BACHELOR'S DEGREE THESIS

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Title: Green Functions of the Damped Wave Equation and Applications

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Thanks to Joan Solà-Morales for tutoring and advising me so well in this exciting work, his comments and ideas were enlightening, always suggesting new and different approaches. Finally, thanks to my family for their invaluable support and patience all the time I was working on this bachelor's degree thesis.

Abstract

This work presents some results on the Green Functions of the Damped Wave Equation. We exhibit its fundamental solution and we study it. This fundamental solution allows us to obtain the solution of the initial value problem when the initial conditions are general distributions. We also give conditions on these initial conditions that insure us uniqueness of solution. To this purpose we develop some theory about the semigroup, as an application, and we finally apply our results to the price evolution of an asset in a Financial Model we propose.

Keywords

Partial Differential Equations, Damped Wave Equation, Green Function, Riemann Function

MSC 2010:

35A08, 35L05, 35Q91.

Contents

1 Introduction

The aim of this work is to study the so-called damped wave equation

$$u_{tt} + ku_t = c^2 u_{xx} \tag{1}$$

for $u(x,t)$ and $(x,t) \in \mathbb{R}^2$ and solve it given general initial conditions that could include distributions in the x-variable and not only functions. More precisely, we are going to find the Green function associated to the equation and, from there, we will be able to deduce some interesting properties and applications.

We will see that the damped wave equation can be transformed, via a change of variable, to the PDE

$$v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v \tag{2}$$

Therefore, it will be the case that everything we can say of the original applies to the transformed and vice-versa.

Before attacking the full problem directly, we will first do some heuristics in Chapter 2 so as to familiarize ourselves with the equation and to comprehend which ideas or techniques may be later useful. At this point, we have used some exercises of [1] as a vital source of inspiration.

Once we know more about (2) we look for one of its fundamental solutions, that is the solution of

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = \delta(x) & x \in \mathbb{R} \end{cases} \tag{3}$$

where $\delta(x)$ is the Dirac delta distribution. In Chapter 3 we are able to prove this problem admits a solution by finding it. It is given by

$$\psi(x,t) = \operatorname{sgn}(t) \frac{1}{2c} I_0 \left(2\alpha \sqrt{c^2 t^2 - x^2} \right) \mathcal{X}_{[-c|t|, c|t|]}(x)$$

where I_0 denotes the modified Bessel function of the first kind of order 0. Moreover the solution presents nice properties, a good degree of regularity when considered as an application from one space to another. This solution will be key in our work, the fundamental stone in which almost everything relies on since we will use this solution to generate the sometimes called Green's or Riemann's function.

In fact, we understand Riemann's or Green's functions as synonyms for our case, as it is frequent to associate the Riemann's name to the Green functions for hyperbolic linear partial differential equations in two variables, and so we will use the two terms indistinctly.

In Chapter 4 we will obtain the solution of (3) when we impose $v_t(x,0) = g \in \mathcal{D}'(\mathbb{R})$, where $\mathcal{D}'(\mathbb{R})$ is the usual space of distributions on $\mathbb{R}$, that is, when the initial condition is also a distribution or at least for distributions of some kind. There, we will see that the resulting solution may not be a function but a distribution, too.

At this point, in Chapter 5 we will go back to the original problem and ask ourselves for the solution of

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{4}$$

From the solution of (3) and with $(f,g) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, where $H^1(\mathbb{R})$ is the Sobolev Space of square integrable functions whose weak derivatives are also square integrable, we will be able to prove that the only solution is

$$u(x,t) = e^{\frac{-k}{2}t} \left\{ \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \alpha ct \int_{x-ct}^{x+ct} \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} f(y)dy \right\} +$$

$$+ e^{\frac{-k}{2}t} \left\{ \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{\lambda}) \left[\frac{k}{2} f(y) + g(y)\right] dy \right\}$$

where $\lambda = c^2t^2 - (x-y)^2$, $\alpha = \frac{k}{4c}$ and I_0 is the modified Bessel function of order zero and parameter 1.

We would like to note that when $k = 0$, problem (4) reduces to the classical 1-D wave equation, we have $\alpha = 0$ and the solution is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

which is precisely the well-known D'Alembert formula for the wave-equation.

There, we also present the first application, the semigroup which will help us solve this problem. We will prove interesting semigroup properties of the operator solution.

In Chapter 6 we question ourselves about the uniqueness of the problems we have been dealing with so far, under which circumstances can we expect to have only one solution.

We will give some results on this matter and we will link it to the complete problem (4) when both f and g are general distributions. In fact we will show that the solution of (4) when f and g are distributions is the distribution u defined by

$$\langle u, \varphi \rangle := e^{-\frac{k}{2}t} \left(\left\langle f, \left(\psi_t + \frac{k}{2}\psi\right) * \varphi \right\rangle + \langle g, \psi * \varphi \rangle \right) \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$$

where ψ_t denotes the time derivative of ψ and is given by

$$\psi_t(\cdot,t) = \frac{1}{2}\left[\delta(\cdot - ct) + \delta(\cdot + ct)\right] + \alpha c|t| \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - \cdot^2}\right)}{\sqrt{c^2t^2 - \cdot^2}} \mathcal{X}_{[-c|t|,c|t|]}(\cdot) \in \mathcal{D}'(\mathbb{R})$$

The second application, much more applied, is suggested by a completely different approach to the problem we have studied so far. In Chapter 7 we model the movement of a particle (or the **price of an asset**) under a certain probability law that ends up to obtain a problem like (4) with distributions as initial conditions.

We will solve this particular problem and then we will implement the result in a numerical scheme to observe and study the behaviour of the price of the asset or the position of the particle.

We would like to stress that despite the fact that some of the results we present are already known, we have chosen an original and maybe never-done (up to our knowledge) approach.

For example, the damped wave equation is treated in [2] as the telegraph equation and is very briefly studied using Fourier Transforms and Riemann functions.

The same happens in [3], where the equation is reduced and treated using Laplace and Fourier Transforms. Once a desired result is reached, the author recovers the solution using integral table to undo the transformations.

Another example would be [4] in which the author studies the uniqueness of the problem but without very much detail. We use the idea he gives in the book in order to prove the uniqueness of the problem.

In contrast to all these approaches, mainly Fourier and Laplace Transform, we have developed our theory and checked our results computing almost directly, using the formal definitions of Distribution Theory and avoiding the Transforms.

We believe that this option has allowed us to truly understand how the core of the solution and the equation works. Furthermore, we have been able to construct new solutions of new problems and reach possibly new results thanks to the knowledge we have gained by working directly with the problem.

2 Some Heuristics

As we have said, we first do some heuristic reasoning with the problems we have in order to familiarize with them and get some insights. Let us first consider a point $y \in \mathbb{R}$ and the equation coefficients $k, c \in \mathbb{R}^+$. Our initial goal is to solve

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = \delta(x-y) & x \in \mathbb{R}. \end{cases} \tag{5}$$

This problem arises when we consider a rope of infinite length that moves on a table. We consider there is some friction between the rope and the table. Initially, the rope is resting straight but at the point y there is a sudden impulse denoted by δ, the *Dirac's delta*.

Let us consider $v(x,t) = e^{\frac{k}{2}t}u(x,t)$ another function. If $u(x,t)$ satisfies (5) then $v(x,t)$ satisfies

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4}v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = \delta(x-y) & x \in \mathbb{R} \end{cases} \tag{6}$$

Indeed, we first check the initial conditions of $v(x,t)$

$$v(x,0) = e^{\frac{k}{2}\cdot 0}u(x,0) = u(x,0) = 0$$

$$v_t(x,0) = \frac{k}{2}e^{\frac{k}{2}\cdot 0}u(x,0) + e^{\frac{k}{2}\cdot 0}u_t(x,0) = 0 + u_t(x,0) = \delta(x-y)$$

As for the equation, we have

$$v_t(x,t) = \frac{k}{2}e^{\frac{k}{2}t}u(x,t) + e^{\frac{k}{2}t}u_t(x,t) \Rightarrow$$

$$\Rightarrow v_{tt}(x,t) = \left(\frac{k}{2}\right)^2 e^{\frac{k}{2}t}u(x,t) + ke^{\frac{k}{2}t}u_t(x,t) + e^{\frac{k}{2}t}u_{tt}(x,t)$$

and it is clear that $v_{xx}(x,t) = e^{\frac{k}{2}t}u_{xx}(x,t)$. Then,

$$v_{tt} = c^2 v_{xx} + \frac{k^2}{4}v \Leftrightarrow \left(\frac{k}{2}\right)^2 e^{\frac{k}{2}t}u + ke^{\frac{k}{2}t}u_t + e^{\frac{k}{2}t}u_{tt} = e^{\frac{k}{2}t}u_{xx} + \frac{k^2}{4}e^{\frac{k}{2}t}u \Leftrightarrow$$

$$\Leftrightarrow ke^{\frac{k}{2}t}u_t + e^{\frac{k}{2}t}u_{tt} = e^{\frac{k}{2}t}u_{xx} \Leftrightarrow u_{tt} + ku_t = u_{xx}$$

We note that we are left with the search of an eigenfunction v with eigenvalue $\frac{k^2}{4}$ of the differential operator $\partial_t^2 - c^2\partial_x^2$. We decide to look for solutions of the form $v(x,t) = f(\lambda)$ with $\lambda = c^2t^2 - (y-x)^2$ since this form is is closely related to the characteristics of the operator. This way, f satisfies $\lambda f'' + f' - \alpha^2 f = 0$ with $\alpha = \frac{k}{4c}$. Indeed,

$$v_t = 2c^2tf'(\lambda) \Rightarrow v_{tt} = 4c^4t^2f''(\lambda) + 2c^2f'(\lambda)$$

$$v_x = 2(y-x)f'(\lambda) \Rightarrow v_{xx} = 4(y-x)^2 f''(\lambda) - 2f'(\lambda)$$

Imposing the PDE of problem (6), we have

$$4c^4t^2 f''(\lambda) + 2c^2 f'(\lambda) = c^2\left(4(y-x)^2 f''(\lambda) - 2f'(\lambda)\right) + \frac{k^2}{4} f(\lambda) \Leftrightarrow$$

$$\Leftrightarrow 4c^2\left(c^2t^2 - (y-x)^2\right) f''(\lambda) + 4c^2 f'(\lambda) = \frac{k^2}{4} f(\lambda) \Leftrightarrow$$

$$\Leftrightarrow \lambda f'' + f' - \frac{k^2}{16c^2} f = 0 \Leftrightarrow \lambda f'' + f' - \alpha^2 f = 0$$

At this point, we denote $\xi = 2\alpha\sqrt{\lambda}$ for those $\lambda \geq 0$. It can be seen that

$$v(x,t) = f(\lambda) = g(\xi) = AI_0(2\alpha\sqrt{\lambda}) + BK_0(2\alpha\sqrt{\lambda})$$

since if we consider solutions of the form $f(\lambda) = g(2\alpha\sqrt{\lambda}) = g(\xi)$, we derive

$$f'(\lambda) = g'(2\alpha\sqrt{\lambda})\frac{2\alpha}{2\sqrt{\lambda}} = g'(2\alpha\sqrt{\lambda})\frac{\alpha}{\sqrt{\lambda}}$$

$$f''(\lambda) = g''(2\alpha\sqrt{\lambda})\frac{\alpha^2}{\lambda} + g'(2\alpha\sqrt{\lambda})\left(\frac{-\alpha}{\lambda}\frac{1}{2}\frac{1}{\sqrt{\lambda}}\right) = g''(2\alpha\sqrt{\lambda})\frac{\alpha^2}{\lambda} - g'(2\alpha\sqrt{\lambda})\left(\frac{\alpha}{2\sqrt{\lambda^3}}\right)$$

and we impose the equation

$$0 = \lambda f'' + f' - \alpha^2 f = \alpha^2 g''(\xi) - g'(\xi)\frac{\alpha}{2\sqrt{\lambda}} + g'(\xi)\frac{\alpha}{\sqrt{\lambda}} - \alpha^2 g(\xi) =$$

$$= \alpha^2 g''(\xi) + g'(\xi)\frac{\alpha}{2\sqrt{\lambda}} - \alpha^2 g(\xi) = \frac{1}{\lambda}\left[\alpha^2\lambda g''(\xi) + g'(\xi)\frac{2\alpha\sqrt{\lambda}}{4} - \alpha^2\lambda g(\xi)\right] =$$

$$= \frac{1}{4\lambda}\left[4\alpha^2\lambda g''(\xi) + g'(\xi)2\alpha\sqrt{\lambda} - 4\alpha^2\lambda g(\xi)\right] = \frac{1}{4\lambda}\left[\xi^2 g''(\xi) + \xi g'(\xi) - \xi^2 g(\xi)\right] = 0$$

to be left with a quite particular ODE. Its solution is a linear combination of the modified Bessel equations of order $n = 0$,

$$g(\xi) = AI_0(\xi) + BK_0(\xi)$$

Given the initial conditions of our problem, our solution $v(x,t)$ is not zero at x only if it has passed enough time t for the wave originated at y to propagate from y to x at a speed c. Then, it is reasonable to assume

$$v(x,t) = \begin{cases} AI_0(2\alpha\sqrt{\lambda}) + BK_0(2\alpha\sqrt{\lambda}) & \text{if } (y-x)^2 \leq c^2t^2 \\ 0 & \text{otherwise} \end{cases}$$

Let us now consider the same problem as before but this time with $u_t(x,t) = v_t(x,t) = g(x)$ a function at least continuous. Since

$$g(x) = \int_{\mathbb{R}} g(y)\delta(x-y)dy$$

we can think $g(x)$ as a superposition of impulses of heigh $g(y)$ centred each one at the point y so we have $\delta(x-y)g(y)$ as the initial condition of our solution.. Then, we can construct the solution $v(x,t)$ by superposing the solutions of each problem.

Besides that, we must take into account where our solution is not null. Using what we have deduce from the physical interpretation, our formula is not zero when $(y-x)^2 \leq c^2t^2$ which is equivalent to $y \in [x-c|t|, x+c|t|]$ and also to $x \in [y-c|t|, y+c|t|]$, a property we will use in the future.

Therefore,

$$v(x,t) = \int_{\mathbb{R}} f(\lambda)\mathcal{X}_{[x-ct,x+ct]}(y)g(y)dy = \int_{x-ct}^{x+ct} f(\lambda)g(y)dy =$$

$$= \int_{x-ct}^{x+ct} \left[AI_0(2\alpha\sqrt{\lambda}) + BK_0(2\alpha\sqrt{\lambda})\right] g(y)dy$$

with A and B still to be determined. Let us check the initial conditions, beginning with computing

$$v_t(x,t) = c\left[AI_0(2\alpha\sqrt{0}) + BK_0(2\alpha\sqrt{0})\right][g(x+ct)+g(x-ct)] +$$

$$+ \int_{x-ct}^{x+ct} \left[AI_0'(2\alpha\sqrt{\lambda}) + BK_0'(2\alpha\sqrt{\lambda})\right] \frac{2\alpha c^2 t}{\sqrt{\lambda}} g(y)dy$$

since $\lambda = 0$ when $y = x \pm ct$. We also have that $I_0(0) = 1$ and $K_0(z) \to \infty$ when $z \to 0$ so for v_t to exist and be bounded we impose $B = 0$. Therefore, it reduces to

$$v_t(x,t) = cA[g(x+ct)+g(x-ct)] + \int_{x-ct}^{x+ct} AI_0'(2\alpha\sqrt{\lambda})\frac{2\alpha c^2 t}{\sqrt{\lambda}} g(y)dy$$

Before doing the limit when $t \to 0$ we have (see Appendix A.3)

$$\lim_{z\to 0} \frac{I_0'(2\alpha z)}{z} = \alpha$$

It is then clear that

$$\lim_{t\to 0} \int_{x-ct}^{x+ct} AI_0'(2\alpha\sqrt{\lambda})\frac{2\alpha c^2 t}{\sqrt{\lambda}} g(y)dy = 0$$

because the interval of integration reduces only to the point x and the integrand is not only bounded but also tends to 0 as t does so. Hence, we have

$$\lim_{t\to 0} v_t(x,t) = 2cAg(x) = g(x)$$

assuming $A = \frac{1}{2c}$ and g is continuous. We have just seen $v(x,t)$ satisfies one initial condition. As for the other, let us use all we have deduce so far. From

$$v(x,t) = \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{\lambda})g(y)dy$$

and using the same argument, when $t \to 0$ the integral reduces to the single point x and the integrand is bounded since $I_0(z) \to 1$ as $z \to 0$ we can easily show that $v(x,0) = 0$. In conclusion v is such that

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

All we have done is quite heuristic, not very rigorous. However, it has provided us a useful insight into the equation and its properties, as well as a good candidate for the real solution of the problem. Let us now go back and study consciously the basics of the problem.

