A note on a new general family of deterministic hierarchical networks *

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Abstract

It is known that many networks modeling real-life complex systems are small-word (large local clustering and small diameter) and scale-free (power law of the degree distribution), and very often they are also hierarchical. Although most of the models are based on stochastic methods, some deterministic constructions have been recently proposed, because this allows a better computation of their properties. Here a new deterministic family of hierarchical networks is presented, which generalizes most of the previous proposals, such as the so-called binomial tree. The obtained graphs can be seen as graphs on alphabets (where vertices are labeled with words of a given alphabet, and the edges are defined by a specific rule relating different words). This allows

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us the characterization of their main distance-related parameters, such as the radius and diameter. Moreover, as a by-product, an efficient shortest-path local algorithm is proposed.

Keywords: Hierarchical network, Small-word, Scale-free, Degree, Routing algorithm, Diameter, Clustering.

1 Introduction

1.1 Models for complex networks

Since the papers of Watts and Strogatz [17] on small-world networks and by Barabási and Albert [3] on scale-free networks, there has been a special interest in the theoretical study of complex networks, such as the World Wide Web [2], some kind of social networks [13, 11, 14], communication networks [10], protein networks [9, 18], etc. Two main characteristics of such networks are the presence of a strong local clustering (that is, groups of nodes with mutual interconnections), high modularity, certain distribution of the degrees, and self-similarity. The subject continues to attract the interest of researchers. For some recent papers, see [16, 1, 12, 15].

In this paper, we propose a new deterministic family, which generalizes some previous proposals of hierarchical networks [14, 13, 11, 6, 7]. As a first approach, our family of hierarchical networks is defined recursively from an initial complete graph on n_k vertices. Also, it is shown that the obtained graphs can also be seen as graphs on alphabets [8]. The vertices of such graphs are labeled with words of a given alphabet, and their edges are defined by some specific rules relating different words. This allows us the characterization of their main distance-related parameters, such as the vertex eccentricities, the radius, and the diameter. Moreover, as a by-product, an efficient shortest-path local algorithm is proposed.

1.2 Some basic notation

Here we present the basic notation used throughout the paper. Let $G = (V, E)$ be a finite, simple, and connected graph with vertex set V, order $N = |V|$, edge set E, and size $M = |E|$. If the vertices u and v are adjacent, $uv \in E$, we represent it by $u \sim v$. The *distance* between vertices u and v is denoted by $\mathrm{dist}(u, v)$. The *eccentricity* of a vertex u is $\varepsilon(u) = \min\{\mathrm{dist}(u, v) : v \in V\}$. Hence, the *radius* and *diameter* of G are, respectively, $R = \min_{u\in V} \mathrm{ecc}(u)$, and $D = \max_{u\in V} \mathrm{ecc}(u)$. The

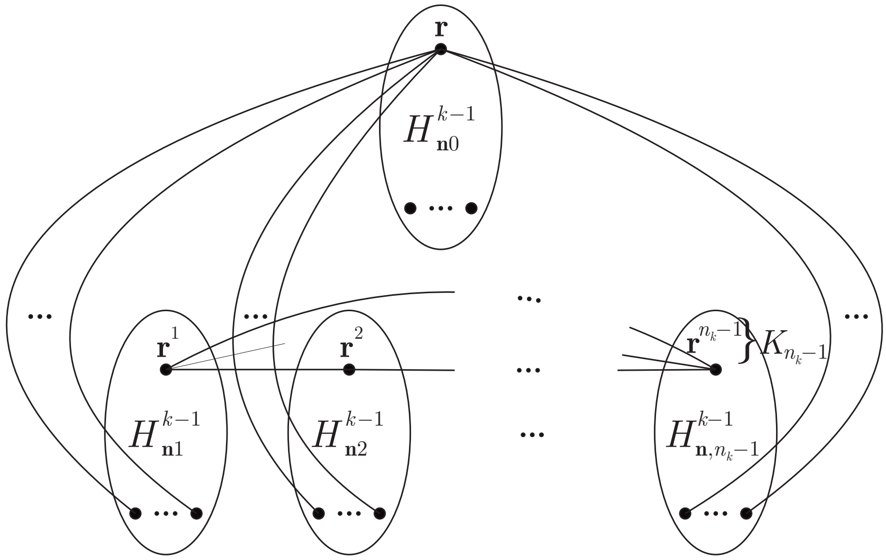

Figure 1: The graph $H^k_{\mathbf{n}}$ from n_k copies of $H^{k-1}_{\mathbf{n}}$.

set of vertices at distance i from a given vertex $u \in V$ is $G_i(u) = \{v : \mathrm{dist}(u, v) = i\}$, for $i = 0, \ldots, D$. Then, the *degree* of a vertex u is just $\delta(u) = |G_1(u)|$.

