polarization wave; that is why it is intuitively called $x$ polarization, $h_x$.

![Figure 4.2: From left to right, evolution of a ring made of dot particles due to the effects of a $h_x$ GW passing through it perpendicularly to the plane of the ring. The phase difference between two circles/ellipses is $\frac{\pi}{4}$.](image)

Note that the 2 polarization states have **transverse, traceless** character, as suggested by the name of the chosen gauge. The demonstration also implies that there only exist these two polarizations, so that a general plane gravitational wave is a superposition of both of them:

\begin{equation}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & h_+ & h_x \\
0 & 0 & h_x & -h_+
\end{bmatrix}.
\tag{4.51}
\end{equation}
Chapter 2

Gravitational waves as a physical phenomenon

5 Sources

Theoretical models predict mainly two types of GWs depending on their origin: it can be cosmological or relativistic astrophysical [24]. They are also differentiated between stochastic and deterministic depending on the possibility of predicting and identifying their waveform[24].

5.1 Deterministic GWs

Relativistic astrophysical gravitational waves (RAGWs) are mainly deterministic, which means that they have a concrete, no-random, potentially predictable waveform. Deterministic sources are divided into two categories:

Predictable

This kind of GWs are generated by compact binary systems in inspiral or merger/ring-down stages and other astronomical events, for which theoretical models for the corresponding gravitational waveform can be constructed before any eventual detection.

- Continuous gravitational waves (CGWs): One spinning massive object, for instance, a neutron star, behaves as a source of continuous gravitational radiation. A spin rate staying constant gives rise to a gravitational wave continuously oscillating in the same frequency –monochromatic in terms of light; single note in terms of sound– and amplitude. This is the reason why they are called “continuous gravitational waves” [4].

- Inspiral gravitational waves (IGWs): The inspiral is an event taking part along millions of years in a compact binary system as the objects are orbiting. They emit GWs as they revolve around each other, at expends of some orbital energy. This makes the objects to get closer together as time passes because of conservation of angular momentum. As they move closer together, they move faster and lose more energy through stronger GWs. This leads the system to an unavoidable fate: a dramatically energetic collision of the two bodies, which generates one the most violent explosions in the observable Universe.

Unpredictable

There are a few astronomical events that are expected to generate gravitational waves. However, the physics underlying some of these phenomena –for example, supernovae– is so difficult to understand that there is still not a good prediction of the waveform they generate. Moreover, it is also expected to detect GWs from systems absolutely undisclosed for us ever before. We refer to both types of unpredictable GWs as ‘Burst Gravitational Waves’ (BGWs).
Figure 5.1: Example of a continuous waveform. The order of magnitude corresponds to the typical order of a hypothetical CGW.

Figure 5.2: Inspiral gravitational waves have a chirped imprint. It means that the frequency changes with time. In the case of IGWs, it increases, as the magnitude of the signals does so. This plot is an example of a chirped waveform, but not an approximation of an IGW. The order of magnitude corresponds to the typical order of a hypothetical IGW.

The detection of BGWs is truly a search for the unexpected due to the large number of still-unrevealed unknowns. In contrast with IGWs and CGWs, it would be an error to assume that BGWs have well-defined properties and waveform. Therefore, analyses cannot be restricted only to GWs’ signatures that are predicted by theorists, but we need to search for any pattern of gravitational waves resembling to what we think a signal may look like. This makes the search of BGWs really difficult but it is also worthy because their discovery will reveal revolutionary information of the Universe; that is the very reason why researchers require being completely open-minded in this field of investigation. References [28] and [29]
offer a description of some of the methodology that has been employed to search for this category of GWs.

5.2 Stochastic GWs

Astronomers predict that, in the Universe, there are very few significant sources of continuous or binary inspiral GWs. This means that two or more events of this nature are very rare to reach a detector at the same time. Nevertheless, many small gravitational waves are probably passing by from all over the Universe all the time, mixing them together randomly. Thus a stochastic signal is made up, because the resulting random pattern may be analyzed statistically but not predicted with precision. It is predicted that part of this stochastic signal may be cosmological; we refer to those waves that could have been produced in the early stages of the Universe – during epochs such as the inflation, preheating and phase transitions, etc. – leaving an imprint on the cosmic microwave background not discovered yet.

![Stochastic waveform](image)

Figure 5.3: Example of a stochastic waveform.

The hypothetical future detection of these relic gravitational waves from the Big Bang, also called primordial gravitational waves, is very interesting because it will give us the necessary knowledge to better understand how the Universe was and evolved farther back than ever before; some theoretical cosmological models of the Universe will be refused against others which are nowadays considered to be equally valid due to our ignorance. This subject will be better discussed at posterior sections.

6 Inspiral GWs from compact binary systems

The compact binary systems, which are interesting from the point of view of the GWs detection, are those systems composed by pairs of white dwarf-white dwarf (WD-WD), neutron star-neutron star (NS-NS), black hole-black hole (BH-BH) or neutron star-black hole (NS-BH). They are assumed to be abundant in our Universe because they are dynamically stable
Binary systems are studied by applying three different available methods (among others): Post-Newtonian Approximation, Numerical Relativity and Black Hole Perturbation Method. Inspiral stage, plunge and merger stage, and post merger stage of the compact binary systems are the most appropriate fields to cover with the previously mentioned methods respectively.