3 The Fundamental Solution

3.1 Existence

The main goal of this chapter is to find the fundamental solution, usually known as the Riemann Function, of the problem

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = \delta(x) & x \in \mathbb{R} \end{cases} \tag{7}$$

To this purpose, we have the following

Theorem 3.1. *Let*

$$v(x,t) = \begin{cases} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right) \mathcal{X}_{[-ct,ct]}(x) & t > 0 \\ 0 & t = 0 \\ -\frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right) \mathcal{X}_{[ct,-ct]}(x) & t < 0 \end{cases} \tag{8}$$

Then, $v(x,t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$ *and solves (7) in the sense of distributions.*

Proof. First of all, let us remark that when $x = \pm ct$ we have $v(x,t) \equiv \pm\frac{1}{2c}$ whether t is positive or negative and notice there is a discontinuity in the straight lines $\{x = ct\}$ and $\{x = -ct\}$, the *characteristics* of our equation.

Hence, given the discontinuity, we can't expect our candidate to be a classical solution of the problem. However, we can extend our notion of solution to those who satisfy some kind of weak formulation.

Let L be the differential operator defined by

$$L(u) = u_{tt} - c^2 u_{xx} - \frac{k^2}{4} u$$

The aim of the proof is to show that $v(x,t)$ satisfies the initial conditions of (7) in the sense of distributions and that

$$\int_{\mathbb{R}^2} L(v)\varphi \, dxdt = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^2\right)$$

from which we will deduce $L(v) = 0$ in the sense of distributions.

In particular, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ be a smooth function with compact support K, say included in a ball centred at the origin, and this ball included inside the rectangle $R = [-a,a] \times [-\frac{a}{c}, \frac{a}{c}] \subset \mathbb{R} \times \mathbb{R}$ for some $a \in \mathbb{R}^+$. Below there's a picture of the situation. The green rectangle denotes R, the red circle denotes the open ball that contain the support of φ and the blue area the region where v is not 0.

We begin with noting that

$$\int_{\mathbb{R}^2} L(v)\varphi \, dxdt := \int_{\mathbb{R}^2} v L^*(\varphi) dxdt = \int_{\mathbb{R}^2} v\left(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\right) dxdt$$

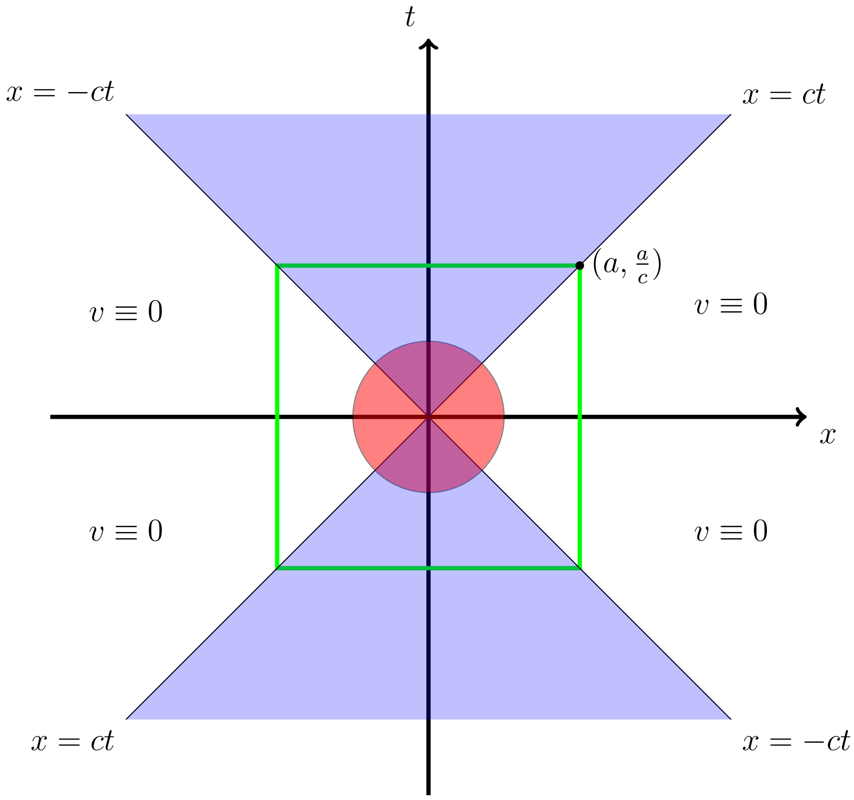


Figure 1: Scheme of the situation when integrating

by definition of the adjoint of the operator, which in this case it is itself. Now, we are left with a well defined equation in which we may only need v to be square integrable in a bounded domain. Therefore, we have to see that our $v(x,t)$ is such that

$$\int_{\mathbb{R}^2} v\left(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\right) dxdt = 0 \;\; \forall \varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^2\right)$$

Let us remark that $v(x,t) \equiv 0$ in $\{|x| > c\,|t|\}$ and that $v(x,t)$ is infinitely differentiable in $\{|x| < c\,|t|\}$.

Given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, due to its compact support $K \subset R$ we have

$$\int_{\mathbb{R}^2} v\left(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\right) dxdt = \int_R v\left(\varphi_{tt} - c^2\varphi_{xx} - \frac{k^2}{4}\varphi\right) dxdt$$

and let us for now focus on

$$\int_R v\left(\varphi_{tt} - c^2\varphi_{xx}\right) = \int_R v\mathrm{div}\underbrace{\begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix}}_{X} = \int_R v\mathrm{div}X =$$

$$= \int_R \mathrm{div}(vX) - \int_R \nabla v \cdot X$$

Let us recall that $v \equiv 0$ when $|x| > c|t|$ so our vector field vX is not null in two distinct triangles inside R. These are

$$\bigtriangledown = \left\{(x,t) \in \mathbb{R}^2 \mid 0 < t < \frac{a}{c},\ -ct < x < ct\right\}$$

and

$$\bigtriangleup = \left\{(x,t) \in \mathbb{R}^2 \mid -\frac{a}{c} < t < 0,\ ct < x < -ct\right\}$$

Inside these triangles our vector field vX is continuously differentiable up to the border because both v and X are so *inside* the two triangles. Hence, we can use the divergence theorem in each of the two triangles.

$$\int_R \operatorname{div}(vX) = \int_{\bigtriangledown} \operatorname{div}(vX) + \int_{\bigtriangleup} \operatorname{div}(vX) =$$

$$= \int_{\partial\bigtriangledown} vX \cdot n\ d\ell + \int_{\partial\bigtriangleup} vX \cdot n\ d\ell = \int_{\vee} vX \cdot n\ d\ell + \int_{\wedge} vX \cdot n\ d\ell$$

where the sets $\vee$ and $\wedge$ are

$$\vee = \left\{(x,t) \in \mathbb{R}^2 \mid 0 < t < \frac{a}{c},\ x = \pm ct\right\}$$

$$\wedge = \left\{(x,t) \in \mathbb{R}^2 \mid -\frac{a}{c} < t < 0,\ x = \pm ct\right\}$$

The boundaries of the triangles also contain the sets $[-a,a] \times \left\{\frac{a}{c}\right\}$ and $[-a,a] \times \left\{-\frac{a}{c}\right\}$. However, the integrals on such sets are 0 because they fall outside the compact support of vX (due to φ) and therefore we just integrate on $\vee$ and $\wedge$ taking n the normal exterior unit vector on such sets. We strongly recommend to look at the illustration again to comprehend this reasoning.

Integral $\vee$

In order to integrate on $\vee$, we divide it according to the sign of x and apply the corresponding values of n in each case. First, we parametrize the line $\sigma(s) = \left(s, \frac{-s}{c}\right)$ with $s \in [-a, 0]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{-1}{c}, -1\right)$. Thus,

$$\int_{-a}^{0} v \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{c} & -1 \end{pmatrix} \left(s, -\frac{s}{c}\right) ds = \int_{-a}^{0} v\left(c\varphi_x - \varphi_t\right)\left(s, -\frac{s}{c}\right) ds =$$

$$= c\int_{-a}^{0} v\left(\varphi_x - \frac{1}{c}\varphi_t\right)\left(s, -\frac{s}{c}\right) ds = c\int_{-a}^{0}\left(v\frac{d\varphi}{ds}\right)\left(s, -\frac{s}{c}\right) ds$$

$$= c\left(\left(v\varphi\right)(0, 0^+) - \int_{-a}^{0}\left(\varphi\frac{dv}{ds}\right)\left(s, \frac{-s}{c}\right) ds\right)$$

where we have used the integration by parts formula, the compact support of φ and the proper directional derivative of a function.

We now parametrize the other part of the set, the line given by $\sigma(s) = \left(s, \frac{s}{c}\right)$ with $s \in [0,a]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{1}{c}, -1\right)$. Thus,

$$\int_0^a v \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{c} & -1 \end{pmatrix} \left(s, \frac{s}{c}\right) ds = \int_0^a v\left(-c\varphi_x - \varphi_t\right)\left(s, \frac{s}{c}\right) ds =$$

$$= -c\int_0^a v\left(\varphi_x + \frac{1}{c}\varphi_t\right)\left(s, \frac{s}{c}\right) ds = -c\int_0^a \left(v\frac{d\varphi}{ds}\right)\left(s, \frac{s}{c}\right) ds$$

$$= c\left(\left(v\varphi\right)(0,0^+) + \int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds\right)$$

where we have used the integration by parts formula, the compact support of φ and the directional derivative of a function. Consequently, we have that

$$\int_{\vee} vX \cdot n \, d\ell =$$

$$= 2c(v\varphi)(0,0^+) + c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{-s}{c}\right) ds$$

Integral $\wedge$

As before, we divide our set into two subsets. The first one is parametrized by $\sigma(s) = \left(s, \frac{s}{c}\right)$ with $s \in [-a, 0]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{-1}{c}, 1\right)$. Thus,

$$= \int_{-a}^0 v \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{c} & 1 \end{pmatrix} \left(s, \frac{s}{c}\right) ds = \int_{-a}^0 v\left(c\varphi_x + \varphi_t\right)\left(s, \frac{s}{c}\right) ds =$$

$$= c\int_{-a}^0 v\left(\varphi_x + \frac{1}{c}\varphi_t\right)\left(s, \frac{s}{c}\right) ds = c\int_{-a}^0 \left(v\frac{d\varphi}{ds}\right)\left(s, \frac{s}{c}\right) ds$$

$$= c\left(\left(v\varphi\right)(0,0^-) - \int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds\right)$$

where we have used the integration by parts formula, the compact support of φ and the directional derivative of a function.

We parametrize the other subset using $\sigma(s) = \left(s, \frac{-s}{c}\right)$ with $s \in [0,a]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{1}{c}, 1\right)$. Thus,

$$\int_0^a v \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{c} & 1 \end{pmatrix} \left(s, -\frac{s}{c}\right) ds = \int_0^a v\left(-c\varphi_x + \varphi_t\right)\left(s, -\frac{s}{c}\right) ds =$$

$$= -c\int_0^a v\left(\varphi_x - \frac{1}{c}\varphi_t\right)\left(s, -\frac{s}{c}\right) ds = -c\int_0^a \left(v\frac{d\varphi}{ds}\right)\left(s, -\frac{s}{c}\right) ds$$
$$= c\left((v\varphi)\left(0, 0^-\right) + \int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, -\frac{s}{c}\right) ds\right)$$
where we have used the integration by parts formula, the compact support of φ and the directional derivative of a function. For $\wedge$ we have

$$\int_\wedge vX \cdot n\, d\ell =$$

$$= 2c(v\varphi)(0, 0^-) + c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, -\frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds$$

$\diamond$

Going back to the proof, we have found that

$$\int_{\partial R} vX \cdot n\, d\ell = 2c(v\varphi)(0, 0^+) + 2c(v\varphi)(0, 0^-) +$$

$$+c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{-s}{c}\right) ds$$
$$+c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, -\frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds$$

The notation 0^+ means the limit to zero going from above and 0^- is the limit to zero going from below. This distinction is crucial since we have v defined differently whereas $t > 0$ or not. In our case, we have $v(0, 0^+) \to \frac{1}{2c}$ and $v(0, 0^-) \to -\frac{1}{2c}$. Due to the fact that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have $2c(v\varphi)(0, 0^+) + 2c(v\varphi)(0, 0^-) = 0$. Then,

$$\int_{\partial R} vX \cdot n\, d\ell =$$

$$= c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{-s}{c}\right) ds$$
$$+c\int_0^a \left(\varphi\frac{dv}{ds}\right)\left(s, -\frac{s}{c}\right) ds - c\int_{-a}^0 \left(\varphi\frac{dv}{ds}\right)\left(s, \frac{s}{c}\right) ds$$
Let us leave this result there and move on to the other parts of the calculus since later on we will recover and use this result.

We have not computed yet

$$\int_R \nabla v \cdot X = \int_R \begin{pmatrix} v_x & v_t \end{pmatrix} \cdot \begin{pmatrix} -c^2\varphi_x \\ \varphi_t \end{pmatrix} = \int_R \underbrace{\begin{pmatrix} -c^2 v_x & v_t \end{pmatrix}}_{Y} \cdot \begin{pmatrix} \varphi_x \\ \varphi_t \end{pmatrix} =$$

$$= \int_R Y \cdot \nabla\varphi = \int_R \operatorname{div}(\varphi Y) - \int_R \varphi \operatorname{div} Y$$

Using the same argument of the triangles as before, we apply the divergence theorem in each of the two triangles (see Figure 1)

$$\int_R \operatorname{div}(\varphi Y) = \int_\bigtriangledown \operatorname{div}(\varphi Y) + \int_\bigtriangleup \operatorname{div}(\varphi Y) =$$

$$= \int_{\partial\bigtriangledown} \varphi Y \cdot n \, d\ell + \int_{\partial\bigtriangleup} \varphi Y \cdot n \, d\ell = \int_\vee \varphi Y \cdot n \, d\ell + \int_\wedge \varphi Y \cdot n \, d\ell$$

where the compact support of φ has been used again.

Integral $\vee$

One again, we parametrize one subset by $\sigma(s) = \left(s, \frac{-s}{c}\right)$ with $s \in [-a, 0]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{-1}{c}, -1\right)$. Thus,

$$\int_{-a}^{0} \varphi \begin{pmatrix} -c^2 v_x \\ v_t \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{c} & -1 \end{pmatrix} \left(s, -\frac{s}{c}\right) ds = \int_{-a}^{0} \varphi \left(c v_x - v_t\right) \left(s, -\frac{s}{c}\right) ds =$$

$$= c \int_{-a}^{0} \varphi \left(v_x - \frac{1}{c} v_t\right) \left(s, -\frac{s}{c}\right) ds = c \int_{-a}^{0} \left(\varphi \frac{dv}{ds}\right) \left(s, -\frac{s}{c}\right) ds$$

where we have used the directional derivative of a function.

The other subset is parametrized by $\sigma(s) = \left(s, \frac{s}{c}\right)$ with $s \in [0, a]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector becomes $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{1}{c}, -1\right)$. Thus,

$$\int_{0}^{a} \varphi \begin{pmatrix} -c^2 v_x \\ v_t \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{c} & -1 \end{pmatrix} \left(s, \frac{s}{c}\right) ds = \int_{0}^{a} \varphi \left(-c v_x - v_t\right) \left(s, \frac{s}{c}\right) ds =$$

$$= -c \int_{0}^{a} \varphi \left(v_x + \frac{1}{c} v_t\right) \left(s, \frac{s}{c}\right) ds = -c \int_{0}^{a} \left(\varphi \frac{dv}{ds}\right) \left(s, \frac{s}{c}\right) ds$$

where we have used the directional derivative of a function. Therefore,

$$\int_\vee \varphi Y \cdot n \, d\ell = c \int_{-a}^{0} \left(\varphi \frac{dv}{ds}\right) \left(s, -\frac{s}{c}\right) ds - c \int_{0}^{a} \left(\varphi \frac{dv}{ds}\right) \left(s, \frac{s}{c}\right) ds$$

Integral $\wedge$

We parametrize the first subset using $\sigma(s) = \left(s, \frac{s}{c}\right)$ with $s \in [-a, 0]$ so that $d\ell = \left(\sqrt{1+\frac{1}{c^2}}\right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1+\frac{1}{c^2}}\right)^{-1} \left(\frac{-1}{c}, 1\right)$. Thus,

$$\int_{-a}^{0} \varphi \begin{pmatrix} -c^2 v_x \\ v_t \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{c} & 1 \end{pmatrix} \left(s, \frac{s}{c}\right) ds = \int_{-a}^{0} \varphi \left(c v_x + v_t\right) \left(s, \frac{s}{c}\right) ds =$$

$$= c\int_{-a}^{0} \varphi \left(v_x + \frac{1}{c} v_t \right) \left(s, \frac{s}{c} \right) ds = c\int_{-a}^{0} \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{s}{c} \right) ds$$

where we have used the directional derivative of a function.

