Published paper:

Quintanilla, R.; Ueda, Y. Decay structures for the equations of porous elasticity in one-dimensional whole space. "Journal of dynamics and differential equations", 12 Juny 2019. doi:10.1007/s10884-019-09767-w

Decay structures for the equations of porous elasticity in one-dimensional whole space

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Abstract: This paper investigates the solutions of the porous-elastic materials with dissipation in the case of the whole real line. We consider three different cases. First we consider the case when there are dissipation mechanisms at the elastic structure and the porous structure and we prove the decay structure is standard type. Second we consider the cases when the dissipation is only on the elastic structure or on the porous structure. In these cases we show that the decay structure is regularity-loss type. Furthermore, we will show the optimality for the decay estimates for all cases.

1. Introduction

Elasticity problems have attracted the attention of researchers from different fields. In particular the time decay of solutions is one goal to study. In this article we are concerned with the theory of porous viscoelastic materials. Porous elasticity is one of the easier extensions of the classical theory of elasticity. It can be seen as a continuum theory of materials with interstitial voids. Besides the usual elastic effects, these materials have a microstructure with an important property: the mass in each point can be obtained as the product of the mass density of the material matrix by the volume fraction. This idea was used by Nunziato and Cowin to develop a theory of elastic materials with voids. It is worth recalling the book of Iesan where this theory is deeply discussed [3]. The relevance of the porous elastic materials has been demonstrated amply by the huge quantity of papers (from different fields) dedicated to this kind of materials. In fact in the last decades this theory has been considered to study solids with small distributed pores such as rocks, soils, woods, ceramics, pressed powders or biological materials as bones.

From a physical point of view we can see the case of the porous-elastic materials as the juxtaposition of two structures: the elastic part and the porous part. The elastic part can be seen as the macroscopic structure and the porous part as the microscopic structure. They both are linked by means of the coupling terms. Therefore it is relevant to clarify how several dissipation mechanisms influence in the global behavior of the material. In this sense our paper is directed in this line. Three cases are going to be considered. One case corresponds to the situation where we have viscoelasticity and porous dissipation. The other two cases correspond to the case when we only have viscoelasticity or porous dissipation.

In the present article, we study the porous dissipation elasticity problem in one-dimensional whole space. It is worth noting that this kind of problem has been studied in a deep way in the case of bounded domains [6, 8, 9, 10, 11, 12]. However, the case when the domain is the whole space has been very forgotten in the literature and as far as we know we can only recall the contributions [2, 14]. Unfortunately, the decay estimates introduced in [2] are not optimal and they considered only the dissipation mechanisms at the elastic structure. In [14] the authors studied the porous elastic problem when viscosity and non-classical thermal effects were present and they obtained results concerning slow decay and regularity-loss type. In contrast to [2], we study not only the elastic structure but also the porous structure and derive the optimal decay estimates. The most interesting property of the porous dissipation elasticity problem is the regularity-loss structure. If we consider this problem with the elastic dissipation or the porous dissipation, the corresponding decay structure is the regularity-loss type.

The typical feature of the standard type is that the high-frequency part decays exponentially while the low-frequency part decays polynomially with the rate of the heat kernel. On the other hand, because of the degeneracy for the high-frequency part, the regularity-loss structure causes more regularity for the initial data when we derive the decay estimate of solutions.

For the whole space problem, we had already known several results concerned with the regularity-loss structure. For example, the dissipative Timoshenko system was discussed in [4, 5], the plate equation was in [7, 13], and the Euler-Maxwell system was studied in [20, 21]. Moreover, artificial models which have the several kinds of the regularity-loss structure are constructed in [19]. Under this situation, Ueda-Duan-Kawashima in [17, 18] tried to construct the new stability condition in order to analyze the regularity-loss structure for the general symmetric hyperbolic systems.

The field equations of porous elastic solids are derived as follows. We start with the following evolution equations

$$\rho u_{tt} = T_x, \qquad \kappa\phi_{tt} = h_x + g,$$

where u and ϕ describe the displacement of a solid elastic material and the volume fraction, respectively. The positive parameters ρ and κ are the mass density and the product of the mass density by the equilibrated inertia, respectively. Furthermore, T, h and g denote the stress, the equilibrated stress and the equilibrated body force, respectively. For these equations, we consider the following constitutive equations.

$$T = \alpha u_x + \beta\phi + \gamma u_{tx} + \varepsilon_1\phi_t, \qquad h = \delta\phi_x, \qquad g = -\beta u_x - \eta\phi - \tau\phi_t - \varepsilon_2 u_{tx},$$

where the parameters α, β, γ, δ, η ε_1, ε_2 and τ are constitutive coefficients. If we introduce the constitutive equations into the evolution equations we find the field equations

$$\begin{aligned} \rho u_{tt} - \alpha u_{xx} - \beta\phi_x - \gamma u_{txx} - \varepsilon_1\phi_{tx} &= 0, \\ \kappa\phi_{tt} - \delta\phi_{xx} + \beta u_x + \eta\phi + \tau\phi_t + \varepsilon_2 u_{tx} &= 0 \end{aligned} \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}$. Here, $u = u(t,x)$ and $\phi = \phi(t,x)$ are unknown scalar functions. We note that the internal energy is given by

$$\mathcal{E} = \alpha u_x^2 + \eta\phi^2 + 2\beta u_x\phi + \delta\phi_x^2,$$

and the dissipation is given by

$$\mathcal{D} = \gamma u_{tx}^2 + \tau\phi_t^2 + (\varepsilon_1 + \varepsilon_2)u_{tx}\phi_t.$$

Therefore when we assume that the internal energy is positive (which is related with the elastic stability) we have to assume that

$$\alpha > 0, \qquad \delta > 0, \qquad \alpha\eta - \beta^2 > 0,$$

meanwhile if we assume that the dissipation is positive we need to impose that

$$\begin{aligned} &\gamma\tau - \varepsilon^2 > 0 \qquad \text{for} \qquad && \gamma > 0 \quad \text{and} \quad \tau > 0, \\ &\varepsilon_1 = \varepsilon_2 = 0 \qquad \text{for} \qquad && \gamma = 0 \text{ (and } \tau > 0) \quad \text{or} \quad \tau = 0 \text{ (and } \gamma > 0), \end{aligned}$$

where we denote $\varepsilon := (\varepsilon_1 + \varepsilon_2)/2$. As we assume the coupling between the field equations we must assume that $\beta \neq 0$ for $\gamma = 0$ or $\tau = 0$.

In this article, we focus on the Cauchy problem defined by the system (1.1) with the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x) \tag{1.2}$$

for $x \in \mathbb{R}$, where $u_0 = u_0(x)$, $u_1 = u_1(x)$, $\phi_0 = \phi_0(x)$ and $\phi_1 = \phi_1(x)$ are given scalar functions.

2. Basic Equations and Theorems

To analyze (1.1) systematically, we rewrite these equations to the symmetric system. Precisely, putting $v = u_t$, $w = u_x$, $\sigma = \phi_t$, $\psi = \phi_x$, we obtain

$$\begin{aligned} \rho v_t - \alpha w_x - \beta\phi_x - \gamma v_{xx} - \varepsilon_1\sigma_x &= 0, \\ \alpha w_t + \beta\phi_t - \alpha v_x - \beta\sigma &= 0, \\ \kappa\sigma_t - \delta\psi_x + \beta w + \eta\phi + \tau\sigma + \varepsilon_2 v_x &= 0, \\ \delta\psi_t - \delta\sigma_x &= 0, \\ \beta w_t + \eta\phi_t - \beta v_x - \eta\sigma &= 0, \end{aligned} \tag{2.1}$$

and

$$\phi_x - \psi = 0. \tag{2.2}$$

Then (2.1) and (2.2) are rewritten as

$$A^0 U_t + AU_x - BU_{xx} + LU + MU_x = 0, \tag{2.3}$$

$$QU_x + RU = 0, \tag{2.4}$$

where $U = (v, w, \sigma, \psi, \phi)^T$ and

$$A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \beta \\ 0 & 0 & \kappa & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & \beta & 0 & 0 & \eta \end{pmatrix}, \qquad A = -\begin{pmatrix} 0 & \alpha & 0 & 0 & \beta \\ \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & \delta & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 \\ 0 & \beta & \tau & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & -\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \varepsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$Q = (0\ 0\ 0\ 0\ 1)$ and $R = (0\ 0\ 0\ -1\ 0)$. Under the assumption for the coefficients, the matrix A^0 is positive definite. Furthermore, A^0 and A are symmetric, and B and L are non-negative definite. Applying the Fourier transform to (2.3) and (2.4), we obtain

$$A^0\hat{U}_t + i\xi A\hat{U} + \xi^2 B\hat{U} + L\hat{U} + i\xi M\hat{U} = 0, \tag{2.5}$$

$$i\xi Q\hat{U} + R\hat{U} = 0, \tag{2.6}$$

where $\xi \in \mathbb{R}$ is a Fourier variable.

