Quantum limits of the resolution of imaging and lidar systems

Miren Lamaison Vidarte

Director: Prof. Dr. Juan P. Torres

Departament de Teoria del Senyal i Comunicacions - Universitat Politècnica de Catalunya
ICFO - The Institute of Photonic Sciences
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To my brother,
for making me hate math.
Abstract

In this project we study the ultimate precision of a Lidar system aimed at estimating the position and velocity of a target with quantum light (entangled photons). By comparison, a standard Lidar system determines the position and velocity of an object using a beam of coherent light. Using Quantum Estimation Theory we will show that the use of entangled pairs of photons can improve the precision of the measurement of position and velocity simultaneously.

Several bounds for the error (typical deviation) in the estimation of the parameters are found. We study which of them is the one that gives us more information about the limits of the uncertainty in the estimation. In particular, we have found that the Holevo Cramer-Rao bound is the best one can get for this case.

Finally, an idea of how this bound could be experimentally achieved is given.
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Chapter 1

Introduction

1.1 The Lidar system

A Lidar (Laser Imaging Detection and Ranging) is a measuring system that uses light to determine the distance to a target as well as its velocity (only the component along the direction of propagation of the light beam that hits the target). Its functioning is based on two simple ideas: 1) the time it takes for a light pulse to reach the detector after reflection in the target inform us of its position, and 2) the reflected beam shows a Doppler frequency shift as a consequence of the motion of the target.

![Standard scheme of a Lidar system.

Figure 1.1: Standard scheme of a Lidar system.](image)

The time delay of the light beam, after travelling from the light source to the target and back, is related to the distance at which the target is situated. The Lidar system includes a sensor that measures this temporal delay and allows us to
calculate the distance as:

\[
\text{Distance} = \frac{\text{travelling time} \times \text{speed of light}}{2}.
\]

On the other hand, the Doppler shift is related to the radial velocity (along the propagation direction of the light beam) that the target has at the time of the reflection. In particular, this velocity is

\[
\text{Velocity} = \frac{\text{Doppler shift} \times \text{wavelength}}{2}.
\]

### 1.2 The problem of non-commuting observables

In a Lidar system, as explained above, the objective is to measure to non-commuting observables, position and velocity. If we consider the uncertainty principle of Heisenberg [1], this means that improving the precision in the measurement of one of the observables involves a detriment in the other’s. However, the estimation of these observables is done through two different quantities, the time delay \(\tau\) and the Doppler shift \(\delta\). The uncertainty relation in the estimation of this two parameters will be slightly different.

In 1964, E. Arthurs and J.L. Kelly addressed the necessity of a precise theory for simultaneous measurements of a pair of conjugate observables. The widely known uncertainty principle, as stated by Heisenberg, does only consider the variances of two hypothetical ideal measurements. In their paper [2], the measurement is carried out by considering two one-dimensional systems coupled to the object system. The two new systems commute, so it is possible to perform ideal measurements on them.

The Arthurs-Kelly relation provides a lower bound for the product of the variances in the measurement of the two variables:

\[
\sigma_x \sigma_y \geq 1.
\]  \hspace{1cm} (1.1)

This is the proper uncertainty principle when it comes to joint measurements. The interpretation is the same as in Heisenberg Uncertainty principle: one cannot know with unlimited precision both desired variables at the same time.

In our case, the uncertainty affects the estimation of the delay and the Doppler shift as \(\sigma_{\tau} \sigma_{\delta} \geq 1\).

It seems reasonable to wonder if this limit can be beaten. How can one overcome this problem? The solution is the use of quantum light.
In this project we study the possibility of beating the Arthurs-Kelly limit, stated in 1.2, using pairs of frequency entangled photons instead of a coherent light beam.

2.1 Quantum states

In quantum mechanics the state of a particle (or a photon) is represented by a normalized vector \( |\Psi\rangle \) in the Hilbert space \( \mathcal{L}_2 \). For example, for a continuous set of position basis \( |x\rangle \), this vector is:

\[
|\Psi\rangle = \int dx \Psi(x) |x\rangle.
\] (2.1)

The complex function \( \Psi(x) \) is known as the wave function and it is what determines the state of the particle\(^1\). Born’s statistical interpretation of the wave function says that its squared modulus \( |\Psi(x)|^2 = \Psi^*(x)\Psi(x) \) corresponds with the probability of finding the particle at a point \( x \). It must be normalized so that:

\[
\langle \Psi | \Psi \rangle = \int dx \Psi^*(x)\Psi(x) = 1.
\] (2.2)

Observable quantities, \( Q \), are represented by Hermitian operators, \( \hat{Q} \); the expectation value of \( Q \) in the state \( |\Psi\rangle \) is

\[
\langle Q \rangle = \langle \Psi | \hat{Q} | \Psi \rangle = \int \Psi^*(x)\hat{Q}\Psi(x) dx.
\] (2.3)

\(^1\)If there are two particles or photons the state can be written equivalently as:

\[
|\Psi\rangle = \int dx_1 dx_2 \Psi(x_1, x_2) |x_1\rangle |x_2\rangle.
\]

The interpretation of the wave function \( \Psi(x_1, x_2) \) is the same.


CHAPTER 2. LIDAR WITH ENTANGLED STATES

2.2 Entanglement

When two photons are entangled, one cannot independently describe each of them as separate photons. The state of the pair has to be seen as a whole. In terms of the wave function this means it cannot be separated into two factors depending on each of the variables:

$$\Psi(x_1, x_2) \neq \Psi_1(x_1)\Psi_2(x_2).$$ (2.4)

On the contrary, if it is possible, we say we have separable states.

Experimentally, the entangled pair of photons usually comes from an Spontaneous Parametric Down Conversion (SPDC) process. The pair is generated when a pump beam interacts with a non-linear crystal.

![Figure 2.1: Simple scheme of an SPDC process.](image)

The two photons obtained are called signal and idler and can be entangled in many ways such as polarization or momentum. In our case, we will consider a pair of frequency entangled photons.

Whichever the case, there are two conditions that must always be fulfilled: energy conservation ($\hbar\omega_p = \hbar\omega_s + \hbar\omega_i$) and momentum conservation ($k_p = k_s + k_i$).

2.3 Quantum Lidar System

We consider a Lidar system that makes use of frequency-entangled paired photons: the signal and the idler. The signal is sent towards the moving target and reflected back. This makes it acquire a temporal delay ($\tau_s$) and a Doppler shift ($\delta$).

Although only the signal photon interacts with the target, the information can be extracted from both the signal and idler since we are assuming an entangled pair.

During the fly time of the signal photon, the idler is stored/delayed until the former comes back reflected from the target. Experimentally, this allows to perform a measurement on both photons. Thus we have to include another time delay $\tau_i$. 


2.3. QUANTUM LIDAR SYSTEM

2.3.1 Quantum State

We consider a Lidar system which uses non-degenerate SPDC as proposed in Zhuang et. al. [4]. The biphoton state of the signal and idler photons at the output face of the nonlinear crystal where they are generated is (in the time domain)

\[ |\Psi\rangle = \int dt_s dt_i \Psi(t_s, t_i) |t_s\rangle_s |t_i\rangle_i \]

(2.5)

where \( |t\rangle \) denotes a single photon at time \( t \), \(|t_s\rangle_s\) for the signal and \(|t_i\rangle_i\) for the idler. The normalization constant is \( N = \frac{1}{\sqrt{2\pi\sigma_{coh}\sigma_{cor}}} \).

In this expression \( \sigma_{cor} \) corresponds with the correlation time of the photons (related with the different propagation velocity of modes inside the non-linear crystal) and \( \sigma_{coh} \) is the pump coherence time.

The analogous state in the frequency domain can be written as

\[ |\Psi\rangle = \int d\Omega_s d\Omega_i \Psi(\Omega_s, \Omega_i) |\Omega_s\rangle_s |\Omega_i\rangle_i \]

(2.7)

where \( \Omega_s = \omega_s + \omega_o^s \) and \( \Omega_i = \omega_i + \omega_o^i \) depend on the frequency of the photons, \( \omega_s \) and \( \omega_i \), and their central frequencies, \( \omega_o^s \) and \( \omega_o^i \). The normalization constant in this case is \( N = \sqrt{\frac{2\sigma_{coh}\sigma_{cor}}{\pi}} \).
CHAPTER 2. LIDAR WITH ENTANGLED STATES

Two important quantities are the rms temporal duration of a single photon (signal or idler)

\[ T = \sqrt{\sigma_{\text{coh}}^2 + \sigma_{\text{cor}}^2/4} \]  

and the rms bandwidth

\[ W = \sqrt{1/16\sigma_{\text{coh}}^2 + 1/4\sigma_{\text{cor}}^2}. \]  

The product of this two parameters informs of the degree of entanglement between signal and idler. The entanglement entropy of the state is \( S_E = \log_2(2TW) \).