### 6.1 Post-Newtonian approximation

In General Relativity, the phase evolution of a binary system orbiting with relative velocity low enough –i.e. $\frac{v}{c} \ll 1$ can be computed using a post-Newtonian (PN) expansion, which is a perturbative expansion in powers of the orbital velocity $v/c$ [30]. Thus, this approximation fails when reaching velocities close to $c$ –this is approximately: for $v > 0.6c$, which happens when the inspiral stage comes to an end, as the pair of compact bodies move faster and faster when they are closer together –see fig. 6.1.

![Figure 6.1: Schematic representation of a pair of black holes orbiting in different stages. The inspiral phase lasts billions of years, while the time difference between merger and ringdown is very few seconds long. Note that only the late inspiral stage is shown. The magnitude of the gravitational radiation is indicated on the ordinate axis as the strain, in other words: the relative distance of the elongation computed as the difference of an arbitrary length divided by itself. Source: [5] DOI: https://doi.org/10.1103/PhysRevLett.116.061102, Copyright (CC BY 3.0), https://creativecommons.org/licenses/by/3.0/legalcode](image)

In the PN approximation, the gravitational wave frequency $f$ evolution is described by the differential equation [6]:

$$f \approx \frac{96}{5} \pi^{8/3} \left(\frac{GM}{c^3}\right)^{5/3} f^{11/3}, \quad (6.1)$$
where $\mathcal{M}$ determines the **leading order term** in the evolution of the binary system’s orbit and is called ‘chirp mass’ [6]:

$$
\mathcal{M} \approx \left( \frac{m_1 m_2}{m_1 + m_2} \right)^{3/5} 
\frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}.
$$

(6.2)

The two formulae above come from the assumption of Kleperian orbits and the use of the **quadrupole formula**, eq.6.3, first derived by Albert Einstein, which describes the emission of gravitational waves:

$$
\bar{h}_{ij} = \frac{2G}{c^4 r} \ddot{I}_{ij}(t - r/c),
$$

(6.3)

where $\bar{h}_{ij}$ is the spatial part of the trace reversed perturbation of the metric and $\ddot{I}_{ij}$ is the second time derivative of the reduced (trace–free) mass quadrupole moment –see [22], Sean M. Carrol’s Gravitation and cosmology (2004), pages 300-307, or [31], page 7. A related formula, eq. 6.4, which is in agreement to the figure 1.1, relates the mass quadrupole moment third derivative to the energy lost of the system due to the emission of GWs.

$$
dE_{GW} = \frac{G}{5c^5} \dddot{I}_{jk} \dddot{I}^{jk},
$$

(6.4)

The gravitational wave frequency is determined by the orbital frequency –in fact, the wave frequency is twice the orbital one when the orbital is circular–, therefore the chirp mass also determines the frequency evolution of the GW signal emitted during the inspiral phase. If one is able to measure both the frequency $f$ and frequency derivative $\dot{f}$ of a gravitational wave coming from a CBS, the chirp mass can be determined, which means that one can compute the chirp mass by knowing the GW waveform [32]. Additional parameters enter at each of the following PN orders. First, the mass ratio, $q = m_2/m_1 \leq 1$, and the BH spin components parallel to the orbital angular momentum vector $\mathbf{L}$ affect the phase evolution. The full degrees of freedom of the spins enter at higher orders [6]. The so called luminosity distance, $D_L$, is also induced from the detected GWform [33].

During the inspiral and at the leading order we can estimate the GWs related to a binary system [24] as:

$$
h_+ = -\frac{M}{r} 2\Omega^2 R^2 (1 + \cos^2 \theta) \cos[2\Omega(t - r)],
$$

(6.5)

$$
h_\times = -\frac{M}{r} 4\Omega^2 R^2 \cos \theta \sin[2\Omega(t - r)],
$$

(6.6)

where $(r, \theta, \phi)$ is the position of the observer with respect to the binary taking the $z$ axis parallel to the binary’s orbital momentum vector, $M$, $R$ and $\Omega$ are the total mass, the separation and the mutual orbiting frequency of the two components of the binary. $h_+$ and $h_\times$ correspond to the two polarization modes described in section 4.1.

The evolution of CBSs is processed in two parts in the framework of PN approximation: the dynamics of binary components and the GWform. In order to get the explicit GW model, we need to solve the dynamics of the binary system and then put the solution into the waveform theory part. Due to the large nonlinear character of the dynamical equations appearing in the PN approximation, numerical methods are needed to get the solution [24],[34] as the system reaches the late inspiral, merger and ringdown stages.
6.2 Numerical Relativity (NR): a historical review of the BH-BH coalescence problem

If we search for a test of our knowledge about strongly interacting bodies in General Relativity, the BH-BH coalescence is especially important because the system is very likely the simplest strong-field gravitational problem, for it contains only gravitational fields and no matter neither energy distributions [9] and they do not emit electromagnetic waves as they would absorb any radiation of this nature [35]—possible exceptional cases where electromagnetic waves may be emitted [36] are: if there is matter surrounding the BHs, such as remnants of mass lost from the parent star, or if the binary was embedded in a circumbinary disc or a common envelop.