To apply the stability condition introduced in Ueda [16], we consider the corresponding eigenvalue problem.

$$\lambda A^0\varphi + (ir\omega A + r^2 B + L + ir\omega M)\varphi = 0 \tag{2.7}$$

with the constraint condition

$$(ir\omega Q + R)\varphi = 0 \tag{2.8}$$

for $r \geq 0$ and $\omega \in \{-1, 1\}$. Here, the pair $(\lambda, \varphi) = (\lambda, \varphi)(r, \omega) \in \mathbb{C} \times \mathbb{C}^5$ is the eigenvalue and eigenvector of (2.7), respectively.

Owing to the effect of $\gamma > 0$ and $\tau > 0$, the system (2.3) is regarded as a dissipative system. Furthermore, in the case that $\gamma = 0$ or $\tau = 0$, the matrix M appeared in (2.7) is zero matrix, and it is possible to apply the following stability condition introduced in [16].

Stability Condition under Constraint(SCC): For any $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}_+$,

$$\mathrm{Ker}(\mu I + \mathcal{A}(\nu)) \cap \mathrm{Ker}(B^\sharp) \cap \mathrm{Ker}(L^\sharp) \cap \mathrm{Ker}(i\nu Q + R) = \{0\},$$

where $\mathcal{A}(\nu) := (A^0)^{-1}(A - i\nu B^\flat - i\nu^{-1}L^\flat)$.

Here and hereafter, we use a notation that $\mathbb{R}_+ := (0, \infty)$, I is the identity matrix, and $X^\sharp$ and $X^\flat$ are denoted the symmetric and skew-symmetric part of the matrix X, respectively.

Using Stability Condition (SCC), we can get the following result.

Theorem 2.1. *For $\gamma = 0$ or $\tau = 0$, the system* (2.3) *with the constraint condition* (2.4) *satisfies Condition* (SCC). *Namely, this system is strictly dissipative under constraint.*

Proof. Condition (SCC) suggests to start that $(\mu,\nu)\in\mathbb{R}\times\mathbb{R}_+$ and $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)^T\in\mathbb{C}^5$ satisfy

$$\begin{cases} \mu\rho\varphi_1-\alpha\varphi_2-\beta\varphi_5=0,\\ \mu(\alpha\varphi_2+\beta\varphi_5)-\alpha\varphi_1+i\nu^{-1}\beta\varphi_3=0,\\ \mu\kappa\varphi_3-\delta\varphi_4-i\nu^{-1}\beta\varphi_2-i\nu^{-1}\eta\varphi_5=0,\\ \mu\delta\varphi_4-\delta\varphi_3=0,\\ \mu(\beta\varphi_2+\eta\varphi_5)-\beta\varphi_1+i\nu^{-1}\eta\varphi_3=0,\end{cases}\qquad\text{and}\qquad \gamma\varphi_1=0,\quad \tau\varphi_3=0. \tag{2.9}$$

Furthermore, we get from the constraint condition that

$$i\nu\varphi_5-\varphi_4=0. \tag{2.10}$$

Because of $\gamma>0$ or $\tau>0$ and $\alpha\eta-\beta^2>0$, it is easy from (2.9) to get $\varphi=0$ for $\mu\neq0$. For $\mu=0$, (2.9) can be rewritten as $\varphi_1=\varphi_3=0$ and

$$\alpha\varphi_2+\beta\varphi_5=0,\qquad \delta\varphi_4+i\nu^{-1}\beta\varphi_2+i\nu^{-1}\eta\varphi_5=0.$$

Furthermore, solving the above two equations with (2.10), we obtain

$$\left(\nu^2\alpha\delta+(\alpha\eta-\beta^2)\right)\varphi_5=0.$$

Since $\alpha>0$, $\delta>0$ and $\alpha\eta-\beta^2>0$, we arrive at $\varphi_5=0$. Eventually, we conclude $\varphi_2=\varphi_4=0$ and complete the proof. □

Inspired by Theorem 2.1, we expect that the solutions to our Cauchy problem decay as $t\to\infty$. Therefore, our main purpose of this article is to obtain the optimal decay estimates for the solutions.

Theorem 2.2. *Let $(\hat{u},\hat{\phi})$ be a Fourier image for the solution to* (1.1), (1.2). *Then the solution satisfies the following pointwise estimate in Fourier space.*

$$\begin{aligned}(|\hat{u}_t|^2+\xi^2|\hat{u}|^2+|\hat{\phi}_t|^2+(1+\xi^2)|\hat{\phi}|^2)\\ \le Ce^{-c\lambda(i\xi)t}(|\hat{u}_1|^2+\xi^2|\hat{u}_0|^2+|\hat{\phi}_1|^2+(1+\xi^2)|\hat{\phi}_0|^2),\end{aligned} \tag{2.11}$$

where c and C are certain positive constants which are independent of γ, τ, ε_1 and ε_2, and $\lambda(i\xi)$ is defined by

$$\lambda(i\xi)=\begin{cases}\dfrac{(\gamma\tau-\varepsilon^2)\xi^2}{\tau(1+\varepsilon_2+\gamma\tau)+\gamma(1+\varepsilon_1+\gamma\tau)\xi^2} & \text{for}\quad \gamma>0,\ \tau>0,\\ \dfrac{\gamma\xi^4}{(1+\xi^2)^2(1+\gamma^2\xi^2)} & \text{for}\quad \gamma>0,\ \tau=0,\\ \dfrac{\tau\xi^2}{(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)} & \text{for}\quad \gamma=0,\ \tau>0.\end{cases} \tag{2.12}$$

Here p is defined by $p:=|\alpha\kappa-\rho\delta|$.

From Theorem 2.2, we conclude the following the decay estimates for the solution to (1.1), (1.2).

Corollary 2.3. *Let s be a non-negative integer and q be a real number with $1 \le q \le 2$. Suppose that $u_1, \partial_x u_0, \partial_x \phi_0, \phi_1 \in H^s(\mathbb{R})$ and $u_1, \partial_x u_0, \phi_0, \phi_1 \in L^q(\mathbb{R})$. Then the solution of* (1.1), (1.2) *satisfies the following decay estimates.*

(i) *For $\gamma > 0$ and $\tau > 0$,*

$$\begin{aligned} &\|\partial_x^k(u_t, \partial_x u, \phi_t)(t)\|_{L^2} + \|\partial_x^k \phi(t)\|_{H^1} \\ &\qquad \le C(1+at)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}\|(u_1, \partial_x u_0, \phi_0, \phi_1)\|_{L^q} \\ &\qquad\qquad + Ce^{-cat}\|\partial_x^k(u_1, \partial_x u_0, \partial_x \phi_0, \phi_1)\|_{L^2}, \end{aligned} \tag{2.13}$$

where $0 \le k \le s$, and a is defined by

$$a := \frac{\gamma\tau - \varepsilon^2}{\tau(1+\varepsilon_2+\gamma\tau) + \gamma(1+\varepsilon_1+\gamma\tau)}.$$

(ii) *For $\gamma > 0$ and $\tau = 0$,*

$$\begin{aligned} &\|\partial_x^k(u_t, \partial_x u, \phi_t)(t)\|_{L^2} + \|\partial_x^k \phi(t)\|_{H^1} \\ &\qquad \le C(1+a_\gamma t)^{-\frac{1}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}}\|(u_1, \partial_x u_0, \phi_0, \phi_1)\|_{L^q} \\ &\qquad\qquad + C(1+a_\gamma t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell}(u_1, \partial_x u_0, \partial_x \phi_0, \phi_1)\|_{L^2}, \end{aligned} \tag{2.14}$$

where $k, \ell \ge 0$ and $0 \le k+\ell \le s$, and a_γ is defined by $a_\gamma := \gamma(1+\gamma^2)^{-1}$.