When \( TW = 1/2 \), or equivalently \( \sigma_{\text{cor}} = 2\sigma_{\text{coh}} \) (Figure 2.3a), the biphoton state can be written as a separable state with no entanglement \( (S_E = 0) \):

\[
\Psi(t_s, t_i) \propto \Phi_s(t_s)\Phi_i(t_i) = \exp\left\{ -t_s^2/16\sigma_{\text{coh}}^2 - i\omega_o t_s \right\} \exp\left\{ -t_i^2/16\sigma_{\text{coh}}^2 - i\omega_o t_i \right\}  
\]

\[
\Psi(\Omega_s, \Omega_i) \propto \Phi_s(\Omega_s)\Phi_i(\Omega_i) = \exp\left\{ -2\sigma_{\text{coh}}^2 \Omega_s^2 \right\} \exp\left\{ -2\sigma_{\text{coh}}^2 \Omega_i^2 \right\}.  
\]

If this is not the case and \( TW \neq 1/2 \) the state corresponds to an entangled pair \( (S_E > 0) \). This will be the case we will be most interested in.

The most common case in SPDC is having \( \sigma_{\text{coh}} \gg \sigma_{\text{cor}} \), this is, \( T \approx \sigma_{\text{coh}} \gg 1/W \approx 2\sigma_{\text{cor}} \). The photons are highly entangled \( (S_E \gg 1) \) in an anticorrelated state. This is the case in Figure 2.3d.

2.3.2 State after hitting the target

To perform the measurement, the signal photon is sent towards the moving target and the idler is delayed in order to be recombined with the signal photon when it comes back after reflection on the target. This reflection on a moving object introduces a Doppler shift \((\delta)\). In addition, a temporal delay in both signal and idler \((\tau_s, \tau_i)\) has to be considered. The wave function in the time domain after hitting the target becomes

\[
\Psi(t_s, t_i) \propto \exp\left\{ -\frac{(t_s - \tau_s + t_i - \tau_i)^2}{16\sigma_{\text{coh}}^2} - \frac{(t_s - \tau_s - t_i + \tau_i)^2}{4\sigma_{\text{cor}}^2} \right\} \times  
\]

\[
\times \exp\left\{ -i\omega_o^s(t_s - \tau_s) - i\omega_o^i(t_i - \tau_i) - i\delta(t_s - \tau_s) \right\}.  
\]

The entanglement entropy of the biphoton state corresponds with the von Neumann entropy \( S_E \) of the signal’s reduced density operator \( \rho_S \):

\[
S_E = -\text{Tr}\{\rho_S \ln \rho_S\}.  
\]
2.3. QUANTUM LIDAR SYSTEM

(a) Separable: $2\sigma_{coh} = \sigma_{cor}$

(b) Anticorrelated: $2\sigma_{coh} > \sigma_{cor}$

(c) Correlated: $2\sigma_{coh} \ll \sigma_{cor}$

(d) Anticorrelated: $2\sigma_{coh} \gg \sigma_{cor}$

Figure 2.3: Square modulus of the wave function, $|\Psi|^2$, for different values of $\sigma_{cor}$ and $\sigma_{coh}$.

In the frequency domain the state corresponds with the following expression:

$$\Psi(\Omega_s, \Omega_i) \propto \exp\left\{-\sigma_{coh}^2 (\Omega_s + \Omega_i + \delta)^2 - \frac{\sigma_{cor}^2}{4} (\Omega_s - \Omega_i + \delta)^2 - i\delta \sqrt{\frac{\sigma_{cor}}{2}} - i\omega_i \tau_s - i\omega_s \tau_i\right\}.$$  

(2.14)

In the next chapters we will use these expressions to study the limits in the estimation of the two parameters of interest: $\tau_s$ and $\delta$. The objective is to see if we can improve the limit given in section 1.2 for the case of coherent light.
Chapter 3

Estimation Theory

Estimation Theory has the goal of determining the value of a parameter (or a set of parameters) from a data set with the greatest possible accuracy. We want to study the precision limits in this estimation.

Both the classical and the quantum estimation theory are revised in this chapter. The main difference that concerns us between both of them is the generality of the results. The precision limits that one can extract from the classical estimation theory are always associated with the results of a given experiment. On the other side, the results obtained from its quantum analogue are valid for any experiment one could devise and only depend on the state of the system or the model itself.

3.1 Quantum Estimation Theory

The aim in Quantum Estimation Theory is to statistically estimate a set of real parameters $\theta = (\theta_i)_{i=1}^n$ from a given quantum statistical model. A quantum statistical model is a family of density operators $\rho_\theta$ defined on a certain separable Hilbert space $\mathcal{H}$ [6].

$$S = \{\rho_\theta; \theta \in \Theta \in \mathbb{R}^n\}.$$ (3.1)

For the sake of simplicity, we will omit the dependence on $\theta$ and write the density operators $\rho_\theta$ plainly as $\rho$.

In general, after an interaction, the density matrix of any state undergoes changes. We want to know what is the best we can do in the estimation of parameters involved in the interaction.
Quantum Fisher Information Matrix

One can define a family of operators analogous to the logarithmic derivative that must fulfill the following conditions:

$$\frac{\partial \rho}{\partial \theta_j} = \frac{1}{2} \left( \rho L_j + L_j^* \rho \right), \quad \text{Tr}\{\rho L_j\} = 0. \quad (3.2)$$

Different solutions for this operators have been proposed. Helstrom introduces in [7] the symmetric logarithmic derivative $L_j^S$ (SLD). This is a special case in which:

$$L_j^S = L_j^{S*}, \quad \frac{\partial \rho}{\partial \theta_j} = \frac{1}{2} \left( \rho L_j^S + L_j^S \rho \right). \quad (3.3)$$

Another option for the logarithmic derivative operator is the right logarithmic derivative $L_j^R$ (RLD) [8]. This operators fulfill the following:

$$\frac{\partial \rho}{\partial \theta_j} = \rho L_j^R. \quad (3.4)$$

From the general operators in (3.2) one can define the complex Fisher information matrix $J$:

$$J = [(L_j, L_k)_{\rho}] = \left[ \text{Tr}\{\rho L_k L_j^*\} \right]. \quad (3.5)$$

Additionally, using the SLDs, one can define the real Fisher information matrix or SLD-Fisher information matrix. This is:

$$J^S = [(L_j^S, L_k^S)_{\rho}] = \left[ \frac{1}{2} \text{Tr}\{\rho (L_k^S L_j^S + L_j^S L_k^S)\} \right]. \quad (3.6)$$

Both matrices are related so that $J = \text{Re} J^S$.

Using the conditions in (3.2) and (3.3) one can rewrite the elements of this matrices as:

$$(J)_{jk} = 4 \text{Tr} \left\{ \rho \frac{\partial \rho}{\partial \theta_j} \frac{\partial \rho}{\partial \theta_k} \right\} \quad (3.7)$$

$$(J^S)_{jk} = 2 \text{Tr} \left\{ \frac{\partial \rho}{\partial \theta_j} \frac{\partial \rho}{\partial \theta_k} \right\}. \quad (3.8)$$

We will focus on the estimation theory for pure states. The density operator can be written as $\rho = |\Psi\rangle \langle \Psi|$. Therefore, the final expressions we will keep for the complex and the real Fisher Information matrices are:

$$(J)_{jk} = 4 \left\{ \left( \frac{\partial |\Psi\rangle}{\partial \theta_j} \right| \langle \Psi| \right\} + \left\langle \Psi \left| \frac{\partial |\Psi\rangle}{\partial \theta_j} \right\rangle \left\langle \Psi \left| \frac{\partial |\Psi\rangle}{\partial \theta_k} \right\rangle \right\} \quad (3.9)$$
3.1. QUANTUM ESTIMATION THEORY

\[(J^S)_{jk} = 4 \text{Re} \left\{ \left\langle \frac{\partial \Psi}{\partial \theta_j} \right| \frac{\partial \Psi}{\partial \theta_k} \right\rangle + \left\langle \Psi \left| \frac{\partial \Psi}{\partial \theta_j} \right\rangle \left\langle \Psi \left| \frac{\partial \Psi}{\partial \theta_k} \right\rangle \right\} \right\} \quad (3.10)\]

The matrices depend on the state, \( |\Psi\rangle \), and its derivatives with respect to the parameters to estimate, \( \langle \frac{\partial \Psi}{\partial \theta_i} \rangle \).

Crâmer-Rao bound

The estimation of parameters is done through an estimator \( \hat{\theta} \). An estimator for \( \theta \) is identified to a generalized measurement \( M \) that takes values on \( \Theta \).

This measurement is called unbiased if the expectation vector is equal to the parameter vector \( \theta \). This is:

\[E[M] = \int \hat{\theta} P^M_{\hat{\theta}}(d\hat{\theta}) = \theta. \quad (3.11)\]

We will be interested in determining the error in the measurement, how far it is form the real value, so is is useful introduce the covariance matrix. The elements of this matrix \( V \) can be defined as:

\[v_{jk} = \int (\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)P^M_{\hat{\theta}}(d\hat{\theta}). \quad (3.12)\]

Given a general weight matrix \( G \) (which is real and positive definite), it is interesting to have a bound of the form

\[\text{Tr}\{GV\} \geq C. \quad (3.13)\]

This expression would give a bound for the variance with which we can estimate the vector parameter \( \theta \). If there exist a constant \( C \) that fulfills (3.13), it is called a Crâmer-Rao (CR) bound. This constant depends on the election of \( G \).

Several different expressions for this bound exist, which are valid or not depending on the specific model one is considering. Apart from being valid, we will be interested in how informative the bounds are, this is, how close they are to the ultimate or attainable bound. The objective is to find the most informative among them.