As for the second subset, let $\sigma(s) = \left(s, \frac{-s}{c} \right)$ with $s \in [0, a]$ so that $d\ell = \left(\sqrt{1 + \frac{1}{c^2}} \right) ds$ and the normal exterior unit vector is $n = \left(\sqrt{1 + \frac{1}{c^2}} \right)^{-1} \left(\frac{1}{c}, 1 \right)$. Thus,

$$\int_0^a \varphi \begin{pmatrix} -c^2 v_x \\ v_t \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{c} & 1 \end{pmatrix} \left(s, -\frac{s}{c} \right) ds = \int_0^a \varphi \left(-c v_x + v_t \right) \left(s, -\frac{s}{c} \right) ds =$$

$$= -c \int_0^a \varphi \left(v_x - \frac{1}{c} v_t \right) \left(s, -\frac{s}{c} \right) ds = -c \int_0^a \left(\varphi \frac{dv}{ds} \right) \left(s, -\frac{s}{c} \right) ds$$

where we have used the directional derivative of a function. Therefore,

$$\int_{\wedge} \varphi Y \cdot n \, d\ell = c \int_{-a}^{0} \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{s}{c} \right) ds - c \int_0^a \left(\varphi \frac{dv}{ds} \right) \left(s, -\frac{s}{c} \right) ds$$

$\diamond$

Finally, we have that

$$\int_R \operatorname{div}(\varphi Y) = \int_{\partial R} \varphi Y \cdot n \, d\ell =$$

$$= c \int_{-a}^{0} \left(\varphi \frac{dv}{ds} \right) \left(s, -\frac{s}{c} \right) ds - c \int_0^a \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{s}{c} \right) ds$$

$$+ c \int_{-a}^{0} \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{s}{c} \right) ds - c \int_0^a \left(\varphi \frac{dv}{ds} \right) \left(s, -\frac{s}{c} \right) ds$$

Mixing all things found so far, we have

$$\int_R v \left(\varphi_{tt} - c^2 \varphi_{xx} \right) = \int_R v \operatorname{div} \underbrace{\begin{pmatrix} -c^2 \varphi_x \\ \varphi_t \end{pmatrix}}_{X} = \int_R v \operatorname{div} X =$$

$$= \int_R \operatorname{div}(vX) - \int_R \nabla v \cdot X =$$

$$= \int_R \operatorname{div}(vX) - \int_R \operatorname{div} \left(\varphi Y \right) + \int_R \varphi \operatorname{div} Y =$$

Finally, adding the missing term,

$$\int_R v \left(\varphi_{tt} - c^2 \varphi_{xx} - \frac{k^2}{4} \varphi \right) dxdt = \int_R \varphi \left(v_{tt} - c^2 v_{xx} - \frac{k^2}{4} v \right) dxdt$$

$$+ 2c \int_0^a \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{s}{c} \right) ds - 2c \int_{-a}^{0} \left(\varphi \frac{dv}{ds} \right) \left(s, \frac{-s}{c} \right) ds$$

$$-2c\int_{-a}^{0}\left(\varphi\frac{dv}{ds}\right)\left(s,\frac{s}{c}\right)ds+2c\int_{0}^{a}\left(\varphi\frac{dv}{ds}\right)\left(s,-\frac{s}{c}\right)ds$$

Let us recall now a property of our candidate to solution, which is v is either $\frac{1}{2c}$ or $-\frac{1}{2c}$ on the lines $x=\pm ct$. That means v is constant on such lines and therefore its directional derivative is 0. We are just saying that

$$\frac{dv}{ds}\left(s,\frac{s}{c}\right)=\frac{dv}{ds}\left(s,-\frac{s}{c}\right)=0,\ \forall s\in\mathbb{R}$$

Hence, the line integrals are all 0 and what remains to be seen is that

$$\int_{R}\varphi\left(v_{tt}-c^2v_{xx}-\frac{k^2}{4}v\right)dxdt=0,\quad \forall\varphi\in\mathcal{C}_0^{\infty}(\mathbb{R})$$

which is true provided that $v_{tt}-c^2v_{xx}-\frac{k^2}{4}v=0$ in the interior of the two triangles. We just check the result for positive times. For negatives times, the minus sign does not affect at all the result.

Let us once again recall what form has our candidate of solution inside the integrating region for positive times. It is

$$v(x,t)=\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2t^2-x^2}\right)$$

This function is infinitely differentiable in both variables in this region and for comfort let us denote $\lambda=c^2t^2-x^2$. To verify it satisfies the PDE, we use the same reasoning as in the Heuristics Chapter 2. That is, we write $v(x,t)=\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})=f(\lambda)=g(2\alpha\sqrt{\lambda})=g(\xi)$ for some f and g to determine. We have

$$v_t=2c^2tf'(\lambda)\Longrightarrow v_{tt}=4c^4t^2f''(\lambda)+2c^2f'(\lambda)$$

and

$$v_x=-2xf'(\lambda)\Longrightarrow v_{xx}=4x^2f''(\lambda)-2f'(\lambda)$$

Hence, recalling $\alpha=\frac{k}{4c}$,

$$v_{tt}-c^2v_{xx}-\frac{k^2}{4}v=$$

$$=4c^4t^2f''(\lambda)+2c^2f'(\lambda)-c^2\left(4x^2f''(\lambda)-2f'(\lambda)\right)-\frac{k^2}{4}f(\lambda)=$$

$$=4c^2\left[\lambda f''+f'-\alpha^2f\right]$$

Now using the relations between f and g,

$$f'(\lambda)=g'(2\alpha\sqrt{\lambda})\frac{2\alpha}{2\sqrt{\lambda}}=g'(2\alpha\sqrt{\lambda})\frac{\alpha}{\sqrt{\lambda}}$$

$$f''(\lambda)=g''(2\alpha\sqrt{\lambda})\frac{\alpha^2}{\lambda}+g'(2\alpha\sqrt{\lambda})\left(\frac{-\alpha}{\lambda}\frac{1}{2}\frac{1}{\sqrt{\lambda}}\right)=g''(2\alpha\sqrt{\lambda})\frac{\alpha^2}{\lambda}-g'(2\alpha\sqrt{\lambda})\left(\frac{\alpha}{2\sqrt{\lambda^3}}\right)$$

we deduce that

$$\lambda f'' + f' - \alpha^2 f = \alpha^2 g''(\xi) - g'(\xi)\frac{\alpha}{2\sqrt{\lambda}} + g'(\xi)\frac{\alpha}{\sqrt{\lambda}} - \alpha^2 g(\xi) =$$

$$= \alpha^2 g''(\xi) + g'(\xi)\frac{\alpha}{2\sqrt{\lambda}} - \alpha^2 g(\xi) = \frac{1}{\lambda}\left[\alpha^2\lambda g''(\xi) + g'(\xi)\frac{2\alpha\sqrt{\lambda}}{4} - \alpha^2\lambda g(\xi)\right] =$$

$$= \frac{1}{4\lambda}\left[4\alpha^2\lambda g''(\xi) + g'(\xi)2\alpha\sqrt{\lambda} - 4\alpha^2\lambda g(\xi)\right] = \frac{1}{4\lambda}\left[\xi^2 g''(\xi) + \xi g'(\xi) - \xi^2 g(\xi)\right]$$

Finally, we have to check that $0 = \frac{c^2}{\lambda}\left[\xi^2 g''(\xi) + \xi g'(\xi) - \xi^2 g(\xi)\right]$. Fortunately, it is trivial since $g(\xi) = \frac{1}{2c}I_0(\xi)$ is a multiple of the modified Bessel equation of first kind of order 0, which is precisely defined as the function that solves this ODE.

We would like to note that we could have seen this result computing explicitly the partial derivatives of $v(x,t) = \frac{1}{2c}I_0(2\alpha\sqrt{c^2t^2 - x^2})$ and then work with what we got. We decided not to do so and instead follow this approach because not only did we avoid having to calculate long terms but also to emphasize the usefulness of some arguments previously done heuristically that now can be applied rigorously.

We have just seen that our v satisfies the PDE in the sense of distributions. As for the initial conditions, the first one $v(x,0) = 0$ is deduced and in fact imposed from the odd symmetry of the function.

Note, also, that we have a discontinuity there, since from above the function "converges" to $\frac{1}{2c}$ while from below it goes to $-\frac{1}{2c}$. Thoughtfully, being 0 reminds us of the approximation theory and of the use of Fourier series when there are points with discontinuities.

As for the second one, in order to see $v_t(x,0) = \delta(x)$ we have several alternatives. The first one will be presented in few moments. The second one will be a direct consequence from a much more stronger result that we will address later on. Let us now finish the proof here, knowing we still have this issue open. □

3.2 Regularity

Let us now ask ourselves about the regularity of the solution we have just found. More specifically, what can we say about the smoothness of their derivatives? And in which sense? Also, we still have to see that $v_t(x,0) = \delta(x)$. In order to answer all this questions and inspired by [4] we have this powerful theorem.

Theorem 3.2. *Let*

$$v(x,t) = \begin{cases} \frac{1}{2c}I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right)\mathcal{X}_{[-ct,ct]}(x) & t > 0 \\ -\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right)\mathcal{X}_{[ct,-ct]}(x) & t < 0 \end{cases}$$

be the solution of (7). Then,

$$\begin{aligned} v : \mathbb{R} &\longrightarrow \mathcal{D}'(\mathbb{R}) \\ t &\longrightarrow v(\cdot,t) \end{aligned} \quad \textit{is such that } v \in \mathcal{C}^1\left(\mathbb{R}, \mathcal{D}'\left(\mathbb{R}\right)\right)$$

and

$$v_t(\cdot,t)=\frac{1}{2}\left[\delta(\cdot-ct)+\delta(\cdot+ct)\right]+\alpha c|t|\frac{I_0'\left(2\alpha\sqrt{c^2t^2-\cdot^2}\right)}{\sqrt{c^2t^2-\cdot^2}}\mathcal{X}_{[-c|t|,c|t|]}(\cdot)$$

Proof. Clearly $v(\cdot,t)\in\mathcal{D}'(\mathbb{R})$, $\forall t\in\mathbb{R}$, the application is well defined. In order to check the first derivative of this application is continuous, one has to see that there

$$\exists\lim_{h\to 0}\frac{v(\cdot,t+h)-v(\cdot,t)}{h}=v_t(\cdot,t)\in\mathcal{C}^0\left(\mathbb{R},\mathcal{D}'(\mathbb{R})\right)$$

with the limits computed using the topology of the arriving space $\mathcal{D}'(\mathbb{R})$. We will first prove the existence of the derivative for each $t\in\mathbb{R}$ and afterwards we will prove its continuity.

Derivative for $t>0$

Since we have different definitions of v according to the sign of the time t, let us start with $t>0$, we want to see that

$$\left\langle\frac{v(\cdot,t+h)-v(\cdot,t)}{h},\varphi\right\rangle\xrightarrow{h\to 0}\langle v_t(\cdot,t),\varphi\rangle\ \ \forall\varphi\in\mathcal{D}(\mathbb{R})=\mathcal{C}_0^\infty(\mathbb{R})$$

where the notation $\langle\psi(\cdot,t),\varphi\rangle$ refers to $\int_{\mathbb{R}}\psi(x,t)\varphi(x)dx$. Therefore, let us take $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$,

$$\left\langle\frac{v(\cdot,t+h)-v(\cdot,t)}{h},\varphi\right\rangle=\int_{\mathbb{R}}\frac{v(x,t+h)-v(x,t)}{h}\varphi(x)dx=$$

$$=\frac{1}{h}\int_{\mathbb{R}}v(x,t+h)\varphi(x)dx-\frac{1}{h}\int_{\mathbb{R}}v(x,t)\varphi(x)dx$$

So let us now focus on the first integral, where we are going to use the dependence of the interval of definition of v with respect to t to obtain.

$$\frac{1}{h}\int_{\mathbb{R}}v(x,t+h)\varphi(x)dx=\frac{1}{h}\int_{\mathbb{R}}\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2-x^2}\right)\mathcal{X}_{[-c(t+h),c(t+h)]}(x)\varphi(x)dx$$

$$=\frac{1}{h}\int_{-c(t+h)}^{c(t+h)}\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2-x^2}\right)\varphi(x)dx=\frac{1}{h}\left(\int_{-c(t+h)}^{-ct}+\int_{-ct}^{ct}+\int_{ct}^{c(t+h)}\right)$$

And we study the three integrals separately. Let us begin with

$$\lim_{h\to 0}\frac{1}{h}\int_{ct}^{c(t+h)}\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2-x^2}\right)\varphi(x)dx=\frac{0}{0}$$

and hence we use Hopital's Rule to follow our calculations, we need

$$\frac{d}{dh}\left(\int_{ct}^{c(t+h)}\frac{1}{2c}I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2-x^2}\right)\varphi(x)dx\right)=$$

$$= \frac{1}{2c} I_0(0)\varphi(c(t+h))c + \int_{ct}^{c(t+h)} \frac{1}{2c} \frac{I_0'\left(2\alpha\sqrt{c^2(t+h)^2 - x^2}\right)}{2\sqrt{c^2(t+h)^2 - x^2}} 2\alpha 2c^2(t+h)\varphi(x)dx$$

Now, when $h \to 0$, the integral vanishes since the quotient is bounded and the interval reduces to a single point, so we are left with

$$\frac{1}{2}\varphi(ct) = \int_{\mathbb{R}} \frac{1}{2}\delta(x - ct)\varphi(x)dx$$

We observe a similar situation in

$$\lim_{h\to 0} \frac{1}{h} \int_{-c(t+h)}^{-ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2 - x^2}\right)\varphi(x)dx = \frac{0}{0}$$

and, as before, applying Hopital's Rule requires

$$\frac{d}{dh}\left(\int_{-c(t+h)}^{-ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2 - x^2}\right)\varphi(x)dx\right) =$$

$$= -\frac{1}{2c} I_0(0)\varphi(-c(t+h))(-c) + \int_{-c(t+h)}^{-ct} \frac{1}{2c} \frac{I_0'\left(2\alpha\sqrt{c^2(t+h)^2 - x^2}\right)}{2\sqrt{c^2(t+h)^2 - x^2}} 2\alpha 2c^2(t+h)\varphi(x)dx$$

so when $h \to 0$ the integral also vanishes for the same reason as before and we end up with

$$\frac{1}{2}\varphi(-ct) = \int_{\mathbb{R}} \frac{1}{2}\delta(x + ct)\varphi(x)dx$$

Finally, we just have left these two terms to analyse and compute

$$\frac{1}{h}\int_{-ct}^{ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2 - x^2}\right)\varphi(x)dx - \frac{1}{h}\int_{-ct}^{ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right)\varphi(x)dx =$$

$$= \int_{-ct}^{ct} \frac{\left(\frac{1}{2c} I_0\left(2\alpha\sqrt{c^2\left(t+h\right)^2 - x^2}\right) - \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2 - x^2}\right)\right)}{h}\varphi(x)dx$$

So when $h \to 0$ we are essentially computing the classical derivative of this expression with respect to t, which results in

$$\int_{-ct}^{ct} \frac{1}{2c} \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{2\sqrt{c^2t^2 - x^2}} 2\alpha 2c^2 t\varphi(x)dx = \int_{-ct}^{ct} \alpha ct \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{\sqrt{c^2t^2 - x^2}}\varphi(x)dx =$$

$$= \int_{\mathbb{R}} \alpha ct \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{\sqrt{c^2t^2 - x^2}} \mathcal{X}_{[-ct,ct]}(x)\varphi(x)dx = \int_{\mathbb{R}} v_t^c(x,t)\varphi(x)dx$$

Summing up all terms, we have obtained

$$\lim_{h\to 0}\left\langle \frac{v(\cdot, t+h) - v(\cdot, t)}{h}, \varphi\right\rangle =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2} \left[\delta(x-ct) + \delta(x+ct) \right] + v_t^c(x,t) \right) \varphi(x) dx =$$

$$= \left\langle \frac{1}{2} \left[\delta(x-ct) + \delta(x+ct) \right] + v_t^c(x,t), \varphi \right\rangle$$

so we end up concluding that

$$v_t(\cdot,t) = \frac{1}{2} \left[\delta(\cdot - ct) + \delta(\cdot + ct) \right] + v_t^c(\cdot,t) \in \mathcal{D}'(\mathbb{R})$$

Derivative for $t = 0$

In this case, we have to see exactly the same as before, but since $t = 0$ is a critical point in the sense we have extended oddly our solution there, we felt it should be better to pay special attention in this case.

Let us first take $1 \gg h > 0$ and since $v(x,0) = 0$ in the sense of distributions, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ our limit is

$$\left\langle \frac{u(\cdot,h)}{h}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{1}{h} u(x,h) \varphi(x) ds \overset{h \geq 0}{=}$$

$$\overset{h \geq 0}{=} \int_{\mathbb{R}} \frac{1}{2c} \frac{I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{h} \mathcal{X}_{[-ch,ch]}(x) \varphi(x) dx = \frac{1}{h} \int_{-ch}^{ch} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right) \varphi(x) dx$$

As a result, when $h \to 0$, the interval of the integral reduces to 0 while the integrand tends to 1, it is bounded. So, the integral goes to 0 and then the quotient tends to $\frac{0}{0}$. Using Hopital's Rule,

$$\frac{d}{dh} \left(\int_{-ch}^{ch} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right) \varphi(x) dx \right) =$$

$$= \frac{1}{2c} \left\{ I_0(0)\varphi(ch)c - I_0(0)\varphi(-ch)(-c) \right\}$$

$$+ \int_{-ch}^{ch} \frac{1}{2c} \frac{I_0'\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{2\sqrt{c^2h^2 - x^2}} 2\alpha 2c^2 h \varphi(x) dx$$

Applying $I_0(0) = 1$ and simplifying terms,

$$= \frac{1}{2} \left\{ \varphi(ch) + \varphi(-ch) \right\} + \int_{-ch}^{ch} \alpha ch \frac{I_0'\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{2\sqrt{c^2h^2 - x^2}} \varphi(x) dx$$

And now, letting $h \to 0$ the integral vanishes since the integrand goes to 0 and the interval collapses and so we are left with

$$= \frac{1}{2} \left\{ \varphi(0) + \varphi(0) \right\} = \varphi(0) = \int_{\mathbb{R}} \delta(x) \varphi(x) dx = \langle \delta, \varphi \rangle$$

From here we may deduce that $v_t(\cdot,t) = \delta(\cdot) \in \mathcal{D}'(\mathbb{R})$. However we still have to check it when we do the limit with negative h.

