T-Indistinguishability Operators with respect to Ordinal Sums.

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***Abstract*—In this paper we study the class of *T*-indistinguishability operators such that the involved t-norm *T* is an ordinal sum. We show that those *T*-indistinguishability operators can be thought of as families of indistinguishabilities with respect to some Archimedean t-norms. An interpretation in terms of hierarchical clustering is provided.**

Keywords-fuzzy relation; T-indistinguishability operator; t-norm; Archimedean t-norm; ordinal sum; clustering; hierarchical clustering; hierarchical aggregation;

I. INTRODUCTION

Relations that are reflexive, symmetric and transitive are called *equivalence relations*. Back in 1971, Zadeh published his views on how these ought to be fuzzyfied, and termed the new class of fuzzy relations *similarity relations* [1].

Zadeh's original definition is as follows.

Definition 1.1: A Similarity Relation E on a set X is a mapping $E: X \times X \to [0,1]$ such that:

1.1.1 $E(x,x)=1$ for all x in X (fuzzy reflexivity).

1.1.2 $E(x,y)=E(y,x)$ for all x, y in X (fuzzy symmetry).

1.1.3 $MIN(R(x,y),R(y,z)) \leq R(x,z)$ for all x, y and z in X (fuzzy transitivity).

He also opened the door to using t-norms other than the *minimum* to define fuzzy transitivity, which other researchers did very soon. Let us recall what a t-norm is.

Definition 1.2: A t-norm is an operation on the unit interval which is associative, commutative and satisfies the boundary conditions $T(x,0)=0$ and $T(x,1)=x$ for all x in [0,1].

NOTE: within this paper we only deal with t-norms $T(x,y)$ that are continuous in both variables.

Pioneering works from Ruspini [2] dealt with *T=LUK*, the Lukasiewicz t-norm, and the associated relations were called *likeness relations*. Ovchinnikov [3] used *T=PROD*, the standard product t-norm, thus introducing the so-called *probabilistic relations*. Trillas [4,5] studied fuzzy transitivity with respect to a general t-norm *T*, and he proposed the unifying term *indistinguishability* to refer to any such relation, which since then has been widely used (see, for example, Valverde [5,6], Jacas [7] and Recasens [8]). Other terms such as *fuzzy equality* or *identity* (more restrictive), or *fuzzy equivalence relation* (perhaps more general) are also in use.

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So, this paper deals with indistinguishability operators in the sense of the following definition

Definition 1.3: An Indistinguishability Operator with respect to a t-norm *T*, or a *T*-indistinguishability for short, is a mapping $E: X \times X \to [0,1]$ satisfying 1.1.1, 1.1.2 and

1.3.3 $T(E(x,y),E(y,z)) \leq E(x,z)$ for all x, y and z in X (fuzzy transitivity w.r.t. *T*).

The t-norms mentioned so far, except *T=MIN*, are all examples of *Archimedean* t-norms. As we will explain soon, all continuous t-norms fall into one of the three following categories: the minimum t-norm, Archimedean t-norms and *ordinal sums.*

To present, we do not know of any specific study on the class of fuzzy relations which are obtained when an ordinal sum is the t-norm chosen to define fuzzy transitivity. To the best of our knowledge, this paper constitutes a first attempt on the issue. It is justified, apart from the sake of completeness, for the semantic and practical consequences derived from the use of ordinal sums. The most important of such consequences is that their associated indistinguishabilities can be used to model clustering problems in which features are hierarchically structured.

We will skip the definitions of *Archimedean* and *strict* t-norm, and will be using instead a handy characterisation which holds only for continuous t-norms. Both these definitions and the proofs of the following theorems are standard and can be found, for example, in [9,10].

Theorem 1.4 (Characterization of continuous Archimedean t-norms): *T* is a continuous Archimedean t-norm if, and only if, there exists a continuous decreasing function $f:[0,1]\to[0,M]$ with $f(0)=M$ and $f(1)=0$ such that $T(x,y)=f^{[-1]}(f(x)+f(y))$ for all x and y in [0,1].

In such case, T is strict if, and only if, $M = \infty$.

The function f is called an *additive generator* of T, and it is not unique.

$f^{[-1]}$ stands for the *quasi-inverse* of f. If $M = \infty$ then $f^{[-1]}$ is simply the inverse of f extended so that $f^{[-1]}(\infty) = 0$ .If $M < \infty$ then $f^{[-1]}$ is the inverse of f on [0,M), and $f^{[-1]}(a) = 0$ for all $a \geq M$.