In the following, we will revise some of the known CR bounds for pure states, including the conditions for them to be applicable and the cases in which they are the most informative one. They can all be found in the book edited by M. Hayashi [9], along with more detailed explanations and derivations of the quantum estimation theory.
CHAPTER 3. ESTIMATION THEORY

3.1.1 CR bound derived from the SLD

This bound is based on the Symmetric Logarithmic Derivatives and it is always a valid bound. Furthermore, for the case of a one-dimensional model (only one parameter to estimate), it can be proved that this is the ultimate bound [6]. For any other case, special conditions apply.

The SLD Crámer-Rao bound defined in [10] is

\[ \text{Tr}[GV] \geq \text{Tr}[G(J^{S})^{-1}] \]  

(3.14)

Regarding its informativeness, K. Matsumoto states in Theorem 18 of [11] that the SLD CR bound is attainable iff

\[ \tilde{J} \equiv \text{Im} J = \text{Im} \text{Tr}\{\rho L_{j}^{S}L_{k}^{S}\} = 0. \]  

(3.15)

An equivalent condition for this matter is that there exist such SLD’s that for any \( j,k \) the operators \( L_{j}^{S} \) and \( L_{k}^{S} \) commute.

3.1.2 CR bound for coherent models

The following CR bound is derived from the Right Logarithmic Derivatives for the specific case of a coherent model. When the conditions are fulfilled, it corresponds with the ultimate bound. For any other kind of model, the bound is not even valid.

In Proposition 27 of [8], the RLD-bound for a two dimensional coherent model is defined as

\[ \text{Tr}[GV] \geq \text{Tr}[G(J^{S})^{-1}] + \frac{\sqrt{\det G}}{\det J^{S}} \text{Tr}\{\rho[L_{1}^{S}, L_{2}^{S}]\} \]  

(3.16)

A two-dimensional model is said to be locally coherent in \( \theta \) if \( L_{1}^{S}\Psi \) and \( L_{2}^{S}\Psi \) are linearly dependent (condition (β) of Theorem 3 in [12]).

Another equivalent condition for the model to be coherent is stated by Matsumoto on Theorem 34 of [11]. It has to be satisfied that

\[ |\det J^{S}| = |\det \tilde{J}|. \]  

(3.17)

3.1.3 CR bound by Matsumoto

K. Matsumoto states in Theorem 40 of [11] that for any pure state model the following is satisfied,

\[ \text{Tr}[J^{S}V] \geq \sum_{\alpha_{i} \in \text{eig. of } D} \frac{2}{1 + (1 - |\alpha_{i}|^{2})^{1/2}}, \]  

(3.18)
where the $\alpha_i$ are the eigenvalues of $D$. This eigenvalues are equivalent to the eigenvalues of $J^{s-1}J$ (Theorem 28 in [11]).

It is hard to verify in which cases this bound is attainable and how much informative it is.

### 3.1.4 Holevo CR bound

The Holevo bound is the most general and always valid for a pure state model. It is defined by Suzuki in Section II.B of [13] as:

$$ C^H := \min_{\vec{X} \in \chi} h[\vec{X}], $$

and $\chi := (X_1, X_2, ...)$, where $X_j$ are Hermitian operators that satisfy two conditions

$$ \text{Tr}(\rho X_j) = 0 \quad \text{and} \quad \left[ \text{Tr} \left( \frac{\partial \rho}{\partial \theta_j} X_k \right) \right] = \delta_{jk}. \quad (3.20) $$

The function $h[\vec{X}]$ that needs to be minimized is defined as:

$$ h[\vec{X}] := \text{Tr}\left\{ G \text{Re} Z[\vec{X}] \right\} + \text{TrAbs}\left\{ \sqrt{G} \text{Im} Z[\vec{X}] \sqrt{G} \right\}, \quad (3.21) $$

where the function $Z[\vec{X}]$ is:

$$ Z[\vec{X}] := \left[ \text{Tr}\{\rho X_j X_k\} \right]_{jk} = \langle \Psi_0 | X_j X_k | \Psi_0 \rangle. \quad (3.22) $$

This bound should always be greater or equal than the SLD bound and the bound for coherent models, this means that the Holevo CR bound is always the most informative among the three (Lemma 2.2 in [13]).

### 3.2 Classical Estimation Theory

In Classical Estimation Theory the starting point is the probability density function (PDF), $f(x; \theta)$, associated to the experimental data $x$ obtained in a given measurement. The shape of this normalized function is affected by the value of the parameter $\theta$ to estimate. This estimation is done through an estimator $\hat{\theta}$ which assigns to each realization of the experiment, $x$, a value of $\theta$. Since $x$ is a random variable, it is different for each realization.

When choosing an estimator $\hat{\theta}$, we want to choose the one that performs the best, this is, the one that approaches the most to the real value of $\theta$. Thus, an statistical approach is necessary to define what does “good estimator” mean.
3.2.1 Minimum Variance Unbiased Estimation

An unbiased estimator is that that, on average, gives the actual parameter value, \( E(\hat{\theta}) = \theta \). This is, the bias\(^1\) is equal to zero. If this holds for all possible values of the parameter \( \theta \), the estimator is considered unbiased.

In any case, being unbiased does not guarantee that the estimator is good. It seems natural to choose an estimator with the minimum mean square error (MSE). Since we are focusing only on unbiased estimators, we have that the MSE coincides with the variance of the estimator:

\[
\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = \text{var}(\hat{\theta}).
\] (3.23)

Therefore, we are looking for what is called the minimum variance unbiased (MVU) estimator. The existence of the MVU is not always guaranteed and even when it does exist, finding it is not straightforward.

3.2.2 Crámer-Rao Lower Bound

The estimation accuracy depends directly on the behavior of the PDF, \( f(x; \theta) \). If this function changes abruptly for small variations of the parameter, intuitively, it will be easier to estimate it. This idea is formally stated in the Crámer-Rao Lower Bound theorem as it appears in [14].

**Theorem 3.1 (CRLB Theorem)** Let the PDF \( f(x; \theta) \) satisfy the regularity condition

\[
E \left[ \frac{\partial \ln f(x; \theta)}{\partial \theta} \right] = 0 \quad \forall \ \theta,
\] (3.24)

where the expectation value is taken with respect to \( f(x; \theta) \). Then, the variance of any unbiased estimator \( \hat{\theta} \) must satisfy

\[
\text{var}(\hat{\theta}) \geq \frac{1}{-E \left[ \frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} \right]},
\] (3.25)

where the derivative is evaluated at the true value of \( \theta \). Furthermore, an unbiased estimator may be found that attains the bound for all \( \theta \) iff

\[
\frac{\partial \ln f(x; \theta)}{\partial \theta} = I(\theta)(g(x - \theta)),
\] (3.26)

for some functions \( g \) and \( I \). That estimator is the MVU and is found as \( \hat{\theta} = g(x) \). The minimum of the variance is \( 1/I(\theta) \).

\(^1\)The bias is defined as \( b(\theta) = E(\hat{\theta}) - \theta \).
3.2. CLASSICAL ESTIMATION THEORY

The expectation value\(^2\) in \((3.25)\) can be rewritten as:

\[
- E \left[ \frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} \right] = E \left[ \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 \right]. \tag{3.27}
\]

Therefore, the following is also a representation of the CR Lower Bound:

\[
\text{var}(\hat{\theta}) \geq \frac{1}{E \left[ \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 \right]} . \tag{3.28}
\]

The function \(I(\theta)\) in Theorem \(3.1\) is known as the \textit{Fisher Information}, and is defined form the denominator in \((3.25)\) (or equivalently in \((3.28)\)). The CRLB can be thus simplified to

\[
\text{var}(\theta) \geq \frac{1}{I(\theta)}. \tag{3.30}
\]

The intuitive notion about “information” here must be related with how close one can get to the real value of the parameter, \(\theta\). The greater the information the smaller the uncertainty.

Finally, an estimator \(\hat{\theta}\) that attains the CRLB it is said to be efficient since it manages to do the best possible estimation with the data set available, \(x\). The MVU estimator mentioned earlier does not necessarily attain the CRLB. It is generally a good estimator but it does not have to be the best possible.

3.2.3 Maximum Likelihood Estimation

The PDF is called the \textit{likelihood function} when it is seen as a function of \(\theta\) with fixed \(x\) rather than the other way around. As we already know, the Minimum Variance Unbiased estimator is usually not straightforward to derive. The Maximum Likelihood Estimation (MLE) is more practical and quite efficient for large enough sets of data (asymptotically efficient).

The MLE is defined as the value of \(\theta\) that maximizes the likelihood function.

3.2.4 Multi-Parameter Vector

In general, we will not be interested in estimating a single parameter. The likelihood function will depend on a set of parameters \(\theta = [\theta_1, \theta_2, ..., \theta_n]\). Assuming the

\[
2\text{As a reminder, this expectation value should be calculated considering the integral}
\]

\[
E \left[ \frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} \right] = \int \frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2} f(x; \theta)dx. \tag{3.29}
\]
estimator \( \hat{\theta} \) is unbiased, the CRLB sets a bound in the variance of each element:

\[
\text{var}(\hat{\theta}_i) \geq [I^{-1}(\theta)]_{ii}.
\] (3.31)

The previous Fisher Information function becomes an \( n \times n \) matrix defined as

\[
[I(\theta)]_{ij} = -E\left[ \frac{\partial^2 \ln f(x; \theta)}{\partial \theta_i \partial \theta_j} \right],
\] (3.32)

which is positive semidefinite by definition. Notice that the bound for the variance of each element, \( \text{var}(\theta_i) \), depends on the true value of all the elements of the vector parameter \( \theta \) and not just on the parameter itself.