2 The hierarchical graph $H^k_{\mathbf{n}}$

In this section, we generalize the constructions of deterministic hierarchical graphs introduced by Ravasz and Barabási [13], Ravasz, Somera, Mongru, Oltvai, and Barabási [14], Noh [11], and Barrière, Comellas, Dalfó and Fiol [7]. Roughly speaking, our graphs are constructed in the following way: Given k integers $n_1, n_2, \ldots, n_k$, we first connect a selected root vertex of the complete graph K_{n_1} to some vertices of $n_2 - 1$ copies of K_{n_1}, and then we add some new edges between such copies. This gives a graph with $n_1 n_2$ vertices. Next, $n_3 - 1$ replicas of the new whole structure are added, again with some edges between them and to the same root vertex. At this step, the graph has $n_1 n_2 n_3$ vertices. Then we iterate the process until the desired graph, with order $N = n_1 n_2 \cdots n_k$, is obtained. We give below two formal definitions.

2.1 A recursive definition

A recursive definition of the considered networks is as follows.

Definition 2.1. *Let* $n_1, n_2, \ldots$ *a sequence of positive integers,* $n_i \geq 2$*, whose finite* k*-subsequences* $n_1, \ldots, n_k$ *are abbreviated by the symbols* ${}^k_{\mathbf{n}}$*,* $k = 1, 2, \ldots$*. The* hierarchical graph $H_{n_1,\ldots,n_k}$*, also denoted by* $H^k_{\mathbf{n}}$*, has vertex set* $V^k_{\mathbf{n}}$*, with* $N = n_1 n_2 \cdots n_k$ *vertices, each denoted by a* k*-string* $x_1x_2x_3 \ldots x_k$*, where* $x_i \in \mathbb{Z}_{n_i}$ *for* $i = 1, \ldots, k$*, and edge set* $E^k_{\mathbf{n}}$ *defined recursively as follows:*

- $H^1_{\mathbf{n}} \equiv H_{n_1}$ *is the complete graph* K_{n_1}*.*
- *For* $k > 1$*,* $H^k_{\mathbf{n}}$ *is obtained from the union of* n_k *copies of* $H^{k-1}_{\mathbf{n}}$*, each denoted by* $H^{k-1}_{\mathbf{n}\alpha}$*, and with vertices* $x_1x_2 \ldots x_{k_1}\alpha$*, for every fixed* $\alpha = 0, \ldots, n_k - 1$*, by adding the following new edges:*

$$00\ldots00 \quad \sim \quad x_1x_2\ldots x_{k-1}x_k, \qquad x_i \neq 0, \textit{ for } i = 1, \ldots, k; \tag{1}$$

$$00\ldots0x_1 \quad \sim \quad 00\ldots0y_1, \qquad x_1, y_1 \neq 0, \textit{ and } \; x_1 \neq y_1. \tag{2}$$

To illustrate the recursive procedure, Fig. 1 shows the hierarchical graph $H^k_{\mathbf{n}}$ obtained by joining n_k copies of $H^{k-1}_{\mathbf{n}}$. Notice that vertex $\mathbf{r} := 00\ldots0$, which we distinguish and call the *root*, is adjacent by (1) to vertices $x_1x_2\ldots x_k$, for $x_i \neq 0$ and $i = 1, \ldots, k$, which we call *peripheral vertices*.

In particular, if all the numbers n_i, $i = 1, \ldots, k$, are equal to, say n, we will denote the corresponding graph as H_{n^k}.

2.2 A definition as a graph on an alphabet

To give a more direct definition of $H^k_{\mathbf{n}}$, it is convenient to introduce the following notation. The length ℓ (number of elements) of a given string $\mathbf{x} = x_1 \ldots x_\ell$ will be denoted by $|\mathbf{x}|$. If all its elements are nonzero, we represent it by $\mathbf{x}^* = x^*_1 \ldots x^*_\ell$. A string with all its elements 0 is denoted by $\mathbf{0}$. Then, the edge set $E^k_{\mathbf{n}}$ is characterized by the following adjacency rules (the substrings of each vertex have appropriate lengths and, hence, they sum up to k):

$$x_1\mathbf{x} \quad \sim \quad y_1\mathbf{x}, \qquad \text{where } x_1 \neq y_1; \tag{3}$$

$$\mathbf{0x} \quad \sim \quad \mathbf{y}^*\mathbf{x}, \qquad \text{where } |\mathbf{y}^*| = |\mathbf{0}|; \tag{4}$$

$$\mathbf{0}x^*_i\mathbf{x} \quad \sim \quad \mathbf{0}y^*_i\mathbf{x}, \qquad \text{where } y^*_i \neq x^*_i. \tag{5}$$

As a concrete example, Fig. 2 shows two different drawings of the graph $H_{2,3,4}$.