To obtain the Fourier-Brouhat proof of local stability and uniqueness of this initial value problem, certain smoothness assumptions—see [9] and references therein—and the use of harmonic coordinates (which means they satisfy the d’Alembert equation, eq.(6.7) [37]) were needed to specify the evolving spacetime. By doing this, Einstein’s equations turn out into a set of ten quasi-linear wave-like equations with favorable (hyperbolic) stability properties.

\[ \nabla_b \nabla^b x^a = 0. \] (6.7)

After this, the search for a stable, accurate numerical method to solve Einstein’s equations passed through difficulties along many decades of investigation. The target was to solve the Cauchy problem by means of a step iterative process from initial spacetime conditions. In 1962 Arnowitt, Misner and Deser (ADM), while originally studying quantum gravity, released the so-called 3+1 approach, which handled the equations with time separated from space. In 1979, York rewrote the ADM prescription so that it would potentially work in the Einstein equations for a dynamically evolving system, such as in the case of a merger process, instead of quantum gravity.

The main idea of ADM approach is decomposing spacetime in a stack of hypersurfaces containing only the three-dimensional spatial metric \(-\gamma_{ij}\), with i and j indexes varying from 1 to 3, each hypersurface \(\Sigma_t\) having a fixed time coordinate. This is what we refer to as spacelike foliations. The invariant spacetime interval in the 3+1 description is:

\[ dt^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \] (6.8)

In a numerical approach, the system evolution is represented by its transition from one foliation to the next. In eq.(6.8), \(\alpha\) is representing the proper time lapse at which we change from one foliation to the next and \(\beta^j\) is the shift vector, meaning the change on spatial coordinates between foliations; changing \(\alpha\) and \(\beta\) in fact means that we are choosing the coordinates as we proceed (\(\alpha\) and \(\beta^j\) are gauge variables, i.e., they are representing the coordinate freedom inherent in Einstein’s equations). By choosing a good time lapse that avoids singularities, speeds up computations and improves the convergence the whole simulation performance is enhanced.

The way to deal with the ‘spacetime slices’ in the ADM approach is taking the real Extrinsic Curvature Tensor \(K_{ij}\): because the foliations \(\Sigma_t\) are embedded in the overall spacetime manifold \(M\), it is used to describe the nature of the embedding. From the definition of \(K_{ij}\) as the covariant derivative of a surface normal vector—with respect to a foliation—along the integral curves described by itself, this tensor is aimed to separate the intrinsic curvature of
the foliation $\gamma_{ij}$ from the extrinsic curvature due to the way it is embedded in the overall spacetime. Eq.(6.9), where $D$ means the spatial covariant derivative and $\partial_t$ is the partial derivative with respect to the proper time, can be considered a definition of $K_{ij}$.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i.$$ \hspace{1cm} (6.9)

In spite of the effort spent on the ADM procedure, evolving a BH-BH system until the merger remained elusive due to the weakly hyperbolic nature of its equations. While hyperbolic equations give rise to finite speed wave-like solutions, the real parabolic and elliptic behavior of the dynamic equations made the system impossible to converge into merger. A version of ADM that showed much better stability was presented in a work of Nakamura, Oohara and Kajima in 1987. The 1995, Shibata and Nakamura and, three years later, Baumgarte and Shapiro confirmed and extended those results. The entire model that they created is commonly known as the BSSNOK approach, it is in wide use today and, in fact, it was essential to achieving full 3-dimensional simulations of BH–BH coalescences.

Stable numerical coalescence of a BH-BH pair is completely differently described by Garfinkle, and Szilágyi and Winicour, Pretorius taking an earlier work by Friedrich. Lindblom extended this treatment soon. The idea is a generalized form of harmonic coordinates by means of the usage of independent functions representing source terms, $H^a$, that are added to the coordinates’ harmonic wave equations:

$$\nabla^b \nabla_b x^a = H^a.$$ \hspace{1cm} (6.10)

In this manner, arbitrary gauge conditions are chosen in a similar way as it happens with the time-lapse and the shift in the ADM prescription. The result of Lindblom et al.’s work is a fully-first order –both in time and space– generalized harmonic evolution development, while, in the ADM treatment equations are first-order in time and up to second order in space.

**Matched-filtering**

The optimal requirements of a numerical relativity simulation for best utility to GW astronomy [34] are the following:

1. sufficient accuracy,
2. realistic orbital eccentricity,
3. sufficient number of orbits to connect properly to PN waveforms
4. and sufficient coverage of the parameter space.

Theoretical template banks of numerical simulations satisfying the stated requirements –see the one presented at [34]– make possible the identification of GWs hidden in the noisy signals received by the interferometers. Parameter estimation of the binary’s components [38] is feasible thanks to matched-filtering methods involving Markov Chain Monte Carlo algorithms.
6.3 Black Hole Perturbation Method

When the merger stage of a binary system ends, the resulting mixture of black-holes ringdowns and eventually settles down to a Kerr black hole (KBH). If the system was formed by white dwarfs and/or neutron stars, the final stable object could be a neutron star. We will only talk about the case ending with a Kerr black hole.

The black hole perturbation method (BHPM) [39] is a way to describe the ringdown stage from the perturbation of a Kerr black hole. It is useful to investigate topics in astrophysics, fundamental problems in gravity and high energy physics. Pioneered by Regge, Wheeler and Zerilli, who deduced the Regge-Wheeler-Zerilli equation of the perturbation around a Schwarzschild black hole, the KBH perturbation method was started by Teukolsky as he developed the Teukolsky equation, which is also valid to Schwarzschild black holes for they are a particular case of KBHs.