The Theorem 3.1 that was valid for a single parameter can be reformulated and adapted to the multiparameter case ([14]).

**Theorem 3.2 (CRLB Theorem - Parameter Vector)** Let the PDF \( f(x; \theta) \) satisfy the regularity condition

\[
E\left[ \frac{\partial \ln f(x; \theta)}{\partial \theta} \right] = 0 \quad \forall \theta,
\] (3.33)

where the expectation value is taken with respect to \( f(x; \theta) \). Then, the covariance matrix \( C_{\hat{\theta}} \) of any unbiased estimator \( \hat{\theta} \) must satisfy

\[
C_{\hat{\theta}} - I^{-1}(\theta) \geq 0,
\] (3.34)

where \( \geq 0 \) is understood as the matrix being positive semidefinite. Furthermore, an unbiased estimator may be found that attains the bound in that \( C_{\hat{\theta}} = I^{-1}(\theta) \) iff

\[
\frac{\partial \ln f(x; \theta)}{\partial \theta} = I(\theta)(g(x - \theta)),
\] (3.35)

for some \( n \)-dimensional function \( g \) and some \( n \times n \) matrix \( I \). That estimator is the MVU and is found as \( \hat{\theta} = g(x) \).

---

The covariance matrix is defined equivalently to the quantum case as

\[
[C_{\hat{\theta}}]_{jk} = \int (\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)f(x; \theta)d\theta.
\] (3.36)
3.3 Quantum and classical Cramer-Rao bounds: an overview

For the sake of clarity, it seems important to summarize what the quantum and classical Cramer-Rao bounds address.

The quantum Cramer Rao bound tells us that, given a certain interaction between our light beam and the target of interest, for whatever measurement we could possibly make, the error in the estimation will be always greater than the quantum CR bound. Therefore, this bound does not depend on performing any particular experiment. It is a bound valid for all possible experiments.

The classical CR bound is a bound for a particular experiment. We measure the light beam reflected from the target and do it for several possible outcomes, following a probability density function (PDF) that comes from the characteristics of the experiment. The classical CR bound tell us the best that we can do with these data. The classical CR bound is always equal or above the quantum CR bound:

$$\text{classical CR bound} \geq \text{quantum CR bound}$$

Finally, we use a specific variable $A$ to estimate the value of the variable of interest $\theta$. The actual error in the estimation process is given by

$$\text{var}(\theta) = \frac{\text{var}(A)}{(\partial \langle A \rangle / \partial \theta)^2}$$

(3.37)

Notice that this is the real error of the estimation, in comparison with the Cramer-Rao bounds, that are this, bounds. The error is always equal or greater than the limit given by the classical and quantum CR bounds:

$$\text{actual error} \geq \text{classical CR bound} \geq \text{quantum CR bound}$$
Figure 3.1: Overview of what are the Quantum and Classical Cramer-Rao bounds, and how they are related to the actual error made when estimating a parameter.
Chapter 4

Lower bounds for a quantum Lidar system

In this chapter we will apply all the CR bounds described in Chapter 3 to the Quantum Lidar system as described in section 2.3. The final objective is to compare all and find the ultimate bound. If this is not possible, we will be content to find which one is the most informative among them. This is section 4.1.

Furthermore, the CR bound for a system of two variables has an expression such as
\[ g_1 \cdot \text{var}(\theta_1) + g_2 \cdot \text{var}(\theta_2) \geq C, \]
where \( g_1 \) and \( g_2 \) are general weights. In order to be able to compare this bounds with the Arthurs-Kelly relation in (1.1), we are interested in a bound for the product of the form
\[ \text{var}(\theta_1)\text{var}(\theta_2) \geq C. \]

In the second part of this chapter (section 4.2), we will calculate the bound for the product derived from each of the different CR bounds previously studied.

4.1 Quantum Fisher Information

We will calculate the QFI of the state with the wave function given in (2.13). This is:
\[
\Psi(t_s, t_i) \propto \exp \left\{ -\frac{(t_s - \tau_s + t_i - \tau_i)^2}{16\sigma^2_{coh}} - \frac{(t_s - \tau_s - t_i + \tau_i)^2}{4\sigma^2_{cor}} \right\} \times \exp \left\{ -i\omega_\delta(t_s - \tau_s) - i\omega_\delta(t_i - \tau_i) - i\delta(t_s - \tau_s) \right\}.
\]
Once we know the state, the matrix is given by
\[
(J)_{jk} = 4 \left\{ \langle \frac{\partial \Psi}{\partial \theta_j} \mid \frac{\partial \Psi}{\partial \theta_k} \rangle + \langle \Psi \mid \frac{\partial \Psi}{\partial \theta_j} \rangle \langle \Psi \mid \frac{\partial \Psi}{\partial \theta_k} \rangle \right\}.
\tag{4.4}
\]

The derivatives of our state with respect to the two parameters that we want to estimate (\(\tau_s\) and \(\delta\)) are:
\[
\begin{align*}
|\Psi_1\rangle &= \| \frac{\partial \psi}{\partial \delta} \| = i \frac{\delta_s}{2} |\Psi_0\rangle - i \int dt_s dt_i \Psi(t_s, t_i) |t_s\rangle_s |t_i\rangle_i \\
|\Psi_2\rangle &= \| \frac{\partial \Psi}{\partial \tau_s} \| = i \left( \omega_s^0 + \frac{\delta}{2} \right) |\Psi_0\rangle \\
&- i \int dt_s dt_i \left[ (a + b)(t_s - \tau_s) + (a - b)(t_i - \tau_i) \right] \Psi(t_s, t_i) |t_s\rangle_s |t_i\rangle_i
\end{align*}
\tag{4.5}
\]

The inner products combining the state and its derivatives are:
\[
\begin{align*}
\langle \Psi_0 | \Psi_0 \rangle &= \langle \Psi | \Psi \rangle = 1 \\
\langle \Psi_0 | \Psi_1 \rangle &= \left\langle \Psi \mid \frac{\partial \Psi}{\partial \delta} \right\rangle = -i \frac{\tau_s}{2} \\
\langle \Psi_1 | \Psi_1 \rangle &= \left\langle \frac{\partial \Psi}{\partial \delta} \mid \frac{\partial \Psi}{\partial \delta} \right\rangle = \frac{\tau_s^2}{4} + T^2 \\
\langle \Psi_0 | \Psi_2 \rangle &= \left\langle \Psi \mid \frac{\partial \Psi}{\partial \tau_s} \right\rangle = i \left( \omega_s^0 + \frac{\delta}{2} \right) \\
\langle \Psi_1 | \Psi_2 \rangle &= \left\langle \frac{\partial \Psi}{\partial \tau_s} \mid \frac{\partial \Psi}{\partial \tau_s} \right\rangle = W^2 + \left( \omega_s^0 + \frac{\delta}{2} \right)^2 \\
\langle \Psi_2 | \Psi_2 \rangle &= \left\langle \frac{\partial \Psi}{\partial \tau_s} \mid \frac{\partial \Psi}{\partial \tau_s} \right\rangle = \frac{\tau_s^2}{2} + \frac{\tau_s^2}{2} \left( \omega_s^0 + \frac{\delta}{2} \right)
\end{align*}
\tag{4.6}
\]

Performing all the calculations we obtain the following Fisher information matrices,
\[
J^S = \text{Re} \, J = \begin{pmatrix}
4W^2 & 0 \\
0 & 4T^2
\end{pmatrix}
\quad \tilde{J} = \text{Im} \, J = \begin{pmatrix}
0 & -2 \\
2 & 0
\end{pmatrix}
\tag{4.7}
\]

We want to know which Cramér-Rao bound of those listed in Chapter 3 is applicable (or the ultimate) in this case.

### 4.1.1 CR bound based on SLD

We choose a general weight matrix \(G = \text{diag}[W^2, zT^2]\), with \(z > 0\), so that we obtain an adimensional relation. The SLD CR bound as described by (3.14) is:
\[ W^2 \text{var}(\tau) + zT^2 \text{var}(\delta) \geq \frac{1 + z}{4} \] (4.8)

where var(\tau) and var(\delta) are the variances in the estimation of \( \tau \) and \( \delta \) respectively.

This bound based on the Symmetric Logarithmic Derivatives is always a valid bound. However, it corresponds with the ultimate bound only if \( L_1 \) and \( L_2 \) commute. In our case, they do not.

\[ \text{Tr}(\rho[L_1, L_2]) = \langle \Psi | [L_1, L_2] | \Psi \rangle = -4i \] (4.9)

The fact that \( \tilde{J} \neq 0 \) also indicates that this is not an attainable bound.

### 4.1.2 CR bound based on RLD for Coherent models

The RLD CR bound is difficult to calculate as a general rule. However, when the model is coherent not only there is a simple expression for the bound but also it can be proved that it is the most informative. We have to check if this is the case.