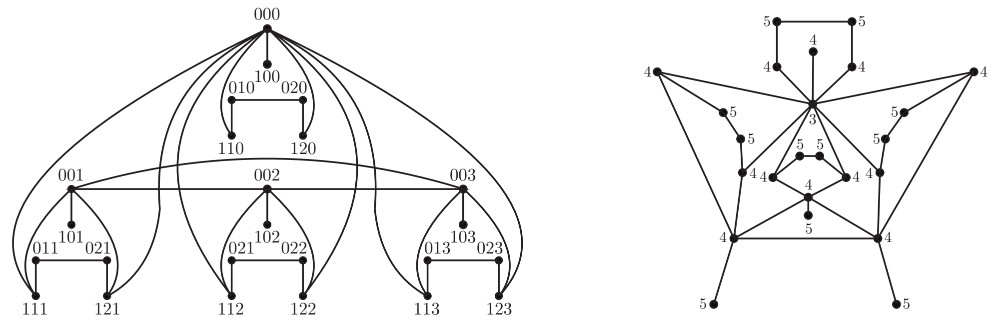

Figure 2: Two representations of the graph $H_{2,3,4}$: On the left a non-planar drawing with the labels of its vertices, and on the right a planar drawing with the eccentricities of its vertices.

2.3 Some particular cases

2.3.1 The hierarchical network $H_{n,k}$

The hierarchical network introduced by Barrière, Comellas, and the authors (see [7]), which was denoted by $H_{n,k}$ is just H_{n^k} (that is, $n_1 = \cdots = n_k = n$). In turn, H_{n^k} generalizes the deterministic hierarchical network introduced by Ravasz, Somera, Mongru, Oltvai, and Barabási [14], see also Barabási and Oltvai, [4] (which corresponds to H_{4^k}). Moreover, the deterministic hierarchical networks introduced by Ravasz and Barabási [13] and generalized by Noh [11], constitute a subgraph of H_{5^k} (some edges are not present).

2.3.2 The binomial tree B_k

As it is well-known, a binomial tree is used in Computer Science to model a recursive data structure. A tree of degree zero B_0 is a singleton, in our context, allowing sequences of length zero, we could denote it as $B_0 = H_{2^0}$. A tree of degree k, B_k, is constructed from two trees of degree $k-1$, by joining their two roots. Then, within our notation, $B_k = H_{2^k}$. Figure 3 shows the binomial trees of order $k = 0, \ldots, 4$. Notice that B_k can also recursively defined by saying that its root node has as children the roots of binomial trees of orders $k-1, k-2, ..., 2, 1, 0$.

The binomial tree can also be seen as the hierarchical product of k copies of the complete graph on 2 vertices K_2, denoted $B_k = K_2 \sqcap \overset{k}{\cdots} \sqcap K_2$ (see Barrière, Comellas, Dalfó, and Fiol [5]).

In due course, we will show that the number of vertices $\mathbf{x}$ of H_{2^k} at distance

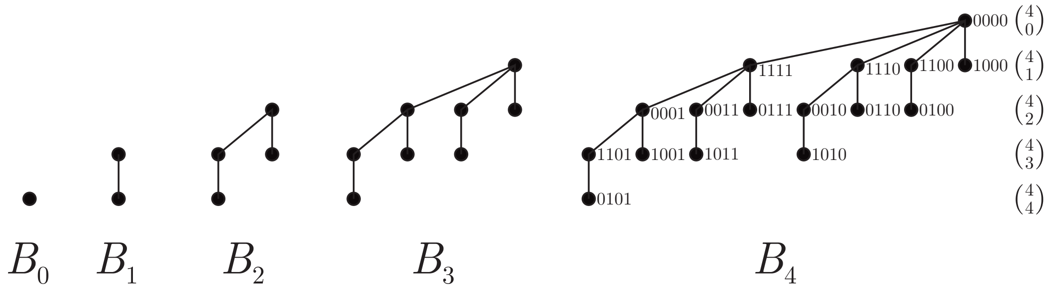

Figure 3: The first binomial trees B_i, $i = 0, \dots, 4$.

i from the root $\mathbf{r} = \mathbf{0}$ is the binomial coefficient $\binom{k}{i}$ (see Fig. 3). Of course, this property is the reason for the name of such a structure.

