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THESIS TITLE: Ejection-Collision Orbits in the Modified Hill's Problem

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Overview

Restricted three-body problem is a special version of n-body problem where an infinitesimal mass is attracted by the gravitation of two positive masses, called primaries, that follow a solution of the Kepler problem. When not specified, restricted three-body problem means the primaries follow a circular orbit with respect to their shared center of mass, that in a rotating reference frame can be seen as two fixed points. Numerical methods are used in these problems since analytical solutions do not exist. Hill's problem is a modification of the restricted three-body problem where the third body is close to the secondary primary. Within this project, we started reviewing the Hill's problem -equilibrium points, zero velocity curves, etc.- when the zero velocity curve is a closed region around the origin, since we are interested in ejection-collision orbits. After that we introduced a perturbation due to the solar radiation pressure. This gives us a more realistic model if we can apply to an asteroid. After that, we studied the collision manifold. To do that, first the equations of motion were regularized. We described the flow on the collision manifold. The equilibrium points play an important role together with their stable and unstable invariant manifolds. Finally, we studied the intersection of the previous invariant manifolds. This theoretical work aims to describe the orbits that have a close approach to secondary primary. The study of the ejection-collision orbits were the backbone of the close approaches. The characteristic changes of the desired orbits depending on various parameters are also examined.

Dedicated to my nephew Ekim Demir...

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CHAPTER 1. INTRODUCTION

1.1. Historical Background

The history of orbital mechanics starts with the cooperation of two men; Tycho Brahe and Johann Kepler. Their work lays the foundation of Newton's most significant discoveries nearly half a century after. Tycho Brahe was a Danish nobleman who had a great understanding of mechanics and an opportunity record the positions of planets accurately. Where he lacked in theory and mathematics, came in Johann Kepler. Kepler was unable to make reliable observations because of his health problems, but he had a keen talent for mathematics and enough patience to use it on Tycho's data. Since the time of Aristotle it was assumed that planetary motion was build on circles. Working to fit geometrical shapes to Tycho's data, Kepler struggled between 1601-1606. In 1609, Kepler found the orbit solution as an ellipse. Later he published his laws of planetary motion in that same year and later in 1619.[1]

Isaac Newton laid the foundations of some of his most important work during the long break he had to take a two year long break from University of Cambridge caused by the plague outbreak. Amongst these work were the fundamental concepts of differential calculus where he was explaining the motion of the Moon. The importance of those papers could only be discovered after two decades. He was to explain the core of the description made by Kepler. A question brought by a simple bet and the coincidence of Edmund Halley bringing the question to Newton surfaced Newton's work which explained and proved the planetary motion. With Halley's suggestion, after two years of preparation Newton published *The Mathematical Principles of Natural Philosophy, or the Principia*, in 1687. With *Principia*, Newton explained his laws of motion and the law of universal gravitation, which paved the road to modern orbital mechanics.[2]

In 1767, Euler proposed a special form for the general three-body problem.[3] He explained that three finite mass bodies aligned on a straight line with proper initial conditions and this straight line rotating on their center of mass, these three bodies could have periodic orbits following ellipses. In 1772, Lagrange proposed another form, in which the three bodies would rest on the edges of a triangle with equal sides and they could revolve on elliptic orbits while keeping their original configuration intact.[4] These two solutions are known as particular solutions for the three-body problem. The formulation of circular restricted three-body problem was lead by Euler in 1767.[3] Soon after, Lagrange followed this study and came up with his Lagrange points which are five equilibrium points where the gravity forces are in balance. In 1836, Jacobi developed an integral for the circular restricted three-body problem using the rotating coordinate system introduced by Euler.[5] This integral is named after him. Between 1877 and 1878, Hill developed zero velocity curves using Jacobi's integral.[6] [7] These curves determine the areas in which the bodies could travel in space. Hill also introduced Hill's problem where two bodies are remarkably smaller than the first primary and established another class of periodic solutions.

After 1850s, Poincaré worked on and improved the understanding of the three-body problem. His biggest contribution has been his three volume book *Les Méthodes Nouvelles de la Mécanique Céleste* which was published between 1892-1899.[8] His new qualitative methods solved differential equations to establish and analyze prospective orbits. His work

also described the unpredictable nature of the problem. This led to a new understanding called chaos.

Heinrich Bruns in 1887 and Henri Poincaré in mid-1890's proved that there is no general solution of the three-body problem using algebraic formulas and integrals. Karl Sundman found an infinite series that could be the ground work for such a solution in 1912.[9] Yet this solution was converging extremely slowly and it further proved that there was no sophisticated solutions to this particular problem.[10]

Poincaré generalized the definition of periodic orbits put by Hill. He managed to establish solutions for the restricted three-body problem for certain initial conditions. Bendixson further studied one degree of freedom dynamical systems and their periodic solutions.[11] His formulation of Poincaré-Bendixson theorem proves the existence of such systems. Poincaré's work made way for further studies from Darwin[12] [13], Moulton[14], Ström-gren, Pedersen[15], Levi-Civita[16] and Lyapunov[17]. Between 1912 and 1915, Birkhoff kept the work on generalizing and extending the ideas of Poincaré going.[18] [19] [20] He developed recurrent motion concept, the explanation of existence of infinite periodic orbits surrounding stable periodic orbits and a topological model for a specific restricted problem.

Singularities were a big problem when looking into solutions for the three-body problem. They would end the solutions suddenly and they had to be eliminated. The process of eliminating singularities is called regularization. The work on regularization was led by Painlevé[21] [22] and continued by Levi-Civita[23], Bisconcini[24], Sundman[25] [26] [9], Siegel and Moser[27] between 1896 and 1991.

Since closed-form solutions were not possible, infinite series solutions were the next focus of researchers. Many tried to find such solutions. Some being disproven by Poincaré, all of them failed to do so. Although Painlevé also did fail to find such a solution, he was sure that in principle such solutions were possible. Sundman proved Painlevé right by developing a power series solution which was mentioned earlier.

Hamiltonian systems conserve their total energy. These systems can be integrable or non-integrable. Even non-integrable systems can have periodic orbit solutions given certain initial questions. Researchers call these quasi-periodic solutions. The effects of perturbation in such systems were explained by Kolmogorov in 1954.[28] Two separate works by Moser in 1962[29] and Arnold in 1963[30], proved the claims of Kolmogorov. These three works form the KAM theorem.

Using the advances in computational simulations, Hénon and Szebehely identified many periodic orbits between 1965 and 1974.[31] [32] [33] Moore discovered 8-type periodic orbit in 1993.[34] More discoveries followed by Chenciner, Montgomery, Suvakov and Dmitrasinovic in 2000s.[35] [36] Xiaoming Li and Shijun Liao published more than eighteen hundred families of orbits in 2017.[37] [38] Li and Liao also published 234 new families for unequal-mass free-fall three-body problem in 2018.[39] The three-body problem is finding more and more applications in modern space science. These applications vary from the examination of the motion of the Moon to exoplanets.

1.2. State of the Art

1.2.1. General Three-Body Problem

The general three-body problem has been implemented in various astronomical settings. Some of these settings are determination of mission trajectories, stellar formations and galactic dynamics. When all three bodies in a three-body problem interacts with each other without limitations on their orbit eccentricities, it is considered a general three-body problem. An example of such problem is the triple stellar systems without hierarchy. They are most likely to have close approaches and these situations must be taken with great care. Some problems have large sources of numerical error and this makes it harder to obtain meaningful results. Reipurth and Mikkola [2012][40] worked on such systems where there exists a finite sized cloud around the three masses. Interaction with the cloud makes the bodies increase their masses. Another problematic system example has wide separation binary stars which have dynamical instabilities. The formation of close binaries is also examined under the general three-body problem in the works of Fabrycky and Tremaine [2007][41], Lidov [1962][42] and Kozai [1962][43]. Some stellar binary systems may host planets. The examination of such systems were made in Gonczi and Froeschle [1981][44], Rabl and Dvorak [1988][45], Dvorak et al. [1989][46], Holman and Wiegert [1999][47], Pilat-Lohinger and Dvorak [2002][48], Pilat-Lohinger et al. [2003][49], Dvorak et al. [2004][50], Musielak et al. [2005][51], Pilat-Lohinger and Dvorak [2008][52], Haghhighipour et al. [2010][53] and Cuntz [2014][54].

NASA's Kepler data has been used to give exoplanet status to exoplanet candidates. A statistical method has been used to identify false positive candidates. Such works are present in Lissauer et al. [2014][55] and Rowe et al. [2014][56]. Before this method, there was another method called transit timing variations (TTVs) to determine systems with numerous planets. This method was used in Lissauer et al. [2011][57], Carter et al. [2012][58], Holman et al. [2010][59] and Winn [2011][60]. A good example to be presented is the discovery of Neptune with the orbit variations of Uranus in the works of Adams [1846][61], Airy [1846][62], Challis [1846][63] and Galle [1846][64]. Coming to a more recent work, Nesvorný et al. [2012][65] detected a non-transiting planet with the observations of a transiting one. And again, Nesvorný [2013][66] discovered the behaviour of planet pair on their interactions with each other. The possibility of discovery and the required observation sensitivity related to perturbations were examined in Miralda-Escudé [2002][67], Holman and Murray [2005][68] and Agol et al. [2005][69]. Some examples of the application of general three-body problem in spaceflight missions are Apollo 8-10, Hubble Space Telescope, CHANDRA X-Ray Observatory, SPITZER Space Telescope, Kepler Space Telescope and the future James Webb Space Telescope.[70]

1.2.2. Restricted Three-Body Problem

The most practical simplification of the general three-body problem is the restricted three-body problem. Thus it has been implemented in many researches and missions. The examination of Earth-Moon-spacecraft system is based on an n-body problem with an approximation of restricted three-body problem. If the space craft is desired to travel further, the method used is decoupling an n-body problem into a number of restricted three-body

problems. A good example of this is the Jupiter-Ganymede-Europa-spacecraft system decoupled into two restricted three-body problems in Gomez et al. [2001][71]. Another case is Sun-Earth-Moon system. Since its inception by Newton [1687][72], this case influenced the theory of gravity. Newton's work approximately 8 percent error margin. Hill [1877,1878][6] [7] developed a general stability criterion. Gladman [1993] [73] improved this criterion. After that, many works included this criterion such as in Cuntz and Yeager [2009][74], Satyal et al. [2013][75] and Szenkovits and Mako [2008][76].

After Sun-Earth-Moon, Sun-Jupiter-asteroid system is the most studied restricted three-body problem. Since asteroids have a really small mass, this system fits the restricted three-body problem quite well. Nesvorny et al. [2002][77] studied the resonances of asteroids with Jupiter and likely three-body resonances in the Solar System.

Another possible system is a single star with giant and terrestrial exoplanets. Mayor and Queloz [1995][78] and Marcy and Butler [1995][79] discovered the first exoplanet. After that, our knowledge of exoplanet expanded to thousands. For terrestrial planets in the habitable zone of various planets, regions of stability of orbits were found in Noble et al. [2002][80]. Barnes and Greenberg [2006][81] established more exoplanets of solar planetary systems. Kasting et al. [1993][82] established habitable zones for different spectral stars. The possibility of habitable zones of stellar systems with giant exoplanets having terrestrial planets were examined in Jones et al. [2005][83]. Kopparapu and Barnes [2010][84] established 4 extra-solar planetary systems that have the possibility of hosting terrestrial planets. Goldreich and Schlichting [2014][85] detailed the resonances in planar circular restricted three-body problems. Other works on the detection of exoplanets are Howard et al. [2012][86], Fressin et al. [2013][87] and Santerne et al. [2013][88].

Another setup for the restricted three-body problem is a single star with a giant exoplanet and exomoon. Kasting et al. [1993][82] and Kopparapu and Barnes [2010][84] proposed more systems with Jupiter-like planets in the habitable zone of host stars. Giant exoplanets can make the exploration of terrestrial planets harder because of their perturbation, the smaller mass may have an unstable orbit. Another possibility is seen as exomoons orbiting giant exoplanets in a habitable zone. Williams et al. [1997][89] discussed the possibility of a Jupiter-like exoplanet with a Earth-sized moon. For further possibilities, there are works of Canup and Ward [2006][90], Kipping [2009][91], Kipping et al. [2012, 2013b.a][92] [93] [94] and Cuntz et al. [2013][95].

The last implementation to be mentioned here is binary stellar systems with a giant or terrestrial exoplanet. This is the most extreme study of the subject. Exoplanet can orbit one or both stars in the system. If it orbits only one star, this is called a satellite type (S-type) and if it orbits both stars, this is called a planetary type (P-type) exoplanet. The works focused on these are Dvorak [1982][96], Eggl et al. [2012][97], Kane and Hinkel [2013][98], Kaltenegger and Haghighipour [2013][99], Haghighipour and Kaltenegger [2013][100] and Cuntz [2014][54]. Doyle et al. [2011][101] and Pilat-Lohinger and Funk [2010][102] focused on P-type exoplanets. Topological methods were discussed in Musielak et al. [2005][51] and Cuntz et al. [2007][103]. Previous statistical methods were implemented in the works of Holman and Wiegert [1999][47] and Dvorak [1984][104]. Eberle and Cuntz [2010a][105] worked on the stability of planets implementing Hamilton's Hodograph in Hamilton [1847][106]. The stability of smaller planets were studied in Eberle and Cuntz [2010b][107], Quarles et al. [2012a][108] and Gozdziewski et al. [2013][109]. Quarles et al. [2012b][110] also studied the possibility of implementing an Earth-mass

exoplanet into the system to discover the possibility of habitability.

1.2.3. Relativistic Three-Body Problem

The assumptions made on the previous section were spherical bodies and masses considered as point like particles. These assumptions of Newton's gravity theory does not apply when it comes to Einstein's general relativity. Newton's gravity can be considered as a weak gravity field approximation of the GTR. GTR replaces the flat space-time with a 4D space-time that is curved under the effect of gravity. Brumberg [1972] [1991][111] [112] and Kopeikin et al. [2011][113] formulated relativistic celestial mechanics and applied its methods. GTR can be only solved for one-body problem. N-body problems require approximations. Some works focusing on that are Brumberg [1991][112], Kopeikin et al. [2011][113], Einstein et al. [1938][114] and Infeld [1957][115].

When there is no restrictions on masses, it is considered a general relativistic three-body problem. The post-Newtonian equations for this problem are developed in Brumberg [1972][111], Brumberg [1991][112], Kopeikin et al. [2011][113], Saha and Tremaine [1992][1994] [116] [117] and Kidder [1995][118]. Yamada and Asada [2010a][119] proved the existence of post-Newtonian collinear solution. Yamada and Asada [2010b][120] also proved the uniqueness of the relativistic collinear solution. Relativistic solution of the Lagrange solution can be found in Krefetz [1967][121], Ichita et al. [2011][122] and Yamada and Asada [2012][123]. Imai et al. [2007][124] and Lousto and Nakano [2008][125] have the relativistic 8-type periodic solution. Suvakov and Dmitrasinovic [2013][36] found the relativistic version of 13 new periodic orbits.

The restricted relativistic three-body problem was first obtained by Brumberg [1972][111]. Douskos and Perdios [2002][126] obtained the equations of motion. Contopoulos [1976] [127] and Bhatnagar and Hallan [1998][128] found the libration points. Linear stability of these points were studied by Ragos et al. [2000][129], Douskos and Perdios [2002][126], Bhatnagar and Hallan [1998][128] and Ahmed et al. [2006][130]. Brumberg [1991][112], Kopeikin et al. [2011][113], Mandl and Dvorak [1984][131], Wanex [2003][132], Yamada and Asada [2010a][119], Palit et al. [2009][133] and Migaszewski and Gozdziwski [2009][134] explored the possible applications.

1.2.4. Hill's Problem

The development of the Hill's problem starts with the work of Hill [1878][7] on lunar theory. Brown finished the theory Hill started over the years. The compilation of their work can be seen in the work of Wilson [2010][135]. A power series solution was later proposed for Hill's problem by Henrard [1978][136]. Henrard [1979][137] also proposed a better solution for the problem, this work was more accurate because it implemented the effects of mass ratio which usually were neglected before. Meletlidou et al. [2001][138] proved that independant from the Hamiltonian, Hill's problem does not have a second analytic integral of the motion. Llibre and Roberto [2011][139] examined the integrability of the regularized Hill's problem.

When it comes to methodological developments, Cabral and Castilho [2001][140] studied a critical solution for the system. Delshams et al. [2008][141] computed a scattering map

for the three-dimensional Hill's problem. A symplectic integrator based on a generalized leapfrog for efficient collision detection was developed in Quinn et al. [2009][142]. Hussein and Santos [2013][143] developed explicit high order composition methods. These methods were used to integrate Hill's problem based on a second order symmetric method. Kozlov and Polekhin [2016][144] used planar restricted three-body problem to establish the areas of a system that is always uncovered in a Hill's region. Bhakta et al. [2017][145] used numerical methods to produce target bands as best as possible.

As mentioned above, some works focused on three-dimensional orbits. Mikhalodimitrakis [1979][146], Zagouras and Markellos [1984][147], Gomez et al. [2001][71] and Gopalakrishnan and Muthusamy [2018][148] are examples of such studies.

An important aspect of applications lies in the study of collinear points of the system. The works that focus on collinear points are Chauvineau and Mignard [1989][149], Simo and Stuchi [1999][150], Arona and Masdemont [2007][151], Douskos [2010][152] and Gopalakrishnan and Muthusamy [2018][148].

One of the most popular aspects of Hill's problem is the possibility of working towards periodic orbits. As they are obtained on this project like others, periodic orbits are highly practical in mission design and celestial mechanics exploration. Some works that work on periodic orbits are already mentioned but the rest of them can be listed as Perko [1982a] [1982b] [1983][153] [154] [155], Hénon [1974][32] and Batkhin [2012a] [2012b][156] [157].

Different aspects and variations of the Hill's problem have also been studied throughout the years. Ichtiaroglu [1980][158] focused on elliptical Hill's problem to find periodic continuations. Villac and Scheeres [2004][159] studied the concept of periapsis on Hill's problem, examining close approaches of the system. Hénon [2005][160] focused on the asymmetric periodic orbits. As a result, this work found some orbits that are symmetric on y axis instead of x and some other that are not symmetric at all. Heggie [2000][161] and Villac and Scheeres [2003][162] focused on the dynamics of escape in Hill's problems and low-energy escape trajectories respectively.

Relating to this project more, the studies that focus on ejection-collision orbits must be mentioned. Lacombe and Llibre [1987][163] focused on transversal ejection-collision orbits. Soon after, Delgado-Fernandez [1988][164] improved the study of transversal ejection-collision orbits on a specific C range. Ollé et al. [2017][165] was the most recent study on ejection-collision orbits and has been used as a guide reference for this project.