In case it is a coherent model we have to apply (3.16). Using the same matrix \( G \) we obtain:

\[ W^2 \text{var}(\tau) + zT^2 \text{var}(\delta) \geq \frac{1 + z}{4} + \frac{\sqrt{z}}{4WT} \] (4.10)

In order to check that this is the most informative bound (this is, that we are considering a coherent model) we compare the determinants of the real and the imaginary part of the matrix \( J_\theta \) to check if they are equal.

\[ \det J^S = 16W^2T^2 \quad \det \tilde{J} = 4 \] (4.11)

The model is coherent only for \( TW = 1/2 \) (\( \det J^S = \det \tilde{J} = 4 \)). Remember that this is the value for a separable state. In this particular case the bound is the most informative. For any other value of \( TW \) it is not even a valid bound.

This can also be shown verifying the independence of \( L_1 \) and \( L_2 \). The condition that must be fulfilled for them to be linearly dependent implies the existence of a set \( \alpha, \beta \) for which:

\[ \alpha L_1 |\Psi\rangle + \beta L_2 |\Psi\rangle = 0 \] (4.12)

In this particular case, we know that

\[ L_1 |\Psi\rangle = 2 \left( \left| \frac{\partial \Psi}{\partial \tau} \right| - i \left( \frac{\omega_\tau}{2} + \frac{\delta}{2} \right) |\Psi\rangle \right) \]
\[ L_2 |\Psi\rangle = 2 \left( \left| \frac{\partial \Psi}{\partial \delta} \right| + i \frac{\tau_s}{2} |\Psi\rangle \right) \] (4.13)
Therefore, it is needed to show that:

\[
2\alpha \left( \frac{\partial \Psi}{\partial \tau} - i \left( \omega_s^0 + \frac{\delta}{2} \right) |\Psi\rangle \right) + 2\beta \left( \frac{\partial \Psi}{\partial \delta} + i \frac{\tau_s}{2} |\Psi\rangle \right) = 0 \quad (4.14)
\]

\[
\int dt_s dt_i \left\{ 2\alpha \left[ \frac{t_s - \tau_s + t_i - \tau_i}{8\sigma_{coh}^2} + \frac{t_s - \tau_s - t_i + \tau_i}{2\sigma_{cor}^2} \right] + 2\beta \left[ -i(t_s - \tau_s) \right] \right\} \times \Psi(t_s, t_i) |t_s\rangle_s |t_i\rangle_i = 0 \quad (4.15)
\]

Finally, separating the dependence with \( t_s \) from the dependence on \( t_i \), the condition in (4.12) transforms into:

\[
(t_s - \tau_s) \left[ 2\alpha \left( \frac{1}{8\sigma_{coh}^2} + \frac{1}{2\sigma_{cor}^2} \right) - i2\beta \right] + (t_i - \tau_i) \left[ 2\alpha \left( \frac{1}{8\sigma_{coh}^2} - \frac{1}{2\sigma_{cor}^2} \right) \right] = 0
\]

\[
(4.16)
\]

Once again, the model is coherent only for \( TW = 1/2 \) (or equivalently, \( \sigma_{cor}^2 = 4\sigma_{coh}^2 \)). In this case, (4.16) is fulfilled if \( \beta = i\alpha \left( \frac{1}{8\sigma_{coh}} + \frac{1}{2\sigma_{cor}} \right) \).

### 4.1.3 CR bound by Matsumoto

The bound is defined for a weight matrix \( G = J^S \). Firstly, we need the eigenvalues, \( \alpha_i \), of the matrix \( J^{S^{-1}J} \).

\[
J^{S^{-1}J} = \begin{pmatrix} 0 & -1/2W^2 \\ 1/2T^2 & 0 \end{pmatrix} \quad \rightarrow \quad \alpha_{1,2} = \pm \frac{i}{2TW} \quad (4.17)
\]

The bound in (3.18) is

\[
W^2\text{var}(\tau) + T^2\text{var}(\delta) \geq \frac{2TW}{2TW + \sqrt{4T^2W^2 - 1}} \quad (4.18)
\]

Notice that in this case there is no dependence with \( z \) since we are not considering a general weight matrix. We cannot check how informative this bound is but we know it is valid for any pure state.
4.1. QUANTUM FISHER INFORMATION

4.1.4 Holevo CR bound

In general, the calculation of the Holevo bound is not a simple problem, since it consist in a minimization over Hermitian operators that depend on the state. In this particular case, to minimize the function \( h(\vec{X}) \) defined in (3.21), we can follow the same procedure used by Bradshaw, Assad and Lam in [15].

First of all, we find an orthonormal basis \( \{ |e_0 \rangle, |e_1 \rangle, |e_2 \rangle \} \) that fulfills the inner products conditions. In order to do so, we redefine the variables as adimensional ones, \( \frac{\tau}{T} \) and \( \frac{\delta}{W} \). Also, since the Holevo CR bound is asymptotically attainable with an adaptive measurement scheme, we evaluate at \( (\tau, \delta) \to 0 \). The inner products become

\[
\begin{align*}
\langle \Psi_0 | \Psi_0 \rangle &= 1 \\
\langle \Psi_0 | \Psi_1 \rangle &= 0 \\
\langle \Psi_1 | \Psi_1 \rangle &= T^2 W^2 \\
\langle \Psi_0 | \Psi_2 \rangle &= i \omega^o s T \\
\langle \Psi_2 | \Psi_2 \rangle &= W^2 T^2 + \omega^o s^2 T^2 \\
\langle \Psi_1 | \Psi_2 \rangle &= i \frac{T}{2} W
\end{align*}
\]

(4.19)

With this basis \( \{ |e_0 \rangle, |e_1 \rangle, |e_2 \rangle \} \) we can rewrite the state and its derivatives as

\[
\begin{align*}
|\Psi_0 \rangle &= |e_0 \rangle \\
|\Psi_1 \rangle &= \sqrt{\frac{WT}{2}} \cosh r |e_1 \rangle + \sqrt{\frac{WT}{2}} \sinh r |e_2 \rangle \\
|\Psi_2 \rangle &= i \omega^o s T |e_o \rangle + i \sqrt{\frac{WT}{2}} \cosh r |e_1 \rangle - i \sqrt{\frac{WT}{2}} \sinh r |e_2 \rangle
\end{align*}
\]

(4.20)

where \( \cosh 2r = 2WT \).

Applying the constraints in (3.20), this are the new conditions that the operators \( X_1 \) and \( X_2 \) must fulfill

\[
\begin{align*}
\langle \Psi_0 | X_1 | \Psi_0 \rangle &= 0 \\
\langle \Psi_0 | X_2 | \Psi_0 \rangle &= 0 \\
\end{align*}
\]

(4.21)

\[
\begin{align*}
\text{Re}\{ \cosh r \langle e_0 | X_1 | e_1 \rangle + \sinh r \langle e_0 | X_1 | e_2 \rangle \} &= \frac{1}{\sqrt{\cosh 2r}} \\
\text{Re}\{ \cosh r \langle e_0 | X_2 | e_1 \rangle + \sinh r \langle e_0 | X_2 | e_2 \rangle \} &= 0 \\
\text{Re}\{ i \cosh r \langle e_0 | X_1 | e_1 \rangle - i \sinh r \langle e_0 | X_1 | e_2 \rangle \} &= 0 \\
\text{Re}\{ i \cosh r \langle e_0 | X_2 | e_1 \rangle - i \sinh r \langle e_0 | X_2 | e_2 \rangle \} &= \frac{1}{\sqrt{\cosh 2r}}
\end{align*}
\]

(4.22)
Considering (3.22), the elements of the matrix $Z[\vec{X}]$ can be written as:

\[
\begin{align*}
Z_{11} &= |\langle e_0 | X_1 | e_1 \rangle|^2 + |\langle e_0 | X_1 | e_2 \rangle|^2 \\
Z_{12} &= \langle e_0 | X_1 | e_1 \rangle \langle e_1 | X_2 | e_0 \rangle + \langle e_0 | X_1 | e_2 \rangle \langle e_2 | X_2 | e_0 \rangle \\
Z_{21} &= \langle e_0 | X_2 | e_1 \rangle \langle e_1 | X_1 | e_0 \rangle + \langle e_0 | X_2 | e_2 \rangle \langle e_2 | X_1 | e_0 \rangle \\
Z_{22} &= |\langle e_0 | X_2 | e_1 \rangle|^2 + |\langle e_0 | X_2 | e_2 \rangle|^2
\end{align*}
\]

One can define each of the elements by its real and its imaginary parts:

\[
\begin{align*}
\langle e_0 | X_1 | e_1 \rangle &= t_1 + j_1 \\
\langle e_0 | X_1 | e_2 \rangle &= s_1 + k_1 \\
\langle e_0 | X_2 | e_1 \rangle &= t_2 + j_2 \\
\langle e_0 | X_2 | e_2 \rangle &= s_2 + k_2
\end{align*}
\]

Therefore, the elements of the matrix $Z[\vec{X}]$ are:

\[
\begin{align*}
Z_{11} &= t_1^2 + j_1^2 + s_1^2 + k_1^2 \\
Z_{12} &= t_1 t_2 + j_1 j_2 + s_1 s_2 + k_1 k_2 + i(j_2 t_1 + k_2 s_1 - j_1 t_2 - k_1 s_2) \\
Z_{21} &= t_1 t_2 + j_1 j_2 + s_1 s_2 + k_1 k_2 - i(j_2 t_1 + k_2 s_1 - j_1 t_2 - k_1 s_2) \\
Z_{22} &= t_2^2 + j_2^2 + s_2^2 + k_2^2
\end{align*}
\]