Theorem 1.5 (Characterization of continuous t-norms): T is a continuous t-norm if, and only if, there exists a family of functions $f_i : [a_i, b_i] \to [0, M_i]$ defined on disjoint intervals $(a_i, b_i)_{i \in I} \subseteq (0,1)$ such that:

$$T(a,b) = \begin{cases} f_i^{[-1]}\left(f_i(a) + f_i(b)\right) & \text{if } a,b \in (a_i, b_i) \\ MIN(a,b) & \text{otherwise} \end{cases}$$

The functions $f_i^{[-1]}$ are defined similarly to $f^{[-1]}$, with a_i acting as 0 and b_i as1, and they will be also referred to as additive generators of T.

Note that if the family $(a_i, b_i)_{i \in I}$ is empty then $T=MIN$, while if it reduces to a single element $(a_1, b_1) = (0,1)$ then T is an Archimedean t-norm.

Definition 1.6: In the conditions of theorem 1.5, a continuous t-norm is called an ordinal sum if the family $(a_i, b_i)_{i \in I}$ has more than one interval, or if it has only one then $(a_1, b_1) \neq (0,1)$.

Intuitively, we like to think of ordinal sums as t-norms made up of some other t-norms, rather than simply of intervals and additive generators. This can be achieved by considering a family of continuous increasing functions $\varphi_i : [0,1] \to [a_i, b_i]$ and the associated Archimedean t-norms T_i which are induced on the unit interval by the additive generators $f_i \circ \varphi_i : [0,1] \to [0, M_i]$.

The standard choice for the functions φ_i is $\varphi_i(a) = a_i + a(b_i - a_i)$, although others could be considered. Accordingly, ordinal sums are noted by $T = \underset{i \in I}{\oplus} T_i$, and referred to as the *ordinal sum of the t-norms* T_i . We will avoid when possible any reference to either the intervals $(a_i, b_i)_{i \in I}$ or the generating functions in order to keep the notation simple.

Also for the sake of simplicity, we will concern ourselves with only a special class of ordinal sums, namely those satisfying:

$$a_0 = 0,\ b_0 = a_1\ \ \ldots\ b_i = a_{i+1}\ \ldots\ b_n = 1 \qquad [1.6.1]$$

That is, ordinal sums with a finite number of intervals leaving no gaps between them.

II. A CHARACTERIZATION THEOREM

The main idea in this section is that T-indistinguishability operators with T an ordinal sum of the restricted class [1.1.6] are nothing but families of stratified indistinguishabilities with respect to Archimedean t-norms, all of them defined on the same set X.

Theorem 2.1 A fuzzy relation E on a set X is an indistinguishability operator w.r.t. an ordinal sum $T = \underset{i \in I}{\oplus} T_i$ if, and only if, there exists a family $\{E_i\}_{i \in I}$ of indistinguishability operators w.r.t. Archimedean t-norms T_i and a family of functions $\varphi_i : [0,1] \to [a_i, b_i]$ such that $E = \inf_{i \in I} e_i$ with

$$e_i(x,y) = \begin{cases} \varphi_i \circ E_i(x,y) & \text{if } E_i(x,y) < 1 \\ 1 & \text{otherwise} \end{cases}$$

Before proceeding to prove theorem 2.1, let us comment on a few things.

First, the functions φ_i in theorem 2.1 are the same as those appearing in the definition of $\underset{i \in I}{\oplus} T_i$, so we intendedly use the same notation to designate both.

Second, the blocks e_i from which E is constructed are built by compressing the range of every E_i (generally [0,1]) into narrower slices $[a_i, b_i]$ which are then piled on top of each other, the only exception being the pairs $E_i(x,y) = 1$ which remain unchanged. Thus defined, the blocks e_i become also indistinguishability operators with respect to some t-norms t_i which are ordinal sums of only one Archimedean t-norm, T_i .

Finally, theorem 2.1 justifies the expression *ordinal sum of* T_i*-indistinguishability operators*, as well as the notation $E = \underset{i \in I}{\oplus} E_i$. Whenever possible, we will keep both the intervals $(a_i, b_i)_{i \in I}$ and the functions $\varphi_i : [0,1] \to [a_i, b_i]$ at an implicit level, so that notation does not become a burden.