The underlying technique leading to the Teukolsky equation is to linearize the Einstein or Maxwell-Einstein equations - ie. if the energy-momentum tensor is that of an electromagnetic field in vacuum - around a known stationary black hole solution with the Kerr metric. To obtain the linearized equations, the metric is chosen as $g_{ab} = g_{ab}^A + h_{ab}^B$, with the superscript $A$ denoting the background - in this case, the Kerr metric - and $B$ denoting the perturbation. The procedure to solve the field equations is analogous to one we performed in section 4.1 in the case of almost Minkowski metric, but now, the Kerr metric is the background instead. In a series of papers started with [40], Teukolsky deals with astrophysical applications of the rotating black hole problem such as the stability of the KBH and gravitational-wave processes. The way he tackles the problem is by means of the Newman-Penrose formalism developed in [41] and the result [40] is that the equations decouple in a single gravitational, a single gravitational and a single neutrino equation, each of them completely separable into ordinary differential equations.

7 Detection of gravitational waves

7.1 Gravitational waves’ physics and parameters: black holes

IGWs from neutron stars and stellar-mass black holes are the main targets for the current generation of GW detectors [34], such as Advanced LIGO, Advanced Virgo and KAGRA. A single BH is described by only its mass and spin - we expect the electric charge of astrophysical BHs to be negligible. Eight intrinsic parameters describe a BH-BH system: the masses $m_1, m_2$ (defined as the gravitational masses of the BHs in isolation) and spins $S_1, S_2$ of each BH.

For a BH of mass $m$, the spin modulus is bounded by:

$$S \leq \frac{Gm^2}{c}; \quad (7.1)$$

hence, it is habitual to use the dimensionless spin magnitude:

$$a = \frac{c|S|}{Gm^2} \leq 1. \quad (7.2)$$
We need nine more parameters to completely describe the binary: the location (luminosity distance $D_L$, right ascension $\alpha$, and declination $\delta$); orientation (the binary’s orbital inclination $i$ and the polarization $\psi$, which is defined in a sense referred to the orientation of the CBS with respect to the observer); time $t_c$ and phase $\phi_c$ of coalescence, and two parameters describing eccentricity (the magnitude $e$ and the argument of periapsis) of the system.

As seen in subsection 4.1, far away from the source, –i.e. at distance $R\lambda_{GW}$– gravitational radiation is described by two independent, and time-dependent polarizations, $h_+$ and $h_\times$ produced by the binary [42] are contained in the perturbation matrix $h_{ab}$ –the wave itself– added to the Minkoski metric corresponding to the flat spacetime:

$$g_{ab} \approx \eta_{ab} + h_{ab}. \quad (7.3)$$

A GW-detector $k$ measures the strain:

$$h_k = F_k^{(+)} h_+ + F_k^{(\times)} h_\times, \quad (7.4)$$

a linear combination of the two possible polarization states weighted by the antenna beam patterns $F_k^{(+,\times)}(\alpha, \delta, \psi)$, which depend on the source location in the sky and the polarization $\psi$. Going back to eqs. 6.5 and 6.6, they are rewritten in a more compact form:

$$h_+ = A_{GW}(t)(1 + \cos^2 i) \cos \phi_{GW}(t), \quad (7.5)$$

$$h_\times = -2A_{GW}(t) \cos i \sin \phi_{GW}(t). \quad (7.6)$$

Eqs. 7.5 and 7.6 tell us that when binaries are viewed face-on ($\cos i = \pm 1$), GWs have circular polarization, whereas for binaries observed edge-on ($\cos i = 0$), GWs have linear polarization. Please, watch the video [43], https://www.youtube.com/watch?v=Y6tSFk5E5Ao, to get a visual understanding of the meaning of this.
7.2 Confirmed events

GW150914

The LIGO interferometers were the first GWs detectors to find a source of gravitational radiation. GW150914 is the first BH-BH binary system ever detected and is also the first direct detection of a black hole itself. Since black holes do not emit any kind of detectable electromagnetic signal, before this event, the presumable existence of black holes was backed up by the study of the gravitational behaviour of clusters of stars, galaxies, gravitational lensing, etc. that could be explained by means of the assumption that there was an object behaving as a black hole in the studied system; but any kind of signal proceeding from a black hole was never before detected. This means that the detection of GW150914 demonstrated the existence of black holes. As it was also the first time that a gravitational signal was detected, it initiated a new era of the astronomy: the gravitational astronomy.

In this detection it was observed that the radiation reaction was efficient in circularizing orbits before the signal enters the sensitive frequency band of the LIGO instruments (20–1000Hz); no evidence for residual eccentricity was found in the analysis performed by the authors of [6].

By the time it was confirmed as astrophysical in origin, this event was the most energetic phenomenon ever detected –apart from the Big Bang– and attests that binary black holes do form and merge within a Hubble time. [6]

![GW150914 Image](https://doi.org/10.1103/PhysRevLett.116.061102, Copyright (CC BY 3.0), https://creativecommons.org/licenses/by/3.0/legalcode)

The gravitational waveform observed for GW150914 comprises 10 cycles during the inspiral
phase from 30 Hz, followed by the merger and ringdown. The properties of the binary affect the phase and amplitude evolution of the observed GWs, allowing us to measure the source parameters.