The constraint in (4.22) becomes:

\[
\begin{align*}
t_2 &= -s_2 \tanh r \\
j_1 &= k_1 \tanh r \\
t_1 &= \frac{\text{sech} r}{\sqrt{\cosh 2r}} - s_1 \tanh r \\
j_2 &= -\frac{\text{sech} r}{\sqrt{\cosh 2r}} + k_2 \tanh r
\end{align*}
\]

Finally, considering a general weight matrix $G = \text{diag}[1, z]$ (with $z > 0$) the
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function \( h[\vec{X}] \) becomes:

\[
h = f + 2|g|\sqrt{z}
\]

\[
f = t_1^2 + j_1^2 + s_1^2 + k_1^2 + z(t_2^2 + j_2^2 + s_2^2 + k_2^2) = (s_1^2 + k_1^2 + zk_2^2 + zs_2^2)(1 + \tanh^2 r) + (1 + z)\frac{\text{sech}^2 r}{\cosh 2r} - 2(s_1 + zk_2)\frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}}
\]

\[
g = -j_1t_2 - k_1s_2 + j_2t_1 + k_2s_1 = (k_2s_1 - k_1s_2)(1 - \tanh^2 r) - \frac{\text{sech}^2 r}{\cosh 2r} + \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}}(s_1 + k_2)
\]

(4.27)

This is the function that needs to be minimized in order to obtain the Holevo bound. To do so, one can use the Karush-Kuhn-Tucker (KKT) conditions (16), (17).

The conditions that must be imposed in this case are

\[
\begin{align*}
-\nabla h &= \nabla g \\
g &\leq 0 \\
\lambda &\geq 0 \\
g\lambda &= 0
\end{align*}
\]

(4.28)

Although finding an analytical solution for every \( z \) and \( r \) is too complicated, calculations are much more simple if we fix \( z = 1 \), which is a case we are interested in. Later on, in section 4.2, we will focus in the minimum of the product \( V(\tau)V(\delta) \). In Appendix A, it is numerically shown this minimum is attained for this value of \( z \).

The details in the minimization of the function \( h[\vec{X}] \) for \( z=1 \) can be also found in Appendix A. The resultant expression is

\[
W^2V(\tau) + T^2V(\delta) \geq \cosh 2r \ e^{-r}
\]

(4.29)

Once again, there is no way of finding out how informative this bound is. However, we know that it is always as good or better than the SLD and the RLD CR bounds.

As it happens, this bound coincides with Matsumoto’s bound in (4.18). We will show this now.

The condition for \( r \) gives the following equation for \( x = e^{\pm 2r} \),

\[
\cosh 2r = \frac{e^{-2r} + e^{2r}}{2} = \frac{x + 1/x}{2} = 2WT \quad \rightarrow \quad x + \frac{1}{x} = 4TW \quad \rightarrow \quad x^2 - 4TWx + 1 = 0
\]

(4.30)
We obtain two solutions for $x$,

$$x = e^{\pm 2r} = 2WT \pm \sqrt{4T^2W^2 - 1}$$  \hspace{1cm} (4.31)$$

We can check that

$$2WT + \sqrt{4T^2W^2 - 1} = 1/(2WT - \sqrt{4T^2W^2 - 1}).$$

One of the previous solutions (the greatest) corresponds to $e^{2r}$ and the other one (the smallest) corresponds to $e^{-2r}$. We have,

$$e^{2r} = 2WT + \sqrt{4T^2W^2 - 1} \quad e^{-2r} = 2WT - \sqrt{4T^2W^2 - 1}$$  \hspace{1cm} (4.32)$$

Therefore, the Holevo CR bound can be written equivalently as

$$W^2\text{var}(\tau) + T^2\text{var}(\delta) \geq \cosh 2r e^{-r}$$

$$W^2\text{var}(\tau) + T^2\text{var}(\delta) \geq \frac{2WT}{2WT + \sqrt{4T^2W^2 - 1}}$$

$$W^2\text{var}(\tau) + T^2\text{var}(\delta) \geq 2WT \left(2WT - \sqrt{4T^2W^2 - 1}\right)$$  \hspace{1cm} (4.33)$$

As they coincide, from now on we will only refer to this as the Holevo bound and forget about Matsumoto’s formula, which was valid only for a particular choice of $G$.

### 4.2 Bound for the product of the variances

The objective in this last part is to rewrite this bounds as $\text{var}(\tau)\text{var}(\delta) \geq C$. This way, we will be able to compare all the previous bounds with the one given by the Arthurs-Kelly relation (1.1).

In order to do so, we consider the minimization problem of the function

$$f = \text{var}(\tau)\text{var}(\delta) = V_1V_2$$  \hspace{1cm} (4.34)$$

subject to the general condition

$$W^2V_1 + zT^2V_2 \geq \gamma(z) \quad \rightarrow \quad g = \gamma(z) - W^2V_1 - zT^2V_2 \leq 0$$  \hspace{1cm} (4.35)$$

where $\gamma(z)$ is any of the previous CR bounds.

The Karush-Kuhn-Tucker (KKT) conditions ([16] [17]) can be applied to find the optimal set $(V_1^*, V_2^*)$. After that, the dependence with $z$ is eliminated by finding the minimum of the function.
4.2. BOUND FOR THE PRODUCT OF THE VARIANCES

The conditions that must be fulfilled are:

\[
\begin{align*}
\nabla F &= \nabla (f + \lambda g) = 0 \\
g &= \gamma(z) - W^2 V_1 - z T^2 V_2 \leq 0 \\
\lambda &\geq 0 \\
g \lambda &= 0
\end{align*}
\]  

(4.36)

From the first condition we obtain that

\[
V_1^* = z \lambda T^2 \\
V_2^* = \lambda W^2
\]

(4.37)

There are two options to satisfy the last condition:

- If \( \lambda = 0 \):

\[
g = \gamma(z) \geq 0
\]

(4.38)

Since we are also imposing that \( g \leq 0 \), we cannot consider \( \lambda = 0 \) a valid solution.

- If \( g = 0 \):

\[
g = \gamma(z) - 2 z \lambda T^2 W^2 = 0 \quad \rightarrow \quad \lambda = \frac{\gamma(z)}{2 z T^2 W^2}
\]

(4.39)

Therefore, the minimum is found in

\[
V_1^* V_2^* = z \lambda^2 T^2 W^2 = \frac{\gamma^2(z)}{z T^2 W^2}
\]

(4.40)

In order to eliminate the dependence on \( z \) this function can be minimized again, this time as a function of \( z \).

\[
\frac{\partial}{\partial z} \left[ \frac{\gamma^2(z)}{z} \right] = z \left[ 2 \gamma \frac{\partial \gamma}{\partial z} - \gamma^2 \right] = 0 \quad \rightarrow \quad 2 \frac{\partial \gamma(z)}{\partial z} = \gamma(z)
\]

(4.41)

The product will be minimum when \( z \) satisfies \( 2 \frac{\partial \gamma(z)}{\partial z} = \gamma(z) \).

4.2.1 CR Bound based on the SLDs

In this first case we have found in (4.8) that \( \gamma(z) = \frac{1 + z}{4} \). The minimum will be reached when (4.41) is satisfied, this is:

\[
2 \frac{1}{4} = \frac{1 + z}{4} \quad \rightarrow \quad z = 1
\]

(4.42)

The resulting bound is

\[
V(\tau) V(\delta) \geq \frac{1}{16 T^2 W^2}
\]

(4.43)
4.2.2 CR Bound for coherent models

Now the CR bound calculated in (4.10) is \( \gamma(z) = \frac{1+z}{4} + \frac{\sqrt{z}}{2TW} \).

\[
2\left(\frac{1}{4} + \frac{1}{\sqrt{z}8TW}\right) = \frac{1+z}{4} + \frac{\sqrt{z}}{4TW} \rightarrow z = 1
\]

The bound for the product in the case of a coherent model is

\[
V(\tau)V(\delta) \geq \frac{(1+2TW)^2}{64T^2W^4}
\] (4.44)

We know this is valid only for \( TW = 1/2 \). In that case, the relation for the product of the variances is:

\[
V(\tau)V(\delta) \geq 1.
\] (4.45)

This coincides with Arthurs-Kelly relation.

4.2.3 Holevo CR bound

In Appendix A it is shown that the minimum of the product in this case is always attained when \( z = 1 \), no matter the value of the bandwidth product \( TW \). Therefore, we can use the analytical expression derived in section 4.1 to calculate the minimum of the product as in all the other cases.

In this case, the expression for \( \gamma(z = 1) \) is

\[
\gamma(z = 1) = 2TW \left(2TW - \sqrt{4T^2W^2 - 1}\right)
\] (4.46)

And the resulting bound for the product is

\[
V(\tau)V(\delta) \geq \left(2TW - \sqrt{4T^2W^2 - 1}\right)^2
\] (4.47)

4.2.4 Comparison of all the bounds

Finally, we can compare how this three bounds for the product behave as a function of \( TW \) and draw conclusions about their validity and informativeness. This is shown in figure 4.1.