Proof (Th.2.1): [$\Rightarrow$] The fuzzy relations E_i are defined by

$$E_i(x,y) = \begin{cases} 0 & \text{if } E(x,y) \leq a_i \\ \varphi_i^{-1} \circ E(x,y) & \text{if } a_i < E(x,y) < b_i \\ 1 & \text{if } E(x,y) \geq b_i \end{cases}$$

First, we need to prove that these are indistinguishability operators w.r.t. the Archimedean t-norms T_i obtained from the additive generators $f_i \circ \varphi_i : [0,1] \to [0, M_i]$.

Both reflexivity [1.1.1] and symmetry [1.1.2] are straightforward. To prove transitivity with respect to T_i [1.3.3], for any given three elements x, y and z there are only three different cases to be considered:

If $E_i(x,y) = 0$ then $0 = T_i\left(E_i(x,y), E_i(x,y)\right) \leq E_i(x,y)$ so that [1.3.3] holds.

If $0 < E_i(x,y) \le 1$ and $E_i(y,z) = 1$ then $a_i \le E(x,y) \le b_i$ but $E(y,z) \ge b_i$ and the ordinal sum T reduces to MIN in that particular case, which means

$$T(E(x,y),E(y,z)) = MIN(E(x,y),E(y,z))$$
$$= E(x,y) \le E(x,z)$$

(the last inequality owing to the T-transitivity of E)

As a consequence, $E_i(x,y) \le E_i(x,z)$, and

$$T_i(E_i(x,y),E_i(y,z)) = T_i(E_i(x,y),1)$$
$$= E_i(x,y) \le E_i(x,z)$$

which proves that [1.3.3].

Finally, if $0 < E(x,y),\ E(y,z) < 1$ then

$$T_i(E_i(x,y),E_i(y,z)) = T_i(\varphi_i^{[-1]} \circ E_i(x,y), \varphi_i^{[-1]} \circ E_i(y,z))$$
$$= \varphi_i^{[-1]} \circ f_i^{[-1]}(f_i \circ \varphi_i \circ \varphi_i^{[-1]} \circ E_i(x,y) + f_i \circ \varphi_i \circ \varphi_i^{[-1]} \circ E_i(y,z))$$
$$= \varphi_i^{[-1]} \circ f_i^{[-1]}(f_i \circ E_i(x,y) + f_i \circ E_i(y,z))$$
$$= \varphi_i^{[-1]} \circ T(E_i(x,y),E_i(y,z)) \le \varphi_i^{[-1]} \circ E_i(x,z) = E_i(x,z)$$

Again [1.3.3] holds.

Any other case can be reduced to one of the former three, and this settles the transitivity of E_i with respect to T_i.

Finally, to conclude this side of the proof, we need to show that $E = \inf_{i \in I} e_i$. To this end, let us express e_i in terms of E instead of E_i:

If $a_i < E(x,y) < b_i$ then $E_i(x,y) = \varphi_i^{[-1]} \circ E(x,y) < 1$ and $e_i(x,y) = \varphi_i \circ E_i(x,y) = E(x,y)$.

If $E(x,y) \le a_i$ then $E_i(x,y) = 0$ and $e_i(x,y) = a_i$.

If $E(x,y) \ge b_i$ then $E_i(x,y) = 1$ and $e_i(x,y) = 1$.

Now, suppose $a_i \le E(x,y) < b_i$ for a certain pair $E(x,y)$. For any j and k such that $j < i < k$ we shall have that $e_j(x,y) = 1$, $e_i(x,y) = E(x,y)$ and $e_k(x,y) = a_k$, and therefore $E(x,y) = \inf_{i \in I} e_i(x,y)$.

Note that nothing changes if i corresponds to the first or the last interval. Also, note that the conditions $a_i \le E(x,y) < b_i$ account for all the possible values of $E(x,y)$ except $E(x,y) = 1$ because we are assuming $b_i = a_{i+1}$ ([1.6.1]) in which case $e_i(x,y) = 1$.

[$\Leftarrow$] Consider the t-norm T which is the ordinal sum of the Archimedean t-norms T_i with the system of intervals (a_i, b_i). It has to be proved that $E = \inf_{i \in I} e_i$ is an indistinguishability operator with respect to $T = \oplus_{i \in I} T_i$. As usually, both fuzzy reflexivity and symmetry are straightforward, so we turn our attention to transitivity.