So, as before, let us take $-1 \ll h < 0$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, our limit is

$$\left\langle \frac{u(\cdot,h)}{h}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{1}{h} u(x,h)\varphi(x)ds \overset{h\leq 0}{=}$$

$$\overset{h\leq 0}{=} \int_{\mathbb{R}} -\frac{1}{2c} \frac{I_0\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{h} \mathcal{X}_{[ch,-ch]}(x)\varphi(x)dx = -\frac{1}{h}\int_{ch}^{-ch} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2h^2-x^2}\right)\varphi(x)dx$$

As before, the limit tends to $\frac{0}{0}$ for the same reasons and again we use Hopital's Rule to get

$$\frac{d}{dh}\left(-\int_{ch}^{-ch} \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2h^2-x^2}\right)\varphi(x)dx\right) =$$

$$= \frac{1}{2c}\left\{-I_0(0)\varphi(-ch)(-c) + I_0(0)\varphi(ch)c\right\}$$

$$-\int_{ch}^{-ch} \frac{1}{2c} \frac{I_0'\left(2\alpha\sqrt{c^2h^2-x^2}\right)}{2\sqrt{c^2h^2-x^2}} 2\alpha 2c^2h\varphi(x)dx$$

Applying $I_0(0) = 1$ and simplifying terms,

$$= \frac{1}{2}\left\{\varphi(-ch) + \varphi(ch)\right\} - \int_{ch}^{-ch} \alpha ch \frac{I_0'\left(2\alpha\sqrt{c^2h^2-x^2}\right)}{2\sqrt{c^2h^2-x^2}}\varphi(x)dx$$

And now, letting $h \to 0$ the integral vanishes since the integrand goes to 0 and the interval collapses and so we are left with

$$= \frac{1}{2}\left\{\varphi(0) + \varphi(0)\right\} = \varphi(0) = \int_{\mathbb{R}} \delta(x)\varphi(x)dx = \langle \delta, \varphi \rangle$$

And so we can finally say that $v_t(\cdot,t) = \delta(\cdot) \in \mathcal{D}'(\mathbb{R})$ since we computed it from the two sides.

Derivative for $t < 0$

Let us take $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $h > 0$,

$$\left\langle \frac{v(\cdot,t) - v(\cdot,t-h)}{h}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{v(x,t) - v(x,t-h)}{h}\varphi(x)dx =$$

$$= \frac{1}{h}\int_{\mathbb{R}} v(x,t)\varphi(x)dx - \frac{1}{h}\int_{\mathbb{R}} v(x,t-h)\varphi(x)dx$$

So let us focus now on the second integral, where we are going to use the dependence of the interval of definition of v with respect to t to obtain.

$$-\frac{1}{h}\int_{\mathbb{R}} v(x,t-h)\varphi(x)dx = -\frac{1}{h}\int_{\mathbb{R}} -\frac{1}{2c} I_0\left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\mathcal{X}_{[c(t-h),-c(t-h)]}(x)\varphi(x)dx$$

$$= \frac{1}{h}\int_{c(t-h)}^{-c(t-h)} \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\varphi(x)dx = \frac{1}{h}\left(\int_{c(t-h)}^{ct} + \int_{ct}^{-ct} + \int_{-ct}^{-c(t-h)}\right)$$

And we study the three integrals separately. Let us begin with

$$\lim_{h\to 0}\frac{1}{h}\int_{-ct}^{-c(t-h)} \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\varphi(x)dx = \frac{0}{0}$$

and hence we use Hopital's Rule to follow our calculations, we need

$$\frac{d}{dh}\left(\int_{-ct}^{-c(t-h)} \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\varphi(x)dx\right) =$$

$$= \frac{1}{2c} I_0(0)\varphi(-c(t-h))c + \int_{-ct}^{-c(t-h)} -\frac{1}{2c}\frac{I_0'\left(2\alpha\sqrt{c^2(t-h)^2-x^2}\right)}{2\sqrt{c^2(t-h)^2-x^2}} 2\alpha 2c^2(t-h)\varphi(x)dx$$

Now, when $h \to 0$, the integral vanishes since the quotient is bounded and the interval reduces to a single point, so we are left with

$$\frac{1}{2}\varphi(-ct) = \int_{\mathbb{R}} \frac{1}{2}\delta(x+ct)\varphi(x)dx$$

We observe a similar situation in

$$\lim_{h\to 0}\frac{1}{h}\int_{c(t-h)}^{ct} \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\varphi(x)dx = \frac{0}{0}$$

and, as before, applying Hopital's Rule

$$\frac{d}{dh}\left(\int_{c(t-h)}^{ct} \frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t-h\right)^2 - x^2}\right)\varphi(x)dx\right) =$$

$$= -\frac{1}{2c} I_0(0)\varphi(c(t-h))(-c) + \int_{c(t-h)}^{ct} -\frac{1}{2c}\frac{I_0'\left(2\alpha\sqrt{c^2(t-h)^2-x^2}\right)}{2\sqrt{c^2(t-h)^2-x^2}} 2\alpha 2c^2(t-h)\varphi(x)dx$$

so when $h \to 0$ the integral also vanishes for the same reason as before and we end up with

$$\frac{1}{2}\varphi(ct) = \int_{\mathbb{R}} \frac{1}{2}\delta(x-ct)\varphi(x)dx$$

Finally, we just have left these two terms to analyse and compute

$$\frac{1}{h}\int_{ct}^{-ct} -\frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2\left(t\right)^2 - x^2}\right)\varphi(x)dx - \frac{1}{h}\int_{ct}^{-ct} -\frac{1}{2c} I_0 \left(2\alpha\sqrt{c^2(t-h)^2 - x^2}\right)\varphi(x)dx =$$

$$= \int_{ct}^{-ct} -\frac{\left(\frac{1}{2c} I_0\left(2\alpha\sqrt{c^2t^2-x^2}\right) - \frac{1}{2c} I_0\left(2\alpha\sqrt{c^2(t-h)^2-x^2}\right)\right)}{h}\varphi(x)dx$$

So when $h \to 0$ we are essentially computing the classical derivative of this expression with respect to t, which results in

$$\int_{ct}^{-ct} -\frac{1}{2c}\frac{I_0'\left(2\alpha\sqrt{c^2t^2-x^2}\right)}{2\sqrt{c^2t^2-x^2}}2\alpha 2c^2 t\varphi(x)dx = \int_{ct}^{-ct} -\alpha ct\frac{I_0'\left(2\alpha\sqrt{c^2t^2-x^2}\right)}{\sqrt{c^2t^2-x^2}}\varphi(x)dx =$$

$$= \int_{\mathbb{R}} -\alpha ct\frac{I_0'\left(2\alpha\sqrt{c^2t^2-x^2}\right)}{\sqrt{c^2t^2-x^2}}\mathcal{X}_{[ct,-ct]}(x)\varphi(x)dx = \int_{\mathbb{R}} v_t^c(x,t)\varphi(x)dx$$

Summing up all terms, we have obtained

$$\lim_{h\to 0}\left\langle \frac{v(\cdot,t)-v(\cdot,t-h)}{h},\varphi\right\rangle =$$

$$= \int_{\mathbb{R}}\left(\frac{1}{2}\left[\delta(x-ct)+\delta(x+ct)\right]+v_t^c(x,t)\right)\varphi(x)dx =$$

$$= \left\langle \frac{1}{2}\left[\delta(x-ct)+\delta(x+ct)\right]+v_t^c(x,t),\varphi\right\rangle$$

so we end up concluding that

$$v_t(\cdot,t) = \frac{1}{2}\left[\delta(\cdot-ct)+\delta(\cdot+ct)\right]+v_t^c(\cdot,t)\in\mathcal{D}'\left(\mathbb{R}\right)$$

We have just seen that there exists a derivative $\forall t\in\mathbb{R}$. Now, we are going to see that this derivative is continuous, that is, we are going to see that

$$\lim_{h\to 0}\left\langle v_t(\cdot,t+h),\varphi\right\rangle = \left\langle v_t(\cdot,t),\varphi\right\rangle,\quad \forall t\in\mathbb{R},$$

which is the meaning of continuity in the sense of distributions. As before, we divide our argument depending on the sign of t.

Continuity for $t>0$

We have that

$$v_t(\cdot,t) = \frac{1}{2}\left[\delta(\cdot-ct)+\delta(\cdot+ct)\right]+v_t^c(\cdot,t)$$

in the sense of distributions, where v_t^c stands for the derivative of v with respect to time t in the classical sense, that is,

$$v_t^c(x,t) = \alpha ct\frac{I_0'\left(2\alpha\sqrt{c^2t^2-x^2}\right)}{\sqrt{c^2t^2-x^2}}\mathcal{X}_{[-ct,ct]}(x)$$

Using the definition of limit in the space of distributions, given $t>0$, $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ and $h>0$ sufficiently small, we compute

$$\left\langle v_t(\cdot,t+h),\varphi\right\rangle =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2} \left[\delta(x - c(t+h)) + \delta(x + c(t+h)) \right] + v_t^c(x, t+h) \right) \varphi(x) dx =$$

$$= \frac{1}{2}\varphi(c(t+h)) + \frac{1}{2}\varphi(-c(t+h)) +$$

$$+ \int_{-c(t+h)}^{c(t+h)} \alpha c(t+h) \frac{I_0'\left(2\alpha\sqrt{c^2(t+h)^2 - x^2}\right)}{\sqrt{c^2(t+h)^2 - x^2}} \varphi(x) dx$$

Now, if we take $h \to 0$ we obtain

$$= \frac{1}{2}\varphi(ct) + \frac{1}{2}\varphi(-ct) + \int_{-ct}^{ct} \alpha ct \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{\sqrt{c^2t^2 - x^2}} \varphi(x) dx =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2} \left[\delta(x - ct) + \delta(x + ct) \right] + v_t^c(x, t) \right) \varphi(x) dx = \langle v_t(\cdot, t), \varphi \rangle$$

from which we deduce that v_t is continuous for positive times.

Continuity for $t = 0$

Since we want

$$\lim_{h \to 0} \langle v_t(\cdot, h), \varphi \rangle = \langle v_t(\cdot, 0), \varphi \rangle$$

and our definition depends on the sign of t, here h, given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we consider first $h > 0$ and then $h < 0$.

For $h > 0$ we have

$$\langle v_t(\cdot, h), \varphi \rangle =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2} \left[\delta(x - ch) + \delta(x + ch) \right] + v_t^c(x, h) \right) \varphi(x) dx =$$

$$= \frac{1}{2}\varphi(ch) + \frac{1}{2}\varphi(-ch) + \int_{-ch}^{ch} \alpha ch \frac{I_0'\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{\sqrt{c^2h^2 - x^2}} \varphi(x) dx$$

Now, if we take $h \to 0$ the integral vanishes as argued as before (the interval collapses and the integrand also tends to 0), so we obtain

$$\frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(0) = \varphi(0) = \int_{\mathbb{R}} \delta(x)\varphi(x) dx = \langle v_t(\cdot, 0), \varphi \rangle$$

and therefore it is continuous, at least going from the right hand side.

On the other hand, for $h < 0$ we have

$$\langle v_t(\cdot, h), \varphi \rangle =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2} \left[\delta(x - ch) + \delta(x + ch) \right] + v_t^c(x, h) \right) \varphi(x) dx =$$

$$= \frac{1}{2}\varphi(ch) + \frac{1}{2}\varphi(-ch) + \int_{ch}^{-ch} -\alpha ch \frac{I_0'\left(2\alpha\sqrt{c^2h^2 - x^2}\right)}{\sqrt{c^2h^2 - x^2}}\varphi(x)dx$$

Now, if we take $h \to 0$ the integral vanishes as argued as before (the interval collapses and the integrand also tends to 0), so we obtain

$$\frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(0) = \varphi(0) = \int_{\mathbb{R}} \delta(x)\varphi(x)dx = \langle v_t(\cdot, 0), \varphi\rangle$$

and hence we can conclude that v_t is continuous at $t = 0$.

Continuity for $t < 0$

The arguments we are going to use are exactly the same as in the case $t > 0$. However, we reproduce them again to be sure everything works out as it should. For $t < 0$ we have

$$v_t(\cdot, t) = \frac{1}{2}\left[\delta(\cdot - ct) + \delta(\cdot + ct)\right] + v_t^c(\cdot, t)$$

in the sense of distributions, where v_t^c stands for the derivative of v with respect to time t in the classical sense, that is,

$$v_t^c(x, t) = -\alpha ct \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{\sqrt{c^2t^2 - x^2}}\mathcal{X}_{[ct,-ct]}(x)$$

Using the definition of limit in the space of distributions, given $t < 0$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $h > 0$ sufficiently small, we compute

$$\langle v_t(\cdot, t-h), \varphi\rangle =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2}\left[\delta(x - c(t-h)) + \delta(x + c(t-h))\right] + v_t^c(x, t-h)\right)\varphi(x)dx =$$

$$= \frac{1}{2}\varphi(c(t-h)) + \frac{1}{2}\varphi(-c(t-h)) +$$

$$+ \int_{c(t-h)}^{-c(t-h)} -\alpha c(t-h) \frac{I_0'\left(2\alpha\sqrt{c^2(t-h)^2 - x^2}\right)}{\sqrt{c^2(t-h)^2 - x^2}}\varphi(x)dx$$

Now, if we take $h \to 0$ we obtain

$$= \frac{1}{2}\varphi(ct) + \frac{1}{2}\varphi(-ct) + \int_{ct}^{-ct} -\alpha ct \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - x^2}\right)}{\sqrt{c^2t^2 - x^2}}\varphi(x)dx =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2}\left[\delta(x - ct) + \delta(x + ct)\right] + v_t^c(x, t)\right)\varphi(x)dx = \langle v_t(\cdot, t), \varphi\rangle$$

from which we deduce that v_t is continuous for negative times.

Finally, we can say that $v_t(\cdot, t) \in \mathcal{C}^0(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ and the proof is complete. □

Remark 3.3. This result allows us to conclude that $v_t(x, 0) = \delta(x)$ rigorously, so that the proof of Theorem 3.1 is now completed.

4 The Riemann Function

4.1 Existence

Once we know the solution of our problem for $\delta \in \mathcal{D}'(\mathbb{R})$ as its initial condition, we now would like to extend this result to a much more general problem. To this purpose, we present the following

Theorem 4.1. *Let $\psi(x,t) = \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2 - x^2})\mathcal{X}_{[-ct,ct]}(x)$ and $g \in \mathcal{D}'(\mathbb{R})$ a distribution. Then, the distribution v defined by*

$$\langle v(t), \varphi \rangle := \langle g, \psi(t) * \varphi \rangle$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ is the solution in the sense of distributions of

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = g & x \in \mathbb{R} \end{cases} \tag{9}$$

Before the proof, let us state these rather useful Remarks.

Remark 4.2. This candidate to solution v may be understood as a resulting distribution on the x variable for each $t \in \mathbb{R}$ or also as a two dimensional distribution. We will use this dual meaning in different parts of the proof.

Remark 4.3. From Theorem 3.1, we know that

$$\int_{\mathbb{R}^2} L(\psi)\phi dxdt := \int_{\mathbb{R}^2} \psi L^*(\phi) dxdt = 0, \quad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

a result that will be key in the proof.

Remark 4.4. In certain cases, we *may* write

$$v(x,t) = \int_{\mathbb{R}} \psi(x-y,t) g(y) dy = \psi(t) * g$$

only when g is such that this expression makes sense (for example, $g \in L^2$ or $g = \delta$)

Remark 4.5. In many cases, $\psi(x,t)$ will be integrated on x. This integration will be usually done in the interval $[-ct, ct]$ thanks to the $\mathcal{X}$ function. We would like to remind here that *this* integration is done in the Riemann sense and hence, the interval must be positive.

We explain this so as to avoid any doubts referring to the minus sign that once appeared in (8) and that now is omitted.

Proof. First of all, let us see that our solution is well-defined. We know that a distribution is characterised by its action on the test functions. In our case, we define the action of $v(t)$ on any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ as the action of the distribution g on the convolution $\psi(t) * \varphi$.

Observe now that $\psi(t)$ is a function with compact support and continuous inside $[-ct, ct]$. Therefore, $\psi(t) * \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and then it makes sense to compute the action of the distribution g on $\psi(t) * \varphi$, so v is well-defined, it is a distribution.