The last subject of focus for Hill's problem to be mentioned is perturbations. Being a crucial part of this project as well, these works are worthy of a mention and study. Starting with the works that focus on radiation and moving on to oblateness, some examples will be presented. Markellos et al. [1999][166] implemented the effects of radiation from both primaries. The results were explored for the comparison of the maximum distance of stable orbits and the predicted maximum solar flares. Markakis et al. [2008][167] focused on the radiation effects on the Lyapunov orbits while considering oblateness as well. Perdiou et al. [2012][168] implemented the radiation perturbation from the first primary and the oblateness of the secondary primary. Yarnoz et al. [2014][169] focused on the specific families of a and g with solar radiation pressure. This work also examines the possibilities of applications to asteroids. Giancotti et al. [2014][170] studies the periodic orbits found under solar radiation pressure to implement them to the mission Hayabusa 2. Vashkov'yak [1998][171] and Markellos et al. [2000][172] focused on the stability of solutions concerning the effects of oblateness. Perdiou et al. [2006][173] is a work on the numerical examination

of the oblateness of the second primary.

Other kinds of perturbation explorations also exist. Chauvineau [1991][174] worked on the generalized Hill's problem with a twist. There exists an external field of force deriving from a central planet. This new potential is different because it is not restricted to Newton's inverse square law. Gentile et al. [2004][175] studies the stability of Hill's equations when they are quasi-periodically perturbed. The more detailed and varied perturbation study grants a better approximation of the problem to the reality.

1.3. N-Body Problem

In order to provide a solid understanding of this work, the background will be presented from the most general problem towards the more specific problem.

N being the number of bodies under inspection, n -body problem aims to inspect the motion of a certain body. The body is under multiple gravitational forces and perturbations, which will be mentioned in more detail later on. The main steps are to apply Newton's law of universal gravitation and to apply Newton's second law of motion. Other elements that affect the system like rocket thrust, atmospheric drag, solar radiation pressure or non-symmetrical/non-spherical shape of the body should be added to the general formula. [176]

Choosing a fitting coordinate system is a crucial job. Since none of the coordinate systems have a full inertial quality certainty.

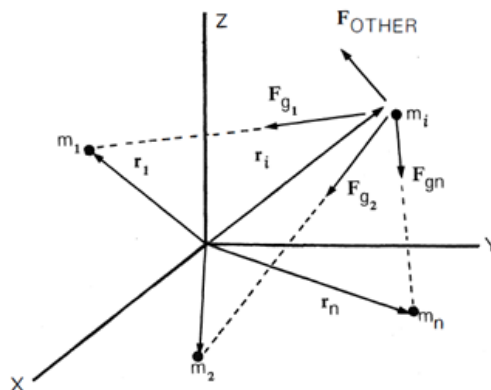


Figure 1.1: The n -body problem [176]

For every i th particle, there are position vectors of q_i and masses of m_i . According to Newton's second law, the sum of forces acting on a particle is the mass times the acceleration, $m_i \cdot \ddot{q}_i$. Newton's law of gravity has two set rules. The magnitude of forces acting on a particle is proportional to the product of the two masses. These forces are inversely proportional to the square of the distance between the two bodies. These forces are directed from one particle towards another along a unit vector. Compiling all of these rules presents the equations of motions. [177]

$$m_i \ddot{q}_i = \sum_{j=1, i \neq j}^N \frac{G m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3} = \frac{\partial U}{\partial q_i}, \quad (1.1)$$

where the self-potential (negative of the potential) is

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|q_i - q_j\|}. \quad (1.2)$$

The gravitational constant here is

$$G = 6.6732 \cdot 10^{-11} m^3 / sec^2 kg. \quad (1.3)$$

These equations represent a general approach and may not support exceptions. Some exceptions are common in space dynamics. A rocket using its fuel is a good example of a system that does not conserve its total mass. Further simplifications can be made with assumptions like a full spherical body or neglecting smaller forces acting on the system namely perturbations.

Having the general expression for n-body problem, it is possible to reach the two-body problem with certain steps. To repeat what was just explained, the assumptions are considering the bodies symmetrically spherical and neglecting internal/external forces other than gravitation. Although in reality a fully inertial reference frame cannot be found, the calculations are first handled with the assumption that such a reference frame is at hand.

(X',Y',Z') is the inertial set of coordinates. (X,Y,Z) is parallel to the former coordinate system, non-rotating and its origin is at the center of M.

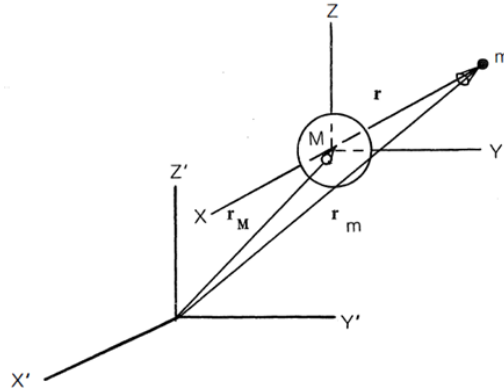


Figure 1.2: Relative motion of two bodies [176]

The directions and magnitudes of r and \dot{r} measured in (X',Y',Z') is equal to their measurements in (X,Y,Z) because of the relation of (X,Y,Z) and (X',Y',Z') . Since the general expression is obtained, (X',Y',Z') coordinate system can be discarded and the measurements can be made in (X,Y,Z) coordinate system alone. A further simplification can be made by considering the second primary greatly smaller than the first primary and using the gravitational parameter.[176]

$$G(M + m) \approx GM, \quad (1.4)$$

$$\mu \equiv GM. \quad (1.5)$$

1.4. Three-Body Problem

Some of the greatest minds of 18th and 19th century focused on the classical Newtonian three-body problem. The general problem remains unsolved today, but the advances in computational science and mathematical methods brought more insight on the subject. Newton was the one with the earliest approach to the three-body problem when he examined the motion of Earth and the Moon while also considering the effects of the Sun over them. Euler brought the first and the simplest exact periodic solution to the problem called the restricted three-body problem in 1767. Shortly after, Lagrange developed the equilateral triangle solution in 1772. For arbitrary masses, these two solutions are the only explicit solutions but for other special cases many solutions do exist. Jacobi, Delaunay and Hill created the foundation of the modern theory for the restricted three-body problem. Poincaré ends the classical period with his exceptional methods and his work was followed by many to this day. [178]

The general three-body problem is presented first. As seen in the two-body problem, the system is examined while the center of mass is the origin. Since from the beginning, the external forces and torques are neglected. This leads to the assumption that the energy and the angular momentum are conserved.

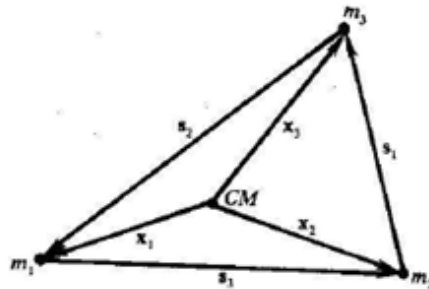


Figure 1.3: General three-body problem [178]

For the first exact solution, Euler's solution is examined. The three bodies follow ellipses with the same eccentricity while always being aligned on a straight line. They orbit around the common center of mass and have the same period. Replacing the mid body position amongst the three, three orbit families are found. These orbit families are merely theoretical, because the smallest perturbations make them unstable.

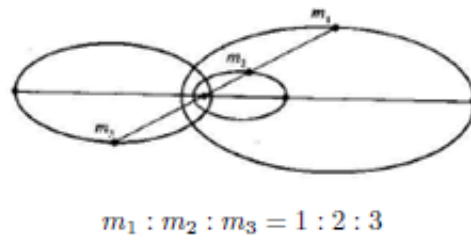


Figure 1.4: Euler's solution [178]

The second exact solution came from Lagrange. This solution was viable when $G = 0$, which meant the relative position vector equations were equal. This meant the bodies would align so that each of them would be on the edges of an equilateral triangle. Triangle could rotate and change size but always keeps the relative positive vector relations. The focal point of three orbits would be at the common center of mass with the orbits sharing the same period as in Euler's solution. This solution would be viable when one of the bodies were much bigger than the other two. Variations of this solutions were made when the bodies shared the same mass or more bodies were involved by Montgomery in 2001. Yet, these solutions were not practically crucial since they demanded very strict initial conditions.

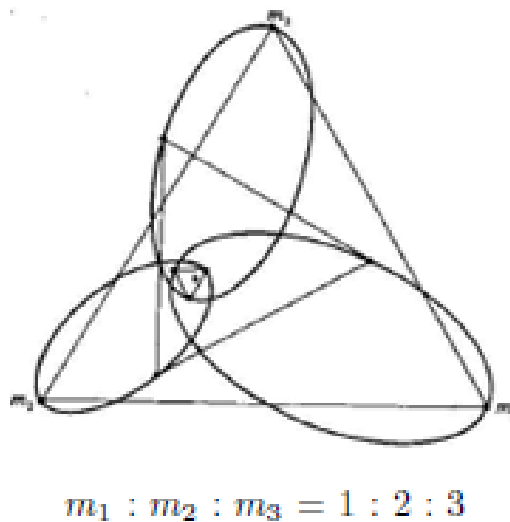


Figure 1.5: Lagrange's solution [178]

Burrau developed a solution for very specific initial configuration. This configuration had three bodies of masses 3, 4 and 5 resting on the edges of a Pythagorean triangle, facing the proportionally same sides of the triangle. The bodies start the configuration motionless and start moving due to their interacting gravitation. In the end, while the bigger two primaries bind in a binary, the third body escapes. This behaviour turned out to be very common given various initial conditions were tested on such systems with similar mass ratio.

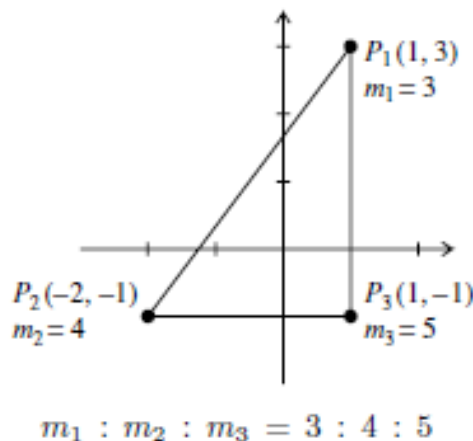


Figure 1.6: Burrau's solution [178]

The general three-body problem has been implemented in many applications including spacecraft missions of different variety. In an astronomical setting, general three-body problem yields a wide array of applications. Three-body problem can be considered general only when all the bodies of the system can interact without orbit eccentricity limitations. An example can be triple stellar systems. These system include close approaches and their calculations must be done precisely. Reaching to meaningful results require a great care of limiting numerical errors. The investigation of the configuration of binary systems are made with the implementation of the three-body problem. The possibility of binary systems hosting planet have been brought to light with these investigations. The classification of exoplanets often include the interaction of three bodies therefore the requirement of three-body problem solutions. The deviation of the orbits comparing with the observed data, creates the opportunity to detect and identify exoplanets. The same method was used when discovering Neptune with the examination of the orbit variations of Uranus. This method is also suggested for the discovery of exosolar moons. Outside of astronomical setting and exoplanets, three-body problem is also implemented in spacecraft trajectory solutions. For small satellites, a special case of the three-body problem is implemented, namely the restricted three-body problem. This is due to the mass of the satellites being greatly smaller than the two primaries' that its under the effect of. Bigger payloads require different variations of the three-body problem. Some of the well known examples of these are NASA's Apollo 8-10 missions, Hubble Space Telescope and James Webb Space Telescopes. [70]

Assumptions and simplifications are made for different versions of the three-body problem. The most practical version is the restricted three-body problem where the third body is of a negligible mass. Some of the example systems that could benefit from the restricted three-body problem are binary stellar systems with a giant or terrestrial exoplanet, a single star with a giant exoplanet and exomoon, a single star with giant and terrestrial exoplanets, Sun-Jupiter system with an asteroid, Sun-Earth-Moon system and Earth-Moon system with a spacecraft.

1.5. Hill's Problem

The symmetric solutions for the Hill's problem can be implemented when looking for solutions of the restricted three-body problem. Around the smaller primary of a system, spacecraft orbits and trajectories can be examined using these solutions.

Hill's problem is a special case of the three-body problem where a zero mass body is examined for its motion around a second primary body which is smaller than a first primary. As in the restricted three-body problem, in default the problem is considered to have a circular orbit. Thus, the two primaries rotate around their common center of mass in a circular orbit. The problem can have their primaries orbit in an ellipse as well. If the zero mass body moves on the plane of the primaries, this problem is called a planar problem.

The origin of the Hill's problem rests with the suggestion of George William Hill on the Sun-Earth-Moon system where he was examining the motion of the Moon. Hill followed Euler's ideas to use trigonometric series to formulate approximate periodic solutions. Many researches studied and contributed to Hill's problem. Although theoretically rich, Hill's problem has practical value in terms of its applications as well. Studying the evolution of natural satellites' orbits, spacecraft mission design and the examination of star cluster dynamics can be listed as some of these applications. The equations of motion go through transformations in order to establish the structure of periodic solutions. Considering their changes under these transformations, periodic solutions can be divided into three categories: asymmetric, singly symmetric and doubly symmetric. Depending on their Hamiltonian, periodic solutions are established. These solutions are not isolated, but rather they form families with equal parameters. Depending on their continuation over Jacobi constant, these families can be categorized as closed, half-open and open families. [156]

The periodic solutions of Hill's problem can be further studied to form periodic solutions for both the general three-body problem and the restricted three-body problem.

Possible regions of space that could be accessed in motion is dictated by the first integral of the Hill's problem. This region is called Hill region. This expression equals zero when the velocity is zero. This specific form of this constraint defines zero velocity curves. [179]

As an example of Hill's problem where Earth and Sun are considered the two big primaries, the assumptions made are taking the solar parallax, the solar eccentricity and the lunar inclination as zero. The importance of Hill's problem comes in its approach to the problem. He presented a new definition for the first approximation of the motion of the moon. These equations had more terms and they converged much faster than the series obtained prior to his work. Prior to this work, the usual steps followed solving the two-body problem and then modifying it. With Hill's problem, the modified problem is solved and then its variations are studied. Even today, new methods are development on the understanding of the Hill's problem.[33]

1.6. Perturbations

Observing the planetary motion, it is commonly observed that the theoretically expected motion does not match the observations fully. The deviation seen on these motions are called perturbations. Although these deviations seem odd at first, after examinations it

has been established that these can be caused by many natural phenomena varying from the gravitational effect of other bodies to additional forces out of Keplerian motion like non-sphericity. Some other examples to known perturbations are solar radiation effects, atmospheric drag-lift, magnetic effects and relativistic effects. Although their effects are crucial, the perturbations that concern orbital flight are predictable and easily calculated.

Perturbations can wield effects as big as the main gravitational forces. Interplanetary missions' accuracy depend on the understanding of the perturbations on effect. Even Newton's work on the motion of the Moon was not accurate because of the perturbations he did not put into account. Adams and Leverrier came up with the presence of Neptune with their analysis on Uranus' perturbed motion. Clairant successfully predicted the return of Halley's Comet with the calculations of perturbations caused by Jupiter and Saturn in 1759. The examination of the eccentricity of Earth's orbit on its perturbation terms led to the establishment of Earth's shape.

Perturbation techniques can be categorized as special perturbations and general perturbations. Equations of motions with all perturbing accelerations are approached with direct numerical integration in special perturbations. This technique is less complex compared to general perturbations. General perturbations are more complex to handle and they implement analytic integration of series expansions of perturbing accelerations. As a result, this technique requires more work but also yield a much better understanding of the problem, its origin and behaviour. The discoveries previously mentioned are a result of the application of general perturbations. [176]

1.7. Solar Radiation Pressure

The radiation pressure was first discovered by Kepler in 1619. He suggested that the reason why the tail of a comet would always point away from the Sun was the radiation pressure pushing it away. When electromagnetic waves fall on a material, they interact with the charges on it. Whether or not the wave was reflected or absorbed, it applies a force on these charges and therefore on the material itself as well. This force can be calculated using Electromagnetic Theory.

Solar radiation pressure is not much on Earth. Although it's not as high as a large bright star could produce, solar radiation pressure has significant effects over long periods of time. An easy example of this is the Viking spacecraft. If the effects of solar radiation pressure were not implemented on the mission calculations, it would miss the orbit of Mars by 15000 km. The solar radiation pressure can be used to manufacture solar sails which can drive probes to immense distances with low costs. [180]

When considering bodies with large area-to-mass ratios, the effects of solar radiation pressure become higher. The orbital flight time and the trajectory itself can be highly impacted by solar radiation pressure since it alters the energy of the system. [181]

When the first works began examining the effects of solar radiation pressure on satellites, it was thought that it was negligible compared to the oblateness of the Earth or lunar-solar gravitation's effect. The mismatch of theory and practice created the need to work on this subject. Further studies proved the significance of the solar radiation pressure. [182]

CHAPTER 2. ANALYTICAL WORK

In this chapter, the methodology used to reach the end goal is explained in detail. The ideas and techniques used in the methodology is supported with the analytical steps taken in this project. Various works on the field has been used as guidelines with the major contribution coming from the lead of the supervisor. Although certain elements of the project are present in this chapter, the results obtained through them are presented in the next chapter.

2.1. Equations of Motion

This project considers the movement of three bodies on a flat surface. The equations for the restricted-three body problem are written as

$$\ddot{x} - 2\dot{y} = \Omega_x, \quad (2.1)$$

$$\ddot{y} + 2\dot{x} = \Omega_y, \quad (2.2)$$

where

$$\Omega = \Omega(x, y) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}(x^2 + y^2) + \frac{1}{2}\mu(1-\mu), \quad (2.3)$$

$$r_1^2 = (x - \mu)^2 + y^2, \quad (2.4)$$

$$r_2^2 = (x + 1 - \mu)^2 + y^2, \quad (2.5)$$

are written, in synodic coordinates, as shown above. More details can be found in Szebehely's work on the theory of orbits.[33]

Afterwards a transfer of origin and scale is made in order to observe the movement near the small body.

$$(x, y) \longrightarrow ((x + 1 - \mu)^{-1/3}, y\mu^{-1/3}). \quad (2.6)$$

The new coordinates are then

$$\ddot{x} - 2\dot{y} = 3x - x(x^2 + y^2)^{-3/2} + \mu^{1/3}(6x^2 + 3y^2) + O(\mu^{2/3}), \quad (2.7)$$

$$\ddot{y} + 2\dot{x} = -y(x^2 + y^2)^{-3/2} - \mu^{1/3}(3xy) + O(\mu^{2/3}). \quad (2.8)$$

After $\mu = 0$ is applied to the equations, the equations of motion for Hill's problem is obtained.

$$\ddot{x} - 2\dot{y} = 3x - x(x^2 + y^2)^{-3/2}, \quad (2.9)$$

$$\ddot{y} - 2\dot{x} = -y(x^2 + y^2)^{-3/2}. \quad (2.10)$$

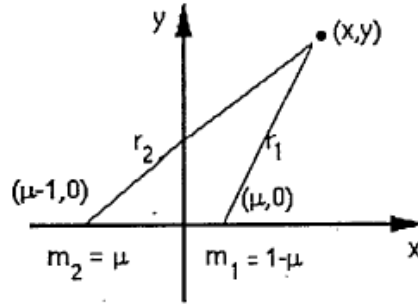


Figure 2.1: Restricted three-body problem in sydonic coordinates [183]

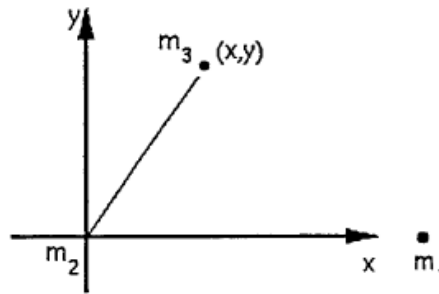


Figure 2.2: Hill's problem in sydonic coordinates [183]

The first integral of the Hill's problem is called Jacobi.

$$C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2), \quad (2.11)$$

where

$$\Omega(x, y) = \frac{1}{2}(3x^2 + 2(x^2 + y^2)^{-1/2}). \quad (2.12)$$

It can be observed that the restricted problem around the small body looks like a disturbance of the Hill's problem. Also it should be noted that the Hill's equations have a singularity at $(x, y) = (0, 0)$. This corresponds to a binary collision.