Before proceeding, let us state a few helpful facts.

First, it is obvious that $e_i(x,y) \le e_j(z,t)$ if $i < j$, except when $e_i(x,y) = 1$.

Second, $E(x,y) = 1$ if and only if $E_i(x,y) = 1$.

Third, if $E(x,y) < 1$ then $E(x,y) = e_i(x,y)$ if and only if $E_i(x,y) < 1$ and $E_j(x,y) = 1$ for all $j < i$.

Finally, for any x, y, z in X, if $E(x,y) = e_i(x,y)$, $E(y,z) = e_j(y,z)$ and $E(x,z) = e_k(x,z)$ then $k \ge \min\{i, j\}$.

To prove the last one, if we had $k < \min\{i, j\}$ then both $E_k(x,y) = 1$ and $E_k(y,z) = 1$. But E_k is transitive with respect to T_k and therefore $E_k(x,z) = 1$, which is against $E(x,z) = e_k(x,z)$.

Now we can proceed to prove transitivity with respect to $T = \oplus_{i \in I} T_i$, that is, to prove that $T(E(x,y),E(y,z)) \le E(x,z)$ for all x, y, z in X. Let us suppose $E(x,y) = e_i(x,y)$, $E(y,z) = e_j(y,z)$ and $E(x,z) = e_k(x,z)$. From the remarks above it follows that only five possible cases deserve attention, which are $i < j < k$, $i < j = k$, $i < k < j$, $i = k < j$ and $i = k = j$. First and third cases are straightforward. As to the remaining ones:

Case $i < j = k$:

$$T(E(x,y),E(y,z)) = MIN(E(x,y),E(y,z))$$
$$= MIN(e_i(x,y),e_k(y,z))$$
$$= e_i(x,y) \le e_k(x,y) = E(x,z)$$

Case $i = k < j$:

$$T(E(x,y),E(y,z)) = MIN(E(x,y),E(y,z))$$
$$= MIN(e_i(x,y),e_j(y,z)) = e_i(x,y)$$

On the other hand, from $T_i(E_i(x,y),E_i(y,z)) \le E_i(x,z)$ and $E_i(y,z) = 1$ it follows that $E(x,y) \le E(x,z)$ and thus $T(E(x,y),E(y,z)) \le E(x,z)$.

Case $i = k = j$:

$$T(E(x,y),E(y,z)) = T(e_i(x,y),e_i(y,z))$$
$$= f_i^{[-1]}(f_i \circ e_i(x,y) + f_i \circ e_i(y,z))$$

$$= f_i^{[-1]}\left(f_i \circ \varphi_i \circ e_i\left(x,y\right) + f_i \circ \varphi_i \circ e_i\left(y,z\right)\right)$$
$$= \varphi_i \circ T_i\left(E_i\left(x,y\right), E_i\left(y,z\right)\right) \leq \varphi_i \circ E_i\left(x,z\right)$$
$$= e_i\left(x,z\right) = E\left(x,z\right)$$

This brings the proof to a conclusion. ■

III. An interpretation of ordinal sums in terms of hierarchical clustering

It is well known that the infimum of any family of *T*-indistinguishability operators is also a *T*-indistinguishability operator.

When we classify elements coming from a set of patterns *X* according to two different criteria, say color and size, then we may break *X* into smaller clusters by combining the two independent classifications via the infimum. Namely, if E_c stands for color and E_s for size, then two elements are indistinguishable by $E = E_c \wedge E_s$ if they are so by both E_c and E_s. Or, to put it the other way round, they are different if they can be discriminated by at least one of the two criteria.

Under this approach, both classifications matter exactly the same. Size and color are equally ranked as valid criteria to sort out the elements of *X*. However, one might want one criterion to play a more important role than the other. For example, if differences in size are sharp enough in order to distinguish two patterns *x* and *y*, then there is no need for color to be taken into account. In that case, color will be considered only when the sizes of the patterns are coincident.

It is worth noting that it makes a big difference whether words such as *coincident*, *discriminated*, etc. are given a crisp meaning or either they stand for fuzzy, graded concepts. For in the former case the two approaches are coincident, and the global outcome is not sensitive to the order or hierarchy considered among criteria, while in the latter it accounts for a great deal.

To better see the differences which arise in the fuzzy case let us develop a little further on the alluded example of colors and sizes.