Let us now interpret our candidate to solution as a two dimensional distribution. In order to see the PDE is satisfied in the sense of distributions we have to check, as before, that for the same operator L,

$$\int_{\mathbb{R}^2} L(v)\phi \, dxdt := \int_{\mathbb{R}^2} vL^*(\phi)dxdt = 0, \quad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

However, this is all notation, since we are now dealing with a general distribution and we do not know how it acts on the tests functions.

In fact, it even does not make sense to multiply and then integrate a general distribution. Nevertheless, assuming we could do it, we can reach an equivalent expression that does make sense.

Therefore, let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we *can* write

$$\int_{\mathbb{R}^2} vL^*(\phi)dxdt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} vL^*(\phi)dx \right) dt$$

Now, for each $t \in \mathbb{R}$, we observe that $L^*(\phi(\cdot, t)) \in \mathcal{C}_0^\infty(\mathbb{R})$ so that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} vL^*(\phi)dx \right) dt = \int_{\mathbb{R}} \langle v(t), L^*(\phi) \rangle \, dt$$

As a result, we define

$$\int_{\mathbb{R}^2} L(v)\phi \, dxdt := \int_{\mathbb{R}} \langle v(t), L^*(\phi) \rangle \, dt$$

with the first term being purely symbolic and notational, while the second one being fully well-defined. We proceed

$$\int_{\mathbb{R}} \langle v(t), L^*(\phi) \rangle \, dt = \int_{\mathbb{R}} \langle g, \psi(t) * L^*(\phi(t)) \rangle \, dt = \left\langle g, \int_{\mathbb{R}} \psi(t) * L^*(\phi(t))dt \right\rangle$$

Notice we can enter the integral inside $\langle \cdot, \cdot \rangle$ because the duality product is indeed a linear continuous form and the integral is a limiting process based on sums that only concerns $\psi(t) * L^{‘*}(\phi(t))$. We now just study

$$\int_{\mathbb{R}} \psi(t) * L^*(\phi(t))dt = \int_{\mathbb{R}^2} \psi(y, t)L^*(\phi(x - y, t))dydt$$

Fixed $x \in \mathbb{R}$, if we write $\phi(x - y, t) = \varphi(y, t)$, we have that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $L^*(\phi(x - y, t)) = L^*(\varphi(y, t)) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ so that

$$\left\langle g, \underbrace{\int_{\mathbb{R}^2} \psi(y, t)L^*(\varphi(y, t))dydt}_{0} \right\rangle = 0$$

thanks to what we commented in Remark 4.3. This way, we make sure that the distribution v is a solution in the sense of distributions of the differential equation.

The first initial condition should be $v(x,0)=0$ in the sense of distributions in x. We have $\langle v(0),\varphi\rangle := \langle g,\psi(0)*\varphi\rangle = \langle g, 0*\varphi\rangle = 0$, $\forall\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$, so that $v(0)=0$ as a distribution.

As for the second initial condition, we now prove $v_t(x,0)=g(x)$ in the sense of distributions. Fixed t, we can interpret $v(x,t)$ as a distribution in x. Due to the fact that these are distributions, the time derivative $v_t(x,0)$ is defined by its action on any $\varphi(x)\in\mathcal{D}(\mathbb{R})$. Therefore, given $\varphi(x)\in\mathcal{C}_0^\infty(\mathbb{R})$

$$\langle v(t),\varphi\rangle := \langle g,(\psi*\varphi)(t)\rangle = \left\langle g,\int_{\mathbb{R}}\psi(x-y,t)\varphi(y)dy\right\rangle$$

Once again we just study

$$\int_{\mathbb{R}}\psi(x-y,t)\varphi(y)dy = \int_{\mathbb{R}}\frac{1}{2c}I_0(2\alpha\sqrt{c^2t^2-(x-y)^2})\mathcal{X}_{[-ct,ct]}(x-y)\varphi(y)dy =$$

$$= \int_{x-ct}^{x+ct}\frac{1}{2c}I_0(2\alpha\sqrt{c^2t^2-(x-y)^2})\varphi(y)dy = \phi(x,t)$$

Deriving ϕ with respect to t, it yields

$$\phi_t = \frac{1}{2}(\varphi(x+ct)+\varphi(x-ct)) + \int_{x-ct}^{x+ct}\alpha ct\frac{I_0'(2\alpha\sqrt{c^2t^2-(x-y)^2})}{\sqrt{c^2t^2-(x-y)^2}}\varphi(y)dy$$

and now, doing $t\to 0$ we get $\phi_t(x,0)=\varphi(x)$ since the integral vanishes (there's a t inside it, the quotient is bounded and the interval of integration collapses) and the limit enters $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$.

Hence, we have that $\langle v(t),\varphi\rangle := \langle g,\phi(t)\rangle$ and deriving v is deriving ϕ thanks to $\langle\cdot,\cdot\rangle$ being the duality operator and the derivation a limiting process that enters inside the duality so that

$$\langle v_t(0),\varphi\rangle = \langle g,\phi_t(0)\rangle = \langle g,\varphi\rangle$$

from which we deduce that $v_t(0)=g$ as a distribution. □

Remark 4.6. With $g=\delta$, we have

$$\langle v(t),\varphi\rangle := \langle g,\psi(t)*\varphi\rangle = \langle\delta,\psi(t)*\varphi\rangle = (\psi(t)*\varphi)(0) =$$

$$= \int_{\mathbb{R}}\psi(-y,t)\varphi(y)dy = \langle\psi(t),\varphi\rangle$$

since $\psi(x,t)$ is even on x and hence $v(t)=\psi(x,t)=\frac{1}{2c}I_0(2\alpha\sqrt{c^2t^2-x^2})\mathcal{X}_{[-ct,ct]}(x)$ as a distribution and is such that satisfies (7). In particular, we now have $v_t(0)=\delta$ rigorously.

Remark 4.7. According to what we have seen during the proof, any distribution v defined by $\langle v(t),\varphi\rangle := \langle g,\psi(t)*\varphi\rangle$ satisfies any PDE in the sense of distributions if its *kernel* or Riemann Function $\psi(t)$ also does so.

4.2 Regularity

This new distribution v that solves (9) inherits the regularity properties we had for the Riemann Function $\psi(x,t)$. In fact, we can extend Theorem 3.2 to our case with the following

Theorem 4.8. *Let $v(t)$ be the distribution defined by $\langle v(t),\varphi\rangle := \langle g,\psi(t)*\varphi\rangle$ that solves (9). Then,*

$$\begin{aligned} v:\mathbb{R} &\longrightarrow \mathcal{D}'(\mathbb{R}) \\ t &\longrightarrow v(\cdot,t) \end{aligned} \quad \textit{is such that } v\in\mathcal{C}^1\left(\mathbb{R},\mathcal{D}'\left(\mathbb{R}\right)\right)$$

Proof. In this case we have the really useful expression $\langle v(t),\varphi\rangle := \langle g,\psi(t)*\varphi\rangle$, with $\psi = \frac{1}{2c}I_0\left(2\alpha\sqrt{c^2t^2-x^2}\right)\mathcal{X}_{[-ct,ct]}(x)$ as a function. As a distribution, thanks to Theorem 3.2, we know that $\psi(t)\in\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$ and hence we can write

$$\langle v_t(t),\varphi\rangle := \langle g,\psi_t(t)*\varphi\rangle$$

where

$$\psi_t(\cdot,t) = \frac{1}{2}\left[\delta(\cdot-ct)+\delta(\cdot+ct)\right] + \alpha c|t|\frac{I_0'\left(2\alpha\sqrt{c^2t^2-\cdot^2}\right)}{\sqrt{c^2t^2-\cdot^2}}\mathcal{X}_{[-c|t|,c|t|]}(\cdot)$$

This expression is well defined since ψ_t is compactly supported both in the distributional and functional sense and therefore $\psi_t*\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$. What is more,

$$\psi_t(t)*\varphi = \int_{\mathbb{R}}\psi_t(y,t)\varphi(x-y)dy = \langle\psi_t(\cdot,t),\varphi(x-\cdot)\rangle$$

Then,

$$\langle v_t(t),\varphi\rangle := \langle g,\langle\psi_t(\cdot,t),\varphi(x-\cdot)\rangle\rangle$$

and since $\psi(t)\in\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$ we have $v(t)\in\mathcal{C}^1(\mathbb{R},\mathcal{D}'(\mathbb{R}))$. □

5 Application 1: The Semigroup

5.1 The complete problem

We have already developed enough tools to tackle down the general problem. We have Theorem 4.1 that guarantees the existence of solutions for g a general distribution. In particular, when $g \in L^2(\mathbb{R})$ and thanks to Remark 4.4 we have

$$v(x,t) = \int_{\mathbb{R}} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2 - (x-y)^2}) \mathcal{X}_{[-ct,ct]}(x-y) g(y) dy =$$

$$= \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2 - (y-x)^2}) g(y) dy$$

is the solution in the sense of distributions of

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = g(x) & x \in \mathbb{R} \end{cases} \tag{10}$$

Recovering the change of functions proposed in the heuristics, we have

$$u(x,t) = e^{-\frac{k}{2}t} \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2 - (y-x)^2}) g(y) dy$$

solves

$$\begin{cases} u_{tt} + k u_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

in the sense of distributions, for $g \in L^2(\mathbb{R})$.

We want now to solve the general problem and to study if the operator solution defines a semigroup. Specifically, we want an expression for the solution of

$$\begin{cases} u_{tt} + k u_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{11}$$

We will construct it from the solution of (10). First, we have this

Lemma 5.1. *For $g \in L^2(\mathbb{R})$, let v be the solution of (10). Then*

$$w = v_t \textit{ is such that } \begin{cases} w_{tt} = c^2 w_{xx} + \frac{k^2}{4} w & x \in \mathbb{R},\ t \in \mathbb{R} \\ w(x,0) = g(x) & x \in \mathbb{R} \\ w_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

in the sense of distributions.

Proof. If we derive the PDE of (10) with respect to time we obtain $v_{ttt} = c^2 v_{xxt} + \frac{k^2}{4} v_t$ and so let us name $w = v_t$. Then, w is such that $w_{tt} = c^2 w_{xx} + \frac{k^2}{4} w$. As for the initial conditions,

$$w(x,0) = v_t(x,0) = g(x)$$

$$w_t(x,0) = v_{tt}(x,0) \overset{\text{PDE (4)}}{=} c^2 v_{xx}(x,0) + \frac{k^2}{4} v(x,0) = 0 + 0 = 0$$

□

With this Lemma we can now state the following

Theorem 5.2. *Given $(f,g) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, the solution to the initial value problem (11) is given by*

$$u(x,t) = e^{\frac{-k}{2}t} \left\{ \frac{1}{2} [f(x+ct) + f(x-ct)] + \alpha ct \int_{x-ct}^{x+ct} \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} f(y) dy \right\} +$$

$$+ e^{\frac{-k}{2}t} \left\{ \int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{\lambda}) \left[\frac{k}{2} f(y) + g(y) \right] dy \right\}$$

with $\lambda = c^2t^2 - (x-y)^2$.

Remark 5.3. Before the proof, let us remind what we said in the Introduction and notice that when $k = 0$ problem (11) reduces to

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

and for this case, with $\alpha = \frac{k}{4c} = 0$ the solution given by Theorem 5.2 is

$$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy$$

which is the celebrated D'Alembert formula for the 1-D wave equation.

Proof. Let us divide the problem into two easier ones. If v and w are such that

$$\begin{cases} v_{tt} + k v_t = c^2 v_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = f(x) & x \in \mathbb{R} \\ v_t(x,0) = 0 & x \in \mathbb{R}. \end{cases} \quad \text{and} \quad \begin{cases} w_{tt} + k w_t = c^2 w_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ w(x,0) = 0 & x \in \mathbb{R} \\ w_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

then $u = v + w$ satisfies (11) by the superposition principle. The problem involving w can be solved using the change of functions and all the theory we have already done, so we have

$$w(x,t) = e^{-\frac{k}{2}t} \int_{x-ct}^{x+ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{\lambda}\right) g(y) dy$$

solves the problem. The same strategy works with the problem on v but care must be taken when it comes to the initial conditions. Let $r(x,t) = e^{\frac{k}{2}t}v(x,t)$, its initial conditions are:

$$r(x,0) = e^{\frac{k}{2}0}v(x,0) = f(x)$$
$$r_t(x,0) = \frac{k}{2}e^{\frac{k}{2}0}v(x,0) + e^{\frac{k}{2}0}v(x,0) = \frac{k}{2}f(x)$$

It is clear then that $r(x,t)$ satisfies

$$\begin{cases} r_{tt} = c^2 r_{xx} + \frac{k^2}{4}r & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = f(x) & x \in \mathbb{R} \\ v_t(x,0) = \frac{k}{2}f(x) & x \in \mathbb{R} \end{cases}$$

To solve this problem, let us use again the superposition principle and consider $r = r^{(1)} + r^{(2)}$, where both solve the PDE and

$$\begin{array}{cc} r^{(1)}(x,0) = f(x) & r_t^{(1)}(x,0) = 0 \\ r^{(2)}(x,0) = 0 & r_t^{(2)}(x,0) = \frac{k}{2}f(x) \end{array}$$

We know how to solve for $r^{(2)}$, its solution being

$$r^{(2)}(x,t) = \int_{x-ct}^{x+ct} \frac{1}{2c} I_0\left(2\alpha\sqrt{\lambda}\right) \frac{k}{2} f(y)dy$$

To solve $r^{(1)}$ we are going to use Lemma 5.1. Let $s(x,t)$ such that

$$\begin{cases} s_{tt} = c^2 s_{xx} + \frac{k^2}{4}s & x \in \mathbb{R},\ t \in \mathbb{R} \\ s(x,0) = 0 & x \in \mathbb{R} \\ s_t(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

we then know that $r^{(1)} = s_t$ satisfies the same PDE with $r^{(1)}(x,0) = f(x)$, $r_t^{(1)}(x,0) = 0$. We can write it as

$$r^{(1)}(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \alpha ct \int_{x-ct}^{x+ct} \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} f(y)dy$$

Finally, gathering all the partial solutions of our original problem $u = v + w = e^{\frac{-k}{2}t}r + w = e^{\frac{-k}{2}t}(r^{(1)} + r^{(2)}) + w$ and so

$$u(x,t) = e^{\frac{-k}{2}t}\left\{\frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \alpha ct \int_{x-ct}^{x+ct} \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} f(y)dy\right\} +$$
$$+ e^{\frac{-k}{2}t}\left\{\int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{\lambda}) \left[\frac{k}{2}f(y) + g(y)\right] dy\right\}$$

□

However, we do not know if this expression is the only solution. That's why we present the following theorem, one that guarantees uniqueness of solutions when u satisfies certain conditions.

Theorem 5.4. *Given $f \in H^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, let us suppose there exists $u(x,t)$ such that $u(\cdot,t) \in H^1(\mathbb{R})$, $u_t(\cdot,t) \in L^2(\mathbb{R})$ a.e $t \in \mathbb{R}$ and*

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{12}$$

Then, (u, u_t) is the unique solution to this problem in $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

Proof. Let us suppose there exists another solution $(w, w_t) \in X$. If we consider $v = u - w$, we have $(v, v_t) \in X$ solves (12) with $f = g = 0$, homogeneous conditions.

Trivially, $v \equiv 0$ is a solution. To see the converse, let us do an energy-based argument. Considering the equation models the movement of an infinite rope with friction with the surface, its energy at time t is

$$E(t) = \int_{\mathbb{R}} \left(\frac{1}{2} v_t^2 + \frac{c^2}{2} v_x^2 \right) dx$$

Let us remark this energy is well defined thanks to the fact that $(v, v_t) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ a.e. $t \in \mathbb{R}$. We compute how it changes with time, to observe that

$$E'(t) = \int_{\mathbb{R}} \left(v_t v_{tt} + c^2 v_x v_{xt} \right) dx \overset{\text{Parts Integration}}{=} \int_{\mathbb{R}} \left(v_t v_{tt} - c^2 v_t v_{xx} \right) dx =$$

$$= \int_{\mathbb{R}} v_t \left(v_{tt} - c^2 v_{xx} \right) dx \overset{\text{PDE}}{=} \int_{\mathbb{R}} -k v_t^2 dx \leq 0$$

Therefore, $E(t)$ is strictly decreasing in time. Also, notice that $E(t) \geq 0$ trivially and that $E(0) = 0$ due to the homogeneous initial conditions. Hence, we have that $\forall t > 0$, $0 = E(0) \geq E(t) \geq 0$ so we deduce $E(t) = 0$, $\forall t \geq 0$.

What is more, we observe

$$E'(t) = \int_{\mathbb{R}} -k v_t^2 dx \geq -\int_{\mathbb{R}} 2k \left(\frac{1}{2} v_t^2 + \frac{1}{2} v_x^2 \right) dx = -2kE(t)$$

from which we deduce that $E'(t) + 2kE(t) \geq 0$ and then

$$e^{2kt} E'(t) + 2k e^{2kt} E(t) \geq 0 \Longrightarrow \frac{d}{dt} \left(e^{2kt} E(t) \right) \geq 0$$

And so, we have that $e^{2kt_1} E(t_1) \leq e^{2kt_2} E(t_2) \Longleftrightarrow t_1 \leq t_2$. In particular, for $t_1 \leq 0$ we have $0 \leq e^{2kt_1} E(t_1) \leq E(0) = 0$, from which we deduce that $E(t) = 0$, $\forall t \leq 0$ and, consequently, $E(t) \equiv 0$.