(iii-i) *For $\gamma = 0$, $\tau > 0$ and $p = 0$,*

$$\begin{aligned} &\|\partial_x^k(u_t, \partial_x u, \phi_t)(t)\|_{L^2} + \|\partial_x^k \phi(t)\|_{H^1} \\ &\qquad \le C(1+\tilde{a}_\tau t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}\|(u_1, \partial_x u_0, \phi_0, \phi_1)\|_{L^q} \\ &\qquad\qquad + Ce^{-c\tilde{a}_\tau t}\|\partial_x^k(u_1, \partial_x u_0, \partial_x \phi_0, \phi_1)\|_{L^2}, \end{aligned} \tag{2.15}$$

where $0 \le k \le s$, and $\tilde{a}_\tau$ is defined by $\tilde{a}_\tau := \tau(2+\tau^2)^{-1}$.

(iii-ii) *For $\gamma = 0$, $\tau > 0$ and $p \neq 0$,*

$$\begin{aligned} &\|\partial_x^k(u_t, \partial_x u, \phi_t)(t)\|_{L^2} + \|\partial_x^k \phi(t)\|_{H^1} \\ &\qquad \le C(1+a_\tau t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}\|(u_1, \partial_x u_0, \phi_0, \phi_1)\|_{L^q} \\ &\qquad\qquad + C(1+a_\tau t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell}(u_1, \partial_x u_0, \partial_x \phi_0, \phi_1)\|_{L^2}, \end{aligned} \tag{2.16}$$

where $k, \ell \ge 0$ and $0 \le k+\ell \le s$, and a_τ is defined by $a_\tau := \tau(1+p)^{-1}(2+\tau^2+p)^{-1}$.

Here c and C are certain positive constants which are independent of γ, τ, ε_1 and ε_2.

Remark 1. (i) *In Corollary 2.3, the cases* (i) *and* (iii-i) *are the standard type, and the cases* (ii) *and* (iii-ii) *are the regularity-loss type.* (ii) *Because of the asymptotic expansion of the eigenvalues in Section 4, we deduce that the pointwise and decay estimates in Theorem 2.2 and Corollary 2.3 are optimal.*

3. Energy estimates

In this section, we give proofs of Theorem 2.2 and Corollary 2.3 by using the energy method in Fourier space. Furthermore, as a byproduct of this proof, we can derive the following energy estimate for the problem (1.1), (1.2).

Proposition 3.1. *Let s be non-negative integers. Then the solution of* (1.1), (1.2) *satisfies the following energy estimates.*

(i) *For $\gamma > 0$ and $\tau > 0$,*

$$\begin{aligned}&\|(u_t, u_x, \phi_t)(t)\|_{H^{s+1}} + \|\phi(t)\|_{H^{s+2}} \\ &\qquad + \int_0^t (\|u_{xx}(t')\|_{H^s} + \|(u_{tx}, \phi_t, \phi_x)(t')\|_{H^{s+1}})\, dt' \\ &\leq C\left(\|(u_1, u_{0,x}, \phi_1)\|_{H^{s+1}} + \|\phi_0\|_{H^{s+2}}\right).\end{aligned} \tag{3.1}$$

(ii) *For $\gamma > 0$ and $\tau = 0$,*

$$\begin{aligned}&\|(u_t, u_x, \phi_t)(t)\|_{H^{s+3}} + \|\phi(t)\|_{H^{s+4}} \\ &\qquad + \int_0^t \left(\|\phi_{txx}(t')\|_{H^s} + \|\partial_x^2(u_x, \phi)(t')\|_{H^{s+1}} + \|u_{tx}(t')\|_{H^{s+3}}\right) dt' \\ &\leq C\left(\|(u_1, u_{0,x}, \phi_1)\|_{H^{s+3}} + \|\phi_0\|_{H^{s+4}}\right).\end{aligned} \tag{3.2}$$

(iii-i) *For $\gamma = 0$, $\tau > 0$ and $p = 0$,*

$$\begin{aligned}&\|(u_t, u_x, \phi_t)(t)\|_{H^{s+1}} + \|\phi(t)\|_{H^{s+2}} \\ &\qquad + \int_0^t (\|\partial_x(u_t, u_x)(t')\|_{H^s} + \|(\phi_t, \phi_x)(t')\|_{H^{s+1}})\, dt' \\ &\leq C\left(\|(u_1, u_{0,x}, \phi_1)\|_{H^{s+1}} + \|\phi_0\|_{H^{s+2}}\right).\end{aligned} \tag{3.3}$$

(iii-ii) *For $\gamma = 0$, $\tau > 0$ and $p \neq 0$,*

$$\begin{aligned}&\|(u_t, u_x, \phi_t)(t)\|_{H^{s+2}} + \|\phi(t)\|_{H^{s+3}} \\ &\qquad + \int_0^t (\|\partial_x(u_t, u_x)(t')\|_{H^s} + \|(\phi_t, \phi_x)(t')\|_{H^{s+2}})\, dt' \\ &\leq C\left(\|(u_1, u_{0,x}, \phi_1)\|_{H^{s+2}} + \|\phi_0\|_{H^{s+3}}\right).\end{aligned} \tag{3.4}$$

Here C is a certain positive constant.

Proof of Theorem 2.2 and Proposition 3.1. We first derive the basic energy equality for the system (2.5) in the Fourier space. We take an inner product $\hat{U}$ with the first equation of (2.5), and take the real part for the resultant equation. Then, by virture the properties for A^0, A, B and L, this yields

$$\frac{1}{2}\frac{\partial}{\partial t}\langle A^0\hat{U}, \hat{U}\rangle + \xi^2\langle B\hat{U}, \hat{U}\rangle + \langle L^\sharp\hat{U}, \hat{U}\rangle + i\xi\langle M^\flat\hat{U}, \hat{U}\rangle = 0.$$

This means

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\big(\rho|\hat{v}|^2+\alpha|\hat{w}|^2+\kappa|\hat{\sigma}|^2+\delta|\hat{\psi}|^2+\eta|\hat{\phi}|^2+2\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})\big)\\&\qquad+\gamma\xi^2|\hat{v}|^2+\tau|\hat{\sigma}|^2+2\varepsilon\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})=0.\end{aligned} \tag{3.5}$$