**GW151012 or LVT151012**

A lot of doubts surrounded the origin of this event until the results on the paper [44] (February, 2019) gave clearer notice of its origin. The first computations lead to a false alarm rate, FAR, of 2.3 per year—which is very high when compared to the other registered events, which all have a $FAR < 10^{-7}$, except for GW151012 with $FAR < 7.92 \cdot 10^{-3}$ and GW170818 with $FAR < 4.20 \cdot 10^{-5}$. Nevertheless, the confidence on the astrophysical origin of this signal increased with each improvement on the data analysis methods [26] up to reach a $FAR = 0.18$. Finally, this lead us to estimate that [26] GW151012—the initially and still so called LVT151012, giving notice of its large $FAR$—is astrophysical in origin with 97.59% probability. The waveform and spectogram of this event are available at GW-catalogues contained in sources such as the one in [26], because no article exclusively aimed to characterize it was released due to the uncertainty around it.

**GW151226**

![Figure 7.2: GW151226. Image source: [45] DOI:https://doi.org/10.1103/PhysRevLett.116.241103, Copyright (CC BY 3.0),https://creativecommons.org/licenses/by/3.0/legalcode]
GW170104

By the time of its detection, it was the farthest detection of black holes at about $3 \cdot 10^9$ light-years.

Figure 7.3: GW170104. Image source: [46] DOI:10.1103/PhysRevLett.118.221101, Copyright (CC BY 4.0).https://creativecommons.org/licenses/by/4.0/legalcode
GW170814

It was first detected by Livingston (LIGO); 8 ms later, by Handford (LIGO) and 14 ms, by Advanced Virgo, thus becoming the first to be detected by 3 interferometers, which implied a more precise locating of the source. The detection from the three different locations allowed to analyze the possible polarizations, leading to the conclusion that they correspond to those predicted by the General Relativity.

Figure 7.4: GW170814. Image source: [17], DOI: https://doi.org/10.1103/PhysRevLett.119.141101, Copyright (CC BY 4.0), https://creativecommons.org/licenses/by/4.0/legalcode
GW170608

Since Hanford detector was in test, it was announced after GW170814 even though being detected actually before it, because there was not automatic alert. It was identified by visual inspection, thus giving notice of the possibility of having detected many other signals that were not identified yet.

![GW170608 spectrograms](Image Source: [47], DOI:10.3847/2041-8213/aa9f0c, Copyright (CC BY 3.0), https://creativecommons.org/licenses/by/3.0/legalcode)

Figure 7.5: GW170608 spectrograms. Image source: [47], DOI:10.3847/2041-8213/aa9f0c, Copyright (CC BY 3.0), https://creativecommons.org/licenses/by/3.0/legalcode
GW170817

In August the 17th, 2017, for the first time in history, many observatories all around the world simultaneously detected the gravitational and electromagnetic radiation proceeding from the same source. The signals came from a pair of neutron stars colliding in a galaxy placed at about 130 million light years far from Earth.

The time difference between the gravitational signal and the electromagnetic one was just $1.7s$. From this, the relative velocity is computed, involving a velocity difference smaller that $10^{-15}$, a number that is in perfect agreement with the Einstein General Relativity stating that GWs propagate through the spacetime exactly at the velocity of light $c$, as exposed in section 4—in fact, the small time delay between the signals can be explained because of the mechanism triggering the explosion. This discovery is of great importance since other existing theories of gravity that could have explained the accelerated expansion of the Universe do not state the same: they state GWs do not propagate at $c$. Consequently, this NS-NS merger has enabled us to reject many theoretical models aiming to explain the dark energy—see section 8. The role of Vigo added to LIGO interferometers proved vital in pinpointing the host galaxy.

Figure 7.6: GW170817 spectrograms. Image source: [48], DOI: https://doi.org/10.1103/PhysRevLett.119.161101, Copyright (CC BY 4.0), https://creativecommons.org/licenses/by/4.0/legalcode

for GW170817: NGC4993 [49]. The electromagnetic signal received from this source allowed
astronomers to detect a kilonova, a kind of ‘radioactive oven’ where elements heavier than iron, such as gold, platinum and uranium are produced. This finding helps explain the abundance of these elements in the Universe.

The data analysis also demonstrated that during the event GW170817 a magnetar was produced and existed for 5s.

**Other confirmed sources**

Four additional BH-BH coalescence events were found after more precise analysis of the registered signal obtained during the first and second observation runs of LIGO. Virgo also detected those sources dated after GW170814.