3 Hierarchical properties

The main structural properties of the graphs $H^k_{\mathbf{n}}$ are the following:

3.1 Order and size

We have already seen that the order of $H^k_{\mathbf{n}}$ is $N = |V^k_{\mathbf{n}}| = \prod_{i=1}^k n_i$. Concerning its size, we have the following result:

Lemma 3.1. *Let* $\mathbf{n} \equiv n_1, n_2, \dots, n_k$*. Then, the number of edges of* $H^k_{\mathbf{n}}$ *is*

$$M = |E^k_{\mathbf{n}}| = \binom{n_1}{2} \prod_{i=2}^k n_i + \sum_{i=2}^k \prod_{j=1}^i (n_j - 1) \prod_{j=i+1}^k n_j + \sum_{i=2}^k \binom{n_i - 1}{2} \prod_{j=i+1}^k n_j. \quad (6)$$

Proof. Let M_h be the size of $H^h_{\mathbf{n}}$. Then, by using the recursive Definition 2.1, we have $M_1 = n_1$ and

$$M_h = n_h M_{h-1} + \prod_{i=1}^h (n_h - 1) + \binom{n_h - 1}{2}.$$

Note that the first term corresponds to the number of edges of the n_h copies of $H^{h-1}_{\mathbf{n}}$, whereas the second and third terms account for the edges joining such copies, according to the conditions (1) and (2), respectively. Using this, the computation of (6) is immediate. □

Of course, the 1st, 2nd, and 3rd terms in (6) also correspond to the adjacency conditions (3), (4), and (5), respectively. For instance, in the graph $H_{2,3,4}$ of Fig. 2, we get $M = 12 + 18 + 3 = 33$.

3.2 Subgraphs

The following lemma, whose proof follows easily by seeing $H^k_{\mathbf{n}}$ as a graph on an alphabet, shows the hierarchical nature of our networks.

Lemma 3.2. *(i) For each given sequence $\alpha_2 \dots \alpha_k$, with $\alpha_i \in \mathbb{Z}_{n_i}$, $i = 2, \dots, k$, the vertex set $\{x_1\alpha_1\alpha_2 \dots \alpha_k : x_1 \in \mathbb{Z}_{n_1}\}$ induces a subgraph isomorphic to K_{n_1}.*

(ii) Let $\mathbf{n} \equiv n_1, \dots, n_k$. For every i, $1 \le i \le k-1$, $H^k_{\mathbf{n}}$ can be decomposed into n^i vertex-disjoint subgraphs isomorphic to $H_{x_1,\dots,x_{k-i}}$. Each of such subgraphs is denoted by $H^{k-i}_{\mathbf{n}\boldsymbol{\alpha}}$, and has vertex labels $x_1x_2 \dots x_{k-i}\boldsymbol{\alpha}$, with $\boldsymbol{\alpha} \equiv \alpha_{k-i+1}, \dots, \alpha_k \in \mathbb{Z}_{k-i+1} \times \cdots \times \mathbb{Z}_k$ being a fixed sequence.

(iii) The root vertex of the subgraph $H^{k-i}_{\mathbf{n}\boldsymbol{\alpha}}$ is $\mathbf{0}\boldsymbol{\alpha}$, where $|\mathbf{0}| = k - i$, whereas its peripheral vertices *are of the form $x^*_1x^*_2 \dots x^*_{k-i}\boldsymbol{\alpha}$.*

(iv) By collapsing in $H^k_{\mathbf{n}}$ each of the $n_{k-i+1} \cdots n_k$ subgraphs $H^i_{\mathbf{n}\boldsymbol{\alpha}}$, with a fixed $\boldsymbol{\alpha} \in \mathbb{Z}_{k-i+1} \times \cdots \times \mathbb{Z}_k$, into a single vertex and all multiple edges into one, we obtain a graph isomorphic to $H_{n_{k-i+1},\dots,n_k}$.

(v) For every fixed i, $1 \le i \le k$, and a given sequence $\boldsymbol{\alpha} \in \mathbb{Z}_{i+1} \times \cdots \times \mathbb{Z}_k$, the $n_i - 1$ vertices labeled $\mathbf{0}x^*_i\boldsymbol{\alpha}$ with $x^*_i \in \mathbb{Z}^*_{n_i}$ (that is, the root vertices of $H^{i-1}_{\mathbf{n}x^*_i\boldsymbol{\alpha}}$) induce a complete graph isomorphic to K_{n_i-1}.