The first important thing to notice is that, as expected, the Holevo bound (represented by the continuous orange line in figure 4.1) is a better bound than the one
4.2. BOUND FOR THE PRODUCT OF THE VARIANCES

Figure 4.1: Different Crámer-Rao bounds for the uncertainty of the product as a function of $TW$.

derived from the SLD operators (blue). From theory we know it has to be better than both the SLD and the bound for coherent models. It is for this reason that the plots reassures us that the model is not coherent (the bound for coherent models is represented by the dashed line in figure 4.1).

Another remarkable result is that the Holevo bound coincides with the bound for coherent models at $TW = 1/2$. This is significant, since we know now that at this point the Holevo bound corresponds with the ultimate and most informative possible bound. We still do not know how informative it is for the rest of the cases, $TW > 1/2$, and if it is attainable but the above observations are a good sign.

Furthermore, the value that the bound takes for separable states is exactly the one given by the Arthurs-Kelly relation in (1.1). As the the degree of entanglement grows (larger values of $TW$), all the bounds tend to zero. This means that the use of entangled states improves the bound for the problem in the estimation of two non-commuting observables. The idea of a quantum Lidar is proved to be effective.
Chapter 5

Future work

In Chapter 4 we have found that the Holevo CR bound is the best limit we can obtain using Quantum Estimation Theory. The next step would be to propose an experiment in which this limit is achieved, thus demonstrating that it corresponds with the ultimate bound.

Our proposal consists in the decomposition of the state in two orthogonal basis of modes, one for the signal photon and one for the idler photon. This mode distribution can be then experimentally measured using filters to project the pair of photons.

![Scheme of an experimental set-up that could be used to measure the decomposition in a general orthonormal basis of modes.](image)

In general, the state can be rewritten as

$$\Psi(\Omega_s, \Omega_i) = \sum_{m,n=0}^{\infty} C_{mn} \cdot f_m(\Omega_s)g_n(\Omega_i),$$

(5.1)

Figure 5.1: Scheme of an experimental set-up that could be used to measure the decomposition in a general orthonormal basis of modes.
with coefficients

\[ C_{mn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\Omega_s d\Omega_i \ \Psi(\Omega_s, \Omega_i) f_m^*(\Omega_s) g_n^*(\Omega_i). \]  

(5.2)

This coefficients are complex and depend on the parameters to estimate, \( C_{mn}(\tau_s, \delta) \).

The probability of coincidences detected in an experiment such as the one in Figure 5.1 will be directly the weight of each mode \( m, n \), that is, \( C_{mn} \).

The election of this orthogonal basis is in principle arbitrary. However, to saturate the quantum CR bound, it can be shown that the best choice is a basis in which the resultant phase of the coefficients \( C_{mn} \) does not depend on the parameters to estimate \( \tau_s \) and \( \delta \), [18].
Chapter 6

Conclusions

The main goal of his project is to show that the use of paired photons entangled in frequency can improve the resolution limit of a Lidar system. This has been accomplished making use of Quantum Estimation Theory.

Focusing on the results for the quantum Lidar, we have found a way to calculate the best possible bound, the Holevo CR bound. The comparison between the bound obtained and the Arthurs-Kelly relation for the case of a classical Lidar has successfully shown that as long as we do not have entanglement the precision limit cannot be beaten. However, the correlations between signal and idler of an entangled pair lead to an improvement in the uncertainty of the joint measurement of time delay and Doppler shift.

Furthermore, we have been able to prove that the state of the pair of photons involved does not correspond with a coherent model, which makes the assumptions of Zhuang et. al. in [4] incorrect.

An exhaustive revision of Quantum Estimation Theory has been carried out. This has provide us with useful tools that could be applied to many interesting cases in quantum sensing and quantum imaging.
Bibliography


Appendix A

Additional calculations for the Holevo CR bound

A.1 Minimum of $h[\vec{X}]$ for $z=1$

The objective is to minimize the function $h[\vec{X}]$ using the Karush-Kuhn-Tucker (KKT) conditions ([16],[17]). We are considering the case $z=1$, so $h = f + 2|g|$

\[
f = t_1^2 + j_1^2 + s_1^2 + k_1^2 + t_2^2 + j_2^2 + s_2^2 + k_2^2 = \]
\[= (s_1^2 + k_1^2 + k_2^2 + s_2^2)(1 + \tanh^2 r) + 2 \frac{\text{sech}^2 r}{\cosh 2r} - 2(s_1 + k_2) \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}}\]

\[
g = -j_1 t_2 - k_1 s_2 + j_2 t_1 + k_2 s_1 = \]
\[= (k_2 s_1 - k_1 s_2)(1 - \tanh^2 r) - \frac{\text{sech}^2 r}{\cosh 2r} + \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}}(s_1 + k_2)\]  

(A.1)

The KKT conditions that must be imposed in this case to find the minimum are

\[
\begin{cases}
-\nabla h = \nabla g \\
g \leq 0 \\
\lambda \geq 0 \\
g\lambda = 0
\end{cases} \quad \text{or} \quad
\begin{cases}
-\nabla h = -\nabla g \\
g > 0 \\
\lambda \geq 0 \\
g\lambda = 0
\end{cases}
\]  

(A.2)

We need to consider two different cases:
APPENDIX A. ADDITIONAL CALCULATIONS FOR THE HOLEVO CR BOUND

• If $g < 0$:

The function to minimize is $h = f - 2g$. The KKT conditions in (4.28) impose that $-\nabla(f - 2g) = 0$. The resultant system is:

\[
\begin{pmatrix}
-2 \cosh 2r & 2 & 0 & 0 \\
2 & -2 \cosh 2r & 0 & 0 \\
0 & 0 & -2 \cosh 2r & -2 \\
0 & 0 & -2 & -2 \cosh 2r \\
\end{pmatrix}
\begin{pmatrix}
s_1^* \\
k_2^* \\
k_1^* \\
s_2^* \\
\end{pmatrix}
= 
\begin{pmatrix}
-4 \frac{\sinh r}{\cosh 2r} \\
0 \\
0 \\
-4 \frac{\sinh r}{\cosh 2r} \\
\end{pmatrix}
\]

(A.3)

The possible point for the minimum of this function is:

\[
\begin{pmatrix}
s_1^* \\
k_2^* \\
k_1^* \\
s_2^* \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\cosh r}{\cosh 2r} \\
\frac{\cosh r}{\cosh 2r} \\
0 \\
0 \\
\end{pmatrix}
\]

(A.4)

Substituting in $g$ we obtain $g = \frac{\cosh r}{\cosh 2r} > 0$. Since we were assuming $g < 0$, the solution is not valid.

• If $g \geq 0$:

The function to minimize is $h = f + 2g$. The KKT conditions in (4.28) impose that

\[
\begin{cases}
-\nabla(f + 2g) = -\lambda \nabla g \\
g\lambda = 0 \\
\lambda \geq 0
\end{cases}
\]

(A.5)

The system in this case is:

\[
\begin{pmatrix}
-2 \cosh 2r & \lambda - 2 & 0 & 0 \\
\lambda - 2 & -2 \cosh 2r & 0 & 0 \\
0 & 0 & 2 \cosh 2r & 2 - \lambda \\
0 & 0 & 2 - \lambda & 2 \cosh 2r \\
\end{pmatrix}
\begin{pmatrix}
s_1^* \\
k_2^* \\
k_1^* \\
s_2^* \\
\end{pmatrix}
= 
\begin{pmatrix}
-\lambda \frac{\sinh r}{\cosh 2r} \\
0 \\
0 \\
-\lambda \frac{\sinh r}{\cosh 2r} \\
\end{pmatrix}
\]

(A.6)

The possible point for the minimum of $h[\{X\}]$ is:

\[
\begin{pmatrix}
s_1^* \\
k_2^* \\
k_1^* \\
s_2^* \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{\cosh 2r} \frac{\lambda \sinh r}{4 \cosh^2 r - \lambda} \\
\frac{1}{\cosh 2r} \frac{\lambda \sinh r}{4 \cosh^2 r - \lambda} \\
0 \\
0 \\
\end{pmatrix}
\]

(A.7)

There are two options to fulfill the condition $g\lambda = 0$:
-- If $\lambda = 0$:

The solution for the minimum is $s_1^* = k_2^* = k_1^* = s_2^* = 0$. With this values
the function $g$ is always negative

$$g = -\frac{\text{sech}^2 r}{\cosh 2r} < 0$$  (A.8)

Therefore, $\lambda = 0$ is not a valid solution.

-- If $g = 0$:

We substitute the possible minimum point in $g$ and obtain an equation
for $\lambda$.

$$g = -\frac{\text{sech}^2 r}{\cosh 2r} + \frac{\text{sech}^2 \lambda^2 \sinh^2 r}{\cosh 2r (4 \cosh^2 r - \lambda)^2} + 2 \frac{\text{sech}^2 r}{\cosh 2r} \frac{\lambda \sinh^2 r}{4 \cosh^2 r - \lambda} = 0$$  (A.9)

There is a solution for $\lambda = 4 \cosh r e^{\pm r}$ which implies

$$s_1^* = k_2^* = \frac{\mp e^{\pm r}}{\sqrt{\cosh 2r}}$$  (A.10)

From this two possible solutions for $(s_1^*, k_2^*, k_1^*, s_2^*)$ we take the one that
minimizes $h[\vec{X}] = f + 2g$. Finally, the minimum of the function $h[\vec{X}]$ is

$$\min h[\vec{X}] = \cosh 2r \ e^{-r}$$  (A.11)

with $r$ satisfying $\cosh 2r = 2WT$.