Finally,

$$0 = \int_{\mathbb{R}} \left(\frac{1}{2} v_t^2 + \frac{c^2}{2} v_x^2 \right) dx \Longleftrightarrow v_t^2 = 0,\ v_x^2 = 0 \Longleftrightarrow v \equiv \text{ctn}$$

since we are integrating positive quantities (they are squares) and so given $v(x,0) = 0$ we deduce that $v \equiv 0 \Rightarrow u - w \equiv 0 \Rightarrow u \equiv w$ and we have uniqueness. $\square$

5.2 The Semigroup

Definition 5.5. A family $\{\Gamma_t\}_{t\geq 0}$ of linear operators on a Banach space X into itself is said to have the semigroup property if

$$\Gamma_0 = I_d \text{ and } \Gamma_t(\Gamma_s) = \Gamma_{t+s} \quad \forall t, s \geq 0$$

Let us consider $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and the operator that for each $t \geq 0$ and for each $(f, g) \in X$ it gives us $\Gamma_t(f, g) =$ the solution and its time derivative of the complete problem (11), with f, g as the initial conditions. To ease the notations, we write $u(t)$ and $u_t(t)$ to refer to such solutions. We have the following

Theorem 5.6. *Let $(f, g) \in X$. Then,*

$$\Gamma_t(f, g) = (u(t), u_t(t)) \in X,\ \forall t \geq 0 \textit{ and } \Gamma_t \textit{ has the semigroup property.}$$

Proof. We have to check mainly two things. The first one is $\Gamma_t \in X,\ \forall t \geq 0$ while the second one is the semigroup property.

Let us start with the latter assuming that $\Gamma_t \in X,\ \forall t \geq 0$, which will be proven afterwards. Firstly, it is trivial that $\Gamma_0 = I_d$ since u is the solution of (11) such that $u(0) = f$ and $u_t(0) = g$.

As for the other property, we notice we satisfy the hypothesis of Theorem 5.4 which gives us uniqueness of solution of problem (11). Indeed, given $s \geq 0$ we have that $\Gamma_{t+s} = (u(t+s), u_t(t+s))$ solves the PDE of (11) and is such that for $t = 0$, $\Gamma_{0+s} = (u(0+s), u_t(0+s)) = (u(s), u_t(s)) = \Gamma_0(\Gamma_s)$, since $\Gamma_0 = I_d$.

What is more, since $\Gamma_t \in X$ we can use Theorem 5.4 when the initial conditions are $(u(s), u_t(s)) \in X$ and, consequently, given that $\Gamma_t(\Gamma_s)$ also solves the PDE of (11) and has the same initial condition as Γ_{t+s}, by uniqueness of solutions it must be $\Gamma_t(\Gamma_s) = \Gamma_{t+s}$.

Hence Γ_t has the semigroup property.

$\diamond$

We still have to prove that $\Gamma_t \in X$. This means we have to see that a.e. $t \geq 0$, $u(x, t) \in L^2(\mathbb{R})$, $u_t(x, t) \in L^2(\mathbb{R})$ and $u_x(x, t) \in L^2(\mathbb{R})$ as functions of the space variable x.

Proof of $u \in L^2(\mathbb{R})$

Let us remember

$$u(x,t) = e^{\frac{-k}{2}t}\left\{\frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \alpha ct\int_{x-ct}^{x+ct}\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}f(y)dy\right\} +$$

$$+e^{\frac{-k}{2}t}\left\{\int_{x-ct}^{x+ct}\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})\left[\frac{k}{2}f(y) + g(y)\right]dy\right\}$$

needs to be square integrable. We will bound its norm by some constants depending on t and by the norms of f and g that we remember they are already both in $L^2(\mathbb{R})$. First of all, it is easy to see that

$$\left\| \frac{1}{2}\left(f(x+ct)+f(x-ct)\right)\right\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$$

For the integrals, we will write things in a proper so way that it will be quite simple to obtain what we desire. For example, let $h = \frac{k}{2}f + g \in L^2(\mathbb{R})$ with $\|h\|_{L^2(\mathbb{R})} \leq \frac{k}{2}\|f\|_{L^2(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})}$, we have

$$\int_{x-ct}^{x+ct} \frac{1}{2c} I_0(2\alpha\sqrt{\lambda})\left[\frac{k}{2}f(y)+g(y)\right]dy =$$

$$= \int_{\mathbb{R}} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-(x-y)^2})\mathcal{X}_{[-ct,ct]}(x-y)h(y)dy = \int_{\mathbb{R}} \gamma^t(x-y)h(y)dy$$

which results in a convolution between γ^t and h. We use now Young's Theorem which gives us

$$\|\gamma^t * h\|_{L^2(\mathbb{R})} \leq \|\gamma^t\|_{L^1(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}$$

provided that $\gamma^t \in L^1(\mathbb{R})$. Indeed,

$$\|\gamma^t\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-x^2})\mathcal{X}_{[-ct,ct]}(x)dx$$

$$= \int_{-ct}^{ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-x^2})dx \leq \int_{-ct}^{ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2})dx = tI_0(2\alpha ct) < \infty$$

On the other hand, we also have

$$\alpha ct \int_{x-ct}^{x+ct} \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}} f(y)dy =$$

$$= \int_{\mathbb{R}} \alpha ct \frac{I_0'(2\alpha\sqrt{c^2t^2-(x-y)^2})}{\sqrt{c^2t^2-(x-y)^2}}\mathcal{X}_{[-ct,ct]}(x-y)f(y)dy = \int_{\mathbb{R}} \rho^t(x-y)f(y)dy$$

which also results in a convolution between ρ^t and f. Once again, Young's Theorem yields

$$\|\rho^t * f\|_{L^2(\mathbb{R})} \leq \|\rho^t\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}$$

provided that $\rho^t \in L^1(\mathbb{R})$, too. Indeed, since $\frac{I_0'(2\alpha z)}{z}$ is a monotonous increasing function for $z \geq 0$ we have

$$\|\rho^t\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \alpha ct \frac{I_0'(2\alpha\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}\mathcal{X}_{[-ct,ct]}(x)dx =$$

$$= \int_{-ct}^{ct} \alpha ct \frac{I_0'(2\alpha\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}dx \leq \int_{-ct}^{ct} \alpha ct \frac{I_0'(2\alpha\sqrt{c^2t^2})}{\sqrt{c^2t^2}}dx = 2\alpha ct I_0'(2\alpha ct) < \infty$$

Gathering all results one gets

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})} \leq e^{-\frac{k}{2}t}\left(\|f\|_{L^2(\mathbb{R})} + \|\gamma^t * h\|_{L^2(\mathbb{R})} + \|\rho^t * f\|_{L^2(\mathbb{R})}\right) \leq$$

$$\leq e^{-\frac{k}{2}t}\left(\|f\|_{L^2(\mathbb{R})} + tI_0(2\alpha ct)\|h\|_{L^2(\mathbb{R})} + 2\alpha ctI_0'(2\alpha ct)\|f\|_{L^2(\mathbb{R})}\right) \leq$$

$$\leq e^{-\frac{k}{2}t}\left((1 + 2\alpha ct(I_0(2\alpha ct) + I_0'(2\alpha ct))\|f\|_{L^2(\mathbb{R})} + tI_0(2\alpha ct)\|g\|_{L^2(\mathbb{R})}\right) =$$

$$\leq e^{-\frac{k}{2}t}\left(M_1(t)\|f\|_{L^2(\mathbb{R})} + N_1(t)\|g\|_{L^2(\mathbb{R})}\right) < \infty$$

so $u \in L^2(\mathbb{R})$ and its dependence on both f and g is continuous.

Proof of $u_t \in L^2(\mathbb{R})$

We need some kind of expression for u_t before we start doing anything. Deriving the u with respect to time and doing some tedious calculations we end up with

$$u_t(x,t) = e^{-\frac{k}{2}t}\left\{\frac{c}{2}[f'(x+ct) - f'(x-ct)] + \alpha^2c^2t[f(x+ct) + f(x-ct)]\right.$$

$$+\frac{1}{2}[g(x+ct) + g(x-ct)] + \alpha ct\int_{x-ct}^{x+ct}\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}g(y)dy$$

$$+\alpha c\int_{x-ct}^{x+ct}\frac{1}{\lambda}\left[2\alpha c^2t^2I_0''(2\alpha\sqrt{\lambda}) - (x-y)^2\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\right]f(y)dy$$

$$\left.-\frac{k}{2}\int_{\mathbb{R}}\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})\left[\frac{k}{2}f(y) + g(y)\right]dy\right\}$$

We will analyse term by term their square integrability. Let us first start with these three

$$\left\|\frac{c}{2}\left(f'(x+ct) - f'(x-ct)\right)\right\|_{L^2(\mathbb{R})} \leq c\|f'\|_{L^2(\mathbb{R})}$$

$$\left\|\alpha^2c^2t\left[f(x+ct) + f(x-ct)\right]\right\|_{L^2(\mathbb{R})} \leq 2\alpha^2c^2t\|f\|_{L^2(\mathbb{R})}$$

$$\left\|\frac{1}{2}\left(g(x+ct) + g(x-ct)\right)\right\|_{L^2(\mathbb{R})} \leq \|g\|_{L^2(\mathbb{R})}$$

As for the integral terms, we use previous results such as

$$\left\|\alpha ct\int_{x-ct}^{x+ct}\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}g(y)dy\right\|_{L^2(\mathbb{R})} = \|\rho^t * g\|_{L^2(\mathbb{R})} \leq$$

$$\leq \|\rho^t\|_{L^1(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} \leq 2\alpha ctI_0'(2\alpha ct)\|g\|_{L^2(\mathbb{R})}$$

and also

$$\left\|-\frac{k}{2}\int_{\mathbb{R}}\frac{1}{2c}I_0(2\alpha\sqrt{\lambda})\left[\frac{k}{2}f(y) + g(y)\right]dy\right\|_{L^2(\mathbb{R})} = \frac{k}{2}\|\gamma^t * h\|_{L^2(\mathbb{R})} \leq$$

$$\leq \frac{k}{2}\|\gamma^t\|_{L^1(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} \leq \frac{k}{2}tI_0(2\alpha ct)\|h\|_{L^2(\mathbb{R})} = 2\alpha ctI_0(2\alpha ct)\|h\|_{L^2(\mathbb{R})} \leq$$

$$\leq k\alpha ctI_0(2\alpha ct)\|f\|_{L^2(\mathbb{R})} + 2\alpha ctI_0(2\alpha ct)\|g\|_{L^2(\mathbb{R})}$$

The last missing term

$$\alpha c\int_{x-ct}^{x+ct}\frac{1}{\lambda}\left[2\alpha c^2t^2I_0''(2\alpha\sqrt{\lambda}) - (x-y)^2\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\right]f(y)dy =$$

$$= \alpha c\int_{\mathbb{R}}\frac{1}{\lambda}\left[2\alpha c^2t^2I_0''(2\alpha\sqrt{\lambda}) - (x-y)^2\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\right]\mathcal{X}_{[-ct,ct]}(x-y)f(y)dy =$$

$$= \alpha c\int_{\mathbb{R}}\vartheta^t(x-y)f(y)dy = \alpha c(\vartheta^t * f)(x)$$

is again a convolution so that by Young's Theorem we have

$$\|\alpha c(\vartheta^t * f)\|_{L^2(\mathbb{R})} \leq \alpha c\|\vartheta^t\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}$$

provided that $\vartheta^t \in L^1(\mathbb{R})$. In order to check this condition, for $\lambda_0 = c^2t^2 - x^2$ we have

$$\|\vartheta^t\|_{L^1(\mathbb{R})} = \alpha c\int_{-ct}^{ct}\frac{1}{\lambda_0}\left|2\alpha c^2t^2I_0''(2\alpha\sqrt{\lambda_0}) - x^2\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\right|dx$$

The integrand is continuous inside the domain and bounded (see Appendix A.4) when approaching the limits ct and $-ct$. Hence, it has a maximum denoted by $P(t)$ and we have the following estimate

$$\|\vartheta^t\|_{L^1(\mathbb{R})} \leq 2\alpha c^2tP(t)$$

Adding all the estimates for each term we have

$$e^{\frac{k}{2}t}\|u_t\|_{L^2(\mathbb{R})} \leq c\|f'\|_{L^2(\mathbb{R})} + 2\alpha^2c^2t\|f\|_{L^2(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})}$$

$$+2\alpha ctI_0'(2\alpha ct)\|g\|_{L^2(\mathbb{R})} + 4\alpha^2c^2tI_0(2\alpha ct)\|f\|_{L^2(\mathbb{R})}$$

$$+2\alpha ctI_0(2\alpha ct)\|g\|_{L^2(\mathbb{R})} + 2\alpha c^2tP(t)\|f\|_{L^2(\mathbb{R})}$$

Rearranging terms we obtain

$$\|u_t\|_{L^2(\mathbb{R})} \leq e^{\frac{-k}{2}t}\left\{c\|f'\|_{L^2(\mathbb{R})} + (1 + 2\alpha ct[I_0(2\alpha ct) + I_0'(2\alpha ct)])\|g\|_{L^2(\mathbb{R})}\right.$$

$$\left.+2\alpha c^2t(\alpha + 2\alpha I_0(2\alpha ct) + P(t))\|f\|_{L^2(\mathbb{R})}\right\} \Rightarrow$$

$$\|u_t\|_{L^2(\mathbb{R})} \leq e^{\frac{-k}{2}t}\left\{c\|f'\|_{L^2(\mathbb{R})} + M_1(t)\|g\|_{L^2(\mathbb{R})} + N_2(t)\|f\|_{L^2(\mathbb{R})}\right\}$$

and thus $u_t(\cdot, t) \in L^2(\mathbb{R})$, bounded again by the L^2 norms of f, g, and f'.

Proof of $u_x \in L^2(\mathbb{R})$

We need some kind of expression for u_x before we start doing anything. Deriving u with respect to space x and doing some tedious calculations we end up with

$$u_x(x,t) = e^{-\frac{k}{2}t}\left\{\frac{1}{2}\left[f'(x+ct) - f'(x-ct)\right] + (\alpha^2 ct + \alpha)\left[f(x+ct) - f(x-ct)\right]\right.$$

$$+\frac{1}{2c}\left[g(x+ct) - g(x-ct)\right] - \int_{x-ct}^{x+ct} \frac{\alpha}{c}(x-y)\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\left(\frac{k}{2}f(y) + g(y)\right)dy$$

$$\left. - \int_{x-ct}^{x+ct} \alpha ct\frac{x-y}{\lambda}\left(2\alpha I_0''(2\alpha\sqrt{\lambda}) - \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\right) f(y)dy\right\}$$

and, as before, we analyse term by term if they are in L^2. We start with

$$\left\|\frac{1}{2}\left(f'(x+ct) - f'(x-ct)\right)\right\|_{L^2(\mathbb{R})} \leq \|f'\|_{L^2(\mathbb{R})}$$

$$\left\|(\alpha^2 ct + \alpha)\left[f(x+ct) + f(x-ct)\right]\right\|_{L^2(\mathbb{R})} \leq 2\alpha(\alpha ct + 1)\|f\|_{L^2(\mathbb{R})}$$

$$\left\|\frac{1}{2c}\left(g(x+ct) + g(x-ct)\right)\right\|_{L^2(\mathbb{R})} \leq \frac{1}{c}\|g\|_{L^2(\mathbb{R})}$$

For the integral terms we write them as convolutions and using similar strategies as before, we first study

$$\left\|\int_{x-ct}^{x+ct} \frac{\alpha}{c}(x-y)\frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\left(\frac{k}{2}f(y) + g(y)\right)dy\right\|_{L^2(\mathbb{R})} = \left\|\eta^t * h\right\|_{L^2(\mathbb{R})}$$

which will be bounded by $\|\eta^t\|_{L^1(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}$ using Young's Theorem provided that, indeed, $\eta^t \in L^1(\mathbb{R})$. Let us observe that

$$\eta^t(x) = \frac{\alpha}{c}x\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\mathcal{X}_{[-ct,ct]}(x)$$

and so

$$\left\|\eta^t\right\|_{L^1(\mathbb{R})} = \int_{-ct}^{ct} \frac{\alpha}{c}|x|\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}dy \leq \int_{-ct}^{ct} \frac{\alpha}{c}|ct|\frac{I_0'(2\alpha\sqrt{c^2t^2})}{\sqrt{c^2t^2}}dy =$$

$$= 2ct\frac{\alpha}{c}|ct|\frac{I_0'(2\alpha ct)}{ct} = 2\alpha t I_0'(2\alpha ct) < \infty$$