Example 3.1: Let $c: X \to [0,1]$ and $s: X \to [0,1]$ be the functions 'color' and 'size' respectively, which are fuzzy sets of some universe of patterns *X*. In order to model the similarity among patterns we choose the Lukasiewicz t-norm, $T_L\left(x,y\right) = \max\left\{0, x+y-1\right\}$. Therefore, the two induced T_L-indistinguishabilities on *X* are $E_c\left(x,y\right) = 1 - |c\left(x\right) - c\left(y\right)|$ and $E_s\left(x,y\right) = 1 - |s\left(x\right) - s\left(y\right)|$, and the joint indistinguishability is:

$$E\left(x,y\right) = E_c\left(x,y\right) \wedge E_s\left(x,y\right)$$
$$= \min\left\{1 - |c\left(x\right) - c\left(y\right)|,\ 1 - |s\left(x\right) - s\left(y\right)|\right\}$$

E is a *T*-indistinguishability with respect to the same t-norm *T* as E_c and E_s (T=T_L in this case). Such construction of an indistinguishabilty operator starting from a set of fuzzy criteria (*color*, *size* or any other graded notions) is standard, and it constitutes the core of Valverde's Representation Theorem [6]. (See [8] for a more comprehensive explanation).

It is also standard to provide a geometrical representation of the indistinguishability *E* where *c* and *s* are the axis and *x*,*y*... the points with coordinates $x = \left(x_s, x_c\right) = \left(s\left(x\right), c\left(x\right)\right)$

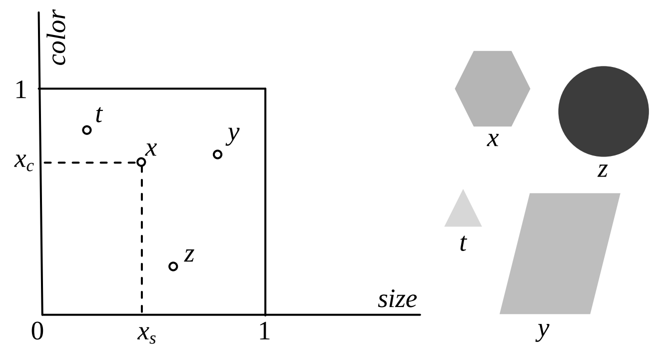

Figure 3.1

Such representations implicitly assume that the patterns *x*,*y*... are part of a universe bigger than *X*, which in our example is $X' = [0,1]^2$. Therefore, the degree of indistinguishability among patterns is inherited from that in *X'*, *E'*, and the ultimate responsibility for perceiving pairs of patterns in *X* as indistinguishable falls on the metric balls, or *α*-cuts, of *E'*.

The *α-cut* centered at *x* is the set of all points *y* in *X'* whose levels of indistinguishability with respect to *x* are above *α*. In the present case, $E = E_c \wedge E_s$ and they are squared in shape.

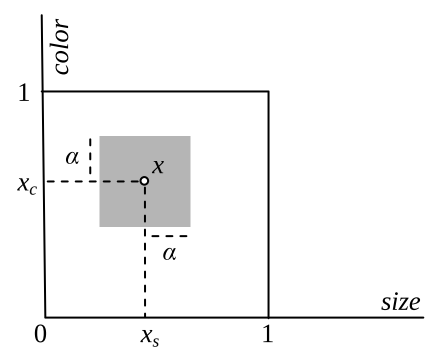

Figure 3.2

How will the balls look like in the hierarchical case? If we are giving size priority over color, then the differences in size determine the indistinguishability between pairs, and the color is only determinant when sizes are exactly the same. The balls will look like vertical bands for some radius (close to zero), and like vertical segments for some others (close to one). See figures 3.3 and 3.4 below.

The rest of this section is devoted to justify figures 3.3 and 3.4. The indistinguishability $E = E_s \oplus E_c$ is *T*-transitive with respect to the t-norm $T = T_L \oplus T_L$ with intervals $a_0 = 0$, $b_0 = a_1 = 0.5$ and $b_1 = 1$. The two basic indistinguishabilities E_s and E_c from which we construct *E* have been already defined above. The mappings φ_i are $\varphi_s:[0,1]\to[0,0.5]$ defined by $\varphi_s(a)=a/2$ for all a in $[0,1]$ and $\varphi_c:[0,1]\to[0.5,1]$ defined by $\varphi_c(a)=(1+a)/2$ for all a in $[0,1]$. Then E is obtained as $E=e_s\wedge e_c$ where

$$e_s=\begin{cases}\varphi_s\circ E_s(x,y) & \text{if } E_s(x,y)<1\\ 1 & \text{otherwise.}\end{cases}$$

and

$$e_c=\begin{cases}\varphi_c\circ E_c(x,y) & \text{if } E_c(x,y)<1\\ 1 & \text{otherwise.}\end{cases}$$