4 Distance parameters

First, we introduce some notation concerning $H^k_{\mathbf{n}}$. Let $\mathrm{dist}_k(\mathbf{x}, \mathbf{y})$ denote the distance between vertices $\mathbf{x}, \mathbf{y} \in V^k_{\mathbf{n}}$ in $H^k_{\mathbf{n}}$; and $\mathrm{dist}_k(\mathbf{x}, U) := \min_{\mathbf{u}\in U}\{\mathrm{dist}_k(\mathbf{x}, \mathbf{u})\}$. Let $\mathbf{r}^\alpha = 00 \dots 0$ be the root vertex of $H^{k-1}_{\mathbf{n}\alpha}$, $\alpha \in \mathbb{Z}_n$ (as stated before, $\mathbf{r}$ stands for the root vertex of $H^k_{\mathbf{n}}$). Let P and P^α for $\alpha \in \mathbb{Z}_{n_k}$, denote the set of peripheral vertices of $H^k_{\mathbf{n}}$ and $H^{k-1}_{\mathbf{n}\alpha}$, respectively.

4.1 The distance function

In [7] Barrière, Comellas, and the authors proved that the radius R_k of $H_{n,k}$, the eccentricity $\mathrm{ecc}_k(\mathbf{r})$ of its root $\mathbf{r}$, and its diameter D_k are $R_k = \mathrm{ecc}(\mathbf{r}) = k$, and $D_k = 2k-1$. Here we will be more precise and will give both a formula for the distance between vertices, and a proof that the routing algorithm of the next section always gives the shortest path.

Given a k-sequence $\mathbf{x} = x_1 \dots x_k \in V_{\mathbf{n}}^k$, consider the 'expanded' $(k+1)$-sequence $\mathbf{x}^+ = \mathbf{x}0$ (that is, with $x_{k+1} = 0$). Then, the *alternating number* of $\mathbf{x}$, denoted by $\mathrm{alt}(\mathbf{x})$, is the number of changes in $\mathbf{x}^+$ from a zero element, say x_i, to a nonzero element x_{i+1}^* (or vice versa). For instance, if $\mathbf{x} = \mathbf{x}_1^*\mathbf{0}_2\mathbf{x}_3^*\mathbf{0}_4\mathbf{x}_5^*$, then $\mathrm{alt}(\mathbf{x}) = 5$ (here the $\mathbf{0}_i$'s denote zero strings of some length), and if $\mathbf{x} = \mathbf{x}_1^*\mathbf{0}_2\mathbf{x}_3^*\mathbf{0}_4$, then $\mathrm{alt}(\mathbf{x}) = 3$. Moreover, given two k-sequences $\mathbf{x}$, $\mathbf{y}$, we denote its maximum common suffix as $\mathbf{x} \cap \mathbf{y}$, with length $\ell = |\mathbf{x} \cap \mathbf{y}|$, for $0 \le \ell \le k$.

By looking at the structure of adjacent vertices given by the adjacency conditions (3)-(5), the following result is clear.

Lemma 4.1. *Let $\mathbf{x} = x_1 \dots \mathbf{u}$ and $\mathbf{y} = y_1 \dots \mathbf{u}$ be two adjacent vertices of $H_{\mathbf{n}}^k$ (with $\mathbf{u}$ possibly being the empty string). Then,*

- *If condition (3) holds, then, either $\mathrm{alt}(\mathbf{x}) = \mathrm{alt}(\mathbf{y}) \pm 1$ if $x_1 = 0$ or $y_1 = 0$, or $\mathrm{alt}(\mathbf{x}) = \mathrm{alt}(\mathbf{y})$ otherwise $(x_1, y_1 \neq 0)$.*
- *If condition (4) holds, then, $\mathrm{alt}(\mathbf{x}) = \mathrm{alt}(\mathbf{y}) \pm 1$.*
- *If condition (5) holds, then $\mathrm{alt}(\mathbf{x}) = \mathrm{alt}(\mathbf{y})$.*

Corollary 4.2. *Let $\mathbf{x}$ be a vertex of $H_{\mathbf{n}}^k$ with $\mathrm{alt}(\mathbf{x}) \neq 0$. Then, there exists a vertex $\mathbf{y} \sim \mathbf{x}$ such that $\mathrm{alt}(\mathbf{y}) = \mathrm{alt}(\mathbf{x}) - 1$, but no vertex $\mathbf{y}' \sim \mathbf{x}$ satisfies $\mathrm{alt}(\mathbf{y}') < \mathrm{alt}(\mathbf{x}) - 1$.*

Corollary 4.3. *Let $\mathbf{x} = x_1 \dots x_k$ be a vertex of $H_{\mathbf{n}}^k$, with root $\mathbf{r} = \mathbf{0}$. Then,*

$$\mathrm{dist}(\mathbf{x}, \mathbf{r}) = \mathrm{alt}(\mathbf{x}, \mathbf{r}). \tag{7}$$