As for the other integral,

$$\left\|\int_{x-ct}^{x+ct} \alpha ct\frac{x-y}{\lambda}\left(2\alpha I_0''(2\alpha\sqrt{\lambda}) - \frac{I_0'(2\alpha\sqrt{\lambda})}{\sqrt{\lambda}}\right) f(y)dy\right\|_{L^2(\mathbb{R})} = \left\|\theta^t * f\right\|_{L^2(\mathbb{R})}$$

and so by Young's Theorem it will be bounded by $\|\theta^t\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}$. We check that $\theta^t \in L^1(\mathbb{R})$ and give an estimate for its norm. Recalling that

$$\theta^t(x) = \alpha ct\frac{x}{\lambda_0}\left(2\alpha I_0''(2\alpha\sqrt{\lambda_0}) - \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\right)\mathcal{X}_{[-ct,ct]}(x)$$

we have

$$\left\|\theta^t\right\|_{L^1(\mathbb{R})} = \int_{-ct}^{ct}\left|\alpha ct\frac{x}{\lambda_0}\left(2\alpha I_0''(2\alpha\sqrt{\lambda_0}) - \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\right)\right| dx$$

Let us remark that the integrand is continuous inside the domain and bounded (see Appendix A.5) when approaching the limits ct and $-ct$. Hence, it has a maximum denoted by $Q(t)$ and we have the following estimate

$$\left\|\theta^t\right\|_{L^1(\mathbb{R})} \leq 2ctQ(t)$$

Gathering all the estimates, we have

$$\|u_x\|_{L^2(\mathbb{R})} \leq e^{-\frac{k}{2}t}\left\{\|f'\|_{L^2(\mathbb{R})} + 2\alpha(\alpha ct+1)\|f\|_{L^2(\mathbb{R})} + \frac{1}{c}\|g\|_{L^2(\mathbb{R})}\right.$$

$$\left.+2\alpha tI_0'(2\alpha ct)\frac{k}{2}\|f\|_{L^2(\mathbb{R})} + 2\alpha tI_0'(2\alpha ct)\|g\|_{L^2(\mathbb{R})} + 2ctQ(t)\|f\|_{L^2(\mathbb{R})}\right\}$$

which can be written as

$$\|u_x\|_{L^2(\mathbb{R})} \leq e^{-\frac{k}{2}t}\left\{\|f'\|_{L^2(\mathbb{R})} + \left(\frac{1}{c} + 2\alpha tI_0'(2\alpha ct)\right)\|g\|_{L^2(\mathbb{R})}\right.$$

$$\left.+\left[2\alpha(\alpha ct+1) + \alpha ktI_0'(2\alpha ct) + 2ctQ(t)\right]\|f\|_{L^2(\mathbb{R})}\right\}$$

and we can finally conclude with saying that $u_x \in L^2(\mathbb{R})$, again bounded by the L^2 norms of f, f' and g. □

Remark 5.7. In all three cases we end up having our terms bounded by the L^2 norms of f, f' and g. A strong consequence of this is that the operator solution is continuous with respect to the initial data. This means close enough initial conditions will produce close enough solutions.

6 The Riemann Function revisited

6.1 Uniqueness

With the results we have found in the Chapter of the semigroup, we are now able to say something about the uniqueness of problem

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = g & x \in \mathbb{R} \end{cases} \tag{13}$$

when g is a general distribution and, in particular, when $g = \delta$. Let us begin with the following

Proposition 6.1. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $g \in \mathcal{D}'(\mathbb{R})$ and v be the solution in the sense of distributions that we have found in Theorem 4.1 of problem (13). Then,*

the function $s = v * \varphi := \langle v, \varphi(x - \cdot)\rangle \in \mathcal{C}^\infty(\mathbb{R})$ is well defined and it is a solution in the sense of distributions of

$$\begin{cases} s_{tt} = c^2 s_{xx} + \frac{k^2}{4} s & x \in \mathbb{R},\ t \in \mathbb{R} \\ s(x,0) = 0 & x \in \mathbb{R} \\ s_t(x,0) = (g * \varphi)(x) & x \in \mathbb{R} \end{cases}$$

Proof. The new candidate to solution is a convolution between a distributions and a test function. Usually, a convolution involves an integration, but once again we are dealing with a general distribution and it may not make sense to multiply it and then integrate it.

However, if we proceed formally we reach an expression that is correct and makes sense for a general distribution. Indeed, for each $x \in \mathbb{R}$ we have $\varphi(x - \cdot) \in \mathcal{C}_0^\infty(\mathbb{R})$ and

$$(v * \varphi)(x) = \underbrace{\int_{\mathbb{R}} v(y,t)\varphi(x-y)dy}_{\text{Notation}} := \langle v, \varphi(x - \cdot)\rangle < \infty.$$

This way, $v * \varphi \in \mathcal{C}^\infty(\mathbb{R})$ due to the fact that φ is infinitely differentiable, the derivatives enter the duality product. Both initial conditions are satisfied easily thanks to the initial conditions that v satisfies. As for the differential equation, we need

$$\int_{\mathbb{R}^2} L(s)\phi \, dxdt := \int_{\mathbb{R}^2} sL^*(\phi)dxdt = 0, \ \ \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

Let us remark that this time the integration is already well defined, since $s \in \mathcal{C}^\infty(\mathbb{R})$ and not a general distribution. Therefore, let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we can write

$$\int_{\mathbb{R}^2} sL^*(\phi)dxdt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} sL^*(\phi)dx \right) dt$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle v, \varphi(x - \cdot) \rangle \, L^*(\phi) dx \right) dt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle v, L^*(\phi)\varphi(x - \cdot) \rangle \, dt \right) dx =$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle v, L^*(\phi(x,t))\varphi(x - \cdot) \rangle \, dt \right) dx \overset{x-\cdot=z}{\underset{dx=dz}{=}}$$

$$\overset{x-\cdot=z}{\underset{dx=dz}{=}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle v, L^*(\phi(z + \cdot, t))\varphi(z) \rangle \, dt \right) dz =$$

$$= \int_{\mathbb{R}} \varphi(z) \left(\int_{\mathbb{R}} \langle v, L^*(\phi(z + \cdot, t)) \rangle \, dt \right) dz =$$

Fixed $z \in \mathbb{R}$, if we write $\phi(z + y, t) = \gamma(y, t)$, we have that $\gamma \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $L^*(\phi(z + y, t)) = L^*(\gamma(y, t)) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ so that

$$\int_{\mathbb{R}} \varphi(z) \underbrace{\left(\int_{\mathbb{R}} \langle v(t), L^*(\gamma) \rangle \, dt \right)}_{0} dz = 0$$

thanks to what we saw in Theorem 4.1. This way, we make sure that $v * \varphi$ is a solution in a weak sense of the differential equation. □

Remark 6.2. The links between u, v and their respective problems are such that we have uniqueness in u if and only if we have uniqueness in v. The following theorem uses this property.

Theorem 6.3. *Let $g \in \mathcal{D}'(\mathbb{R})$ such that $(g * \varphi) \in L^2(\mathbb{R})$, $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Then, problem*

$$\begin{cases} v_{tt} = c^2 v_{xx} + \frac{k^2}{4} v & x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(x, 0) = 0 & x \in \mathbb{R} \\ v_t(x, 0) = g & x \in \mathbb{R} \end{cases}$$

admits uniqueness of solution.

Proof. We already know this problem admits existence thanks to Theorem 4.1. What is more, thanks to the last Proposition, we have that for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $s = v * \varphi \in \mathcal{C}^\infty(\mathbb{R})$ is a solution in the sense of distributions of

$$\begin{cases} s_{tt} = c^2 s_{xx} + \frac{k^2}{4} s & x \in \mathbb{R}, \ t \in \mathbb{R} \\ s(x, 0) = 0 & x \in \mathbb{R} \\ s_t(x, 0) = (g * \varphi)(x) & x \in \mathbb{R} \end{cases} \tag{14}$$

Let us assume there exists another solution w of (13), we construct $r = w * \varphi \in \mathcal{C}^\infty(\mathbb{R})$ another solution in the sense of distributions of (14). For both s and r we have that $s_t(x, 0) = (g * \varphi)(x) = \widetilde{g} \in L^2(\mathbb{R})$.

Now, using Theorem 5.6 with initial conditions $(0, \widetilde{g}) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ we can apply Theorem 5.4 since the solutions and the initial conditions satisfy the required properties so that $s = r$ and

$$s = r \Longrightarrow (v - w) * \varphi = 0, \ \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}) \Longrightarrow v = w$$

from which we obtain the much desired uniqueness. □

Remark 6.4. Considering $g = \delta \in \mathcal{D}'(\mathbb{R})$ we have $g * \varphi = \delta * \varphi = \varphi \in L^2(\mathbb{R})$, $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ so that we can use Theorem 6.3 and guarantee the uniqueness of the problem when $g = \delta$.

Remark 6.5. We can also use Theorem 6.3 for those problem in which g has compact support or $g \in L^2(\mathbb{R})$.

6.2 The complete problem revisited

We now intend to obtain a solution u for the general problem

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f & x \in \mathbb{R} \\ u_t(x,0) = g & x \in \mathbb{R}. \end{cases} \tag{15}$$

when f and g are both distributions. We first present the following

Lemma 6.6. *Let v be the distribution that solves problem (13) with g a distribution as its initial condition. Then, the distribution $w = v_t$ defined by*

$$\langle w, \varphi \rangle := \langle g, \psi_t * \varphi \rangle$$

solves

$$\begin{cases} w_{tt} = c^2 w_{xx} + \frac{k^2}{4} w & x \in \mathbb{R},\ t \in \mathbb{R} \\ w(x,0) = g & x \in \mathbb{R} \\ w_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

in the sense of distributions.

Proof. It is reasonable to define w this way since we will use Lemma 5.1. Let us remember that

$$\psi_t(\cdot, t) = \frac{1}{2}\left[\delta(\cdot - ct) + \delta(\cdot + ct)\right] + \alpha c|t| \frac{I_0'\left(2\alpha\sqrt{c^2t^2 - \cdot^2}\right)}{\sqrt{c^2t^2 - \cdot^2}} \mathcal{X}_{[-c|t|, c|t|]}(\cdot) \in \mathcal{D}'(\mathbb{R})$$

Notice ψ_t is compactly supported as a distribution and, as a result, $\psi_t * \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Let us now recall Remark 4.7 which says that the distribution w satisfies the PDE if $\psi_t(t)$ does so.

We know $\psi(t)$ does solve the PDE, and hence, applying Lemma 5.1 we have ψ_t also solves it. Consequently, w satisfies the PDE in the sense of distributions.

As for the initial conditions, they are easily deduced from the application of Lemma 5.1 and the equation the solution solves (remember that $\psi_t(0) = \delta$). □

We are now able to state an equivalent result that of Theorem 5.2 with the following

Theorem 6.7. *Given $f, g \in \mathcal{D}'(\mathbb{R})$ distributions, the solution to the initial value problem (15) in the sense of distributions is the distribution u defined as*

$$\langle u, \varphi \rangle := e^{-\frac{k}{2}t} \left(\left\langle f, \left(\psi_t + \frac{k}{2}\psi \right) * \varphi \right\rangle + \langle g, \psi * \varphi \rangle \right) =$$

$$= e^{-\frac{k}{2}t} \left(\langle f, \psi_t * \varphi \rangle + \left\langle \frac{k}{2} f + g, \psi * \varphi \right\rangle \right)$$

Proof. The proof is identical to the one proposed for Theorem 5.2, when (f, g) were functions in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. We divide the general problem into smaller ones. We use Lemma 6.6 for the problem that has a distribution as its first initial condition.

Each smaller problem has then been solved and we know how they act on test functions. The resulting solution is the sum of all the partial solutions acting on the same test function.

Using the same notation as before but now considering them as distributions, we find that

$$u = e^{-\frac{k}{2}t} \left(r^{(1)} + r^{(2)} \right) + w \Longrightarrow \langle u, \varphi \rangle := \left\langle e^{-\frac{k}{2}t} \left(r^{(1)} + r^{(2)} \right) + w, \varphi \right\rangle$$

using the linearity of the duality $\langle \cdot, \cdot \rangle$ we get

$$e^{-\frac{k}{2}t} \left(\left\langle r^{(1)}, \varphi \right\rangle + \left\langle r^{(2)}, \varphi \right\rangle + \langle v, \varphi \rangle \right) :=$$

$$e^{-\frac{k}{2}t} \left(\langle f, \psi_t * \varphi \rangle + \left\langle \frac{k}{2} f, \psi * \varphi \right\rangle + \langle g, \psi * \varphi \rangle \right) =$$

$$= e^{-\frac{k}{2}t} \left(\langle f, \psi_t * \varphi \rangle + \left\langle \frac{k}{2} f + g, \psi * \varphi \right\rangle \right)$$

and also

$$= e^{-\frac{k}{2}t} \left(\left\langle f, \left(\psi_t + \frac{k}{2}\psi \right) * \varphi \right\rangle + \langle g, \psi * \varphi \rangle \right)$$

□

Remark 6.8. As a matter of fact, given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we can use Proposition 6.1 to see that $u * \varphi$ defined by $s(x) = (u * \varphi)(x) := \langle u, \varphi(x - \cdot) \rangle \in \mathcal{C}^\infty(\mathbb{R})$ is a solution in the sense of distributions of

$$\begin{cases} s_{tt} + k s_t = c^2 s_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ s(x, 0) = (f * \varphi)(x) & x \in \mathbb{R} \\ s_t(x, 0) = (g * \varphi)(x) & x \in \mathbb{R}. \end{cases} \tag{16}$$

via the change $u = e^{-\frac{k}{2}t} v$. With this, we have the following

Theorem 6.9. *Let $f, g \in \mathcal{D}'(\mathbb{R})$ such that $(f * \varphi) \in H^1(\mathbb{R})$ and $(g * \varphi) \in L^2(\mathbb{R})$, $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Then, problem*

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R} \\ u(x,0) = f & x \in \mathbb{R} \\ u_t(x,0) = g & x \in \mathbb{R}. \end{cases} \tag{17}$$

admits uniqueness of solution.

Proof. It is really similar to that proposed for Theorem 6.3. Let us assume there exists another solution w of (17), we construct $r = w * \varphi \in \mathcal{C}^\infty(\mathbb{R})$ is another solution in the sense of distributions of (16). For both s and r we have $r(x,0) = s(x,0) = (f * \varphi)(x) = \widetilde{f} \in H^1(\mathbb{R})$ and $r_t(x,0) = s_t(x,0) = (g * \varphi)(x) = \widetilde{g} \in L^2(\mathbb{R})$.

Now, using Theorem 5.6 with initial conditions $(\widetilde{f}, \widetilde{g}) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ we can apply Theorem 5.4 since the solutions and the initial conditions satisfy the required properties so that $s = r$ and

$$s = r \Longrightarrow (u - w) * \varphi = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}) \Longrightarrow u = w$$

$\square$

7 Application 2: A Financial Model

7.1 The Model

We are going to deduce our modified wave equation using an approach different from the classical mechanical one of the moving rope. Let us assume a particle moves on the discrete set $\{k\Delta x \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ at time intervals of Δt.

Let us suppose that with probability p it repeats the same move as in the previous jump and with probability $1-p$ does the contrary move. In the limiting process, when Δx and Δt are small, we will assume that p is near 1, representing this way some kind of inertia in the movement.

In order to deduce which laws do the movement follow, let us denote the pair (k,n) =the particle is in the position $x = k\Delta x$ at time $t = n\Delta t$. Let us also define

$$\alpha(k,n) = Prob\,(\text{the particle is in } (k,n) \text{ and comes from } (k-1,n-1))$$

$$\beta(k,n) = Prob\,(\text{the particle is in } (k,n) \text{ and comes from } (k+1,n-1))$$

We are interested in deducing a law for $\gamma(k,n) = \alpha(k,n) + \beta(k,n)$, which is the probability of the particle being in (k,n). Let us observe that

$$\alpha(k,n+1) = p\alpha(k-1,n) + (1-p)\beta(k-1,n) =$$

$$= p\gamma(k-1,n) + (1-2p)\beta(k-1,n)$$

and

$$\beta(k,n+1) = (1-p)\alpha(k+1,n) + p\beta(k+1,n) =$$
$$= p\gamma(k+1,n) - (1-2p)\alpha(k+1,n)$$

and

$$\gamma(k,n+1) = \alpha(k,n+1) + \beta(k,n+1) =$$
$$= p(\gamma(k-1,n) + \gamma(k+1,n)) + (1-2p)(\beta(k-1,n) + \alpha(k+1,n)) =$$
$$= p(\gamma(k-1,n) + \gamma(k+1,n)) + (1-2p)\gamma(k,n-1)$$

Hence, we have that

$$\gamma(k,n+1) - 2p\gamma(k,n) + (2p-1)\gamma(k,n-1) =$$

$$= p(\gamma(k-1,n) - 2\gamma(k,n) + \gamma(k+1,n))$$

Let us now take $c = \frac{\Delta x}{\Delta t}$, we rearrange terms and divide both sides by $(\Delta x)^2$ to get

$$\frac{\gamma(k,n+1) - 2\gamma(k,n) + \gamma(k,n-1)}{c^2(\Delta t)^2} + \frac{2-2p}{c^2\Delta t}\frac{\gamma(k,n) - \gamma(k,n-1)}{\Delta t} =$$

$$= p\frac{\gamma(k-1,n) - 2\gamma(k,n) + \gamma(k+1,n)}{(\Delta x)^2}$$

Now, doing the limits we obtain

$$\frac{1}{c^2}\gamma_{tt} + \frac{k}{c^2}\gamma_t = \gamma_{xx}$$

where $k = \lim_{\Delta t \to 0} \frac{2-2p}{\Delta t} \geq 0$. Reorganizing and denoting $\gamma = u$ we are left with.

$$u_{tt} + ku_t = c^2 u_{xx}$$

which is precisely the PDE we have been studying so far.