<table>
<thead>
<tr>
<th>GW</th>
<th>$m_1$ [$M_\odot$]</th>
<th>$m_2$ [$M_\odot$]</th>
<th>$M_f$ [$M_\odot$]</th>
<th>$a_f$</th>
<th>$E_{\text{rad}}$ [$M_\odot c^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>150914</td>
<td>35.6$^{+4.8}_{-3.0}$</td>
<td>30.6$^{+3.0}_{-4.4}$</td>
<td>63.1$^{+3.3}_{-3.0}$</td>
<td>0.69$^{+0.05}_{-0.04}$</td>
<td>3.1$^{+0.4}_{-0.3}$</td>
</tr>
<tr>
<td>151012</td>
<td>23.3$^{+3.0}_{-4.8}$</td>
<td>13.6$^{+4.1}_{-3.8}$</td>
<td>35.7$^{+9.9}_{-4.4}$</td>
<td>0.67$^{+0.11}_{-0.2}$</td>
<td>1.5$^{+0.5}_{-0.1}$</td>
</tr>
<tr>
<td>151226</td>
<td>13.7$^{+10.0}_{-5.6}$</td>
<td>7.1$^{+2.2}_{-4.5}$</td>
<td>20.5$^{+1.5}_{-2.9}$</td>
<td>0.74$^{+0.07}_{-0.05}$</td>
<td>1$^{+0.1}_{-0.2}$</td>
</tr>
<tr>
<td>170104</td>
<td>31.0$^{+7.2}_{-5.6}$</td>
<td>20.1$^{+4.9}_{-4.5}$</td>
<td>49.1$^{+5.2}_{-3.9}$</td>
<td>0.66$^{+0.08}_{-0.10}$</td>
<td>2.2$^{+0.5}_{-0.2}$</td>
</tr>
<tr>
<td>170608</td>
<td>10.9$^{+5.3}_{-1.7}$</td>
<td>7.6$^{+1.3}_{-2.1}$</td>
<td>17.8$^{+3.2}_{-0.7}$</td>
<td>0.69$^{+0.04}_{-0.03}$</td>
<td>0.9$^{+0.05}_{-0.9}$</td>
</tr>
<tr>
<td>170729</td>
<td>50.6$^{+10.6}_{-10.2}$</td>
<td>34.3$^{+10.1}_{-9.1}$</td>
<td>80.3$^{+14.6}_{-14.0}$</td>
<td>0.81$^{+0.13}_{-0.07}$</td>
<td>4.8$^{+1.7}_{-1.7}$</td>
</tr>
<tr>
<td>170809</td>
<td>35.2$^{+8.3}_{-6.0}$</td>
<td>23.8$^{+5.2}_{-4.5}$</td>
<td>56.4$^{+3.2}_{-3.7}$</td>
<td>0.70$^{+0.08}_{-0.03}$</td>
<td>2.7$^{+0.6}_{-0.6}$</td>
</tr>
<tr>
<td>170814</td>
<td>30.7$^{+5.7}_{-3.0}$</td>
<td>25.3$^{+2.9}_{-4.1}$</td>
<td>53.4$^{+3.2}_{-2.4}$</td>
<td>0.72$^{+0.07}_{-0.05}$</td>
<td>2.7$^{+0.4}_{-0.3}$</td>
</tr>
<tr>
<td>170817</td>
<td>1.46$^{+0.12}_{-0.10}$</td>
<td>1.27$^{+0.09}_{-0.09}$</td>
<td>≤ 2.8</td>
<td>≤ 0.89</td>
<td>≥ 0.04</td>
</tr>
<tr>
<td>170818</td>
<td>35.5$^{+4.7}_{-4.7}$</td>
<td>26.8$^{+4.3}_{-5.2}$</td>
<td>59.3$^{+4.8}_{-3.8}$</td>
<td>0.67$^{+0.07}_{-0.08}$</td>
<td>2.7$^{+0.5}_{-0.5}$</td>
</tr>
<tr>
<td>170823</td>
<td>39.6$^{+10.0}_{-6.6}$</td>
<td>29.4$^{+6.3}_{-7.1}$</td>
<td>65.6$^{+9.4}_{-6.6}$</td>
<td>0.71$^{+0.08}_{-0.10}$</td>
<td>3.3$^{+0.9}_{-0.8}$</td>
</tr>
</tbody>
</table>

Table 1: These are data about the system parameters induced from all the available confident GWs’ detections: $m_1$ and $m_2$ are the black holes masses, $M_f$ is the mass of the resulting KBH, $a_f$, the dimensionless spin magnitude of the resulting KBH and $E_{\text{rad}}$ is the radiated. The data source is: [26]

### 7.3 Parallel search for non-solar-mass-IGW sources

Apart from the successful search for solar mass compact binary coalescence events performed by the LIGO and Virgo Collaborations with data from the LIGO de Virgo and GEO600 interferometers, the two collaborations also work together in a parallel search for other possible sources that are expected to have frequencies in the frequency range of the apparatuses.

- Reference [50] is about the study of the possible detection of sub-solar compact binary mergers. A detection of such an event would mean the need for new theoretical physics because the most accepted theories underlying them state that NSs’ and BHs’ masses are bounded by $M_{NS} \gtrsim 0.9M_\odot$ and $M_{BH} \gtrsim 1.4M_\odot$. At least for now, no positive results of sub-solar compact binaries have been found. Tight constraints of sub-solar mass ultracompact binaries are set up from this search, which will follow to inform cosmological and particle physics scenarios.
Table 2: These are data about the location from all the available confident GWs’ detections: $D_L$ as the luminosity distance (it is given the mean value in the 90% credible interval), $\Delta \Omega$ as the solid angle of the sky localization (it is the area of the 90% credible region). The data source is: [26]

- References [28] and [29] show no event candidates for BGWs. The searches set better limits on rate density of this kind of sources per unit time and volume and the rate of BGWs at Earth.

7.4 Detectors

7.4.1 LIGO, Virgo, KAGRA and GEO600

These four detectors all have a frequency band from about 10Hz to 10kHz. The LIGO detectors consists of 2 interferometers placed at 2 extremes of United States (Hanford and Livingston) with two arms 4 kilometers long. Virgo is a single interferometric gravitational-wave antenna consisting of two 3-kilometre-long arms. Despite having roughly half the sensitivity of LIGO’s interferometers, its contribution to the detection of GWs has been crucial for locating the radiating source.