We remark that

$$\begin{aligned}&\alpha|\hat{w}|^2+\eta|\hat{\phi}|^2+2\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})\\&\geq\frac{1}{2}(\alpha\eta-\beta^2)(\frac{1}{\eta}|\hat{w}|^2+\frac{1}{\alpha}|\hat{\phi}|^2)+\frac{\alpha}{2}(|\hat{w}|-\frac{|\beta|}{\alpha}|\hat{\phi}|)^2+\frac{\eta}{2}(|\hat{\phi}|-\frac{|\beta|}{\eta}|\hat{w}|)^2\\&\geq c_*(|\hat{w}|^2+|\hat{\phi}|^2)\end{aligned} \tag{3.6}$$

for $\alpha>0$ and $\eta>0$, where c_* is defined by

$$c_*=\frac{1}{2}(\alpha\eta-\beta^2)\min\left\{\frac{1}{\alpha},\frac{1}{\eta}\right\}$$

and c_* is a positive constant provided by $\alpha\eta-\beta^2>0$. Similarly,

$$\begin{aligned}&\gamma\xi^2|\hat{v}|^2+\tau|\hat{\sigma}|^2+2\varepsilon\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})\\&\geq\frac{1}{2}(\gamma\tau-\varepsilon^2)(\frac{1}{\tau}\xi^2|\hat{v}|^2+\frac{1}{\gamma}|\hat{\sigma}|^2)+\frac{\gamma}{2}(|\xi||\hat{v}|-\frac{\varepsilon}{\gamma}|\hat{\sigma}|)^2+\frac{\tau}{2}(|\hat{\sigma}|-\frac{\varepsilon}{\tau}|\xi||\hat{v}|)^2\\&\geq\frac{1}{2}(c_\gamma^*\xi^2|\hat{v}|^2+c_\tau^*|\hat{\sigma}|^2)\end{aligned} \tag{3.7}$$

for $\gamma>0$ and $\tau>0$, where c_γ^* and c_τ^* are defined by $c_\gamma^*:=\gamma-\varepsilon^2/\tau$ and $c_\tau^*:=\tau-\varepsilon^2/\gamma$. Here, c_γ^* and c_τ^* are positive constants provided by $\gamma\tau-\varepsilon^2>0$, and $c_\gamma^*=\gamma$ and $c_\tau^*=\tau$ for $\varepsilon=0$.

Next, to construct the dissipation terms, we recall our problem in Fourier space. Namely, we start the proof at

$$\begin{aligned}\rho\hat{v}_t-\alpha i\xi\hat{w}-\beta i\xi\hat{\phi}+\gamma\xi^2\hat{v}-\varepsilon_1 i\xi\hat{\sigma}&=0,\\\hat{w}_t-i\xi\hat{v}&=0,\\\kappa\hat{\sigma}_t+\beta\hat{w}+(\eta+\delta\xi^2)\hat{\phi}+\tau\hat{\sigma}+\varepsilon_2 i\xi\hat{v}&=0,\\\hat{\phi}_t-\hat{\sigma}&=0.\end{aligned} \tag{3.8}$$

We multiply the first and second equations in (3.8) by $i\xi\bar{\hat{w}}$ and $-\rho i\xi\bar{\hat{v}}$, respectively. Then, combining the resultant equations and taking a real part, we obtain

$$\rho\xi\frac{\partial}{\partial t}\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\xi^2(\alpha|\hat{w}|^2-\rho|\hat{v}|^2)+\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\phi}})+\gamma\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\varepsilon_1\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})=0. \tag{3.9}$$

For $\gamma=0$ or $\tau=0$, to eliminate $\mathrm{Re}(\hat{w}\bar{\hat{\phi}})$, we multiply the second and third equations in (3.8) by $\kappa\beta\bar{\hat{\sigma}}$ and $\beta\bar{\hat{w}}$, respectively, and combine the resultant equations and take a real part. Then we have

$$\kappa\beta\frac{\partial}{\partial t}\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})+\beta^2|\hat{w}|^2-\kappa\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})+\beta(\eta+\delta\xi^2)\mathrm{Re}(\hat{w}\bar{\hat{\phi}})+\tau\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})=0. \tag{3.10}$$

Combining (3.9) and (3.10) to obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho(\eta+\delta\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})-\kappa\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})\right\}\\&\qquad+(\alpha\eta-\beta^2+\alpha\delta\xi^2)\xi^2|\hat{w}|^2-\rho(\eta+\delta\xi^2)\xi^2|\hat{v}|^2\\&\qquad+\gamma(\eta+\delta\xi^2)\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\kappa\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})-\tau\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})=0.\end{aligned}\tag{3.11}$$

Similarly, we multiply the third and fourth equations in (3.8) by $\bar{\hat{\phi}}$ and $\kappa\bar{\hat{\sigma}}$, respectively. Then, combining the resultant equations and taking a real part, we have

$$\kappa\frac{\partial}{\partial t}\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})+(\eta+\delta\xi^2)|\hat{\phi}|^2-\kappa|\hat{\sigma}|^2+\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})+\tau\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})+\varepsilon_2\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})=0.\tag{3.12}$$

For $\gamma=0$ or $\tau=0$, to eliminate $\mathrm{Re}(\hat{w}\bar{\hat{\phi}})$, we multiply the first and fourth equations in (3.8) by $\beta i\xi\bar{\hat{\phi}}$ and $-\rho\beta i\xi\bar{\hat{v}}$, respectively. Then, combining the resultant equations and taking a real part, we obtain

$$\rho\beta\xi\frac{\partial}{\partial t}\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})+\beta^2\xi^2|\hat{\phi}|^2+\alpha\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\phi}})-\rho\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})+\gamma\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})=0.\tag{3.13}$$

We combine (3.12) and (3.13) to obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\alpha\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})-\rho\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right\}+(\alpha\eta-\beta^2+\alpha\delta\xi^2)\xi^2|\hat{\phi}|^2-\alpha\kappa\xi^2|\hat{\sigma}|^2\\&+\alpha\tau\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})+\rho\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})-\gamma\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})=0.\end{aligned}\tag{3.14}$$

Using these equations, we construct the desired pointwise estimate in Fourier space. In the case $\gamma>0$ and $\tau>0$, we get from (3.9) and (3.12) that

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})\right\}+\xi^2\left(\alpha|\hat{w}|^2+(\eta+\delta\xi^2)|\hat{\phi}|^2\right)-\xi^2(\rho|\hat{v}|^2+\kappa|\hat{\sigma}|^2)\\&+2\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\phi}})+\gamma\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\tau\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})+\varepsilon_1\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})+\varepsilon_2\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})=0.\end{aligned}$$

Using (3.6) and the Hölder inequality, we estimate

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})\right\}+c\xi^2|\hat{w}|^2+c(1+\xi^2)\xi^2|\hat{\phi}|^2\\&-C(1+\varepsilon_2^2+\gamma^2\xi^2)\xi^2|\hat{v}|^2-C(\tau^2+(1+\varepsilon_1^2)\xi^2)|\hat{\sigma}|^2\le 0.\end{aligned}\tag{3.15}$$

Therefore, calculating (3.5) $\times$ $(\tilde{c}_1+\tilde{c}_2\xi^2)$ + (3.15) $\times$ $\omega_0c_\gamma^*c_\tau^*$, we arrive at

$$\partial_tE+D=0\tag{3.16}$$

with

$$\begin{aligned}E:=&\frac{1}{2}(\tilde{c}_1+\tilde{c}_2\xi^2)\left(\rho|\hat{v}|^2+\alpha|\hat{w}|^2+\kappa|\hat{\sigma}|^2+(\eta+\delta\xi^2)|\hat{\phi}|^2+2\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})\right)\\&+\omega_0c_\gamma^*c_\tau^*\Big(\rho\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})+\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})\Big),\\D:=&cc_\gamma^*c_\tau^*\xi^2\left(|\hat{w}|^2+(1+\xi^2)|\hat{\phi}|^2\right)+c(\tilde{c}_1+\tilde{c}_2\xi^2)(\gamma\xi^2|\hat{v}|^2+\tau|\hat{\sigma}|^2),\end{aligned}$$

where we choose a positive number ω_0 suitably small, and

$$\begin{aligned}\tilde{c}_1 &:= (1+\varepsilon_2)c_\tau^* + \tau^2 c_\gamma^* = \tau(1+\varepsilon_2+\gamma\tau)\left(1-\frac{\varepsilon^2}{\gamma\tau}\right),\\ \tilde{c}_2 &:= (1+\varepsilon_1)c_\gamma^* + \gamma^2 c_\tau^* = \gamma(1+\varepsilon_1+\gamma\tau)\left(1-\frac{\varepsilon^2}{\gamma\tau}\right).\end{aligned} \tag{3.17}$$