Let us now consider a financial asset, a stock share, for example, whose price resembles the motion of the described particle, i.e., it shows certain tendency to repeat the movement previously done.

Then, given suitable initial conditions this equation models the probability density function of the price of the asset, a random variable. We are going to study a simplified case of this model, we are going to consider the problem given by

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & x \in \mathbb{R},\ t \in \mathbb{R}^+ \\ u(x,0) = \delta & x \in \mathbb{R} \\ u_t(x,0) = -c\delta' & x \in \mathbb{R}. \end{cases} \tag{18}$$

The first initial condition $u(x,0) = \delta(x)$ means that we are absolutely sure the particle, in this case the price of the asset, is 0 at time $t = 0$. The second one $u_t(x,0) = -c\delta'(x)$ indicates us the particle, or the price of the asset, has an initial tendency to go upwards, to increase its value.

As a matter of fact, let us explain u_t. We assume an initial tendency to go upwards, that is, at time Δt we know the particle is in Δx, so we also have $u(x, \Delta t) = \delta(x - \Delta x)$. By definition,

$$u_t(x,0) = \lim_{\Delta t \to 0} \frac{u(x,\Delta x) - u(x,0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\delta(x - \Delta x) - \delta(x)}{\Delta t}$$

and since we are dealing with distributions, we compute its distributional derivative:

$$\left\langle \frac{\delta(x-\Delta x) - \delta(x)}{\Delta t}, \varphi(x) \right\rangle = \int_{\mathbb{R}} \frac{\delta(x-\Delta x) - \delta(x)}{\Delta t} \varphi(x) dx =$$

$$= \frac{\varphi(\Delta x) - \varphi(0)}{\Delta t} \xrightarrow[\Delta t \to 0]{\Delta x = c\Delta t} c\varphi'(0) =: \langle -c\delta', \varphi \rangle \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$$

and so we shall have $u_t(x,0) = -c\delta'(x)$.

In order to solve problem (18), we can use the formula we found out for the complete problem applying the considered initial conditions. Let us remember the general solution of the problem when the initial conditions are distributions is

$$\langle u, \varphi \rangle := e^{-\frac{k}{2}t} \left(\left\langle f, \left(\psi_t + \frac{k}{2}\psi \right) * \varphi \right\rangle + \langle g, \psi * \varphi \rangle \right) =$$

In our case, we have $f = \delta$ and $g = -c\delta'$ and so we have the following

Theorem 7.1. *The solution to the asset price problem (18) is a probability density function and it is given by*

$$u(x,t) = e^{\frac{-k}{2}t}\left\{\delta(x-ct) + \left(\alpha(x+ct)\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} + \frac{k}{4c}I_0(2\alpha\sqrt{\lambda_0})\right)\mathcal{X}_{(-ct,ct)}(x)\right\}$$

where $\lambda_0 = c^2t^2 - x^2$.

Proof. First of all, let us check that the solution of (18) considered is a probability density function. Clearly, the proposed $u(x,t) \geq 0$, we also have to see that $1 \equiv M(t) = \int_{\mathbb{R}} u(x,t)dx$, $\forall t \geq 0$. We derive

$$M'(t) = \int_{\mathbb{R}} u_t dx \Rightarrow M''(t) = \int_{\mathbb{R}} u_{tt} dx \overset{PDE}{=} \int_{\mathbb{R}} (c^2 u_{xx} - k u_t)dx =$$

$$= -k\int_{\mathbb{R}} u_t dx = -kM'(t) \Rightarrow M''(t) + kM'(t) = 0$$

as $\int_{\mathbb{R}} u_{xx}dx = 0$ because $u(\cdot,t)$ has compact support $\forall t \geq 0$. Now, recalling $k > 0$, the solution of this ODE is

$$M(t) = A + Be^{-kt}$$

and with the initial conditions we will adjust the two constants.

$$1 = \int_{\mathbb{R}} \delta(x)dx = \int_{\mathbb{R}} u(x,0)dx = M(0) = A + B$$

$$0 = (c\cdot 1)' = \int_{\mathbb{R}} -c\delta'(x)dx = \int_{\mathbb{R}} u_t(x,0)dx = M'(0) = -kB$$

and hence we infer that $B = 0$ and $A = 1$ so that $M(t) \equiv 1$, $\forall t \geq 0$.

$\diamond$

As for the deduction of the formula, we simply take $f = \delta$, $g = -c\delta'$ and we use Remark 4.6. The part involving $f = \delta$ can be written as

$$\frac{1}{2}(\delta(x+ct) + \delta(x-ct)) + \left(\alpha ct\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} + \frac{k}{4c}I_0(2\alpha\sqrt{\lambda_0})\right)\mathcal{X}_{(-ct,ct)}(x)$$

The part of $g = -c\delta'$ deserves special attention since it is tricky to develop. By definition, we have

$$\langle g, \psi * \varphi\rangle = \langle -c\delta', \psi * \varphi\rangle := c(\psi * \varphi)'(0) = c\int_{\mathbb{R}} \psi(-y,t)\varphi'(y)dy$$

and we will compute $(\psi * \varphi)'(0) = \int_{\mathbb{R}} \psi(-y,t)\varphi'(y)dy$. In our case, let us define

$$\phi(y) = \frac{1}{2c}I_0(2\alpha\sqrt{c^2t^2 - y^2})\mathcal{X}_{(-ct,ct)}(y) = \psi(y,t)$$

and let us remember that ζ is the weak derivative of γ if

$$\int_{\mathbb{R}} \gamma\varphi' = -\int_{\mathbb{R}} \zeta\varphi, \quad \varphi \in \mathcal{C}_0^{\infty}(\mathbb{R})$$

Hence, we compute

$$\int_{\mathbb{R}} \phi(y)\varphi'(y)dy = \int_{\mathbb{R}} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-y^2})\mathcal{X}_{(-ct,ct)}(y)\varphi'(y)dy =$$

$$= \int_{-ct}^{ct} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-y^2})\varphi'(y)dy \overset{\text{Parts}}{=} \frac{1}{2c} I_0(2\alpha\sqrt{c^2t^2-y^2})\varphi(y)\Big|_{y=-ct}^{y=ct} +$$

$$+ \int_{-ct}^{ct} \frac{1}{2c} \frac{I_0'(2\alpha\sqrt{c^2t^2-y^2})}{2\sqrt{c^2t^2-y^2}} 2\alpha 2y\varphi(y)dy =$$

$$= \frac{1}{2c}\left(\varphi(ct) - \varphi(-ct)\right) + \int_{-ct}^{ct} \frac{\alpha y}{c} \frac{I_0'(2\alpha\sqrt{c^2t^2-y^2})}{\sqrt{c^2t^2-y^2}} \varphi(y)dy =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2c}\left(\delta(y-ct) - \delta(y+ct)\right) + \frac{\alpha y}{c} \frac{I_0'(2\alpha\sqrt{c^2t^2-y^2})}{\sqrt{c^2t^2-y^2}} \mathcal{X}_{(-ct,ct)}(y) \right) \varphi(y)dy$$

From which we deduce that

$$\phi'(y) = \frac{1}{2c}\left(\delta(y+ct) - \delta(y-ct)\right) - \frac{\alpha y}{c} \frac{I_0'(2\alpha\sqrt{c^2t^2-y^2})}{\sqrt{c^2t^2-y^2}} \mathcal{X}_{(-ct,ct)}(y)$$

Notice that $(\psi * \varphi)'(0) = \int_{\mathbb{R}} \psi(-y,t)\varphi'(y)dy = \int_{\mathbb{R}} \psi(-y,t)'\varphi(y)dy$ and $\phi(-y) = \psi(-y,t)$. This way, we have

$$(\psi(-y,t))' = \frac{1}{2c}\left(\delta_0(y-ct) - \delta_0(y+ct)\right) + \frac{\alpha y}{c} \frac{I_0'(2\alpha\sqrt{c^2t^2-y^2})}{\sqrt{c^2t^2-y^2}} \mathcal{X}_{(-ct,ct)}(y)$$

in the weak sense and then we can write $\langle -c\delta', \psi * \varphi \rangle$ as

$$\int_{\mathbb{R}} \left(\frac{1}{2}\left(\delta_0(x-ct) - \delta_0(x+ct)\right) + \alpha x \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \mathcal{X}_{(-ct,ct)}(x) \right) \varphi(x)dx$$

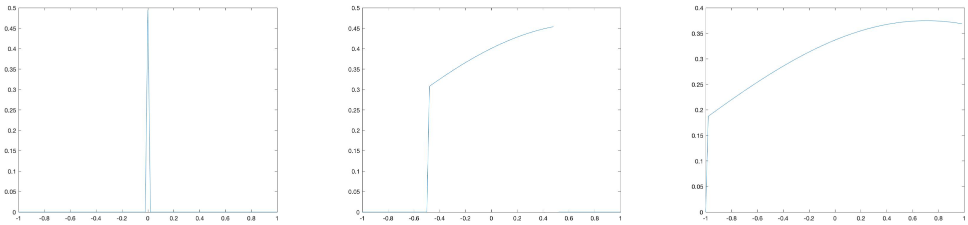
Adding this to the results for $f = \delta$ we notice that one of the *Dirac delta*'s cancels out and we obtain

$$u(x,t) = e^{\frac{-k}{2}t} \left\{ \delta(x-ct) + \left(\alpha(x+ct)\frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} + \frac{k}{4c} I_0(2\alpha\sqrt{\lambda_0}) \right) \mathcal{X}_{(-ct,ct)}(x) \right\}$$

as a distribution and a probability density function. $\square$

Remark 7.2. Let us notice this p.d.f. is a mixed distribution: It has a discrete part governed by the Dirac delta $\delta(x-ct)$ and a continuous part delimited in the interval $[-ct, ct]$.

Example 7.3. We consider a particular case of the model, a particle or the price of an asset that begins to move to the right (or increase) from position (value) 0 and with such probability and speed that $c = 1$ and $k = 2$. We implement our formula with Matlab. (see Appendix B). Below there are some selected figures of the profile of the p.d.f..



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$

Figure 2: Profile of the p.d.f. u for different times

Notice when $t = 0$ we have a peak at 0, this is the δ as the initial condition. When $t = 0.5$ the process has evolved and there is more mass in the right than in the left. This is due to the other initial condition $-c\delta'$.

What is more, the discrete part of the p.d.f. is present when $x = t$, represented by the peak that takes place when $t = 0.5$. The compact support of the solution is also noticeable.

Finally, when $t = 1$ we have a profile that indicate us the tendency of the particle, or the price of the asset, to move right (increase its value) rather than to go left (decrease its value).

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A Bessel Functions

Here we give some definitions form [5] of the Bessel Functions we have used in this work. We also prove several properties of the Bessel Functions that have been used throughout the entire work.

Definition A.1. The modified Bessel equation of the first kind of order 0 is $I_0(x)$ and is such that satisfies

$$x^2 f''(x) + x f'(x) - x^2 f(x) = 0$$

It is an exponentially growing function and bounded in the origin. A common expression for this function and the one we will work with the most is

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cdot\cos\theta} d\theta$$

Lemma A.2.

$$I_0(0) = 1, \quad I_0'(0) = 0, \quad I_0''(0) = \tfrac{1}{2}$$
$$I_0'''(0) = 0, \quad I_0^{IV}(0) = \tfrac{3}{8}$$

Proof. It is a trivial calculation from

$$I_0^{(n)}(x) = \frac{1}{\pi}\int_0^{\pi} (\cos\,\theta)^n e^{x\cdot\cos\theta} d\theta \Longrightarrow I_0^{(n)}(0) = \frac{1}{\pi}\int_0^{\pi} (\cos\,\theta)^n d\theta, \quad \forall n \in \mathbb{N}$$

□

Proposition A.3.

$$\lim_{z\to 0} \frac{I_0'(2\alpha z)}{z} = \alpha$$

Proof. We compute

$$\lim_{z\to 0} \frac{I_0'(2\alpha z)}{z} = \frac{0}{0} \overset{\text{Hopital}}{\Rightarrow} = \lim_{z\to 0} \frac{I_0''(2\alpha z)}{1} 2\alpha = 2\alpha I_0''(0) = \alpha$$

□

Proposition A.4. *Let* $\lambda_0 = c^2t^2 - x^2$,

$$\lim_{x\to ct} \frac{1}{\lambda_0}\left(2\alpha c^2t^2 I_0''(2\alpha\sqrt{\lambda_0}) - x^2 \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\right) = \alpha + \alpha^3 c^2 t^2$$

Proof. We have

$$\lim_{x\to ct} \frac{1}{\lambda_0}\left(2\alpha c^2t^2 I_0''(2\alpha\sqrt{\lambda_0}) - x^2 \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}}\right) =$$

$$= \lim_{x\to ct} \frac{1}{\lambda_0} \left(2\alpha c^2 t^2 I_0''(2\alpha\sqrt{\lambda_0}) + \sqrt{\lambda_0} I_0'(2\alpha\sqrt{\lambda_0}) - c^2t^2 \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \right) =$$

$$= \lim_{x\to ct} \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} + \lim_{x\to ct} \frac{c^2t^2}{\lambda_0} \left(2\alpha I_0''(2\alpha\sqrt{\lambda_0}) - \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \right) =$$

$$= \lim_{z\to 0} \frac{I_0'(2\alpha z)}{z} + \lim_{z\to 0} \frac{c^2t^2}{z^2} \left(2\alpha I_0''(2\alpha z) - \frac{I_0'(2\alpha z)}{z} \right)$$

We already know the first limit (its value is α) thanks to the previous Proposition. As for the second limit, we write it as

$$\lim_{z\to 0} \frac{c^2t^2}{z^2} \left(2\alpha I_0''(2\alpha z) - \frac{I_0'(2\alpha z)}{z} \right) = \lim_{z\to 0} c^2t^2 \left(\frac{2\alpha z I_0''(2\alpha z) - I_0'(2\alpha z)}{z^3} \right) =$$

$$= \frac{0}{0} \overset{\text{Hopital}}{\Rightarrow} = \lim_{z\to 0} c^2t^2 \left(\frac{2\alpha I_0''(2\alpha z) + 4\alpha^2 z I_0'''(2\alpha z) - 2\alpha I_0''(2\alpha z)}{3z^2} \right) =$$

$$= \lim_{z\to 0} c^2t^2 \frac{4\alpha^2 I_0'''(2\alpha z)}{3z} = \lim_{z\to 0} \frac{4\alpha^2c^2t^2}{3} \frac{0}{0} = \overset{\text{Hopital}}{\Rightarrow} = \lim_{z\to 0} \frac{4\alpha^2c^2t^2}{3} 2\alpha I_0^{(IV)}(0) =$$
$$= \alpha^3 c^2 t^2$$

□

Proposition A.5. *Let* $\lambda_0 = c^2t^2 - x^2$ *then,*

$$\lim_{x\to ct} \alpha ct \frac{x}{\lambda_0} \left(2\alpha I_0''(2\alpha\sqrt{\lambda_0}) - \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \right) = \alpha^4 c^2 t^2$$

Proof. We have

$$\lim_{x\to ct} \alpha ct \frac{x}{\lambda_0} \left(2\alpha I_0''(2\alpha\sqrt{\lambda_0}) - \frac{I_0'(2\alpha\sqrt{\lambda_0})}{\sqrt{\lambda_0}} \right) =$$

$$= \alpha \lim_{z\to 0} \frac{c^2t^2}{z} \left(2\alpha I_0''(2\alpha z) - \frac{I_0'(2\alpha z)}{z} \right) = \alpha^4 c^2 t^2$$

thanks to what we have just seen in the proof of the last proposition. □

B Numerics of the Financial Model

Here we present the code in Matlab we have used to simulate the financial model

```
% f=delta g=cdelta'
clear all; clc; close all;
c=1; alpha=.5; A=1/(2*c); xf=1; x0=-xf; n=100;
X=linspace(x0,xf,n+1);
T=linspace(0,1,n/2+1);
for i=1:length(T)
    U=zeros(n+1,1);
    for j=1:n+1
        x=X(j);
        t=T(i);
        lam=c^2*t^2-x^2;
        if lam >=0
            if x==c*t
                U(j)=U(j)+dirac(x-c*t)+2*c*t*alpha^2;
            end
            if x~=-c*t && x~=c*t
                U(j)=U(j)+alpha*(x+c*t)*besseli(1,2*alpha*sqrt(lam))/sqrt(lam);
                U(j)=U(j)+alpha*besseli(0,2*alpha*sqrt(lam));
            end
            U(j)=exp(-2*alpha*c*t)*U(j);
        end
        if i==2 || i==round(length(T)/2) || i==length(T)
            figure(i), plot(X,U)
        end
    end
    figure(1), plot(X,U), legend('u(x,t)')
    pause(1)
end
```