### A.2 Minimum of the product for any z

Now we want to show that the minimum of the product is always attained for $z = 1$.

Since an analytical solution is hard to determine, we proceed to solve it numerically. The procedure is similar to the one used for $z = 1$ but now we have to consider the dependence with $z$ in (4.27). This is:

$$h = f + 2\sqrt{z} |g|$$

$$f = t_1^2 + j_1^2 + s_1^2 + k_1^2 + z(t_2^2 + j_2^2 + s_2^2 + k_2^2) =$$

$$= (s_1^2 + k_1^2 + z k_2^2 + z s_2^2)(1 + \tanh^2 r) + (1 + z) \frac{\text{sech}^2 r}{\cosh 2r} - 2(s_1 + z k_2) \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}}$$

$$g = -j_1 t_2 - k_1 s_2 + j_2 t_1 + k_2 s_1 =$$

$$= (k_2 s_1 - k_1 s_2)(1 - \tanh^2 r) - \frac{\text{sech}^2 r}{\cosh 2r} + \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}} (s_1 + k_2)$$  (A.12)
APPENDIX A. ADDITIONAL CALCULATIONS FOR THE HOLEVO CR BOUND

The KKT conditions to impose are the same that in \(A.2\).

We consider the same two cases:

- If \(g < 0\):

  This is the resultant system:

  \[
  \begin{bmatrix}
  -2 \cosh 2r & 2\sqrt{z} & 0 & 0 \\
  2\sqrt{z} & -2z \cosh 2r & 0 & 0 \\
  0 & 0 & -2 \cosh 2r & -2\sqrt{z} \\
  0 & 0 & -2\sqrt{z} & -2z \cosh 2r
  \end{bmatrix}
  \begin{bmatrix}
  s_1^* \\
  k_2^* \\
  k_1^* \\
  s_2^*
  \end{bmatrix}
  =
  \begin{bmatrix}
  \frac{-\sinh r}{\sqrt{\cosh 2r}} (2\sqrt{z} - 2) \\
  \frac{\sinh r}{\sqrt{\cosh 2r}} (-2\sqrt{z} - 2z) \\
  0 \\
  0
  \end{bmatrix}
  \]

  \(A.13\)

  The possible point for the minimum of \(h[\vec{X}]\) is:

  \[
  \begin{bmatrix}
  s_1^* \\
  k_2^* \\
  k_1^* \\
  s_2^*
  \end{bmatrix}
  =
  \begin{bmatrix}
  - \frac{\sinh r}{\sqrt{\cosh 2r}} \frac{4z + 4z \cosh 2r + 4\sqrt{z}}{4z - 4z \cosh 2r} \\
  \frac{\sinh r}{\sqrt{\cosh 2r}} \frac{4z + 4z \cosh 2r + 4\sqrt{z}}{4z - 4z \cosh 2r} \\
  0 \\
  0
  \end{bmatrix}
  \]

  \(A.14\)

  Substituting in \(g\):

  \[
  g = \frac{\text{sech}^2 r}{\cosh 2r} \left( 1 - \frac{(1 + \sqrt{z})^2}{4z - \sqrt{z}} \right) > 0
  \]

  \(A.15\)

  Again, there is no valid solution when \(g < 0\).

- If \(g \geq 0\):

  The resultant system is:

  \[
  \begin{bmatrix}
  -2 \cosh 2r & \lambda - 2\sqrt{z} & 0 & 0 \\
  \lambda - 2\sqrt{z} & -2z \cosh 2r & 0 & 0 \\
  0 & 0 & -2 \cosh 2r & 2\sqrt{z} - \lambda \\
  0 & 0 & -2\sqrt{z} - \lambda & -2z \cosh 2r
  \end{bmatrix}
  \begin{bmatrix}
  s_1^* \\
  k_2^* \\
  k_1^* \\
  s_2^*
  \end{bmatrix}
  =
  \begin{bmatrix}
  \frac{\sinh r}{\sqrt{\cosh 2r}} (2\sqrt{z} - 2 - \lambda) \\
  \frac{\sinh r}{\sqrt{\cosh 2r}} (2\sqrt{z} - 2z - \lambda) \\
  0 \\
  0
  \end{bmatrix}
  \]

  \(A.16\)
The possible point for the minimum of the function is:

\[
\begin{pmatrix}
  s_1^* \\
  k_2^* \\
  k_1^* \\
  s_2^*
\end{pmatrix} = \begin{pmatrix}
  -\sinh r \sqrt{\cosh 2r} \\
  4z + 4\lambda + 4\sqrt{z} + \lambda^2 + 4z \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4z + 4\lambda + 4 \sqrt{z} + 4 \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4 \lambda \cosh 2r \\
  \sinh r \sqrt{\cosh 2r} \\
  4z - 4\lambda \sqrt{z} + \lambda^2 + 4z \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4z - 4\lambda \sqrt{z} + \lambda^2 + 4z \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4 \lambda \cosh 2r \\
  \sinh r \sqrt{\cosh 2r} \\
  4z - 4\lambda \sqrt{z} + \lambda^2 + 4z \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4z - 4\lambda \sqrt{z} + \lambda^2 + 4z \cosh 2r - 4\sqrt{z} + 4 \cosh 2r + 4 \lambda \cosh 2r
\end{pmatrix}
\]

(A.17)

Now there are two options to fulfill the condition \(g\lambda = 0\):

- If \(\lambda = 0\):

\[
\begin{pmatrix}
  s_1^* \\
  k_2^* \\
  k_1^* \\
  s_2^*
\end{pmatrix} = \begin{pmatrix}
  1 - \sqrt{z} \\
  \frac{1 - \sqrt{z}}{\sinh r \sqrt{\cosh 2r}} \\
  \frac{1 - \sqrt{z}}{\sinh r \sqrt{\cosh 2r}} \\
  0
\end{pmatrix}
\]

(A.18)

The function \(g\) becomes:

\[
g = -\frac{\text{sech}^2 r \cosh 2r}{\cosh 2r} \left( 1 + \text{coth}^2 r \left(1 - \frac{\sqrt{z}}{\sqrt{z}}\right) \right) < 0
\]

(A.19)

The solution for \(\lambda = 0\) is not a valid one.

- If \(g = 0\):

We find that the solution to the minimization problem is obtained when imposing \(g = 0\). The variables \(s_2^*\) and \(k_1^*\) are equal to zero and \(s_1^*, k_2^*\) are the solution of the system:

\[
\begin{align*}
g &= k_2^* s_1^* (1 - \tanh^2 r) - \frac{\text{sech}^2 r \cosh 2r}{\cosh 2r} + \frac{\text{sech} r \tanh r}{\sqrt{\cosh 2r}} (s_1^* + k_2^*) = 0 \\
s_1^* &= -\frac{\sinh r}{\sqrt{2WT}} \left( 4z + 2\lambda z - 4\lambda \sqrt{z} + \lambda^2 + 4z^{3/2} + 8TWz - 8TWz^{3/2} - 4z \lambda \cosh 2r \right) \\
k_2^* &= -\frac{\sinh r}{\sqrt{2WT}} \left( 2\lambda z - 4\lambda \sqrt{z} + \lambda^2 - 4\sqrt{z} + 4TW\lambda + 8TWz - 8TW\sqrt{z} - 4z \lambda \cosh 2r \right)
\end{align*}
\]

(A.20)

Instead of finding the analytical solution for \(\lambda\), we solve it numerically.

At the beginning of section 4.2 it is shown that the minimum of the product of the variances is given by (4.40). The dependence of this function with \(z\) is shown in figure A.1 for different values of \(r\) (different values of the bandwidth product \(TW\)).

In all the plots it can be seen that the function has a minimum for \(z = 1\). Therefore, to find the minimum of the product \(V_1V_2\), the analytical solution found in section A.1 is valid.
Figure A.1: Minimum of $V_1V_2$ as a function of $z$ for different values of $r$. It is clearly seen that the function always has a minimum for $z = 1$, regardless the value of $r$. 
Appendix B

Karush-Kuhn-Tucker conditions

The Karush-Kuhn-Tucker conditions ([16][17]) set the solution for an optimization nonlinear problem. In particular, they are useful to optimize a function $f(x)$ (called the objective function) subject to both inequality and constant constraints of the form

\[
\begin{align*}
g_i(x) &\leq 0 \\
h_j(x) &= 0
\end{align*}
\] (B.1)

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The optimization variable is $x$.

We assume all the functions $f(x)$, $g_i(x)$ and $h_j(x)$ are continuously differentiable at a point $x^\ast$. This point $x^\ast$ is optimal if the functions satisfy some regularity conditions and there exist constants $\mu_i$ and $\lambda_j$ such that

\[
\begin{align*}
\begin{cases}
g_i(x^\ast) \leq 0 & \forall i \\h_j(x^\ast) = 0 & \forall j
\end{cases} \\
\mu_i \geq 0 & \forall i \\
\mu_i g_i(x^\ast) = 0 & \forall i
\end{align*}
\] (B.2)

For maximizing:

\[
\nabla f(x^\ast) = \sum_{i=1}^{m} \mu_i \nabla g_i(x^\ast) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^\ast)
\] (B.5)

For minimizing:

\[
-\nabla f(x^\ast) = \sum_{i=1}^{m} \mu_i \nabla g_i(x^\ast) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^\ast)
\] (B.6)