Furthermore, using the Hölder inequality and letting ω_0 suitably small, we obtain

$$\begin{aligned}&c(\tilde{c}_1+\tilde{c}_2\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right)\\ &\qquad \le E \le C(\tilde{c}_1+\tilde{c}_2\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right).\end{aligned} \tag{3.18}$$

Integrating (3.16) over t and applying (3.18) to the resultant estimate, we get

$$\begin{aligned}&|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\\ &\quad+\int_0^t\left\{\frac{c_\gamma^* c_\tau^*\xi^2}{\tilde{c}_1+\tilde{c}_2\xi^2}\left(|\hat{w}|^2+(1+\xi^2)|\hat{\phi}|^2\right)+c_\gamma^*\xi^2|\hat{v}^2|+c_\tau^*|\hat{\sigma}|^2\right\}dt'\\ &\le C(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2)|_{t=0},\end{aligned}$$

where C is a positive constant which is independent of γ, τ, ε_1 and ε_2. Here, we remark that

$$\frac{c_\gamma^* c_\tau^*\xi^2}{\tilde{c}_1+\tilde{c}_2\xi^2} = \frac{(\gamma\tau-\varepsilon^2)\xi^2}{\tau(1+\varepsilon_2+\gamma\tau)+\gamma(1+\varepsilon_1+\gamma\tau)\xi^2}.$$

We conclude (3.1) from this estimate. On the other hand, we apply the estimate (3.18) to (3.16) again. Then this yields from $c\lambda(i\xi)E \le D$ that $\partial_t E + c\lambda(i\xi)E \le 0$, where $\lambda(i\xi)$ is defined in Theorem 2.2 with $\gamma>0$ and $\tau>0$. Consequently, we obtain

$$E(t,\xi) \le CE(0,\xi)e^{-c\lambda(i\xi)t}, \qquad \lambda(i\xi) = \frac{(\gamma\tau-\varepsilon^2)\xi^2}{\tau(1+\varepsilon_2+\gamma\tau)+\gamma(1+\varepsilon_1+\gamma\tau)\xi^2},$$

where c and C are positive constants which do not depend on γ, τ, ε_1 and ε_2. Namely, we arrive at the desired pointwise estimate (2.11) for $\gamma>0$ and $\tau>0$.

In the case $\gamma>0$ and $\tau=0$, we start from (3.11), (3.12) and (3.13). Namely, we construct the energy estimate by using

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho(\eta+\delta\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})-\kappa\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})\right\}+c(1+\xi^2)\xi^2|\hat{w}|^2\\ &-C(1+\xi^2)(1+\gamma^2\xi^2)\xi^2|\hat{v}|^2-C|\xi|^3|\hat{v}||\hat{\sigma}|\le 0,\end{aligned} \tag{3.19}$$

$$-\kappa\frac{\partial}{\partial t}\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})+c|\hat{\sigma}|^2-C(1+\xi^2)|\hat{\phi}|^2-C|\hat{w}|^2\le 0, \tag{3.20}$$

$$\rho\beta\xi\frac{\partial}{\partial t}\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})+c\xi^2|\hat{\phi}|^2-C\xi^2|\hat{w}|^2-\gamma^2C\xi^4|\hat{v}|^2-C|\xi||\hat{v}||\hat{\sigma}|\le 0. \tag{3.21}$$

Computing (3.19) + (3.21) $\times\, \omega_1\xi^2$, this yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho(\eta+\delta\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})-\kappa\beta\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})+\omega_1\rho\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right\}\\ &+c\xi^4|\hat{\phi}|^2+c(1+\xi^2)\xi^2|\hat{w}|^2-C(1+\xi^2)(1+\gamma^2\xi^2)\xi^2|\hat{v}|^2-C|\xi|^3|\hat{v}||\hat{\sigma}|\le 0.\end{aligned} \tag{3.22}$$

Moreover, computing (3.22) $\times (1+\xi^2)$ + (3.20) $\times \omega_2\xi^4$ and the Hölder inequality, we get

$$\begin{aligned}&\frac{\partial}{\partial t}\Big\{\rho(\eta+\delta\xi^2)(1+\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})-\kappa\beta(1+\xi^2)\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})\\&\qquad+\omega_1\rho\beta(1+\xi^2)\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})-\omega_2\kappa\xi^4\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})\Big\}+c\xi^4|\hat{\sigma}|^2+c(1+\xi^2)\xi^4|\hat{\phi}|^2\\&\quad+c(1+\xi^2)^2\xi^2|\hat{w}|^2-C(1+\xi^2)^2(1+\gamma^2\xi^2)\xi^2|\hat{v}|^2\le 0,\end{aligned}\tag{3.23}$$

where c and C are positive constants which do not depend on γ. Finally, calculating (3.23) $\times (1+\xi^2)^2(1+\gamma^2\xi^2)$ + (3.20) $\times \omega_0$, we arrive at

$$\partial_t E_\gamma + D_\gamma = 0,\tag{3.24}$$

where

$$\begin{aligned}E_\gamma :=\ &\frac{1}{2}(1+\xi^2)^2(1+\gamma^2\xi^2)\left(\rho|\hat{v}|^2+\alpha|\hat{w}|^2+\kappa|\hat{\sigma}|^2+(\eta+\delta\xi^2)|\hat{\phi}|^2+2\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})\right)\\&+\omega_0\gamma\Big(\rho(\eta+\delta\xi^2)(1+\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})-\kappa\beta(1+\xi^2)\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})\\&\qquad+\omega_1\rho\beta(1+\xi^2)\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})-\omega_2\kappa\xi^4\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})\Big),\\D_\gamma :=\ &c\gamma\xi^4\left(|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right)+c\gamma(1+\xi^2)^2\xi^2|\hat{w}|^2+c\gamma(1+\xi^2)^2(1+\gamma^2\xi^2)\xi^2|\hat{v}|^2.\end{aligned}$$

Therefore, using the Hölder inequality and letting ω_0, ω_1 and ω_2 suitably small, we obtain

$$\begin{aligned}&c(1+\xi^2)^2(1+\gamma^2\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right)\\&\qquad\le E_\gamma\le C(1+\xi^2)^2(1+\gamma^2\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right),\end{aligned}\tag{3.25}$$

Integrating (3.24) over t and applying (3.25) to the resultant estimate, we get

$$\begin{aligned}&|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\\&\quad+\int_0^t\left\{\frac{\gamma\xi^4}{(1+\xi^2)^2(1+\gamma^2\xi^2)}\left(|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right)+\frac{\gamma\xi^2}{1+\gamma^2\xi^2}|\hat{w}|^2+\gamma\xi^2|\hat{v}^2|\right\}dt'\\&\le C(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2)|_{t=0},\end{aligned}$$

where C is a positive constant which is independent of γ. We conclude (3.2) from this estimate. Furthermore, employing the same argument as before, we obtain

$$E_\gamma(t,\xi)\le CE_\gamma(0,\xi)e^{-c\lambda_\gamma(i\xi)t},\qquad \lambda_\gamma(i\xi)=\frac{\gamma\xi^4}{(1+\xi^2)^2(1+\gamma^2\xi^2)}.$$

This pointwise estimate means (2.11) for $\gamma>0$ and $\tau=0$.