LIGO and Virgo

The two ground GW-observatories that have detected gravitational radiation during the latest few years are the Laser Interferometer Gravitational-Wave Observatory, in the USA, and the Virgo interferometer, in Italia. They consist of modified Michelson interferometers which basically measure the lengthening and shortening phenomena taking place on the interferometer arms by means of the analysis of the induced changes on the beams’ interaction patterns due to the pass of a GW. Initially, the interaction is destructive so that no signal is detected in the ‘dark fringe’ of the detector. A gravitational wave propagating orthogonally to the instrument’s plane and linearly polarized parallel to the optical cavities will have the effect of lengthening one arm and shortening the other during one half-cycle of the wave; these length changes are reversed during the other half-cycle. The output photodetector records these differential cavity length variations by analyzing the interference pattern of the laser beams. While a detector’s directional response is maximal for this case, it is still significant for most
other angles of incidence or polarizations [5]. The highest sensitivity of these detectors is around 100 Hz [50], near to the frequencies at which Solar-mass binary black holes, binary neutron stars and BH-NS systems in late inspiral and merger stages radiate.

Figure 7.7: Simplified diagram of an Advanced LIGO detector (not to scale). Inset (a): Location and orientation of the LIGO detectors at Hanford, WA (H1) and Livingston, LA (L1). Inset (b): The instrument noise for each detector near the time of the detection of GW150917; this is an amplitude spectral density, expressed in terms of equivalent gravitational-wave strain amplitude. The sensitivity is limited by photon shot noise –see GEO600– at frequencies above 150 Hz, and by a superposition of other noise sources at lower frequencies. Narrow-band features include calibration lines (33–38, 330, and 1080 Hz), vibrational modes of suspension fibers (500 Hz and harmonics), and 60 Hz electric power grid harmonics. Image source: [5] DOI:https://doi.org/10.1103/PhysRevLett.116.061102, Copyright (CC BY 3.0).https://creativecommons.org/licenses/by/3.0/legalcode

Both LIGO and Virgo instruments have many components in common to detect vibrations produced by GWs. Here we name some of them:

- **Fabry Perot cavities.** By adding a mirror with the convenient refractive index in the proper position on each of the arms of an interferometer, we obtain what is known as a Fabry Perot cavity—a good explanation of how do Fabry Perot cavities work is found at [51]. They work in such a way that the laser beam performs multiple reflections before it leaves the cavity to eventually reach the photodetector. This process effectively enlarges the cavity length, which implies an **increase in the interferometer sensitivity** since the relative difference on the arm length makes the travelling-time difference of the laser beam increase, thus magnifying the amplitude of any vibration—including the smallest, interesting ones—happening in the travel direction [52],[53].
• **Power recycling mirror.** Variations of the laser beam intensity are proportional to the GW’s amplitude, so a very high optical power is important because it allows the improvement in the sensitivity of the interferometer. To do so, power recycling mirrors are included between the laser device and the beam splitter –see fig.7.8. That mirror is designed in such a way that its only partially reflective: it behaves a ‘one-way’ mirror. The word ‘recycling’ comes from the fact that laser light coming back to the laser source from the beam splitter strikes the recycling power mirror being reflected back in the interferometer, thus enhancing the optical power by adding those photons to the incident beam [52],[53].

![Figure 7.8: Sensitivity curves of the two LIGO and the Virgo antenna by the time of the detection of GW170814. Image source: [17] DOI: https://doi.org/10.1103/PhysRevLett.119.141101, Copyright (CC BY 4.0), https://creativecommons.org/licenses/by/4.0/legalcode](https://doi.org/10.1103/PhysRevLett.119.141101)

**GEO600**

GEO600 is a detector located near Hannover, Germany. Even though it has not announced any detection, the technology applied to this instrument is of high interest to enhance the sensitivity of any GW-antenna for high frequency light. In fact, the ‘squeezed light’ technology, that has been used in GEO600 from mid-2010, has been installed to the LIGO interferometers before their 3rd observation run, highly increasing their sensitivity for detecting events occurring in the high-frequency range of the apparatuses frequency band. The technique consists on the addition of a few entangled photons per second to the main laser beam producing a sufficiently significant result in the sensitivity. At frequencies around 1000 Hertz –exactly where signals from supernovae and the birth of neutron stars are expected– the quantum nature of light takes a toll. Individual photons will hit the detector at an uneven rate because of quantum fluctuations resulting from the Uncertainty Principle: this ‘shot-noise’ may completely
modify the interference pattern in the dark fringe of the interferometer, which could turn the GW signal undetectable, as it would hide within the noise [54].

KAGRA

KAGRA is a new GW-interferometer with two 3-km baseline arms, located inside the Mt. Ikenoyama, Kamioka, Gifu, Japan. Its design is similar to those of the second generations—such as Advanced LIGO/Virgo—, but it will be operating at the cryogenic temperature with sapphire mirrors. This low temperature feature is advantageous for improving the sensitivity around 100 Hz. It is expected to start its first observing run in late 2019 [18].

![Figure 7.9: A schedule of the detection distance range in Mpc achieved by LIGO, Virgo and KAGRA interferometers as they are updated with the years. O1 and O2 refers to the first and second observing runs. Image source: [36] DOI: https://doi.org/10.1007/lrr-2016-1, Copyright (CC BY 4.0) https://creativecommons.org/licenses/by/4.0/](#)

7.4.2 ET

ET, *Einstein Telescope*, is a proposed ground-based detector in which some institutions of the European Union are working. Similarly to KAGRA, it will have cryogenic optics and will be located underground in order to reduce seismic noise and ‘gravity gradient noise’ caused by nearby moving objects. Its arms will be 10km long, forming a equilateral triangle with two photodetectors at each corner, so that there is not any direction to which the telescope is unsensitive. Each of the three detectors will be composed of two interferometers, one optimized to work under 30Hz and the other designed for higher frequencies [55].