2.2. Implementation of the Solar Radiation Pressure

At this stage, the expressions obtained before need to be altered so that the effect of solar radiation pressure is also counted in the calculations. Certain assumptions are made before acquiring the final equations of motion. The gravitational potential of the second primary is considered a point, neglecting the effects of the shape factor. The second primary orbits the first primary in a circular orbit. A rotating reference frame is used. This

reference frame is fixed on the second primary as its origin and the first primary will reside in the $-x$ direction. The equations of motion are calculated through Szebehely's work and the mentioned assumptions are the following ones. [170]

$$\ddot{x} - 2\dot{y} = -\frac{x}{r^3} + 3x + \beta, \quad (2.13)$$

$$\ddot{y} + 2\dot{x} = -\frac{y}{r^3}, \quad (2.14)$$

$$\ddot{z} = -\frac{z}{r^3} - z. \quad (2.15)$$

The relation of r to x and y is like the expression shown below.

$$r = \sqrt{x^2 + y^2}. \quad (2.16)$$

The remaining unknown variable β represents the effect of the solar radiation pressure. Since in this project the first primary is the Sun, β always points away from the first primary. Its calculation is made through the expression below.

$$\beta = \frac{(1 + \rho)P_0}{B\kappa_s^{2/3}\kappa^{1/3}}. \quad (2.17)$$

The terms used in this expression are;

Reflectance of the spacecraft's surface: ρ

Ratio of spacecraft's mass and surface: $B = M/A$

Solar constant: $P_0 = 10^8 \text{ kg km}^3 \text{ s}^{-2} \text{ m}^{-2}$

Gravitational parameter of the Sun: κ_s

The only element here without an exact value is κ . It depends on the mass of the second primary, which in this case is an asteroid. Since an asteroid's volume varies and its density is an uncertainty, only approximate values can be used. In the work of Giancotti (2014), a range of values has been calculated as $27 < \beta < 55$. The Jacobi constant is then expressed as

$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{r} - \frac{3}{2}x^2 + \frac{1}{2}z^2 - \beta x. \quad (2.18)$$

From this point forward, the examination of the problem is made on x - y plane alone. Therefore, the elements of z plane are neglected.

2.3. The Proof of Two Equilibrium Points

As mentioned before, the equilibrium points are obtained through the modification of the equations of motion. For the sake of a complete explanation here, the mentioned expressions are shown below.

$$\ddot{x} - 2\dot{y} = -\frac{x}{r^3} + 3x + \beta, \quad (2.19)$$

$$\dot{y} + 2\dot{x} = -\frac{y}{r^3}. \quad (2.20)$$

Fixing the first and second derivatives to zero is the first step. This is related to the zero velocity and acceleration conditions on the equilibrium points. From the second expression of the set, it can be seen that this condition applies when $y = 0$. This new condition is then applied to the first expression on the set. This creates a two conditioned expression set that will define the equilibrium points. [183]

$$x > 0 \longrightarrow 3x^3 + \beta x^2 - 1 = 0, \quad (2.21)$$

$$x < 0 \longrightarrow 3x^3 + \beta x^2 + 1 = 0. \quad (2.22)$$

In order to find the extrema points, the derivatives of these expressions are examined. In this case they are the same.

$$9x^2 + 2\beta x = 0, \quad (2.23)$$

$$x = 0, x = -2\beta/9. \quad (2.24)$$

According to Rolle's theorem, on either side of these extremum points are two points of the function that have the same value. On this case it means that either side of the extremum points can have one or zero roots of the function. These roots are the equilibrium points of interest. Each of these sections of the functions cannot have more than one root, because that would mean the existence of more extremum points where there is none.

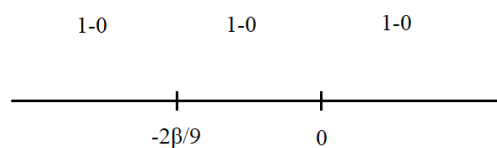


Figure 2.3: Extrema points

The rest of the examination is under the consideration that beta values are always positive, since the goal of this project is to observe the results under beta values between 27-55.

According to Bolzano's theorem, if a function is continuous between a and b, between those values it takes every value between $f(a)$ and $f(b)$. Specifically, this means if the function has values between positive and negative side of the spectrum, it has the zero value at some point.

Now the three sections of this case will be examined with this knowledge. Both in the first (left) and third (right) section, the function gets negative and positive values. This means that function crosses $y = 0$ in these sections, giving two roots – equilibrium points. The second (middle) section has positives values only for the beta values at hand. The lack of

another extremum point in this section proves that the function cannot cross $y = 0$ under this circumstances, solidifying the fact that there is not an equilibrium points here. All of this examination concludes that this case has a total of two equilibrium points. These points are simply found using the two expressions acquired for $x < 0$ and $x > 0$ in the beginning.

2.4. Zero-Velocity Curves

The importance and the function of zero-velocity curves were mentioned briefly on the introduction chapter. This work has two main variables that defines the system. These are energy and solar radiation pressure. For every couple of these values, zero-velocity curves exist. These curves define the allowed regions of travel for a certain body. These regions are separate or connected. Their connections happen around Lagrange points, where the forces acting on a body are balanced.

These Lagrange points are also called libration points. They can be obtained through the expressions of motion. The first step is to make reduction due to the assumptions. These assumptions are that at these points the object has zero velocity and acceleration. These assumption is applied through the reduction of the terms with $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$. The remaining expressions can be used to calculate the libration points. The connections around these points on zero-velocity curves allow objects to move from one region to another region. These transfers are carried out by invariant manifolds of Lyapunov orbits.

The zero-velocity curves are plotted using the modified energy expressions. The same reductions are made as in the libration points. The remaining expression is dependent on x and y when the desired h for energy and β for solar radiation pressure are input.

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{r} - \frac{3}{2}x^2 - \beta x, \quad (2.25)$$

$$H = -\frac{1}{\sqrt{x^2 + y^2}} - \frac{3}{2}x^2 - \beta x. \quad (2.26)$$

The process made on analytical work is used to plot a specific zero-velocity curve. Another approach is fixing the β and plotting the zero-velocity curves of various h values along this solar radiation pressure. [A.3](#).

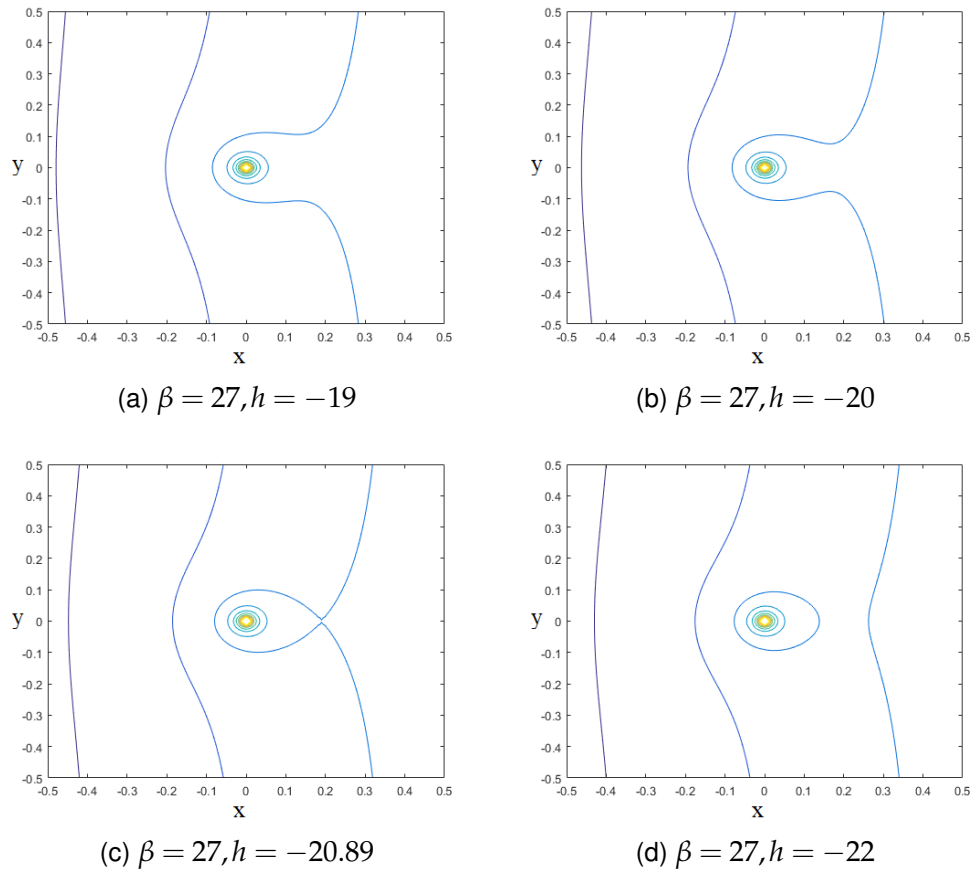


Figure 2.4: The separation of the closed region in zero-velocity curves

These four plots show the evolution of the zero velocity curves for a specific solar radiation pressure. As the energy level decreases, the area around the origin separates itself from the right side and this separate area shows the closed Hill's regions. In this closed regions, it mathematically impossible for an object to leave without a change in the energy. Simulating in these regions is the goal of this project. It serves a great advantage where the possibility of escape for an orbit is mathematically impossible.

2.5. Regularization of the Singularity

At this point the regularization of the problem is needed. This mathematical technique is used to get rid of the r terms on the denominator. Since this project aims to reach periodic orbits through work building on top of ejection-collision orbits, the term “ r ”, namely the distance from the origin, is crucial. The orbits of interest pass through really small r values, even from $r = 0$ in theory. This creates a singularity on the Hamiltonian expression and cripples the solution.

The first change starts with the expressions,

$$q_1 = x, \tag{2.27}$$

$$q_2 = y, \quad (2.28)$$

$$p_1 = \dot{x} - y, \quad (2.29)$$

$$p_2 = \dot{y} + x. \quad (2.30)$$

For the next step, the derivatives of the expressions above are calculated such as,

$$\dot{q}_1 = \dot{x} = p_1 + y = p_1 + q_2 = \frac{\partial H}{\partial p_1}, \quad (2.31)$$

$$\dot{q}_2 = \dot{y} = p_2 - x = p_2 - q_1 = \frac{\partial H}{\partial p_2}, \quad (2.32)$$

$$\dot{p}_1 = \ddot{x} - \dot{y} = \dot{y} - \frac{x}{r^3} + 3x + \beta = p_2 - q_1 - \frac{q_1}{r^3} + 3q_1 + \beta = p_2 + 2q_1 - \frac{q_1}{r^3} + \beta = -\frac{\partial H}{\partial q_1}, \quad (2.33)$$

$$\dot{p}_2 = \ddot{y} + \dot{x} = -\dot{x} - \frac{y}{r^3} = -p_1 - q_2 - \frac{q_2}{r^3} = -\frac{\partial H}{\partial q_2}. \quad (2.34)$$

Using these expressions the Jacobi is calculated as

$$C(q_1, q_2, p_1, p_2), \quad (2.35)$$

$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{r} - \frac{3}{2}x^2 - \beta x, \quad (2.36)$$

$$C = \frac{1}{2}((p_1 + q_2)^2 + (p_2 - q_1)^2) - \frac{1}{r} - \frac{3}{2}q_1^2 - \beta q_1, \quad (2.37)$$

$$\gamma = p_1^2 + q_2^2 + 2p_1q_2 + p_2^2 + q_1^2 - 2p_2q_1, \quad (2.38)$$

$$C = \frac{1}{2}(p_1^2 + p_2^2) + p_1q_2 - p_2q_1 + \frac{q_2^2}{2} - q_1^2 - \frac{1}{r} - \beta q_1, \quad (2.39)$$

$$H = C, \quad (2.40)$$

$$H = \frac{1}{2}(p_1^2 + p_2^2) + p_1q_2 - p_2q_1 + \frac{q_2^2}{2} - q_1^2 - \frac{1}{r} - \beta q_1. \quad (2.41)$$

This expression goes through a simple notation change to fit the ones in Ollé's work in order to make the change to polar coordinates. [184]

$$(q_1, q_2, p_1, p_2) \longrightarrow (x, y, P_x, P_y), \quad (2.42)$$

$$H = \frac{1}{2}(P_x^2 + P_y^2) + yP_x - xP_y - \frac{1}{r} + \frac{y^2}{2} - x^2 - \beta x. \quad (2.43)$$

As mentioned above, the canonical change of polar coordinates are shown below.

$$x = r \cos(\theta), \quad (2.44)$$

$$y = r \sin(\theta), \quad (2.45)$$

$$P_x = P_r \cos(\theta) - \frac{P_\theta}{r} \sin(\theta), \quad (2.46)$$

$$P_y = P_r \sin(\theta) + \frac{P_\theta}{r} \cos(\theta). \quad (2.47)$$

As a result of these expressions, the Hamiltonian becomes

$$\begin{aligned} (P_x^2 + P_y^2) &\rightarrow P_r^2 \cos^2(\theta) + \frac{P_\theta^2}{r^2} \sin^2(\theta) - 2 \frac{P_r P_\theta}{\theta} \sin(\theta) \cos(\theta) + \\ P_r^2 \sin^2(\theta) + \frac{P_\theta^2}{r^2} \cos^2(\theta) + 2 \frac{P_r P_\theta}{\theta} \sin(\theta) \cos(\theta) &= P_r^2 (\cos^2(\theta) + \sin^2(\theta)) + \\ \frac{P_\theta^2}{r^2} (\cos^2(\theta) + \sin^2(\theta)) &= (\cos^2(\theta) + \sin^2(\theta)) (P_r^2 + \frac{P_\theta^2}{r^2}) = P_r^2 + \frac{P_\theta^2}{r^2}, \end{aligned} \quad (2.48)$$

$$\begin{aligned} H(r, \theta, P_r, P_\theta) &= \frac{1}{2} (P_r^2 + \frac{P_\theta^2}{r^2}) + r \sin(\theta) (P_r \cos(\theta) - \frac{P_\theta}{r} \sin(\theta)) - \\ r \cos(\theta) (P_r \sin(\theta) + \frac{P_\theta}{r} \cos(\theta)) - \frac{1}{r} + \frac{r^2 \sin^2(\theta)}{2} - r^2 \cos^2(\theta) - \beta r \cos(\theta) &= \\ \frac{1}{2} (P_r^2 + \frac{P_\theta^2}{r^2}) - P_\theta \sin(\theta)^2 - P_\theta \cos(\theta)^2 - \frac{1}{r} + \frac{r^2 \sin^2(\theta)}{2} - r^2 \cos^2(\theta) - \beta r \cos(\theta), \end{aligned} \quad (2.49)$$

$$H(r, \theta, P_r, P_\theta) = \frac{1}{2} (P_r^2 + \frac{P_\theta^2}{r^2}) - P_\theta - \frac{1}{r} + \frac{r^2 \sin^2(\theta)}{2} - r^2 \cos^2(\theta) - \beta r \cos(\theta). \quad (2.50)$$

The Hamiltonian ODE established through this expression are

$$\dot{r} = \frac{\partial H}{\partial P_r} = P_r, \quad (2.51)$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{r^2} - 1, \quad (2.52)$$

$$\dot{P}_r = -\frac{\partial H}{\partial r} = \frac{P_\theta^2}{r^3} - \frac{1}{r^2} - r \sin(\theta)^2 + 2r \cos(\theta)^2 + \beta \cos(\theta), \quad (2.53)$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -3r^2 \sin(\theta) \cos(\theta) - \beta r \sin(\theta). \quad (2.54)$$

Using the work of McGehee[184], new variables are presented as

$$v = \dot{r}r^{1/2}, \quad (2.55)$$

$$u = r^{3/2}\dot{\theta}. \quad (2.56)$$

and also expressions for the change of time

$$\frac{dt}{d\tau} = r^{3/2}, \quad (2.57)$$

$$' = \frac{d}{d\tau}. \quad (2.58)$$

Making these changes, the set of ODE becomes

$$r' = \frac{dr}{dt} \frac{d}{d\tau} = \dot{r}r^{3/2} = vr^{-1/2}r^{3/2} = vr, \quad (2.59)$$

$$\theta' = \frac{d\theta}{dt} \frac{d}{d\tau} = \dot{\theta}r^{3/2} = ur^{-3/2}r^{3/2} = u, \quad (2.60)$$

$$\begin{aligned} v' &= \frac{dv}{dt} \frac{d}{d\tau} = \dot{v}r^{3/2} = (\ddot{r}r^{1/2} + \frac{1}{2}r^{-1/2}\dot{r}^2)r^{3/2} = \\ \ddot{r}r^2 + \frac{1}{2}\dot{r}^2r &= \frac{v^2}{2} + u^2 - 1 + 2ur^{3/2} + 3r^3 \cos(\theta)^2 + \beta r^2 \cos(\theta), \end{aligned} \quad (2.61)$$

$$\begin{aligned} u' &= \frac{du}{dt} \frac{d}{d\tau} = (\frac{3}{2}r^1/2\dot{\theta} + r^{3/2}\ddot{\theta})r^{3/2} = \\ \frac{3}{2}r^2\dot{\theta} + r^3\ddot{\theta} &= -\frac{1}{2}vu - 2vr^{3/2} - 3r^3 \sin(\theta) \cos(\theta) - \beta r^2 \sin(\theta). \end{aligned} \quad (2.62)$$

Following this set of ODE, the Hamiltonian becomes

$$P_r^2 = \dot{r}^2 = (vr^{-1/2})^2 = v^2r^{-1}, \quad (2.63)$$

$$P_\theta = (\dot{\theta} + 1)r^2 = (ur^{-3/2} + 1)r^2 = ur^{1/2} + r^2, \quad (2.64)$$

$$P_\theta^2 = u^2r + r^4 + 2ur^{5/2}, \quad (2.65)$$

$$H = \frac{1}{2}(v^2r^{-1} + \frac{u^2r + r^4 + 2ur^{5/2}}{r^2}) - ur^{1/2} - r^2 - \frac{1}{r} + \frac{r^2 \sin(\theta)^2}{2} - r^2 \cos(\theta)^2 - \beta r \cos(\theta), \quad (2.66)$$

$$-r^3 + r^3 \sin(\theta)^2 - 2r^3 \cos(\theta)^2 = -3r^3 \cos(\theta)^2, \quad (2.67)$$

$$r2H = v^2 + u^2 - 2 - 3r^3 \cos(\theta)^2 - 2\beta r^2 \cos(\theta), \quad (2.68)$$

$$H' = 2H, \quad (2.69)$$

$$\beta' = 2\beta, \quad (2.70)$$

$$rH' = v^2 + u^2 - 2 - 3r^3 \cos(\theta)^2 - \beta' r^2 \cos(\theta). \quad (2.71)$$

2.6. Collision Manifold

The system's behavior is determined by its energy level. This energy level is H . For every level there is a invariant manifold. This is defined for $r = 0$ and named the collision manifold. Applying the $r = 0$ condition to Hamiltonian, the following expression is obtained.[184]

$$\Lambda = u^2 + v^2 = 2, \theta \in [0, 2\pi]. \quad (2.72)$$

This defines a torus.