In the case $\gamma=0$ and $\tau>0$, we start from (3.10) and (3.13). We add these equations to erase $\xi^2\mathrm{Re}(\hat{w}\bar{\hat{\phi}})$. Then we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\alpha\kappa\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})-\rho\delta\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right\}+\beta^2\left(\alpha|\hat{w}|^2-\delta\xi^2|\hat{\phi}|^2\right)\\&-\beta(\alpha\kappa-\rho\delta)\xi\mathrm{Re}(i\hat{v}\bar{\hat{\sigma}})+\alpha\eta\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})+\alpha\tau\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})=0.\end{aligned}$$

Namely, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\alpha\kappa\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})-\rho\delta\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right\}+c|\hat{w}|^2\\&-C(1+\xi^2)|\hat{\phi}|^2-\tau^2C|\hat{\sigma}|^2-Cp|\xi||\hat{v}||\hat{\sigma}|\le 0,\end{aligned}\tag{3.26}$$

where we define p is defined in Theorem 2.2. Remark that we do not need to control the last term in (3.26) if $p=0$. On the other hand, (3.9) and (3.13) give us

$$-\rho\xi\frac{\partial}{\partial t}\mathrm{Re}(i\hat{v}\bar{\hat{w}})+c\xi^2|\hat{v}|^2-C\xi^2(|\hat{w}|^2+|\hat{\phi}|^2)\le 0,\tag{3.27}$$

$$\begin{aligned}&(1+\chi\xi^2)\frac{\partial}{\partial t}\left\{\alpha\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})-\rho\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right\}+c(1+\chi\xi^2)(1+\xi^2)\xi^2|\hat{\phi}|^2\\&-C(1+\tau^2)(1+p\xi^2)\xi^2|\hat{\sigma}|^2-C(1+p\xi^2)|\xi||\hat{v}||\hat{\sigma}|\le 0.\end{aligned}\tag{3.28}$$

Using (3.26) and (3.28), we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\alpha\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})-\rho\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})+\omega_1(1+p\xi^2)\big(\alpha\kappa\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})-\rho\delta\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\big)\right\}\\&+c\xi^2|\hat{w}|^2+c(1+p\xi^2)(1+\xi^2)\xi^2|\hat{\phi}|^2-C(1+\tau^2)(1+p\xi^2)\xi^2|\hat{\sigma}|^2\\&-C(1+p\xi^2)|\xi||\hat{v}||\hat{\sigma}|\le 0.\end{aligned}\tag{3.29}$$

Furthermore, (3.27) and (3.29) give us

$$\begin{aligned}&\frac{\partial}{\partial t}\Big\{\alpha\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})-\rho\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\\&\qquad+\omega_1(1+p\xi^2)\left(\alpha\kappa\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})-\rho\delta\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right)-\omega_2\rho\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})\Big\}\\&+c\xi^2\left(|\hat{v}|^2+|\hat{w}|^2+(1+p\xi^2)(1+\xi^2)|\hat{\phi}|^2\right)\\&-C(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)|\hat{\sigma}|^2\le 0,\end{aligned}\tag{3.30}$$

where c and C are positive constants which do not depend on τ. Finally, calculating (3.23) $\times$ $(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)$ + (3.30) $\times$ $\omega_0\tau$, we arrive at

$$\partial_t E_\tau+D_\tau=0,\tag{3.31}$$

where

$$\begin{aligned}E_\tau:=&\ \frac{1}{2}(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)\cdot\\&\qquad\cdot\left(\rho|\hat{v}|^2+\alpha|\hat{w}|^2+\kappa|\hat{\sigma}|^2+(\eta+\delta\xi^2)|\hat{\phi}|^2+2\beta\mathrm{Re}(\hat{w}\bar{\hat{\phi}})\right)\\&+\omega_0\tau\Big(\alpha\kappa\xi^2\mathrm{Re}(\hat{\sigma}\bar{\hat{\phi}})-\rho\beta\xi^3\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\\&\qquad+\omega_1(1+p\xi^2)\left(\alpha\kappa\beta\mathrm{Re}(\hat{w}\bar{\hat{\sigma}})-\rho\delta\beta\xi\mathrm{Re}(i\hat{v}\bar{\hat{\phi}})\right)-\omega_2\rho\xi\mathrm{Re}(i\hat{v}\bar{\hat{w}})\Big),\\D_\tau:=&\ c\xi^2\left(|\hat{v}|^2+|\hat{w}|^2+(1+p\xi^2)(1+\xi^2)|\hat{\phi}|^2\right)+c(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)|\hat{\sigma}|^2.\end{aligned}$$

Therefore, using the Hölder inequality and letting ω_0, ω_1 and ω_2 suitably small, we obtain

$$\begin{aligned} &c(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right) \\ &\le E_\tau \le C(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)\left(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2\right), \end{aligned} \tag{3.32}$$

where c and C are positive constants which do not depend on τ. Integrating (3.31) over t and applying (3.32) to the resultant estimate, we get

$$\begin{aligned} &|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2 \\ &\quad + \int_0^t \left\{ \frac{\tau\xi^2\left(|\hat{v}|^2+|\hat{w}|^2\right)}{(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)} + \frac{\tau(1+\xi^2)\xi^2|\hat{\phi}|^2}{1+(1+\tau^2+p)\xi^2} + \tau|\hat{\sigma}|^2 \right\} dt' \\ &\le C(|\hat{v}|^2+|\hat{w}|^2+|\hat{\sigma}|^2+(1+\xi^2)|\hat{\phi}|^2)|_{t=0}, \end{aligned}$$

where C is a positive constant which is independent of τ. We conclude (3.3) and (3.4) from this estimate. Moreover, from the same argument as before, we arrive at

$$E_\tau(t,\xi) \le CE_\tau(0,\xi)e^{-c\lambda_\tau(i\xi)t}, \qquad \lambda_\tau(i\xi) = \frac{\tau\xi^2}{(1+p\xi^2)(1+(1+\tau^2+p)\xi^2)},$$

where c and C are positive constants which do not depend on τ. This pointwise estimate means that (2.11) for $\gamma = 0$ and $\tau > 0$. Therefore, we complete the proofs of Theorem 2.2 and Proposition 3.1. □

Proof of Corollary 2.3. We first prove (2.14) in Corollary 2.3. From (2.12), we have

$$\lambda(i\xi) \ge \begin{cases} \dfrac{\gamma}{4(1+\gamma^2)}\xi^4 & \text{for} \quad |\xi| \le 1, \\ \dfrac{\gamma}{4(1+\gamma^2)}\xi^{-2} & \text{for} \quad |\xi| \ge 1. \end{cases}$$

Then, applying the Plancherel theorem to (2.11), we get

$$\begin{aligned} \|\partial_x^k U(t)\|_{L^2}^2 &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \xi^{2k}|\hat{U}(t,\xi)|^2 d\xi \\ &\le C\int_{|\xi|\le 1} \xi^{2k}e^{-ca_\gamma\xi^4 t}|\hat{U}_0(\xi)|^2 d\xi + C\int_{|\xi|\ge 1} \xi^{2k}e^{-ca_\gamma\xi^{-2}t}|\hat{U}_0(\xi)|^2 d\xi \\ &=: I_1 + I_2, \end{aligned} \tag{3.33}$$

where $a_\gamma := \gamma(1+\gamma^2)^{-1}$. For the low frequency part I_1, we first employ the Hölder inequality to obtain

$$I_1 \le C\|\xi^{2k}e^{-ca_\gamma\xi^4 t}\|_{L^{p_1}(|\xi|\le 1)}\|\hat{U}_0^2\|_{L^{p_2}(|\xi|\le 1)} \le C(1+a_\gamma t)^{-\frac{1}{4p_1}-\frac{k}{2}}\|\hat{U}_0\|_{L^{2p_2}}^2$$

for $1 \le p_1, p_2 \le \infty$ with $1/p_1 + 1/p_2 = 1$. Furthermore, using Hausdorff-Young inequality, we have $\|\hat{U}_0\|_{L^{2p_2}} \le C\|U_0\|_{L^q}$ for $1 \le q \le 2$ with $1/(2p_2) + 1/q = 1$. Thus, combining these estimate, we get $I_1 \le C(1+a_\gamma t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\|U_0\|_{L^q}^2$ for $1 \le q \le 2$. On the other hand, for the high frequency part I_2, we compute

$$I_2 \le C \sup_{|\xi|\ge 1}\{|\xi|^{-2\ell}e^{-ca_\gamma \xi^{-2}t}\}\int_{|\xi|\ge 1}|\xi|^{2(k+\ell)}|\hat{U}_0(\xi)|^2 d\xi \le C(1+a_\gamma t)^{-\ell}\|\partial_x^{k+\ell}U_0\|_{L^2}^2.$$

Finally, substituting the estimates of I_1 and I_2 into (3.33), we arrive at the desired decay estimate (2.14).