Proof. By Corollary 4.2, it is clear that, if $x_k = 0$ there is a shortest path from $\mathbf{x}$ to $\mathbf{r} = \mathbf{0}$ of length $\mathrm{alt}(\mathbf{x})$. Otherwise, if $x_k = x_k^* \neq 0$, there is a shortest path from $\mathbf{x}$ to $\mathbf{x}^* = x_1^* \dots x_k^*$ of length $\mathrm{alt}(\mathbf{x}) - 1$, but $\mathbf{x}^*$ is adjacent to $\mathbf{0}$. □

As an example, note that in the binomial tree $B_k = H_{2^k}$, the vertices at distance i from the root $\mathbf{r} = \mathbf{0}$ are the binary sequences $\mathbf{x}$ satisfying $\mathrm{alt}(\mathbf{x}) = i$. Thus, its number is the binomial coefficient $\binom{k}{i}$, as was commented before.

The following result gives the distance between two generic vertices $\mathbf{x}$, $\mathbf{y}$ of $H^k_{\mathbf{n}}$. To avoid the trivial case of the complete graph, we will suppose that $k \geq 2$. Moreover, without loss of generality, we can assume that $\mathbf{x}$ and $\mathbf{y}$ have no common suffix (so that $x_k \neq y_k$). Otherwise, if $\mathbf{x} = x_1x_2 \ldots x_{k-i}\boldsymbol{\alpha}$ and $\mathbf{y} = y_1y_2 \ldots y_{k-i}\boldsymbol{\alpha}$, that is, $\boldsymbol{\alpha} = \mathbf{x} \cap \mathbf{y}$ and $i = |\boldsymbol{\alpha}| > 0$, we are in the subgraph $H^{k-i}_{\mathbf{n}\boldsymbol{\alpha}}$ (by Lemma 3.2(iii)) and, hence, we can apply the routing algorithm to the vertices $\mathbf{x}' = x_1x_2 \ldots x_{k-i}$ and $\mathbf{y}' = y_1y_2 \ldots y_{k-i}$ of $H^{k-i}_{\mathbf{n}}$.

Proposition 4.4. *Let $\mathbf{x} = x_1 \ldots \mathbf{u}$ and $\mathbf{y} = y_1 \ldots \mathbf{v}$ be two vertices of $H^k_{\mathbf{n}}$, with no common suffix, and where all elements of $\mathbf{u}$ are either zero or nonzero, and all elements of $\mathbf{v}$ are either nonzero or zero, respectively. Let $\mathbf{x}'$ be the sequence obtained from $\mathbf{x}$ by deleting its rightmost element x_k, and similarly for $\mathbf{y}' = \mathbf{y} \setminus y_k$. Then,*

(i) *If either $x_k = 0$ or $y_k = 0$, then*

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{alt}(\mathbf{x}) + \operatorname{alt}(\mathbf{y}).$$

(ii) *If $x_k, y_k \neq 0$, and $|\mathbf{u}|, |\mathbf{v}| > 1$, then*

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{alt}(\mathbf{x}) + \operatorname{alt}(\mathbf{y}).$$

(iii) *If $x_k, y_k \neq 0$, and either $|\mathbf{u}| = 1$ or $|\mathbf{v}| = 1$, then*

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{alt}(\mathbf{x}') + \operatorname{alt}(\mathbf{y}') + 1.$$

Proof. First, notice that, as $|\mathbf{x} \cap \mathbf{y}| = 0$, we always have $x_k \neq y_k$. Then the key idea is that we cannot reach $\mathbf{y}$ from $\mathbf{x}$ without going through vertices with most of their elements being null (roots). Thus, using Corollary 4.3, we have:

(i)-(ii) In these cases, the shortest path must go through the root $\mathbf{r} = \mathbf{0}$ and, hence,

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{dist}(\mathbf{x}, \mathbf{r}) + \operatorname{dist}(\mathbf{r}, \mathbf{y}) = \operatorname{alt}(\mathbf{x}) + \operatorname{alt}(\mathbf{y}).$$

(iii) Now, the shortest path must go through the roots $\mathbf{r}' = \mathbf{0}x^*_k$ and $\mathbf{r}'' = \mathbf{0}y^*_k$ of the subgraphs isomorphic to $H_{x_1 \ldots, x_{k-1}}$ and $H_{y_1 \ldots, y_{k-1}}$, respectively. Then