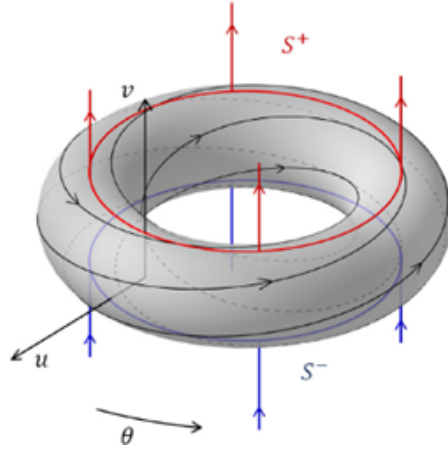


Figure 2.5: The collision manifold [184]

S^+ and S^- are the two circumferences of equilibrium points on the system. These are obtained by applying the conditions to the ODE set. Starting with $r = 0$, these circumferences are

$$S^+ = (0, \theta, \sqrt{2}, 0), \quad (2.73)$$

$$S^- = (0, \theta, -\sqrt{2}, 0). \quad (2.74)$$

The stability of the equilibrium points needs to be determined. This can be achieved by the linearization at the equilibrium points on these circumferences. Matrix M is calculated through

$$r' = R_1, \quad (2.75)$$

$$\theta' = R_2, \quad (2.76)$$

$$v' = R_3, \quad (2.77)$$

$$u' = R_4. \quad (2.78)$$

$$\begin{bmatrix} \frac{\partial R_1}{\partial r} & \frac{\partial R_1}{\partial \theta} & \frac{\partial R_1}{\partial v} & \frac{\partial R_1}{\partial u} \\ \frac{\partial R_2}{\partial r} & \frac{\partial R_2}{\partial \theta} & \frac{\partial R_2}{\partial v} & \frac{\partial R_2}{\partial u} \\ \frac{\partial R_3}{\partial r} & \frac{\partial R_3}{\partial \theta} & \frac{\partial R_3}{\partial v} & \frac{\partial R_3}{\partial u} \\ \frac{\partial R_4}{\partial r} & \frac{\partial R_4}{\partial \theta} & \frac{\partial R_4}{\partial v} & \frac{\partial R_4}{\partial u} \end{bmatrix}$$

Applying the circumference conditions to this matrix, the result for this project becomes

$$\begin{bmatrix} \pm\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & 0 & \mp\frac{\sqrt{2}}{2} \end{bmatrix}$$

Two versions of this matrix corresponds to S^+ and S^- respectively. Next, eigenvalues and eigenvectors of this matrix are calculated.

$$\lambda_1 = \sqrt{2}, \lambda_2 = 0, \lambda_3 = \sqrt{2}, \lambda_4 = -\frac{\sqrt{2}}{2}, \quad (2.79)$$

$$v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0), v_4 = (0, -\sqrt{2}, 0, 1). \quad (2.80)$$

$$\lambda_1 = -\sqrt{2}, \lambda_2 = 0, \lambda_3 = -\sqrt{2}, \lambda_4 = \frac{\sqrt{2}}{2}, \quad (2.81)$$

$$v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0), v_4 = (0, \sqrt{2}, 0, 1). \quad (2.82)$$

$$S^+ = (0, \theta_0, \sqrt{2}, 0), \quad (2.83)$$

$$S^- = (0, \theta_0, -\sqrt{2}, 0), \quad (2.84)$$

$$(0, \theta_0, \pm\sqrt{2}, 0) + a(1, 0, 0, 0) + b(0, 0, 1, 0) = (0, \theta_0, \pm\sqrt{2}, 0) + (a, 0, b, 0). \quad (2.85)$$

It is established that S^+ has a 2d unstable manifold and a 1d stable manifold with S^- having a 2d stable manifold and a 1d unstable one. Any orbit that ejects from and collides into origin must belong to 2d unstable manifold of S^+ and 2d stable manifold of S^- . Thus, every ejection-collision becomes a heteroclinic orbit that connects a point from one of these manifolds to another point from the other one.

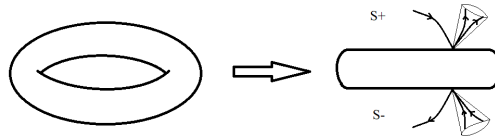


Figure 2.6: The collision manifold with stable-unstable manifolds

2.7. Ejection-Collision Orbits

In order to compute ejection orbits, first a tangent plane on an equilibrium point needs to be established. This is done through eigenvectors. The vectors that correspond to both eigenvalues that make 2d unstable S^+ manifold and 2d stable S^- manifold are $(1,0,0,0)$ and $(0,0,1,0)$. Having constants a and b , these vectors combine to create $(a,0,b,0)$. After having the tangent plane, a perpendicular vector is calculated. First, Hamiltonian is written in the form of [177]

$$0 = -rh + v^2 + u^2 + \dots \quad (2.86)$$

After, this expression is partially derived for four elements (r,θ,v,u) . The condition $r = 0$ is applied. The resulting values form the perpendicular vector for S^+ and S^- .

$$\frac{\partial H}{\partial r} = -h, \quad (2.87)$$

$$\frac{\partial H}{\partial \theta} = 0, \quad (2.88)$$

$$\frac{\partial H}{\partial v} = \pm 2\sqrt{2}, \quad (2.89)$$

$$\frac{\partial H}{\partial u} = 0, \quad (2.90)$$

$$S^+ \longrightarrow (-h, 0, 2\sqrt{2}, 0), \quad (2.91)$$

$$S^- \longrightarrow (-h, 0, -2\sqrt{2}, 0). \quad (2.92)$$

The perpendicular vector and tangent vector's scalar product gives 0 and create this relation.

$$(a, 0, b, 0) \cdot (-h, 0, 2\sqrt{2}, 0) = 0, \quad (2.93)$$

$$-ah + 2\sqrt{2}b = 0. \quad (2.94)$$

$$(a, 0, b, 0) \cdot (-h, 0, -2\sqrt{2}, 0) = 0, \quad (2.95)$$

$$-ah - 2\sqrt{2}b = 0. \quad (2.96)$$

This relation can be used to establish the tangent vector. When $a = 1$ condition is decided, b can be calculated easily. The general form is

$$(0, \theta_0, \pm\sqrt{2}, 0) + s \frac{w}{\|w\|}. \quad (2.97)$$

The specific form for this project is

$$(0, \theta_0, \pm\sqrt{2}, 0) + \alpha \frac{(1, 0, \pm \frac{h}{2\sqrt{2}}, 0)}{\|(1, 0, \pm \frac{h}{2\sqrt{2}}, 0)\|}, \quad (2.98)$$

where α is taken as a small as 10^{-6} . This last step is the bridge between analytical and numerical work. Established form provides the initial conditions used in the next chapter.

2.8. Symmetrical Periodic Orbits

This work aims to find and characterize periodic orbits that are symmetrical. This symmetry is established with the relation of $(t, x, y, \dot{x}, \dot{y}) \rightarrow (-t, x, -y, -\dot{x}, \dot{y})$. The representation of the nature of symmetric periodic orbits can be observed on the x-y coordinates. The symmetry is due to the x axis, this can be also defined as it is due to $y = 0$. This holds great value in this observation.

As a starting point, a point on the $y = 0$ with velocity vector perpendicular to the $y = 0$ is chosen around an intersection point. This means that y and \dot{x} of this point, at the start, is zero. This point is then integrated until it reaches $y = 0$ again. When integration hits $y = 0$, the set of values have changed to different x , \dot{x} and \dot{y} . The importance of observing \dot{x} at this point is crucial. If \dot{x} reaches zero at $y = 0$ again, this means that velocity only has a \dot{y} component, meaning it is perpendicular to $y = 0$ again. Since it is known that symmetry exists, this means the half orbit that was just integrated exists on the other side of $y = 0$, exactly mirrored. These two halves create a symmetrical periodic orbit.

The observation is made with a range of x values around an intersection point established. The full set of values are translated to $[r, \theta, v, u]$ first. Integration is made until it reaches $y = 0$, the crossing point is recorded after the set of values are translated to $[x, y, \dot{x}, \dot{y}]$ again. Starting from different x values and observing the \dot{x} at the crossing, it is possible to see positive and negative values. The sign change is important, because the $\dot{x} = 0$ belongs in between the one previous and the one after integrations when the sign change is observed.

Obtaining a symmetrical periodic orbit is the first goal of this exercise. The rest of the work is about finding orbit families around this periodic orbits. The identity of a periodic orbit consists of its solar radiation pressure, energy level, starting point and the number of passes on $y = 0$ required to obtain this orbit. Keeping everything else fixed but the energy and the starting point is the starting point of the next step. A slight change in energy should result in a slight change in the starting point. An algorithm can be written to follow these changes. The false position method is a powerful tool to implement here. For every step, h is changed to a degree first. The new starting point to be determined is known to be close to the original starting point. Establishing two points around the original starting point is the starting setup. As long as a root is definitely between these two points, it can be found through a number of steps. This is how another member of the same family is obtained.

Finding more and more orbits is how a family of orbits are established. These families are not infinitely available, they appear at certain energy levels and disappear on others.

CHAPTER 3. NUMERICAL RESULTS

This chapter focuses on the explanation of numerical methods used on implementing the analytical work to achieve meaningful results on the subject. These results are presented in a way to support the explanations and conclusions made. A higher number of result output can be found on the appendix.

3.1. Existence of ECO

The numerical work starts with the help of the initial condition values collected during analytical work. This form has the following variables; α , h and θ . α is decided as a small value between 10^{-7} and 10^{-5} . For this project, it is decided to be 10^{-6} . This form supports the general problem but it is known that this project's focus is on the modified problem. Considering solar radiation pressure, another variable $-\beta$ has been implemented as seen on the analytical work. Throughout the numerical work, different h and β values will be tested in order to test the behaviour of the system on various situations. It should be remembered that these two variables represent the energy of the system and the solar radiation pressure constant respectively. Their evaluation corresponds to various real world situations. Setting these two variables to a set of values access the solution of a specific situation. Also, this leaves only θ as a variable for a specific study. This means the orbits of ejection and collision are to be defined by θ .

Full set of initial conditions are obtained by varying θ between 0 and 2π . These conditions belong to the tangent plane to the corresponding manifolds. For S^+ , these set of values belong to the unstable manifold for ejection orbits. The initial conditions are integrated forward in time. The Hamiltonian is checked throughout the process to see if the total energy remains constant. This factor proves or disproves the solution system reliable. Similarly for the S^- , the initial conditions belong to the stable manifold for collision orbits. The integration is made backwards in time for this part. The process still needs to support the conservation of the total energy.

Following the establishment of reliable orbits, Poincaré sections will be used. Considering Poincaré sections is a highly beneficial tool. Examining the passes through the polar coordinate system plane for $v = 0$ provides the moments of maximum and minimum distance to the origin passes in the system. In order to find the exact moment of these passes, Newton's method can be used. These passes can be named as D_n^+ for the integration of S^+ and D_n^- for the integration of S^- . "n" is used to define the number of the pass from the Poincaré section. Both D_n defines a set of points on the Poincaré section, when the integration is made for a set value of initial conditions with varying θ between 0 and 2π . These set of points form enclosed shapes on the Poincaré section.

There are many techniques to determine the existence of ECO. The method used in this work is based on the intersection of D^+ and D^- . These intersection points can be visualized on the Poincaré section. These points show that there are connections between the ejection orbits of S^+ and the collision orbits of S^- . This is the first hint for the existence of ejection-collision orbits. [A.1.1.](#) [A.1.2.](#)

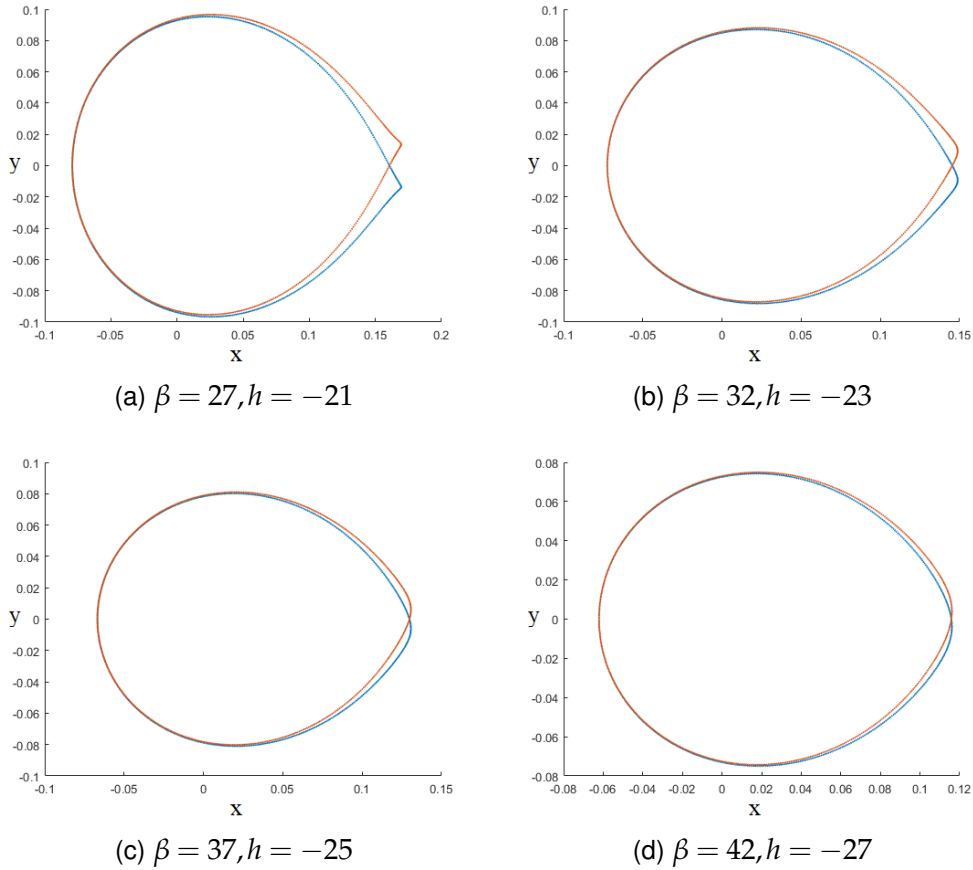


Figure 3.1: Curves D_1^+ (blue) and D_1^- (red) in (x,y) variables

Many examples have been plotted for a variety of energy and solar radiation pressure values. As it can be seen from the examples shown above, all the spectrum under examination has two intersection points at least. These points are the hint of the existence of orbits with close passes to collision and a more detailed examination around these points is a good starting point to find them.

It is easy to notice that there is a clear unsymmetrical aspect to these plots. That is caused by the affect of solar radiation pressure.

3.2. Zooming around an Intersection Point

Now that the intersection points can be established, the next step is to observe the behaviour of the points around these intersection points. In order to observe this, detailed integrations around the close proximity of the intersection points need to be carried out.

The main principle is to start with establishing a grid around a set intersection point. The coordinate system to base this grid on is at (r, θ) . This means that two parameters that will be changed to obtain this grid is r and θ . The other two parameters left for a full set of initial conditions are v and u . Since Poincaré section will be used again, it is established that v is zero in theory. All of the points that will be under inspection will be on Poincaré section, allowing the integration to start on Poincaré section. The set intersection point that will define the grid already has zero as the value of v theoretically. In practice, this

value can be extremely small, not zero. Yet, this value can be also used as the v value of the initial set. Now that three of the four values are established, the remaining is u . The calculation of u is pretty straight forward. For every point on the grid, the value of u will be calculated using the energy expression and the set $[r, \theta, v]$. Every set of $[r, \theta, v, u]$ satisfies the fixed energy requirement. An example that is used for this work is shown below. A.2.

$$u = \sqrt{rH' - v^2 + 2 + 3r^3 \cos(\theta)^2 + \beta' r^2 \cos(\theta)}. \quad (3.1)$$

Having a grid of points around an intersection and full sets of values of them, it is time to start the integration. What was done when trying to find the intersection points was to stop the integration exactly on Poincaré section and record this crossing point. For this examination, the integration is not stopped at Poincaré section, instead it is allowed to pass through this section multiple times. The amount of passes is controlled before the start of the integration and every point for every pass through the Poincaré section is recorded in an array. These points can be plotted and observed, carrying certain characteristics of the orbits of interest.