On the other hand, from the same argument and the fact that

$$\lambda(i\xi) \ge \begin{cases} \dfrac{\gamma\tau-\varepsilon^2}{\tau(1+\varepsilon_2+\gamma\tau)+\gamma(1+\varepsilon_1+\gamma\tau)}\xi^2 & \text{for} \quad |\xi| \le 1, \\ \dfrac{\gamma\tau-\varepsilon^2}{\tau(1+\varepsilon_2+\gamma\tau)+\gamma(1+\varepsilon_1+\gamma\tau)} & \text{for} \quad |\xi| \ge 1 \end{cases}$$

for $\gamma > 0$ and $\tau > 0$, and

$$\lambda(i\xi) \ge \begin{cases} \dfrac{\tau}{2+\tau^2}\xi^2 & \text{for} \quad |\xi| \le 1, \\ \dfrac{\tau}{2+\tau^2} & \text{for} \quad |\xi| \ge 1 \end{cases}$$

for $\gamma = 0$, $\tau > 0$ and $p = 0$, and

$$\lambda(i\xi) \ge \begin{cases} \dfrac{\tau}{(1+p)(2+\tau^2+p)}\xi^2 & \text{for} \quad |\xi| \le 1, \\ \dfrac{\tau}{(1+p)(2+\tau^2+p)}\xi^{-2} & \text{for} \quad |\xi| \ge 1 \end{cases}$$

for $\gamma = 0$, $\tau > 0$ and $p \ne 0$, we conclude the decay estimates (2.13), (2.15) and (2.16). This completes the proof. □

4. Eigenvalue problem

In this section, to discuss the optimality of the pointwise estimates in Theorem 2.2, we consider the eigenvalue problem for (1.1). The eigenvalue problem (2.7) is written as $(\lambda I - \hat{\Phi}(i\xi))\varphi = 0$, where I is an identity matrix and $\hat{\Phi}$ is defined by

$$\hat{\Phi}(i\xi) := -(A^0)^{-1}(i\xi A + \xi^2 B + L + i\xi M).$$

By the direct calculation, the characteristic polynomial of $\hat{\Phi}(i\xi)$ is

$$\begin{aligned} &\det(\lambda I - \hat{\Phi}(i\xi)) \\ &= \frac{1}{\rho\kappa}\lambda\{\rho\kappa\lambda^4 + (\kappa\gamma\xi^2+\rho\tau)\lambda^3 + \big((\alpha\kappa+\rho\delta+\gamma\tau-\varepsilon_1\varepsilon_2)\xi^2+\rho\eta\big)\lambda^2 \\ &\qquad + \big(\delta\gamma\xi^2+\alpha\tau+\eta\gamma-\beta(\varepsilon_1+\varepsilon_2)\big)\xi^2\lambda + (\alpha\delta\xi^2+\alpha\eta-\beta^2)\xi^2\}. \end{aligned} \tag{4.1}$$

The polynomial (4.1) tells us that $\hat{\Phi}(i\xi)$ has a zero eigenvalue. The eigenspace W_ξ of zero eigenvalues is described as

$$W_\xi = \{\varphi = (0, -\frac{\delta\beta}{\alpha\eta-\beta^2}i\xi\varphi_0, 0, \varphi_0, \frac{\alpha\delta}{\alpha\eta-\beta^2}i\xi\varphi_0) \mid \varphi_0 \in \mathbb{C}\}$$

for $\xi \neq 0$. Then, the orthogonal complement of W_ξ is

$$W_\xi^\perp = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \frac{\delta}{\alpha\eta - \beta^2} i\xi(\alpha\varphi_4 - \beta\varphi_2), \varphi_4) \mid \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{C}\}.$$

On the other hand, the constraint condition (2.4) denotes

$$V_\xi = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, i\xi\varphi_4, \varphi_4) \mid \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{C}\}.$$

Then it is easy to obtain $V_\xi = W_\xi^\perp$ for $\xi \in \mathbb{R}$. Therefore, because our eigenvector is in V_ξ under the constraint condition (2.4), the corresponding eigenvalues are solutions of

$$\begin{aligned} &\rho\kappa\lambda^4 + (\kappa\gamma\xi^2 + \rho\tau)\lambda^3 + \left((\alpha\kappa + \rho\delta + \gamma\tau - \varepsilon_1\varepsilon_2)\xi^2 + \rho\eta\right)\lambda^2 \\ &\quad + \left(\delta\gamma\xi^2 + \alpha\tau + \eta\gamma - \beta(\varepsilon_1 + \varepsilon_2)\right)\xi^2\lambda + (\alpha\delta\xi^2 + \alpha\eta - \beta^2)\xi^2 = 0. \end{aligned} \tag{4.2}$$

Furthermore, from Theorem 2.1, the non-zero eigenvalues must satisfy $\mathrm{Re}\lambda(i\xi) < 0$ for $\xi \neq 0$ if $\varepsilon_1 = \varepsilon_2 = 0$. Hence, it is enough to consider the asymptotic expansion of $\lambda = \lambda(i\xi)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$.

Let $\lambda_j(i\xi)$ be the non-zero eigenvalues of $\hat{\Phi}(i\xi)$. We first consider the asymptotic expansion for $|\xi| \to 0$ with the expression

$$\lambda_j(i\xi) = \sum_{k=0}^{\infty} \lambda_j^{(k)} \xi^k. \tag{4.3}$$

Substituting (4.3) into (4.2), we find, after lengthy but straightforward calculations, as follows.

(i) $\gamma > 0$ and $\tau > 0$:

$$\lambda_j(i\xi) = \pm\sqrt{\frac{\alpha\eta - \beta^2}{\rho\eta}}\, i\xi - \frac{\chi_1}{4\rho\eta^2}\xi^2 + O(\xi^3), \tag{4.4}$$

for $j = 1, 2$ and

$$\begin{aligned} \lambda_j(i\xi) &= -\frac{1}{2\kappa}(\tau \pm \sqrt{\tau^2 - 4\kappa\eta}) + O(\xi^2) && \text{if } \tau^2 - 4\kappa\eta \neq 0, \\ \lambda_j(i\xi) &= -\frac{\tau}{2\kappa} \pm \sqrt{\frac{\delta}{\kappa} - \frac{1}{\rho\eta}\left(\beta - \frac{\varepsilon_1\tau}{2\kappa}\right)\left(\beta - \frac{\varepsilon_2\tau}{2\kappa}\right)}\, i\xi + O(\xi^2) && \text{if } \tau^2 - 4\kappa\eta = 0 \end{aligned} \tag{4.5}$$

for $j = 3, 4$, where

$$\chi_1 := \left(\gamma\tau - \varepsilon^2\right)\left(\frac{\eta^2}{\tau} + \frac{\beta^2}{\gamma}\right) + \gamma\left(\eta - \frac{\varepsilon\beta}{\gamma}\right)^2 + \tau\left(\beta - \frac{\varepsilon\eta}{\tau}\right)^2.$$

Remark that $\chi_1 > 0$ for $\gamma\tau - \varepsilon^2 > 0$.