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{dist}(\mathbf{x}, \mathbf{r}') + \operatorname{dist}(\mathbf{r}', \mathbf{r}'') + \operatorname{dist}(\mathbf{r}'', \mathbf{y}) = \operatorname{alt}(\mathbf{x}') + 1 + \operatorname{alt}(\mathbf{y}'),$$

since $\mathbf{r}'$ and $\mathbf{r}''$ are adjacent by (5). This completes the proof. □

4.2 Eccentricity, radius and diameter

As a consequence of the above results, we have the following lemma:

Lemma 4.5. *Let* $\mathbf{x}$ *be a vertex of* $H_{\mathbf{n}}^k$*, as in Proposition 4.4. Then, its eccentricity* $\mathrm{ecc}_k(\mathbf{x})$*, the radius* R_k *of* $H_{\mathbf{n}}^k$*, and its diameter* D_k *are:*

(*i*) *For the eccentricity, we must distinguish two cases:*

(*i*.1) *If* $\mathbf{n} = 2, 2, \dots, 2$ *(that is,* $H_{\mathbf{n}}^k$ *is the binomial tree* B_k*), let* $\overline{\mathbf{x}}$ *denote the sequence obtained from* $\mathbf{x}$ *by interchanging* 0*'s and* 1*'s. Then,*

$$\mathrm{ecc}_k(\mathbf{x}) = \begin{cases} \mathrm{alt}(\mathbf{x}) + k, & \text{if } x_k = 0, \\ \mathrm{alt}(\overline{\mathbf{x}}) + k, & \text{if } x_k = 1. \end{cases}$$

(*i*.2) *If* $H_{\mathbf{n}}^k$ *is not the binomial tree* B_k*, then*

$$\mathrm{ecc}_k(\mathbf{x}) = \begin{cases} \mathrm{alt}(\mathbf{x}) + k - 1, & \text{if } |\mathbf{u}| = 1 \text{ and } x_k \neq 0, \\ \mathrm{alt}(\mathbf{x}) + k, & \text{otherwise.} \end{cases}$$

(*ii*) *The radius of* $H_{\mathbf{n}}^k$ *coincides with the eccentricity of its root:*

$$R_k = \mathrm{ecc}(\mathbf{r}) = k.$$

(*iii*) *The diameter of* $H_{\mathbf{n}}^k$ *is the same as that of* H_{n^k}*:*

$$D_k = 2k - 1.$$

Proof. (*i*) If $x_k = 0$, by Proposition 4.4(*i*), $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \mathrm{alt}(\mathbf{x}) + \mathrm{alt}(\mathbf{y})$, but $\max\{\mathrm{alt}(\mathbf{y})\} = k$. This proves que first case in (*i*.1), and the second one in (*i*.2) (since the same is true when $|\mathbf{u}| > 1$, by Proposition 4.4(*ii*)). The second equality in (*i*.1) follows from the fact that the mapping $\mathbf{x} \mapsto \overline{\mathbf{x}}$ is an isomorphism on the binomial tree. Finally, the first equality in (*i*.2) is proved similarly by using Proposition 4.4(*iii*).

(*ii*) Take $\mathbf{x} = \mathbf{r}$ in (*i*).

(*iii*) Take $\mathbf{x}$ with $\mathrm{alt}(x) = k$ in (*i*). □

5 A shortest path routing algorithm

The reasoning that support Proposition 4.4, leads us to a routing algorithm between two generic vertices $\mathbf{x}$, $\mathbf{y}$ of $H_{\mathbf{n}}^k$, which follows a shortest path.

algorithm: `Routing`
  input: `Sequence` $n_1, \dots, n_k$ `,`
  `Vertices` $\mathbf{x} = x_1 \dots \mathbf{u}$`,` $\mathbf{y} = y_1 \dots \mathbf{v}$ `(with` $\mathbf{u}$ `and` $\mathbf{v}$ `maximal being uniform subsequences)`
  output: `A shortest path between` $\mathbf{x}$ `and` $\mathbf{y}$
  If $x_k = 0$ **or** $y_k = 0$ **or** $|\mathbf{u}|, |\mathbf{v}| > 1$ **then**
    while $\mathrm{alt}(\mathbf{x}) > 0$ **do**
      go to $\mathbf{x} := \mathrm{swap}(\mathbf{x})$
    $\mathbf{r} := \mathbf{0}$
    while $\mathbf{r} \neq \mathbf{y}$ **do**
      go to $\mathbf{r} := \mathrm{swap}(\mathbf{r} \to \mathbf{y})$
  Else
  $\mathbf{x}' := \mathbf{x} \setminus x_k$ `,` $\mathbf{y}' := \mathbf{x} \setminus y_k$ `,`
    while $\mathrm{alt}(\mathbf{x}') > 0$ **do**
      go to $\mathbf{x}' := \mathrm{swap}(\mathbf{x}')$
    $\mathbf{x} := \mathbf{x}' \cup x_k$
    $\mathbf{r} := \mathbf{0}y_k$
    while $\mathbf{r} \neq \mathbf{y}$ **do**
      go to $\mathbf{r} := \mathrm{swap}(\mathbf{r} \to \mathbf{y})$
end

Table 1: A shortest path routing algorithm.