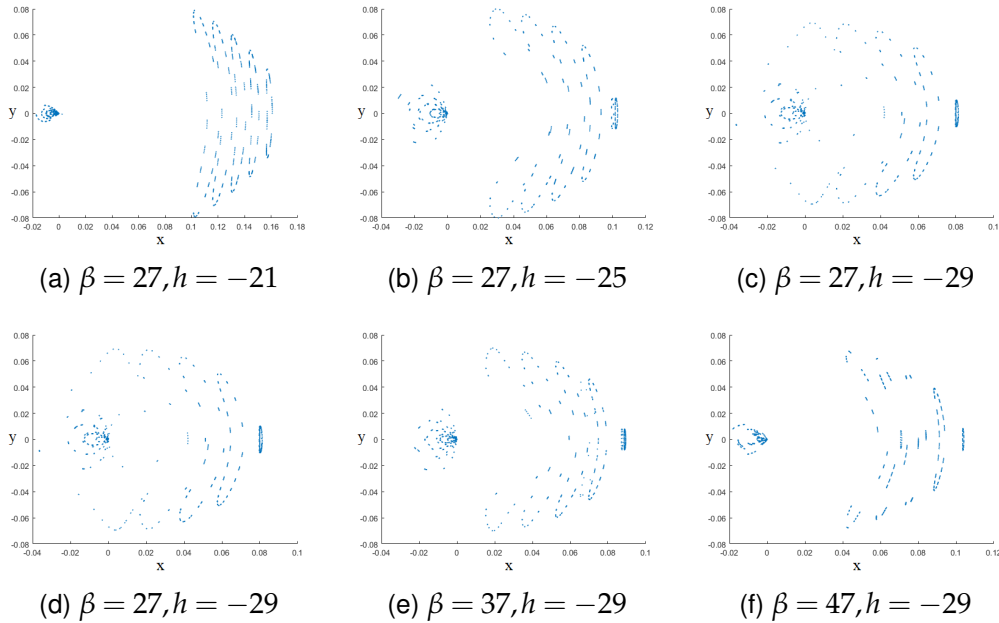


Figure 3.2: Poincaré section $v = 0$ for a higher number of iterates of points around ECO in (x, y) variables

Two sets of plots are used to make observations. In both sets of plots, it can be seen that there are two main areas where periodic orbits lie in a closed shape. The first set is where the solar radiation pressure is fixed and the energy level is changed. Lowering the energy causes two areas to expand in size but get closer in distance. These factors makes it harder to find periodic orbits. This means when the energy decreases, periodic orbits lie in a more collapsed range as well. The second set of plots have a fixed energy level but the solar radiation pressure is increased. This acts as in a manner where the energy would increase. The separate closed curves collapse in size while moving away from one another.

3.3. Periodic Orbit Families

With the methods mentioned during analytical work, many periodic orbits have been found. This exploration is made for four different conditions. Two for the negative and positive starting points and two for direct and retrograde orbits. When a starting point is integrated in order to find a periodic orbits there are many possibilities to find an orbit. It may not pass from $y = 0$ with $\dot{x} = 0$ the first time. Yet, as the integration proceeds this $y = 0$ pass criteria can be met during further passes. Periodic orbits obtained from different number of passes provide periodic passes with differences of appearance. Periodic orbits exist on their multiplication of the number of the passes as well. For examples, a first pass periodic orbit exists on all other passes or a third pass periodic orbit exists on 6th and 9th passes as well. Although a wide range of different pass orbits have been observed, the focus of this work is on 5th and fewer pass periodic orbits.

At this point, having a wide variety of periodic orbits found, some deductions can be made. For each solar radiation pressure value, there are certain energy levels that are the limit for a closed Hill's region. In order to stay in this region, the explorations are made starting from this energy and then decreasing it. As it was mentioned, higher the energy more likely the region opens. The exploration is made between 27 and 55 solar radiation pressure within the focus of this project.

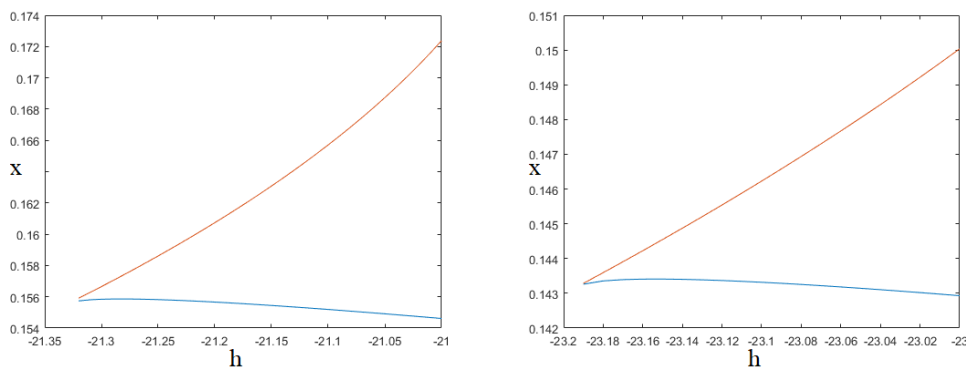
The same patterns are seen for all four different scenarios with different signs of x and u , that is why the remarks will be generally made for all of them. Starting the exploration from the minimum solar radiation pressure and staying close to the limit energy, periodic orbits with all the number of passes can be found. In this case those are first, second, third, fourth and fifth passes. The first pass periodic orbits can be found close to the origin, as they have been hinted when zooming to the intersection points. These orbits can be repeatedly found on other passes as mentioned before. The other periodic orbits with a higher number of pass can be found away from the origin as they have been hinted on the second closed curves of zooming in intersection points. For the same solar radiation pressure, decreasing the energy almost instant causes the disappearance of the orbits with higher number of passes. The first pass orbits keep residing even if the energy decreases. Every time the solar radiation pressure is increased, the exploration starts from the maximum energy limit for closed regions. Observing these limit conditions for increasing solar radiation pressure yields a meaning of its own. As mentioned before, the exploration starts with the lowest solar radiation pressure for all four different scenarios. The first case has all the variety of periodic orbits, but as solar radiation decreases the availability of certain orbits change. The first pass orbits always exists. If their starting point is positive, most of its path goes through negative side. That is the reason why they are harder to constrain with the diminishing allowed range of x . Higher pass orbits are a different story. They reside in the same sign as their starting point. With increasing the solar radiation pressure, the higher level orbits become less and less available. The lower the number of pass, the earlier a type orbits becomes unavailable. In other words, second pass orbits disappear much sooner than third pass orbits and so on. If a certain type of orbit is found on a set of values, it is clear that other types of orbits with higher number of pass can also be found here.

Some other remarks can be made specifically on the starting points of the periodic orbits found. As energy decreases, the absolute value of starting points decrease as well. This relation is the opposite for solar radiation pressure. As it increases, the absolute values

of starting points decrease. The first pass orbits always have starting points close to the origin. On the other hand higher pass orbits always have starting points closer to the limits of the closed region. These closed regions collapse with the decreasing energy or increasing solar radiation pressure. An important note is about the evolution of starting points depending on the number of passes. As the number of passes increase for higher pass orbits, the starting points get smaller absolute values. This is a crucial information, because it explains why higher the number of passes, easier it is to find them. With a smaller value of starting point, any orbit becomes more likely to exist despite the collapsing nature of closed region under the affect of changing parameters. [A.4.1.](#) [A.4.2.](#) [A.4.3.](#) [A.4.4.](#)

After individual orbits, the focus shifts to families of orbits. Just as a reminder, the families of orbits are found with changing the energy of a certain periodic orbit's energy and finding its changed starting point. As it was mentioned on the availability of periodic orbits, the families are also affected by the collapsing closed regions and their limitations. The four conditions have different behaviour of families but their limitations of closed region is common. The closed region collapses with the decrease of energy at all times. It is known now that the higher pass periodic orbits start close their closed region limits already. When the collapse reaches this starting point, the the family of orbits disappear. This is inevitable. This idea is presented with two different cases below. [A.5.1.](#) [A.5.2.](#) [A.5.3.](#) [A.5.4.](#)

In this work positive and negative x starting points are presented with the notation $+x$ and $-x$. The direct and retrograde orbits have the notation $+u$ and $-u$ respectively.



(a) 4th pass family on $\beta = 27, h = -21$

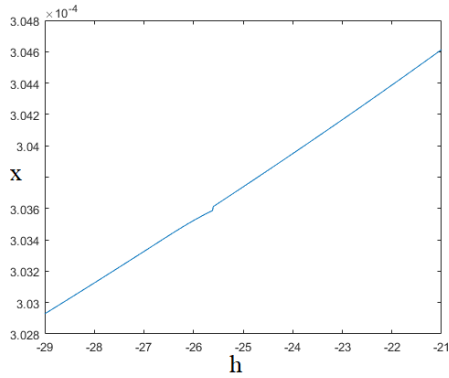
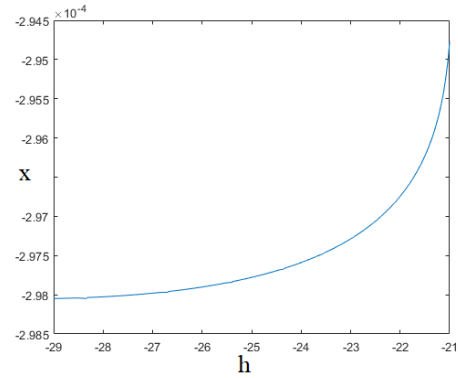
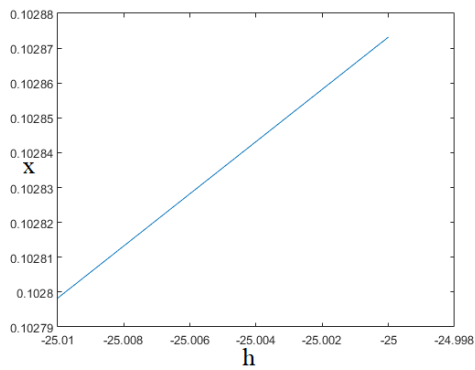
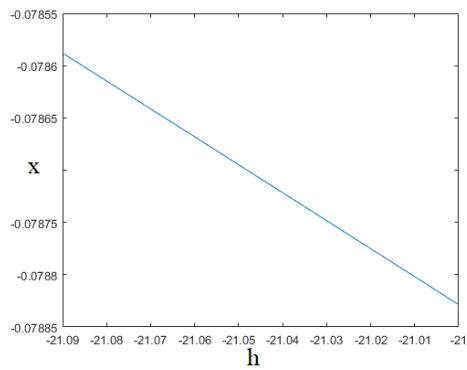
(b) 4th pass family on $\beta = 32, h = -23$

Figure 3.3: The disappearance of orbit families due to closed region limits (red)

The families of orbits are represented with the characteristic curves that track the energy change and the corresponding starting point change. The results are reached with the observation of such curves.

The first pass orbits will be discussed first. These orbits have two points where they cross $y = 0$ perpendicularly. This means they can be found on two different conditions. A retrograde orbit with a positive x starting point can also be found as a retrograde orbits with a negative x starting point. The opposite sign relationship applies on odd-numbered pass orbits and the even-numbered pass orbits have a same sign starting point relationship. Yet although the same orbit can be found in different conditions, the characteristic curves created from different β conditions are not the same. The first pass orbit starting points for positive x – retrograde and negative x – direct conditions are really close to the origin. Yet,

the orbit families found here are bigger in terms of size. This means a certain orbit found here can be tracked for a wider range. The first pass orbit starting points for negative x – retrograde and positive x – direct conditions are the opposite. They are far away from the origin and they have narrower range for their families of orbit.

(a) $+x - u$ (b) $-x + u$ (c) $+x + u$ (d) $-x - u$ Figure 3.4: 1st pass family orbits at $\beta = 27$

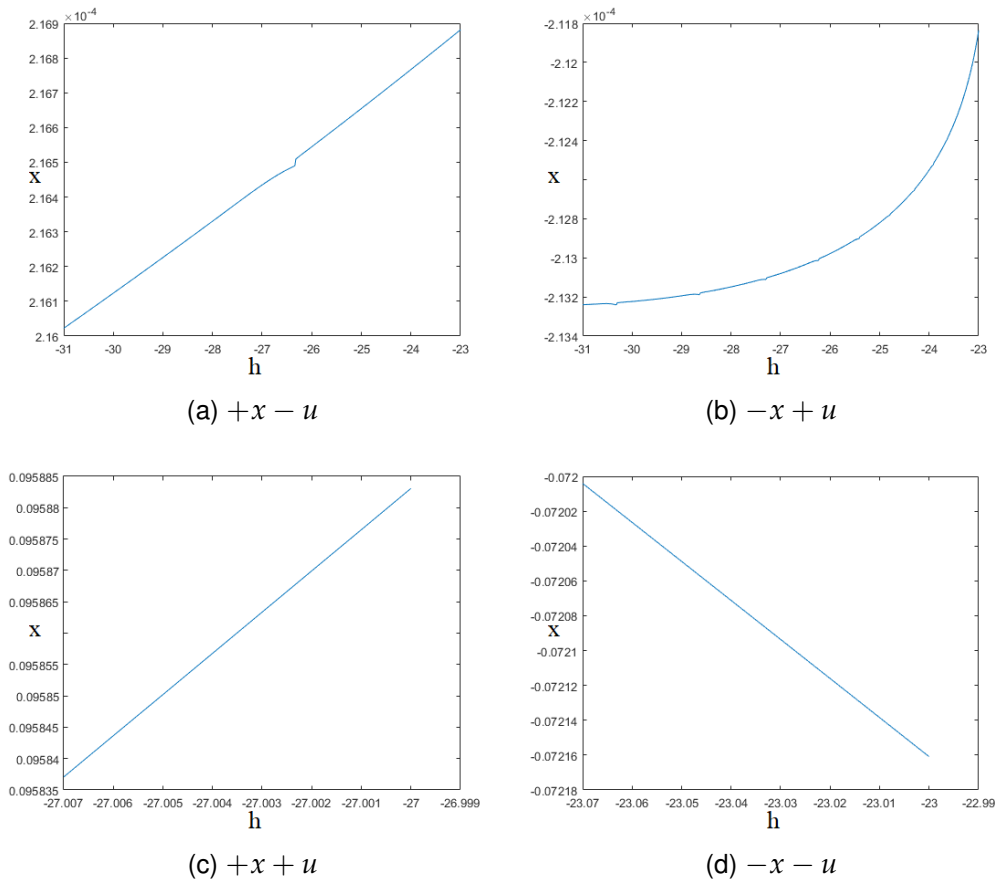
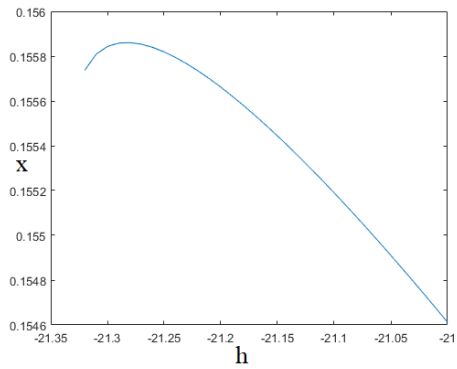
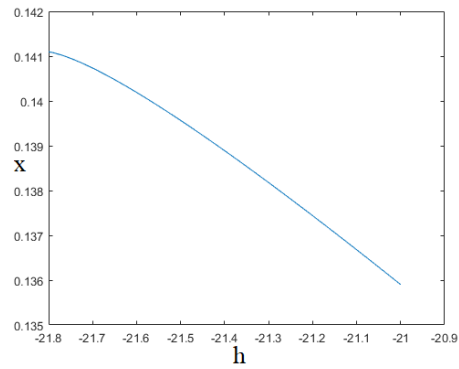
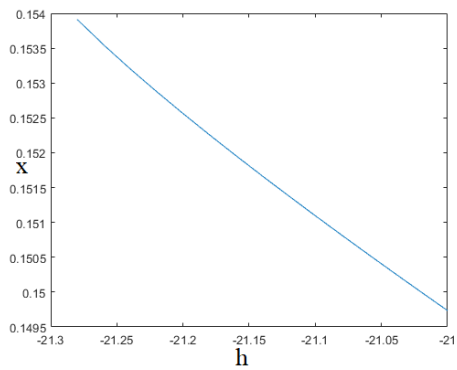
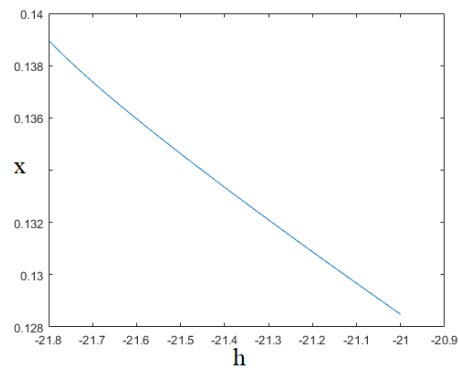
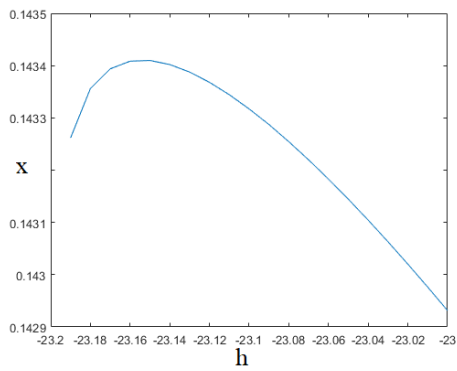
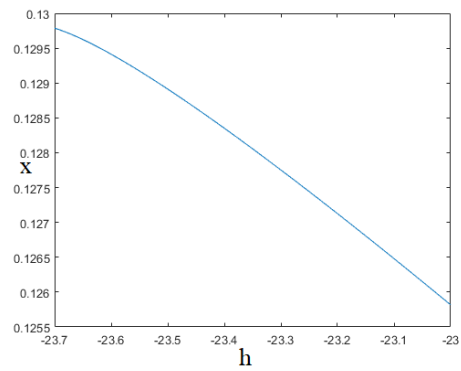
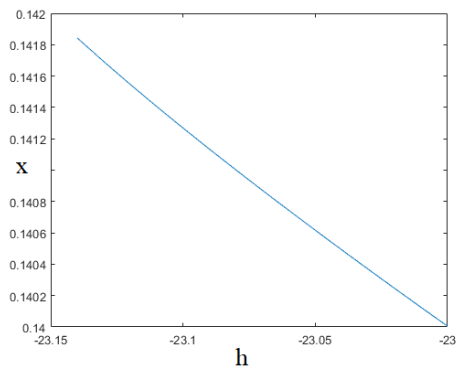
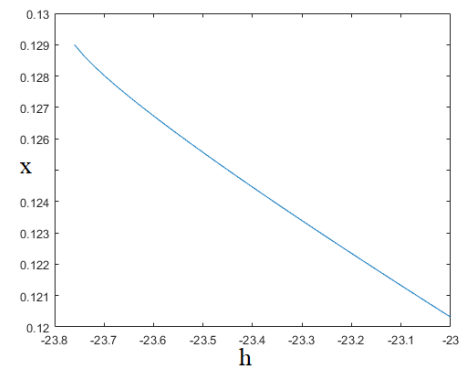


Figure 3.5: 1st pass family orbits at $\beta = 32$

After the first pass orbits, the focus is now on higher pass orbits which more or less act similar between them. As it was established on the availability of periodic orbits, the families of orbits have similar a nature when it comes to the size of the family. As the number of passes increase, the family can be tracked further. Also as the energy decreases or solar radiation pressure increases, the family size decreases in a similar manner.