7.4.3 LISA

LISA, *Laser Interferometer Space Antenna* [58], is planned to be launched in 2034. It will have a triangular, 2-detectors-per-corner structure similar to the one of the Einstein Telescope. However, LISA will operate in a new frequency range, **between 0.1mHz and 1Hz**, as it is aimed to detect GWs produced by compact binary systems and mergers of supermassive
black holes in galactic nuclei. The entire arrangement will be placed in an Earth-like solar orbit at 1AU to the Sun, since the frequency band requires an arm length of 2.5 million kilometers, which is in fact larger than Earth’s diameter. LISA will also be able to detect GWs from extreme mass ratio inspirals—i.e., compact objects being absorbed by supermassive black holes—, compact binaries in our galaxy (mainly white dwarf pairs [56], many of which will be unresolvable)—this source is expected to produce a noisy background because it is predicted [57] that there are 26 million galactic binaries—and compact binaries beyond.

From the NASA’s point of view [58], LISA will be able to detect cosmological GWs produced by quantum fluctuations from the early Universe too. Nevertheless, the main target of LISA concerns the detection of galaxy mergers, which is expected to reveal uncountable mysteries regarding the dark energy and dark matter, which together constitute the 95% of the total mass-energy content of the Universe.

![Figure 7.10: Sensitivity curves of various GW detectors as a function of frequency together with the characteristic strain of a selection of astrophysical sources. KAGRA’s and LiGO’s curves are not shown here because they are very similar to the Virgo’s (AdV) curve and they would overlap in an uncomfortable way. Image credit: Christopher Moore, Robert Cole and Christopher Berry Copyright [CC BY-SA 1.0 (https://creativecommons.org/licenses/by-sa/1.0)](https://creativecommons.org/licenses/by-sa/1.0)](https://creativecommons.org/licenses/by-sa/1.0)
7.4.4  IPTA and SKA

IPTA, *International Pulsar Timing Array* [59], will combine observations of pulsars from both Northern and Southern hemisphere observatories with the main aim of detecting ultra-low frequency \(10^9 \text{ to } 10^8 \text{ Hz}\) GWs by investigating the induced timing residuals caused by gravitational waves on the pulsar timing emission rate. SKA, *Square Kilometre Array* [60], is a project of the construction of squared antenna arrays of hundreds of thousands and eventually up to a million low-frequency antennae that will be located in Western Australia and South Africa. These projects are expected to detect in the GW frequency band corresponding to the stochastic cosmological background.

8  Consequences

Thanks to the detection of compact object merger events, the scientific community has learnt enough so as to better establish the boundaries of theoretical physics. Many theoretical models –which would have potentially explained the nature of dark matter– have been rejected because the new information we have is actually incompatible with them. Some of the ‘surviving’ theories are: a) General relativity with additional spatial dimensions, b) the hypothesis of existence of quintessence [61] and c) the tensor–vector–scalar gravity (TeVeS) [62].

Most of the consequences of the detection of GWs have been exposed along all the text. Here we recall the most relevant:

- Binary black holes are observable by direct means now.
- It has been discovered that GWs travel at speed of light.
- GWs have the polarization states predicted by Einstein.
- It has been observed a kilonova: the origin of many chemical elements has been corrected from this discovery.
- Better bounds in the solar-mass compact binaries population have been established.
- We are closer to solve mysteries related to the dark energy, dark matter and the early Universe.
9 Conclusions

When the General Relativity Theory was released to the world in 1916 by Albert Einstein, a new way to understand space, time and gravity came to stay: From then to now, space and time have been codependent dimensions that are inseparably studied as a whole, the so called ‘spacetime’, and gravity has never more been understood as a conventional physical force, but as the deformation of the spacetime itself.

100 years of theoretical, numerical and technological advance was needed to view a major potential of the General Relativity, as the gravitational astronomy has shown it to be sufficiently precise for the prediction of events involving astrophysical objects that still remain not completely understood.

In Chapter 1 of this thesis, we have seen how the prediction of GWs was performed. Applying loads of algebra, the Linearized Einstein Equations finally lead us to two possible polarization states of plane gravitational waves.

In Chapter 2, the main expected GW sources are presented. Later, we focus on the CBS coalescence problem, which is also divided into its three habitual stages: inspiral, ringdown and merger. Each stage has been properly extended for the case of BH-BH merger so as to understand their significance in the sphere of the current gravitational astronomy. The physical parameters related to a BH-BH coalescence are presented afterwards. Later, the first noticed events are presented, including waveforms, spectrograms and tables with some of the parameters related to all confirmed events up to day. A list of parallel search for other sources follows the previous gw-catalog, which is followed by a summarized description of the currently-operative detectors and future detectors. A synopsis of the most significant consequences ends the thesis.
10 Acronyms

- **BH** Black hole
- **BHPM** Black hole perturbation method
- **CBS** Compact binary system
- **CGW** Continuous gravitational wave
- **GW(form)** Gravitational wave(form)
- **IGW** Inspiral gravitational wave
- **KBH** Kerr black hole
- **LEE** Linearized Einstein’s equations
- **NS** Neutron star
- **WD** White dwarf
Acknowledgements

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