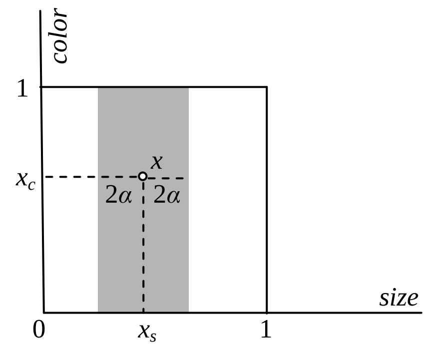

Figure 3.3

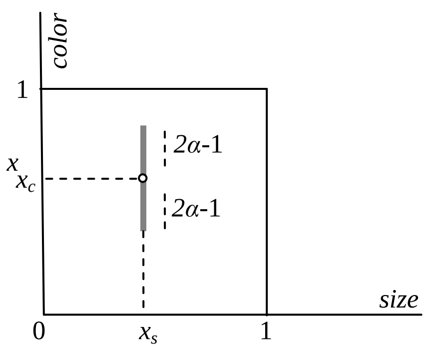

Figure 3.4

The ball or α-cut centered at x is:

$$\begin{aligned}
B(x,\alpha) &= \left\{y\in[0,1]^2 \ \ s.t.\ \ E(x,y)\ge\alpha\right\}\\
&= \left\{y\in[0,1]^2 \ \ s.t.\ \ \mathrm{e}_s(x,y)\wedge\ \mathrm{e}_c(x,y)\ge\alpha\right\}\\
&= \left\{y\in[0,1]^2 \ \ s.t.\ \ \varphi_s\circ E_s(x,y)\ge\alpha \text{ or } E_s(x,y)=1\right\}\\
&\bigcap\left\{y\in[0,1]^2 \ \ s.t.\ \ \varphi_c\circ E_c(x,y)\ge\alpha \text{ or } E_c(x,y)=1\right\}\\
&= \left\{y\in[0,1]^2 \ \ s.t.\ \ E_s(x,y)\ge 2\alpha \text{ or } E_s(x,y)=1\right\}\\
&\bigcap\left\{y\in[0,1]^2 \ \ s.t.\ \ E_c(x,y)\ge 2\alpha-1 \text{ or } E_c(x,y)=1\right\}\\
&= B_1\bigcap B_2
\end{aligned}$$

If $0\le\alpha<0.5$ then $B_1=\left\{y\in[0,1]^2 \ \ s.t.\ \ E_s(x,y)\ge 2\alpha\right\}$ and $B_2=[0,1]^2$, so $B(x,\alpha)=B_1\bigcap B_2=B_1$ (fig. 3.3) .

If $0\le\alpha<0.5$ then $B_2=\left\{y\in[0,1]^2 \ \ s.t.\ \ E_c(x,y)\ge 2\alpha-1\right\}$ and $B_1=\left\{y\in[0,1]^2 \ \ s.t.\ \ E_s(x,y)=1\right\}$, so $B(x,\alpha)=B_1\bigcap B_2$ is simply a segment (fig. 3.4).

IV. Conclusions

We have presented a first insight into the class of indistinguishability operators with respect to t-norms which are ordinal sums of Archimedean t-norms.

This is an important step for both theoretical and practical reasons. From the theoretical point of view, ordinal sums are the only remaining class of continuous t-norms when the Archimedean and the minimum t-norms are discarded. So the study is justified for the sake of completeness.

From the practical side, this class of indistinguishability operators provide a formal tool for dealing with hierarchical clustering. Whenever criteria are organized from more to less relevant, and the corresponding indistinguishabilities are aggregated consistently, we have proved that the outcoming structure is still an indistinguishability operator, only with respect to a different t-norm, namely an ordinal sum.

There are many aspects of this class of fuzzy relations to be investigated yet. Mostly questions related to their representation via fuzzy sets, dimension, etc, but also possible generalizations, their behavior under discretization, as well as comparing them with other methods of hierarchical clustering.

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