(a) 4th pass $+x - u$ (b) 5th pass $+x - u$ (c) 4th pass $+x + u$ (d) 5th pass $+x + u$ Figure 3.6: 4th and 5th pass family orbits at $\beta = 27$

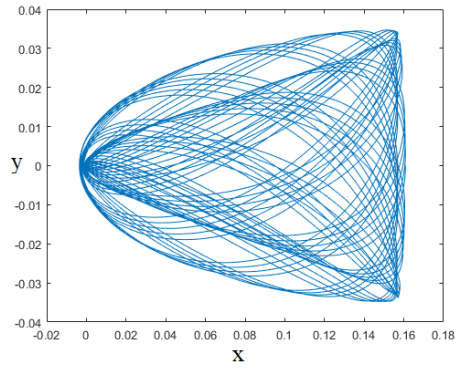
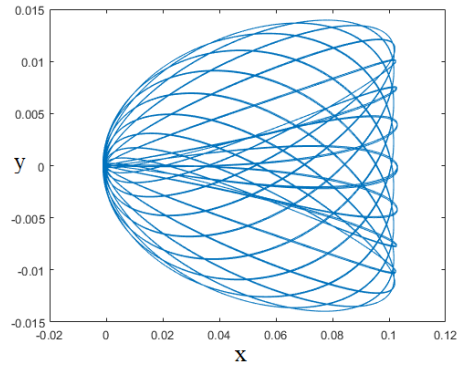
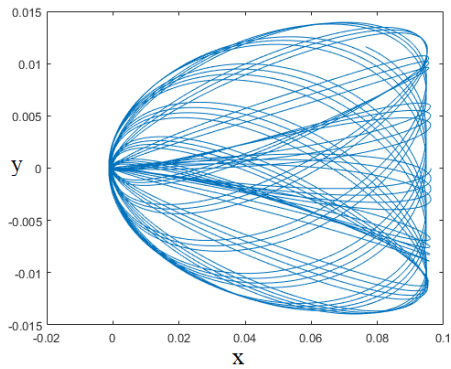
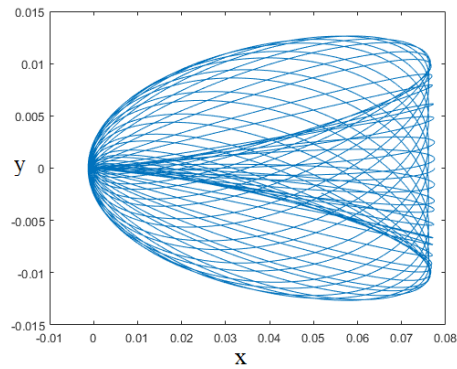
(a) 4th pass $+x - u$ (b) 5th pass $+x - u$ (c) 4th pass $+x + u$ (d) 5th pass $+x + u$ Figure 3.7: 4th and 5th pass family orbits at $\beta = 32$

3.4. Plotting Orbits

After all the preliminary work, this part of the project is the most straight forward part. A script will be written to follow initial set of values over the equations of motion. Two criteria is set for energy and solar radiation pressure. For the first part, the initial conditions are chosen with θ and v as zero. When these two are zero, the x value is equal to the corresponding r value. This value is obtained from the intersection points of the system with the set energy and solar radiation pressure. The value of u is again calculated through the energy relation. At this point, all of the starting conditions are set. They are integrated as long as desired, afterwards the results are presented after they have been translated into $[x, y, \dot{x}, \dot{y}]$. This presentation represents the motion of the system in the real world.

The first part has initial values closest to the intersection points. There are endless couples of energy and solar radiation pressure to be examined. All of these couples have different intersection points that show the existence of periodic orbits. Within this project, the focus is on a certain range of energy level and solar radiation pressure. In short, the real goal is to observe the desired periodic orbits. After the first part, the variations of the prior conditions will be examined to have a better grasp of the behaviour of the system. One of the approaches is moving away from the intersection points. Starting r values more and more away from the intersection points is how these orbits are achieved. The other method is sticking to the r values obtained from the intersection points, but starting the integration with θ values other than zero.

There are orbits to be found outside of symmetrical periodic orbits in this system. Because the system is in a closed region, even random orbits do not escape the system. These orbits are called quasi-periodic orbits. Some examples are presented below with a changing energy level. [A.6](#).

(a) $\beta = 27, h = -21$ at $x = 0.1608$ (b) $\beta = 27, h = -25$ at $x = 0.1030$ (c) $\beta = 27, h = -26$ at $x = 0.0962$ (d) $\beta = 27, h = -30$ at $x = 0.0775$ Figure 3.8: Quasi-periodic orbits starting with $\theta = 0$ for 500 integration time

Quasi-periodic orbits scan a certain area with a certain rate. They may have use for certain mission criteria, but they are not the focus of this project.

Periodic orbits slightly alter their appearance depending on all the varying parameters talked about so far but their general structure is similar within the same number of pass type. The examples of each type until the sixth pass orbits are presented below.

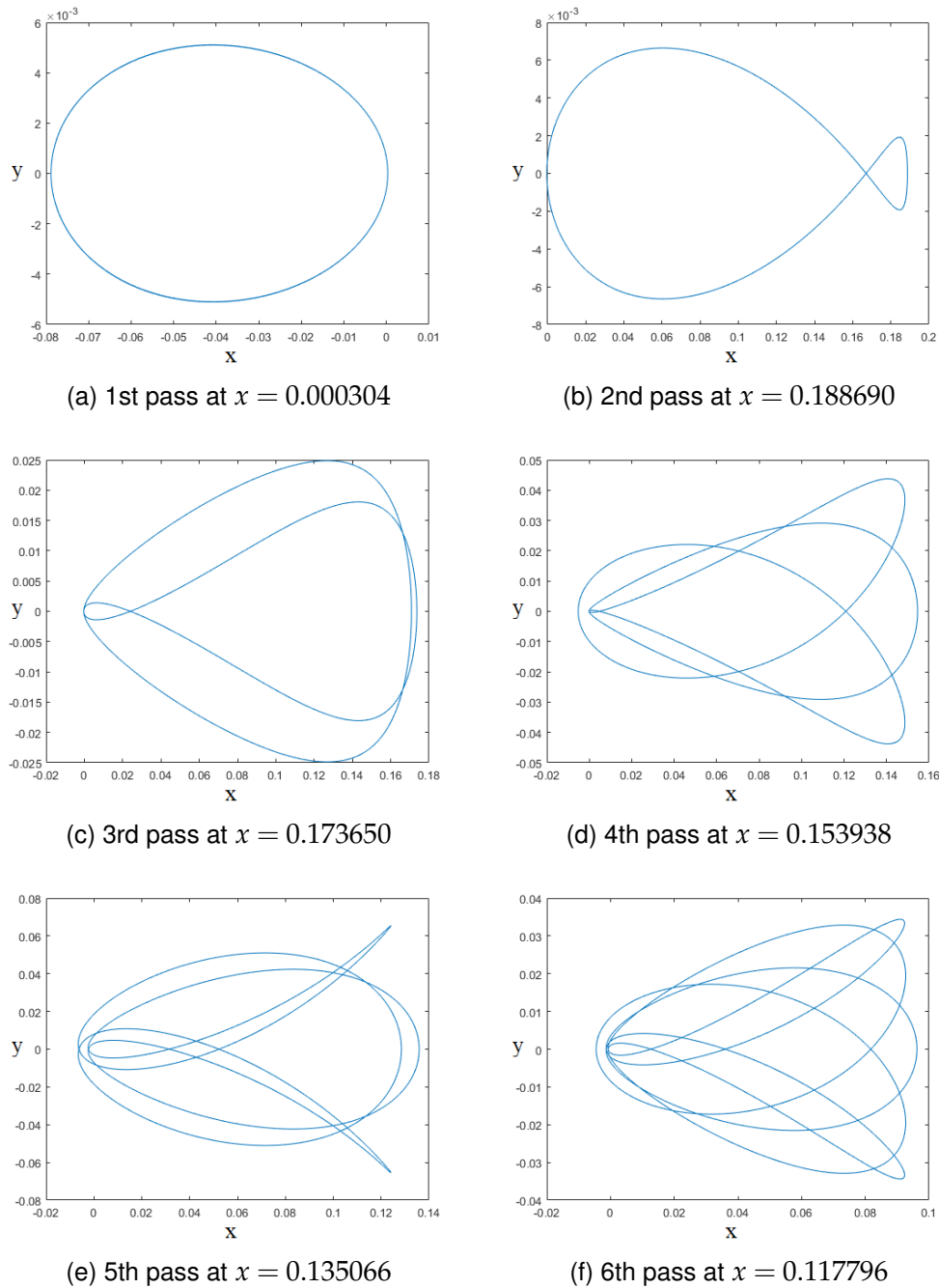


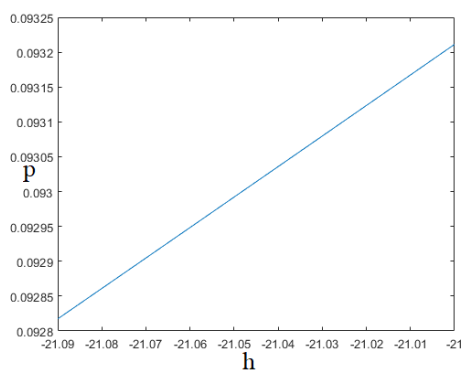
Figure 3.9: Examples of orbits of different passes at $\beta = 27, h = 20.8947$ for $+x - u$

3.5. Period and Range

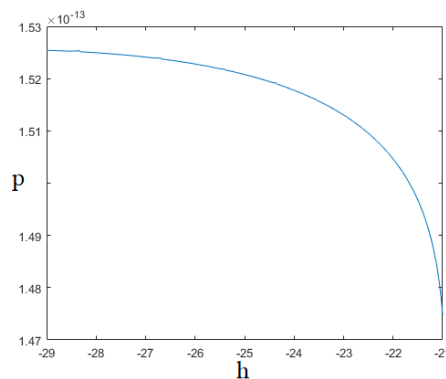
An important aspect of this work is the comparison of the characteristics of the different periodic orbit families. Two of these characteristics which is examined here are the period of an orbit and its maximum-minimum distance from the origin. These comparisons can be made with the starting point-energy couples gathered in the periodic orbit families. These couples make the characteristic curves of such families.

For period, it is necessary to implement a fifth equation for the real time into the set of four equations of motion. The integration is not made in real time, that is why an extra

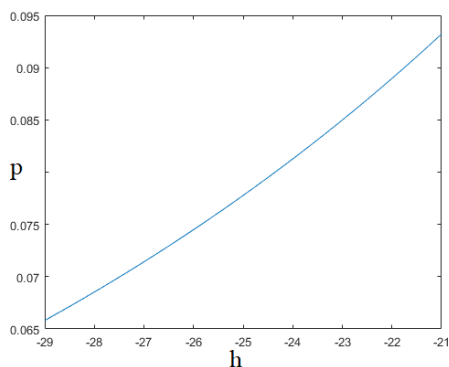
expression is needed. After this implementation, the time passed can be observed through the output array, along $[r, \theta, v, u]$. It should be noted that along the characteristic curves are the points gathered when the orbits reach their half-period. This means that the time observed should be doubled in order to obtain the full period of any orbit. A.7.



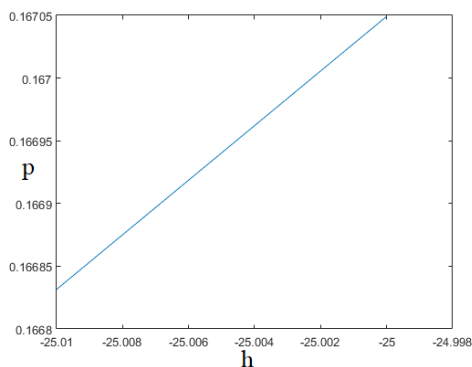
(a) $-x - u$ at 3.4d



(b) $-x + u$ at 3.4b



(c) $+x - u$ at 3.4a



(d) $+x + u$ at 3.4c

Figure 3.10: Periodic characteristic curves of 1st pass family orbits for $\beta = 27$

The same number of pass periodic orbits have similar behaviour. The four examples above are all for $\beta = 27$ and for the first pass. Although the rest have decreasing period with decreasing energy, the condition where starting point is negative on direct orbits is the opposite. For the higher pass orbits, the behaviour becomes also the opposite to the first pass orbits. Fourth and fifth pass examples of positive starting point – direct orbits are presented below, fitting the conclusion.

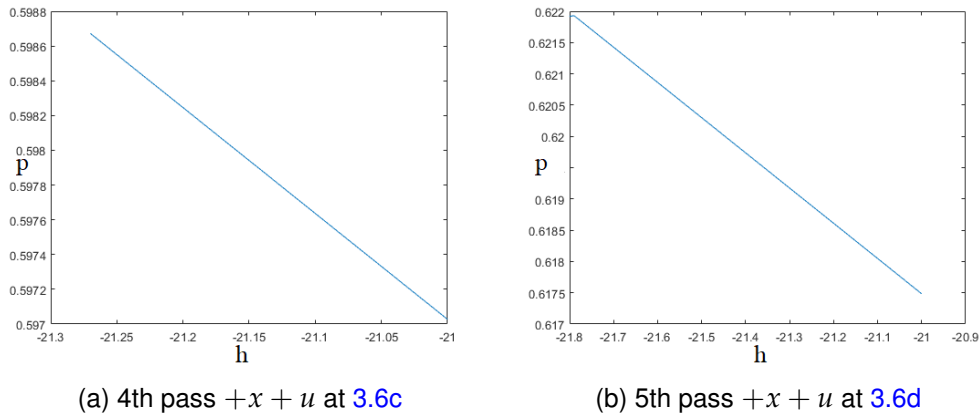
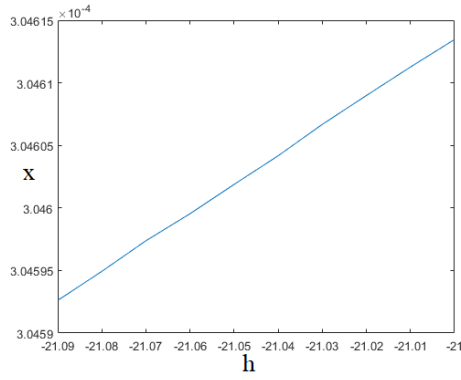


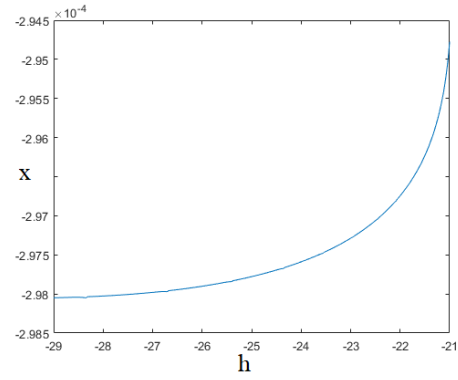
Figure 3.11: Periodic characteristic curves of 4th-5th pass family orbits for $\beta = 27$

For the range, there are two aspects that are crucial. These are the minimum distance and the maximum distance to the origin. Either using the half period or full period are applicable here. Since the code for the orbits plot using roughly every point on the orbits, these set of points are where the maximum and the minimum distance to the origin are obtained. The set of values has the first variable “r”, which depicts the distance to the origin. Certain functions to find the maximum and the minimum values of an array exist in many programming languages. For every set energy level on the family, these points are found this way. Two different plots can be brought together afterwards; the first one being the relationship between the energy level and the maximum distance, the other one being the relationship between the energy level and the minimum distance.

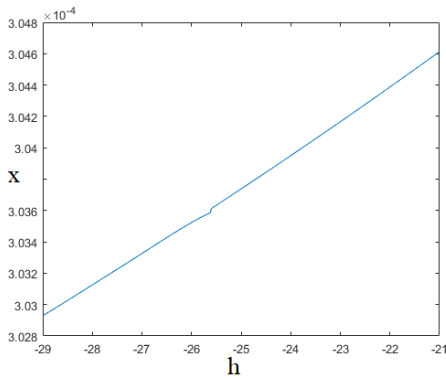
The minimum distance characteristic curves corresponding to the period characteristic curves for the first pass periodic orbit examples above are presented below.



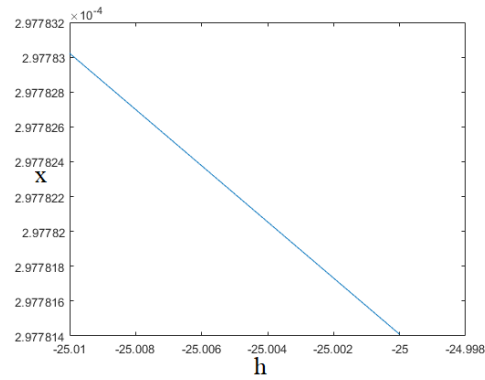
(a) $-x - u$ at 3.4d



(b) $-x + u$ at 3.4b



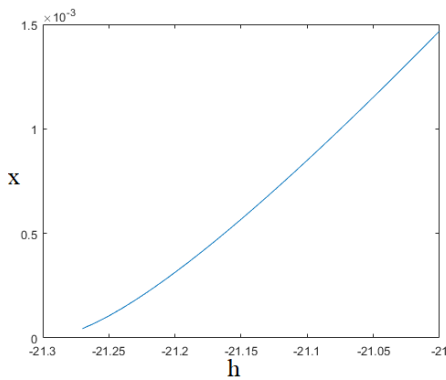
(c) $+x - u$ at 3.4a



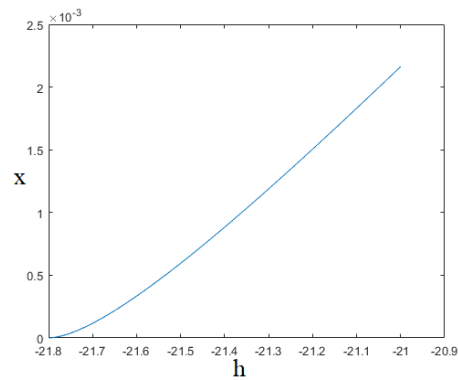
(d) $+x + u$ at 3.4c

Figure 3.12: Min distance characteristic curves of 1st pass family orbits for $\beta = 27$

The minimum distance characteristic curves corresponding to the period characteristic curves for the fourth and fifth pass orbits are below.



(a) 4th pass $+x + u$ at 3.6c



(b) 5th pass $+x + u$ at 3.6d

Figure 3.13: Min distance characteristic curves of 4th-5th pass family orbits for $\beta = 27$

The maximum distance characteristic curves corresponding to the period characteristic curves for the first pass periodic orbit examples above are presented below.