(ii) $\gamma > 0$ and $\tau = 0$:

$$\lambda_j(i\xi) = \pm\sqrt{\frac{\alpha\eta - \beta^2}{\rho\eta}}\, i\xi - \frac{\gamma\eta}{2\rho}\xi^2 + O(\xi^3), \tag{4.6}$$

for $j=1,2$ and

$$\begin{aligned}\lambda_j(i\xi) = \pm\sqrt{\frac{\eta}{\kappa}}i \pm \frac{1}{2}\left(\frac{\delta}{\kappa}+\frac{\beta^2}{\rho\eta}\right)\sqrt{\frac{\kappa}{\eta}}i\xi^2 \\ - \frac{1}{2\rho\eta}\left\{\gamma\left(\delta+\frac{2\kappa\beta^2}{\rho\eta}\right)\pm\chi_2 i\right\}\xi^4 + O(\xi^5),\end{aligned} \tag{4.7}$$

for $j=3,4$, where

$$\chi_2 := \left(\frac{\rho\kappa}{4}\left(\frac{\delta}{\kappa}+\frac{\beta^2}{\rho\eta}\right)^2 + \left(\rho\left(\frac{\delta}{\kappa}+\frac{\beta^2}{\rho\eta}\right)+\alpha\right)\left(\kappa\left(\frac{\delta}{\kappa}+\frac{\beta^2}{\rho\eta}\right)+\delta\right)\right)\sqrt{\frac{\kappa}{\eta}}.$$

(iii) $\gamma = 0$ and $\tau > 0$:

$$\lambda_j(i\xi) = \pm\sqrt{\frac{\alpha\eta-\beta^2}{\rho\eta}}i\xi - \frac{\tau\beta^2}{2\rho\eta^2}\xi^2 + O(\xi^3), \tag{4.8}$$

for $j=1,2$ and

$$\begin{aligned}\lambda_j(i\xi) &= -\frac{1}{2\kappa}(\tau\pm\sqrt{\tau^2-4\kappa\eta}) + O(\xi^2) \qquad \text{if} \quad \tau^2-4\kappa\eta\neq 0,\\ \lambda_j(i\xi) &= -\frac{\tau}{2\kappa}\pm\sqrt{\frac{\delta}{\kappa}-\frac{\beta^2}{\rho\eta}}i\xi + O(\xi^2) \qquad \text{if} \quad \tau^2-4\kappa\eta = 0\end{aligned} \tag{4.9}$$

for $j=3,4$.

We next study the asymptotic expansion for $|\xi|\to\infty$. To this end, we rewrite $\hat{\Phi}(i\xi)$ as $\hat{\Phi}(i\xi) = \xi^2\hat{\Psi}(i\xi)$, where $\hat{\Psi}(i\xi) := -(A^0)^{-1}(i\xi^{-1}A + B + \xi^{-2}L + i\xi^{-1}M)$. Then non-zero eigenvalues $\Lambda_j(i\xi)$ of $\hat{\Psi}(i\xi)$ satisfy

$$\begin{aligned}\rho\kappa\Lambda^4 + (\kappa\gamma+\rho\tau\xi^{-2})\Lambda^3 + \left(\alpha\kappa+\rho\delta+\gamma\tau-\varepsilon_1\varepsilon_2+\rho\eta\xi^{-2}\right)\xi^{-2}\Lambda^2 \\ + \left(\delta\gamma + (\alpha\tau+\eta\gamma-\beta(\varepsilon_1+\varepsilon_2))\xi^{-2}\right)\xi^{-2}\Lambda + \left(\alpha\delta+(\alpha\eta-\beta^2)\xi^{-2}\right)\xi^{-4} = 0.\end{aligned}$$

We consider the asymptotic expansion for $|\xi|\to\infty$ with the expression

$$\Lambda_j(i\xi) = \sum_{k=0}^{\infty}\Lambda_j^{(k)}\xi^{-k}.$$

Similar calculation as before leads the expansion of the eigenvalues $\lambda_j(i\xi) = \xi^2\Lambda_j(i\xi)$ that

(i) $\gamma > 0$ and $\tau > 0$:

$$\begin{aligned}&\lambda_j(i\xi) = \pm\sqrt{\frac{\delta}{\kappa}}i\xi - \frac{1}{2\kappa\gamma}(\gamma\tau-\varepsilon_1\varepsilon_2) \mp \frac{\chi_3}{8\kappa\gamma^2\sqrt{\delta\kappa}}i\xi^{-1} + O(\xi^{-2}),\\ &\lambda_3(i\xi) = -\frac{\alpha}{\gamma} + O(\xi^{-2}), \qquad \lambda_4(i\xi) = -\frac{\gamma}{\rho}\xi^2 + \frac{\alpha\kappa-\varepsilon_1\varepsilon_2}{\kappa\gamma} + O(\xi^{-1})\end{aligned} \tag{4.10}$$

for $j=1,2$, where

$$\chi_3 := \gamma^2(\tau^2-4\kappa\eta) + 8\kappa\gamma\beta\varepsilon + \varepsilon_1\varepsilon_2\left(\varepsilon_1\varepsilon_2 - 2\gamma\tau - 4(\alpha\kappa-\rho\delta)\right).$$

Remark that $\gamma\tau - \varepsilon_1\varepsilon_2 > 0$ for $\gamma\tau - \varepsilon^2 > 0$.

(ii) $\gamma > 0$ and $\tau = 0$:

$$\begin{aligned} \lambda_j(i\xi) &= \pm\sqrt{\frac{\delta}{\kappa}}i\xi \pm \frac{\eta}{2\delta}\sqrt{\frac{\delta}{\kappa}}i\xi^{-1} - \frac{\alpha\eta}{2\delta\gamma}\xi^{-2} + O(\xi^{-3}), \\ \lambda_3(i\xi) &= -\frac{\alpha}{\gamma} + O(\xi^{-2}), \qquad \lambda_4(i\xi) = -\frac{\gamma}{\rho}\xi^2 + \frac{\alpha}{\gamma} + O(\xi^{-1}) \end{aligned} \tag{4.11}$$

for $j = 1, 2$.

(iii-i) $\gamma = 0$, $\tau > 0$ and $p = 0$:

$$\lambda_j(i\xi) = \sqrt{\frac{\alpha}{\rho}}i\xi - \frac{1}{4\kappa}(\tau \pm \sqrt{\tau^2 - 4\kappa\beta^2/\alpha}) + O(\xi^{-1}) \tag{4.12}$$

for $j = 1, 2$ and

$$\lambda_j(i\xi) = -\sqrt{\frac{\alpha}{\rho}}i\xi - \frac{1}{4\kappa}(\tau \pm \sqrt{\tau^2 - 4\kappa\beta^2/\alpha}) + O(\xi^{-1}) \tag{4.13}$$

for $j = 3, 4$.

(iii-ii) $\gamma = 0$, $\tau > 0$ and $p \neq 0$:

$$\lambda_j(i\xi) = \pm\sqrt{\frac{\delta}{\kappa}}i\xi - \frac{\tau}{2\kappa} \mp \frac{1}{2}\left(\frac{\tau^2}{4\delta\kappa} + \frac{\rho\eta}{\alpha\kappa - \rho\delta}\right)\sqrt{\frac{\delta}{\kappa}}i\xi^{-1} + O(\xi^{-2}) \tag{4.14}$$

for $j = 1, 2$ and

$$\lambda_j(i\xi) = \pm\sqrt{\frac{\alpha}{\rho}}i\xi \pm \frac{\rho\eta}{2(\alpha\kappa - \rho\delta)}\sqrt{\frac{\alpha}{\rho}}i\xi^{-1} - \frac{\alpha\rho\eta\tau}{2(\alpha\kappa - \rho\delta)^2}\xi^{-2} + O(\xi^{-3}) \tag{4.15}$$

for $j = 3, 4$.

The expansions (4.11) and (4.15) denote the regularity-loss structure. We conclude from these expansions that the estimates in Theorem 2.2 and Corollary 2.3 should be optimal.

Acknowledgments

The work of the first author is supported by project "Análisis Matemático de Problemas de la Termomecánica" (MTM2016-74934-P), (AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness. The work of the second author is supported by Grant-in-Aid for Scientific Research (C) No. 18K03369 from Japan Society for the Promotion of Science.

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