5.1 Routing

Let us consider two vertices in $H^k_{\mathbf{n}}$, say $\mathbf{x} = x_1x_2 \dots x_k$ and $\mathbf{y} = y_1y_2 \dots y_k$. In Table 1, we show a possible version of the routing algorithm. As commented above, the key idea is to go through, either, the root $\mathbf{r} = \mathbf{0}$ of $H^k_{\mathbf{n}}$, or the roots $\mathbf{r}' = \mathbf{0}k - 1x_k$ and $\mathbf{r}'' = \mathbf{0}k - 1y_k$ of the respective subgraphs. To describe the algorithm it is useful to introduce some further notation. Given a sequence $\mathbf{x} = x_1x_2 \dots$, we denote by $\overline{\mathbf{x}} = \overline{x}_1\overline{x}_2 \dots$ the sequence obtained from $\mathbf{x}$ by changing each zero term x_i by an (arbitrary) nonzero term $\overline{x}_i \in \mathbb{Z}_{n_i}$ and vice versa. Of course, if $n_i = 2$, $\overline{x}_i$ is just the already defined (and determined) conjugate of x_i. Moreover, we say that $\mathbf{x}$ is *uniform* if all its terms are either zero or nonzero. If $\mathbf{x}$ has maximum uniform prefix $\mathbf{u}$, we denote by $\mathrm{swap}(\mathbf{x})$ the sequence obtained from $\mathbf{x}$ by changing $\mathbf{u}$ to $\overline{\mathbf{u}}$. Similarly, given two sequences of the form $\mathbf{x} = \mathbf{u}_1\mathbf{u}_2\mathbf{w}$, and $\mathbf{y} = \mathbf{v}_1\mathbf{v}_2\mathbf{w}$, where both $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2$ and $\mathbf{v}_2$ are maximal uniform subsequences, one of them being $\mathbf{0}$, and $|\mathbf{u}_2| = |\mathbf{v}_2|$, we denote by $\mathrm{swap}(\mathbf{x} \to \mathbf{y})$ the sequence $\mathbf{u}_1\mathbf{v}_2\mathbf{w}$. (For a better understanding, see the examples in the next subsection.)

5.2 Examples

Let us now see three examples of the outputs of our routing algorithm. Each example corresponds to one of the cases of Proposition 4.4. According to this result, in each case we have taken vertices $\mathbf{x}$ and $\mathbf{y}$ with no common suffix, so that $x_k \neq y_k$.

- If $\mathbf{x} = 01010$, and $\mathbf{y} = 10101$ in H_{2^5}, we have $\mathrm{alt}(\mathbf{x}) = 4$ and $\mathrm{alt}(\mathbf{y}) = 5$. Since $x_5 = 0$, by Proposition 4.4(i), $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = 4 + 5 = 9$. Indeed, the shortest path is:

  $01010 \to 11010 \to 00010 \to 11110 \to 00000 \to 11111 \to 00001 \to 11101 \to 00101 \to 10101$.

- If $\mathbf{x} = 0210312$, and $\mathbf{y} = 1221023$ in $H_{2,3,3,3,4,4,4}$, we have $\mathrm{alt}(\mathbf{x}) = 4$ and $\mathrm{alt}(\mathbf{y}) = 3$. Since $|\mathbf{u}|, |\mathbf{v}| > 1$, Proposition 4.4($ii$) tells us again that $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = 4 + 3 = 7$. In this case, one of the possible shortest paths is:

  $0210312 \to 1210312 \to 0000312 \to 1111112 \to 0000000 \to 1111123 \to 0000023 \to 1221023$.

- Let $\mathbf{x} = 102302$, and $\mathbf{y} = 101013$ in $H_{2,2,3,4,2,4}$. Since $x_6, y_6 \neq 0$ and $|\mathbf{u}| = 1$, we must apply Proposition 4.4(iii) with $\mathrm{alt}(\mathbf{x}') = 3$ and $\mathrm{alt}(\mathbf{y}') = 5$, which gives $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = 3 + 5 + 1 = 9$. In this case, one of the shortest paths is:

  $102302 \to 002302 \to 112302 \to 000002 \to 000003 \to 111113 \to 000013 \to 111013 \to 001013 \to 101013$.

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