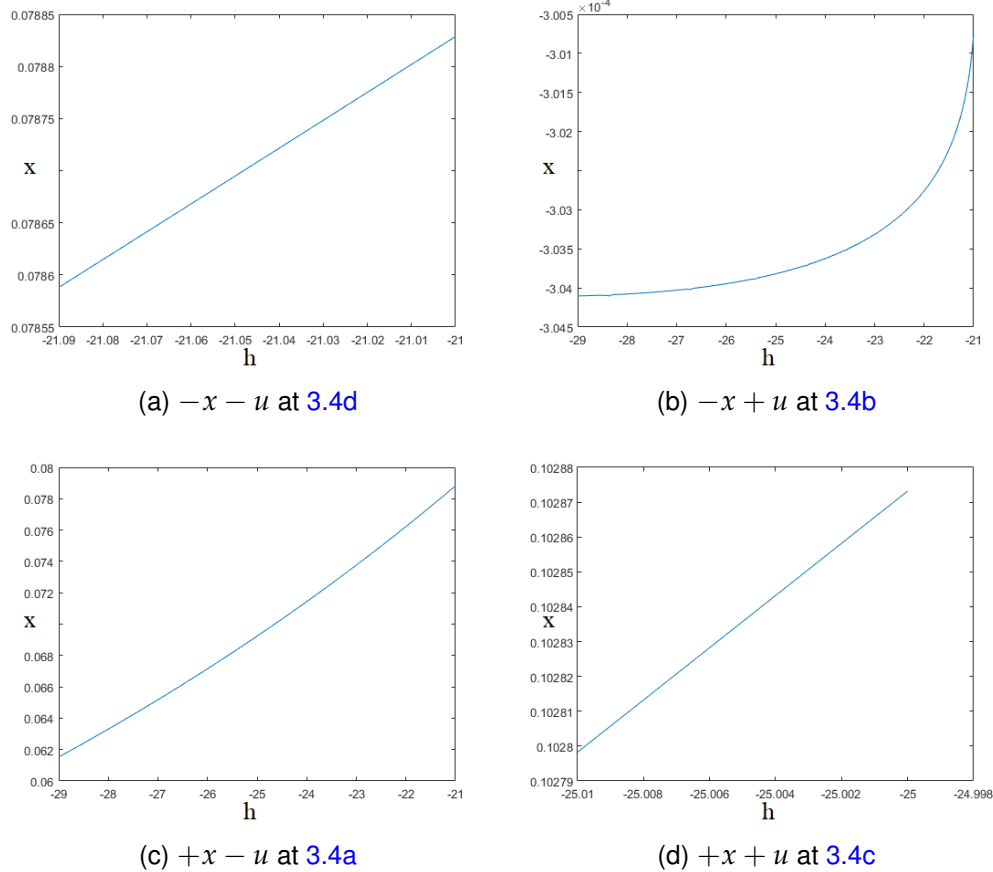


Figure 3.14: Max distance characteristic curves of 1st pass family orbits for $\beta = 27$

The maximum distance characteristic curves corresponding to the period characteristic curves for the fourth and fifth pass orbits are below.

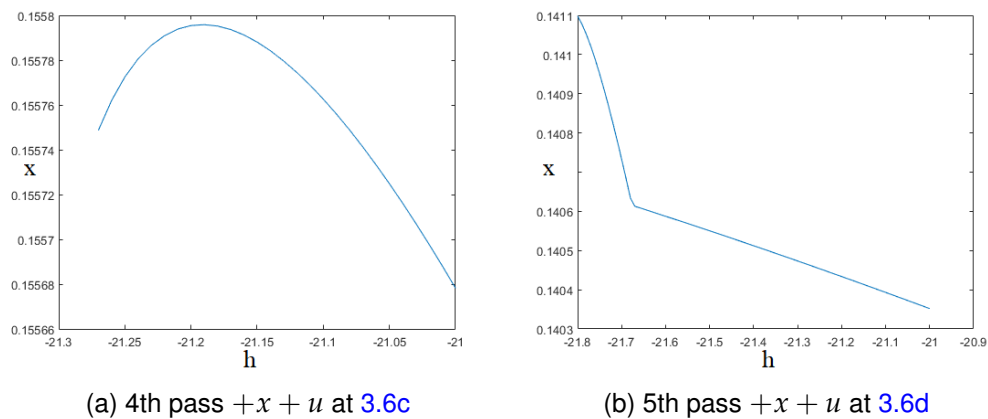


Figure 3.15: Max distance characteristic curves of 4th-5th pass family orbits for $\beta = 27$

All of these plots give some sense of correlation between the range of the orbit and the period of the orbit. Since these orbits have the origin of ejection-collision orbits, periapsis of the orbits are greatly smaller than apoapsis of the orbits with varying difference. This means that a change in the maximum range of an orbit has a bigger impact on the change of the period of the orbit. An increasing minimum range and maximum range usually are

good hints of an increasing period. Yet, the maximum range has a bigger weight as said before. The example of positive x – direct orbit families fit this statement. This means even a decreasing minimum distance and an increasing maximum distance will most likely mean an increasing period.

Minimum range and maximum range hold meaningful values when it comes to mission design, but their effect is not the only one present when it comes to the change of period of an orbit. Although the same pass orbits share a similar structure overall, their specific shape can affect the period of the orbit as well.

CHAPTER 4. CONCLUSIONS AND ACKNOWLEDGEMENTS

4.1. Conclusions

The aim of this project was to examine a perturbed Hill's problem. The effects of solar radiation pressure have been implemented into the system to achieve a more realistic and practical solution of such a system. The base of this work rests on the ejection-collision manifolds on the closed regions. Closed regions are preferred in order to obtain systems which have bodies that are naturally incapable of escape.

A magnitude of symmetrical periodic orbits have been found in the desired nature. These orbits have been classified due to their number of pass and rotation in order to reach some patterns for these classes. The effects of change in energy and solar radiation pressure have been observed. Found orbits belong in orbit families with widely different specifications. These families varied in size and the energy-solar radiation pressure gap of existence. Examining the characteristic curves of orbits families, their reason of disappearance and the common behaviour of specific classes have been established. For the sake of a better understanding of a dynamical system and seeking the possibilities of plausible mission design, the family of orbits have been explored in a way that many other properties have also been examined. The properties like period and range had correlations to the factors at play. Those are also established in this work.

It is known that closed regions open up at the equilibrium points of the system. A future work could be based on open regions. In open regions, the existence of Lyapunov orbits around the equilibrium points and their associate invariant manifolds would play an important dynamical role with the ejection-collision manifolds. The study of such systems would be a fitting continuation of this project and it would have a higher scope with valuable results.

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APPENDICES

APPENDIX A. THE COMPLETE SET OF CODES

A.1. Existence of ECO

A.1.1. Integration - Forwards in Time

```
% Forwards in time

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*(x(1)^3)...
*(cos(x(2))^2)+27*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*(x(1)^(1.5))...
-3*(x(1)^3)*sin(x(2))*cos(x(2))-27*(x(1)^2)*sin(x(2))];

result1=[];

for i=0:0.01:(2*pi)

teta=i;

t0=0;

alfa=1e-6;

h=-21;

n0rm=sqrt(1+((h^2)/8));

[t,xa]=ode45(@(t,x) g(t,x), [t0 (t0+0.1)], [(alfa/n0rm) (teta) ((alfa*...
((h/(2*sqrt(2)))/n0rm)+sqrt(2)) (0))]);

Control=1;

while Control>=0

    %t0=t0+0.1;

    [t,xa]=ode45(@(t,x) g(t,x), [t0 (t0+0.1)], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))]);

    Control=dot(xa(1,3),xa(end,3));

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);
```

```

t1=t0-((xa(1,3))/((xa(1,3)^2)/2+(xa(1,4)^2)-1+2*xa(1,4)*(xa(1,1)^(1.5))...
+3*(xa(1,1)^3)*(cos(xa(1,2))^2)+27*(xa(1,1)^2)*cos(xa(1,2))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) (cross3) (cross4)]);

    t1=t0-((xa(end,3))/((xa(end,3)^2)/2+(xa(end,4)^2)-1+2*xa(end,4)*(xa(end,1)^(1.5))...
+3*(xa(end,1)^3)*(cos(xa(end,2))^2)+27*(xa(end,1)^2)*cos(xa(end,2))));

end

result1=[result1;xa(end,1) xa(end,2) xa(end,3) xa(end,4)];

end

```

A.1.2. Integration - Backwards in Time

```

% Backwards in time

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*(x(1)^3)...
*(cos(x(2))^2)+27*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*(x(1)^(1.5))-3*...
(x(1)^3)*sin(x(2))*cos(x(2))-27*(x(1)^2)*sin(x(2))];

result2=[];

for i=0:0.01:(2*pi)

    teta=i;

    t0=0;

    alfa=1e-6;

    h=-30;

    n0rm=sqrt(1+((h^2)/8));

    [t,xa]=ode45(@g(t,x), [t0 (t0-0.1)], [(alfa/n0rm) (teta) ((alfa*((-h/...
(2*sqrt(2)))/n0rm))-sqrt(2)) (0)]);

    Control=1;

```



```

while Control>=0

    %t0=t0-0.1;

    [t,xa]=ode45(@g(t,x), [t0 (t0-0.1)], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))]);

    Control=dot(xa(1,3),xa(end,3));

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t1=t0-((xa(1,3))/((xa(1,3)^2)/2+(xa(1,4)^2)-1+2*xa(1,4)*(xa(1,1)^(1.5))...
+3*(xa(1,1)^3)*(cos(xa(1,2))^2)+27*(xa(1,1)^2)*cos(xa(1,2))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) (cross3) (cross4)]);

    t1=t0-((xa(end,3))/((xa(end,3)^2)/2+(xa(end,4)^2)-1+2*xa(end,4)*(xa(end,1)^(1.5))...
+3*(xa(end,1)^3)*(cos(xa(end,2))^2)+27*(xa(end,1)^2)*cos(xa(end,2))));

end

result2=[result2;xa(end,1) xa(end,2) xa(end,3) xa(end,4)];

end

```

A.2. Zooming around an Intersection Points

```

% Starting with one intersection point - manually input data set.

r=0.160203411681009;
teta=6.27888412224692;
v=-1.01602472095397e-12;
u=-0.184302691884947;

bt=27;

% The value of h is calculated with the data above for control and in order

```

```

% to use it to calculate u values inside the grid.

%control_h=((v^2)+(u^2)-2-(3*(r^3)*((cos(teta))^2))-(54*(r^2)*(cos(teta))))/r);%

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*(x(1)^3)*...
(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*(x(1)^(1.5))-3*...
(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

% The set of values are transfered into different names to not cause a mess
% when the original set of data is used for the for loops.

r_set=r;
teta_set=teta;
v_set=v;
h=-21;
rings=[];

for i=(r-0.04):0.01:(r)

    r_set=i;

    for j=(teta):0.001:(teta)

        teta_set=j;

        u_set=(-1)*sqrt((r_set*(h))-(v_set^2)+2+(3*(r_set^3)*((cos(teta_set))^2))...
+((2*bt)*(r_set^2)*(cos(teta_set))));

        %now we have [r_set,teta_set,v_set,u_set]%

        t0=0;

        [t,xa]=ode45(@(t,x) g(t,x), [t0 (t0+0.1)], [(r_set) (teta_set) ...
(v_set) (u_set)]);

        Control=1;

        while Control>=0

            %t0=t0+0.1;

            [t,xa]=ode45(@(t,x) g(t,x), [t0 (t0+0.1)], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))]);

            Control=dot(xa(1,3),xa(end,3));

        end

```

```

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t1=t0-((xa(1,3))/((xa(1,3)^2)/2+(xa(1,4)^2)-1+2*xa(1,4)*(xa(1,1)^(1.5))...
+3*(xa(1,1)^3)*(cos(xa(1,2))^2)+bt*(xa(1,1)^2)*cos(xa(1,2))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) (cross4)

    t1=t0-((xa(end,3))/((xa(end,3)^2)/2+(xa(end,4)^2)-1+2*xa(end,4)*(xa(end,
+3*(xa(end,1)^3)*(cos(xa(end,2))^2)+bt*(xa(end,1)^2)*cos(xa(end,2))));

end

rings=[rings;xa(end,1) xa(end,2) xa(end,3) xa(end,4)];

% After the first of crossing in v=0, the process is repeated 99
% more times to have 100 passes per data set.

for k=1:499

    t0=0;

    [t,xa]=ode45(@ (t,x) g(t,x), [t0 (t0+0.1)], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))]);

    Control=1;

    while Control>=0

        %t0=t0+0.1;

        [t,xa]=ode45(@ (t,x) g(t,x), [t0 (t0+0.1)], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))]);

        Control=dot(xa(1,3), xa(end,3));

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

```

```

        t1=t0-((xa(1,3))/((xa(1,3)^2)/2+(xa(1,4)^2)-1+2*xa(1,4)*(xa(1,1)^(1.5)).
+3*(xa(1,1)^3)*(cos(xa(1,2))^2)+bt*(xa(1,1)^2)*cos(xa(1,2))));

        while abs(t1-t0)>1e-6

            t0=t1;

            [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) (cross3) (cross4)]);

            t1=t0-((xa(end,3))/((xa(end,3)^2)/2+(xa(end,4)^2)-1+2*xa(end,4)*(xa(end,1)^(1.5)).
+3*(xa(end,1)^3)*(cos(xa(end,2))^2)+bt*(xa(end,1)^2)*cos(xa(end,2))));

        end

        rings=[rings;xa(end,1) xa(end,2) xa(end,3) xa(end,4)];

    end

end

end
end
end

```

A.3. Zero-Velocity Curves

```

%Checking out the zero-velocity curves%

hh=-21;
btbt=27;

[x,y]=meshgrid(-0.5:0.01:0.5,-0.5:0.01:0.5);
z= -1*(2./sqrt(x.^2+y.^2)+3.*x.^2+2.*btbt.*x);
contourf(x,y,z,100)

[C,h]=contourf(x,y,z,100);
clabel(C,h)

```

A.4. Observation of Velocity on Various Starting Points

A.4.1. Negative x - Negative u

```

h=-21;
bt=27;

```

```

how_many_pass=2;

x0=-0.078829;
check_xdot=[];

options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*(x(1)^3)*...
(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*(x(1)^(1.5))-3*...
(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

for i=(x0-0.0001):0.000001:(x0+0.0001)

    hp=0;

    x0_set=i;

    r=(-1)*x0_set;
    teta=pi;
    v=0;
    u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
    hp=hp+0.1;

    Control=1;

    while Control>=0

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;

```

```

        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    hp=hp-0.1;

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+(xa(1,1)*...
xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@(t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) ...
(cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

    end

    hp=hp+t0;

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) (xa(end,3)) .
(xa(end,4))], options);
        hp=hp+0.1;

        Control=(-1)^(j-1);

        if Control==1

            while Control>=0

```

```

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;

```

```

        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+(xa(1,1)*...
xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) ...
(cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))*...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

```



```

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

fp=hp*2;
check_xdot=[check_xdot;x0_set x y xdot ydot fp];

end

sz=1;

scatter(check_xdot(:,1),check_xdot(:,4),sz)

```

A.4.2. Negative x - Positive u

```

h=-21;
bt=27;

how_many_pass=6;

x0=-0.018226;
check_xdot=[];

options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*...
(x(1)^3)*(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*...
(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

for i=(x0-0.0001):0.000001:(x0+0.0001)

    hp=0;

    x0_set=i;

    r=(-1)*x0_set;
    teta=pi;
    v=0;
    u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

```

```

[t, xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
hp=hp+0.1;

Control=-1;

while Control<=0

    [t, xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+(xa(1,1)*...
xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

```

```

[t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) (cross4)], c

t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%For more than one passes%

for j=2:how_many_pass

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
hp=hp+0.1;

Control=(-1)^j;

if Control==1

while Control>=0

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
hp=hp+0.1;

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

```

```

        end

    else

        while Control<=0

            [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
            hp=hp+0.1;

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

```

```

while abs(t1-t0)>1e-6

    t0=t1;

    [t, xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) ...
(cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

fp=hp*2;
check_xdot=[check_xdot;x0_set x y xdot ydot fp];

end

sz=1;

scatter(check_xdot(:,1),check_xdot(:,4),sz)

```

A.4.3. Positive x - Negative u

```
h=-21;
```

```

bt=33;

how_many_pass=7;

x0=0.02;
check_xdot=[];

options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*...
(x(1)^3)*(cos(x(2)))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*...
(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

for i=(x0-0.01):0.0001:(x0+0.11)

    hp=0;

    x0_set=i;

    r=(1)*x0_set;
    teta=0;
    v=0;
    u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
    hp=hp+0.1;

    Control=-1;

    while Control<=0

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);
    end
end

```

```

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+(xa(1,1)*...
xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x) g(t,x), [0 t0], [(cross1) (cross2) (cross3) ...
(cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@g(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    Control=(-1)^j;

    if Control==1

        while Control>=0

```

```

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

```



```

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+(xa(1,1)*...
xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

```

```

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

fp=hp*2;
check_xdot=[check_xdot;x0_set x y xdot ydot fp];

end

sz=1;

scatter(check_xdot(:,1),check_xdot(:,4),sz)

```

A.4.4. Positive x - Positive u

```

h=-21;
bt=27;

how_many_pass=5;

x0=0.128482;
check_xdot=[];

options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+3*...
(x(1)^3)*(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*x(3)*...
(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

for i=(x0-0.0001):0.000001:(x0+0.0001)

    hp=0;

    x0_set=i;

    r=(1)*x0_set;
    teta=0;
    v=0;
    u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

```

```

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
hp=hp+0.1;

Control=1;

while Control>=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

```

```

t0=t1;

[t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%For more than one passes%

for j=2:how_many_pass

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
hp=hp+0.1;

Control=(-1)^(j-1);

if Control==1

while Control>=0

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
hp=hp+0.1;

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

```

```

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...

```

```

(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

fp=hp*2;
check_xdot=[check_xdot;x0_set x y xdot ydot fp];

end

sz=1;

scatter(check_xdot(:,1),check_xdot(:,4),sz)

```

A.5. Characteristic Curves

A.5.1. Negative x - Negative u

```
%Finding the characteristic curve%

char_curve=[];
period_curve=[];

for k=(-21):(-0.01):(-21.1)

    h=k;
    bt=27;

    how_many_pass=2;

    x0=-0.0788;
    check_xdot=[];

    options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

    g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))...
+3*(x(1)^3)*(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)...
-2*x(3)*(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

    for i=(x0-0.0001):0.0002:(x0+0.0001)

        x0_set=i;

        r=(-1)*x0_set;
        teta=pi;
        v=0;
        u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

        Control=1;

        while Control>=0

            [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);
```

```

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^(j-1);

```



```

if Control==1

    while Control>=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . . .
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . . .
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);

```

```

        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);

```

```

q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

check_xdot=[check_xdot;x0_set x y xdot ydot];

end

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/...
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(-1)*x0_set;
teta=pi;
v=0;
u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

Control=1;

while Control>=0

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;

```

```

        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

    end

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        Control=(-1)^(j-1);

        if Control==1

            while Control>=0

                [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

                r=xa(end,1);
                theta=xa(end,2);
                v=xa(end,3);
                u=xa(end,4);
            end
        end
    end

```

```

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

else

while Control<=0

[t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

end

```

```

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

%Finding the spot with a tolerance%

while abs(xdot)>1e-6

```

```

if xdot<=0

    open=sol_step;
    close=check_xdot(2,1);

else

    open=check_xdot(1,1);
    close=sol_step;

end

check_xdot=[];

for i=(open):(close-open):(close)

    x0_set=i;

    r=(-1)*x0_set;
    teta=pi;
    v=0;
    u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

    Control=1;

    while Control>=0

        [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;

```

```

        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2))...
+ (xa(1,1)*xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)
+ (xa(end,1)*xa(end,4)*cos(xa(end,2))))));

    end

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        Control=(-1)^(j-1);

        if Control==1

            while Control>=0

                [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)
(xa(end,3)) (xa(end,4))], options);

                r=xa(end,1);
                teta=xa(end,2);
                v=xa(end,3);
            end
        end
    end

```



```

        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end
end

```

```

        end

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))...
+ (xa(1,1)*xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2))))));

    end

    %Recording%

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    check_xdot=[check_xdot;x0_set x y xdot ydot];

```

```

end

hp=0;

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(-1)*x0_set;
teta=pi;
v=0;
u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
hp=hp+0.1;

Control=1;

while Control>=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

hp=hp-0.1;

```

```

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    Control=(-1)^(j-1);

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
            hp=hp+0.1;

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

```

```

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

else

while Control<=0

[t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
hp=hp+0.1;

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

end

```

```

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))).
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

```

```

end

char_curve=[char_curve;h sol_step];
fp=hp*2;
period_curve=[period_curve;h fp];

end

plot(char_curve(:,1),char_curve(:,2))
%plot(period_curve(:,1),period_curve(:,2))

```

A.5.2. Negative x - Positive u

```

%Finding the characteristic curve%

char_curve=[];
period_curve=[];

for k=(-21):(-0.0001):(-21.0001)

    h=k;
    bt=27;

    how_many_pass=6;

    x0=-0.018226;
    check_xdot=[];

    options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

    g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+...
3*(x(1)^3)*(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*...
x(3)*(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

    for i=(x0-0.0001):0.0002:(x0+0.0001)

        x0_set=i;

        r=(-1)*x0_set;
        teta=pi;
        v=0;
        u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

```

```

Control=-1;

while Control<=0

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

```



```

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    else

        while Control<=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);

```

```

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

```

```

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

check_xdot=[check_xdot;x0_set x y xdot ydot];

end

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/...
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(-1)*x0_set;
teta=pi;
v=0;
u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

Control=-1;

while Control<=0

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

```

```

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

```

```

if Control==1

    while Control>=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

```

```

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);

```

```

p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

%Finding the spot with a tolerance%

while abs(xdot)>1e-6

    if xdot<=0

        open=sol_step;
        close=check_xdot(2,1);

    else

        open=check_xdot(1,1);
        close=sol_step;

    end

    check_xdot=[];

    for i=(open):(close-open):(close)

        x0_set=i;

        r=(-1)*x0_set;
        teta=pi;
        v=0;
        u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

        Control=-1;

        while Control<=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

```

```

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))...
+(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@g(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

```



```

if Control==1

    while Control>=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);

```

```

        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))...
+ (xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@(t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);

```

```

    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    check_xdot=[check_xdot;x0_set x y xdot ydot];

end

hp=0;

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(-1)*x0_set;
teta=pi;
v=0;
u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
hp=hp+0.1;

Control=1;

while Control<=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

```

```

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    hp=hp-0.1;

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))).
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

    end

    hp=hp+t0;

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        Control=(-1)^j;

        if Control==1

```

```

while Control>=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . .
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

else

while Control<=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . .
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

```

```

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);

```

```

    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

end

char_curve=[char_curve;h sol_step];
fp=hp*2;
period_curve=[period_curve;h fp];

end

plot(char_curve(:,1),char_curve(:,2))
%plot(period_curve(:,1),period_curve(:,2))

```

A.5.3. Positive x - Negative u

```

%Finding the characteristic curve%

char_curve=[];
period_curve=[];

for k=(-21):(-0.02):(-21.3)

    h=k;
    bt=27;

    how_many_pass=4;

    x0=0.1546;
    check_xdot=[];

    options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

    g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+...
3*(x(1)^3)*(cos(x(2)))^2+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*...
x(3)*(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

```

```

for i=(x0-0.0001):0.0002:(x0+0.0001)

    x0_set=i;

    r=(1)*x0_set;
    teta=0;
    v=0;
    u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

    Control=-1;

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...

```



```

(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

```

```

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

```

```

t0=t1;

[t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

check_xdot=[check_xdot;x0_set x y xdot ydot];

end

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/...
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(1)*x0_set;
teta=0;
v=0;
u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

Control=-1;

```

```

while Control<=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

```

```

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    else

        while Control<=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);

```

```

        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

```

```

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

%Finding the spot with a tolerance%

while abs(xdot)>1e-6

    if xdot<=0

        open=sol_step;
        close=check_xdot(2,1);

    else

        open=check_xdot(1,1);
        close=sol_step;

    end

    check_xdot=[];

    for i=(open):(close-open):(close)

        x0_set=i;

        r=(1)*x0_set;
        teta=0;
        v=0;
        u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

        [t,xa]=ode45(@g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

```

```

Control=-1;

while Control<=0

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))...
+(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2)...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

```



```

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^j;

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    else

        while Control<=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

```

```

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
+ (xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

```

```

    %Recording%

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    check_xdot=[check_xdot;x0_set x y xdot ydot];

end

hp=0;

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(1)*x0_set;
teta=0;
v=0;
u=-sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@g(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);
hp=hp+0.1;

Control=-1;

while Control<=0

    [t,xa]=ode45(@g(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);

```

```

    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%For more than one passes%

```

```

for j=2:how_many_pass

    [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    Control=(-1)^j;

    if Control==1

        while Control>=0

            [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
            hp=hp+0.1;

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    else

        while Control<=0

            [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
            hp=hp+0.1;

            r=xa(end,1);

```

```

        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+...
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

```

```

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

end

char_curve=[char_curve;h sol_step];
fp=hp*2;
period_curve=[period_curve;h fp];

end

plot(char_curve(:,1),char_curve(:,2))
%plot(period_curve(:,1),period_curve(:,2))

```

A.5.4. Positive x - Positive u

```

%Finding the characteristic curve%

char_curve=[];
period_curve=[];

for k=(-21):(-0.02):(-21.08)

    h=k;
    bt=27;

```

```

how_many_pass=7;

x0=0.161150;
check_xdot=[];

options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);

g=@(t,x) [x(3)*x(1);x(4);(x(3)^2)/2+(x(4)^2)-1+2*x(4)*(x(1)^(1.5))+...
3*(x(1)^3)*(cos(x(2))^2)+bt*(x(1)^2)*cos(x(2));-0.5*x(3)*x(4)-2*...
x(3)*(x(1)^(1.5))-3*(x(1)^3)*sin(x(2))*cos(x(2))-bt*(x(1)^2)*sin(x(2))];

for i=(x0-0.001):0.002:(x0+0.001)

    x0_set=i;

    r=(1)*x0_set;
    teta=0;
    v=0;
    u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

    Control=1;

    while Control>=0

        [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;
    end
end

```



```

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^(j-1);

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2))...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

```

```

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

else

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . . .
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

end

```

```

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

check_xdot=[check_xdot;x0_set x y xdot ydot];

end

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/...
(check_xdot(2,4)-check_xdot(1,4));

```

```

x0_set=sol_step;

r=(1)*x0_set;
teta=0;
v=0;
u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

Control=-1;

while Control<=0

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

```

```

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))...
+(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%For more than one passes%

for j=2:how_many_pass

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

    Control=(-1)^(j-1);

    if Control==1

        while Control>=0

            [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

```

```

        end

    else

        while Control<=0

            [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

```

```

t0=t1;

[t, xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

t1=t0-(((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

%Finding the spot with a tolerance%

while abs(xdot)>1e-6

    if xdot<=0

        open=sol_step;
        close=check_xdot(2,1);

    else

        open=check_xdot(1,1);
        close=sol_step;

    end

    check_xdot=[];

```

```

for i=(open):(close-open):(close)

    x0_set=i;

    r=(1)*x0_set;
    teta=0;
    v=0;
    u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

    [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

    Control=-1;

    while Control<=0

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

    cross1=xa(1,1);
    cross2=xa(1,2);
    cross3=xa(1,3);
    cross4=xa(1,4);

    t0=0;

    t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2))))...

```



```

+ (xa(1,1)*xa(1,4)*cos(xa(1,2)))));

    while abs(t1-t0)>1e-6

        t0=t1;

        [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)
+ (xa(end,1)*xa(end,4)*cos(xa(end,2))))));

    end

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);

        Control=(-1)^(j-1);

        if Control==1

            while Control>=0

                [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)
(xa(end,3)) (xa(end,4))], options);

                r=xa(end,1);
                teta=xa(end,2);
                v=xa(end,3);
                u=xa(end,4);

                pr=v*r^(-1/2);
                pteta=u*r^(1/2)+r^2;

                q1=r*cos(teta);
                q2=r*sin(teta);
                p1=pr*cos(teta)-(pteta/r)*sin(teta);
                p2=pr*sin(teta)+(pteta/r)*cos(teta);

                x=q1;
                y=q2;
                xdot=p1+y;
                ydot=p2-x;

```

```

        Control=y;

    end

else

    while Control<=0

        [t, xa]=ode45(@ (t, x) g(t, x), [0 0.1], [(xa(end,1)) (xa(end,2))
(xa(end,3)) (xa(end,4))], options);

        r=xa(end,1);
        teta=xa(end,2);
        v=xa(end,3);
        u=xa(end,4);

        pr=v*r^(-1/2);
        pteta=u*r^(1/2)+r^2;

        q1=r*cos(teta);
        q2=r*sin(teta);
        p1=pr*cos(teta)-(pteta/r)*sin(teta);
        p2=pr*sin(teta)+(pteta/r)*cos(teta);

        x=q1;
        y=q2;
        xdot=p1+y;
        ydot=p2-x;

        Control=y;

    end

end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+
(xa(1,1)*xa(1,4)*cos(xa(1,2))));

while abs(t1-t0)>1e-6

```

```

t0=t1;

[t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)
+ (xa(end,1)*xa(end,4)*cos(xa(end,2))))));

end

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

check_xdot=[check_xdot;x0_set x y xdot ydot];

end

hp=0;

sol_step=(check_xdot(1,1)*check_xdot(2,4)-check_xdot(2,1)*check_xdot(1,4))/
(check_xdot(2,4)-check_xdot(1,4));

x0_set=sol_step;

r=(1)*x0_set;
teta=0;
v=0;
u=sqrt(-(-r*h+v^2-2-3*r^3*(cos(teta))^2-2*bt*r^2*cos(teta)));

[t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(r) (teta) (v) (u)], options);

```

```

hp=hp+0.1;

Control=-1;

while Control<=0

    [t,xa]=ode45(@g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
    hp=hp+0.1;

    r=xa(end,1);
    teta=xa(end,2);
    v=xa(end,3);
    u=xa(end,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);
    p1=pr*cos(teta)-(pteta/r)*sin(teta);
    p2=pr*sin(teta)+(pteta/r)*cos(teta);

    x=q1;
    y=q2;
    xdot=p1+y;
    ydot=p2-x;

    Control=y;

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

while abs(t1-t0)>1e-6

    t0=t1;

```

```

        [t,xa]=ode45(@ (t,x) g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

        t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)))+
(xa(end,1)*xa(end,4)*cos(xa(end,2)))));

    end

    hp=hp+t0;

    %For more than one passes%

    for j=2:how_many_pass

        [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
        hp=hp+0.1;

        Control=(-1)^(j-1);

        if Control==1

            while Control>=0

                [t,xa]=ode45(@ (t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) ...
(xa(end,3)) (xa(end,4))], options);
                hp=hp+0.1;

                r=xa(end,1);
                teta=xa(end,2);
                v=xa(end,3);
                u=xa(end,4);

                pr=v*r^(-1/2);
                pteta=u*r^(1/2)+r^2;

                q1=r*cos(teta);
                q2=r*sin(teta);
                p1=pr*cos(teta)-(pteta/r)*sin(teta);
                p2=pr*sin(teta)+(pteta/r)*cos(teta);

                x=q1;
                y=q2;
                xdot=p1+y;
                ydot=p2-x;

                Control=y;

```

```

        end

    else

        while Control<=0

            [t,xa]=ode45(@(t,x) g(t,x), [0 0.1], [(xa(end,1)) (xa(end,2)) . . .
(xa(end,3)) (xa(end,4))], options);
            hp=hp+0.1;

            r=xa(end,1);
            teta=xa(end,2);
            v=xa(end,3);
            u=xa(end,4);

            pr=v*r^(-1/2);
            pteta=u*r^(1/2)+r^2;

            q1=r*cos(teta);
            q2=r*sin(teta);
            p1=pr*cos(teta)-(pteta/r)*sin(teta);
            p2=pr*sin(teta)+(pteta/r)*cos(teta);

            x=q1;
            y=q2;
            xdot=p1+y;
            ydot=p2-x;

            Control=y;

        end

    end

end

cross1=xa(1,1);
cross2=xa(1,2);
cross3=xa(1,3);
cross4=xa(1,4);

hp=hp-0.1;

t0=0;

t1=t0-((xa(1,1)*sin(xa(1,2)))/((xa(1,3)*xa(1,1)*sin(xa(1,2)))+...
(xa(1,1)*xa(1,4)*cos(xa(1,2)))));

```

```

while abs(t1-t0)>1e-6

    t0=t1;

    [t,xa]=ode45(@g(t,x), [0 t0], [(cross1) (cross2) ...
(cross3) (cross4)], options);

    t1=t0-((xa(end,1)*sin(xa(end,2)))/((xa(end,3)*xa(end,1)*sin(xa(end,2)).
+ (xa(end,1)*xa(end,4)*cos(xa(end,2)))));

end

hp=hp+t0;

%Recording%

r=xa(end,1);
teta=xa(end,2);
v=xa(end,3);
u=xa(end,4);

pr=v*r^(-1/2);
pteta=u*r^(1/2)+r^2;

q1=r*cos(teta);
q2=r*sin(teta);
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

end

char_curve=[char_curve;h sol_step];
fp=hp*2;
period_curve=[period_curve;h fp];

end

plot(char_curve(:,1),char_curve(:,2))
%plot(period_curve(:,1),period_curve(:,2))

```

A.6. Plotting Orbits

```
clear all
close all
dr=@(r,th,v,u) r*v;
dth=@(r,th,v,u) u;
dv=@(r,th,v,u) v^2/2+u^2-1+2*u*r^(3/2)+3*r^3*(cos(th))^2-27*r^2*cos(th);
du=@(r,th,v,u) -(1/2)*u*v-2*v*r^(3/2)-3*r^3*cos(th)*sin(th)+27*r^2*sin(th);
deriv=@(t,X) [dr(X(1),X(2),X(3),X(4)); dth(X(1),X(2),X(3),X(4));...
dv(X(1),X(2),X(3),X(4));du(X(1),X(2),X(3),X(4))];
h=-30;
rr=0.06;
thth=pi;
vv=0;
uu=-sqrt(-(-rr*h+vv^2-2-3*rr^3*(cos(thth))^2+54*rr^2*cos(thth)));
Xini=[rr,thth,vv,uu];
%
options=odeset('AbsTol',1.e-12,'RelTol',1.e-10);
%
[t,X]=ode45(deriv,[0,100],Xini,options);
%
length(t)
jacobi=(h*X(:,1))-(X(:,3).^2+X(:,4).^2-2-3*X(:,1).^3.*(cos(X(:,2))).^2...
+54*X(:,1).^2.*cos(X(:,2)));
```

A.7. Period and Range

```
% clear all
% close all

[m,n]=size(char_curve);
the_range=[];
real_time=[];

for z=1:1:m

    h_set=char_curve(z,1);
    rr_set=char_curve(z,2);

    bt=27;

    dr=@(r,th,v,u,T) r*v;
    dth=@(r,th,v,u,T) u;
    dv=@(r,th,v,u,T) v^2/2+u^2-1+2*u*r^(3/2)+3*r^3*(cos(th))^2+bt*r^2*cos(th);
    du=@(r,th,v,u,T) -(1/2)*u*v-2*v*r^(3/2)-3*r^3*cos(th)*sin(th)-bt*r^2*sin(th);
```



```

dT=@(r,th,v,u,T) r^(1.5);
deriv=@(t,X) [dr(X(1),X(2),X(3),X(4),X(5)); dth(X(1),X(2),X(3),X(4),X(5));dv(X(1),X(2),X(3),X(4),X(5));du(X(1),X(2),X(3),X(4),X(5));dT(X(1),X(2),X(3),X(4),X(5))];
h=h_set;
rr=-1*rr_set;
thth=pi;
vv=0;
uu=-sqrt(-(-rr*h+vv^2-2-3*rr^3*(cos(thth))^2-2*bt*rr^2*cos(thth)));
TT=0;
Xini=[rr,thth,vv,uu,TT];
%
options=odeset('AbsTol',1.e-12,'RelTol',1.e-10); % precision of the method
% the default is AbsTol=1.e-6 and RelTol=1.e-3
[t,X]=ode45(deriv,[0,period_curve(z,2)],Xini,options);
%
length(t)
jacobii=(h*X(:,1))-(X(:,3).^2+X(:,4).^2-2-3*X(:,1).^3.*(cos(X(:,2))).^2...
-2*bt*X(:,1).^2.*cos(X(:,2)));

minr=min(X(:,1));
maxr=max(X(:,1));
the_range=[the_range;h minr maxr];
real_time=[real_time; X(end,5)];

end

plot(the_range(:,1),the_range(:,2)) %or (:,3)
plot(the_range(:,1),real_time(:,1));

%Beyond this point is for plotting the orbit%
%plot(X(:,1).*cos(X(:,2)),X(:,1).*sin(X(:,2)))

j=ans;

orbit_xy=[];

for i=1:j

    r=X(i,1);
    teta=X(i,2);
    v=X(i,3);
    u=X(i,4);

    pr=v*r^(-1/2);
    pteta=u*r^(1/2)+r^2;

    q1=r*cos(teta);
    q2=r*sin(teta);

```

```
p1=pr*cos(teta)-(pteta/r)*sin(teta);
p2=pr*sin(teta)+(pteta/r)*cos(teta);

x=q1;
y=q2;
xdot=p1+y;
ydot=p2-x;

orbit_xy=[orbit_xy;x y xdot ydot];

end

plot(orbit_xy(:,1),orbit_xy(:,2))
```