Probabilistic power indices for games with abstention*

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Abstract

In this paper we introduce eight power indices that admit a probabilistic interpretation for voting rules with abstention or with three levels of approval in the input, briefly (3,2) games. We analyze the analogies and discrepancies between standard known indices for simple games and the proposed extensions for this more general context. A remarkable difference is that for (3,2) games the proposed extensions of the Banzhaf index, Coleman index to prevent action and Coleman index to initiate action become non–proportional notions, contrarily to what succeeds for simple games. We conclude the work by providing procedures based on generating functions for (3,2) games, and extensible to (j,k) games, to efficiently compute them.

Key words: Voting rules with abstention; Probabilistic power indices; Banzhaf and Coleman extensions; Success and Decisiveness; Generating functions

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1 Introduction

A list of remarkable power indices have been introduced in the literature for simple games. Among them we would like to recall those introduced by Rae, Banzhaf, Coleman to prevent action, Coleman to initiate action and König and Bräuninger. These authors themselves and other scholars have justified these indices by means of probabilistic approaches. A unifying work in this line of inquiry has been done by Laruelle and Valenciano in [17] where a probabilistic approach has been given for each of the above mentioned power indices.

It is worth noticing that although the philosophy under Banzhaf, Coleman to prevent action and Coleman to initiate action power indices is different in the context of simple

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games, these three notions become proportional and for this reason they have been regarded by scholars as almost equivalent notions without doing distinctions among them. In this work we consider their natural extensions to the context of games with abstention or more generally to (3,2) games. In this slightly more wide context these three extended notions emerge to be non–proportional, and therefore the difference of meaning among them is highlighted in this broader context. Hence the two Coleman's indices, almost disregarded for simple games, excel in the context of games with abstention.

During several decades abstention has almost been omitted by scholars. However, abstention plays a key role in many of the real voting systems that have been modeled by simple games (such as the United Nations Security Council, or the United States federal system).

In the mainstream literature, Fishburn [[9] pp. 53–55] and Rubinstein [21] are isolated and brief exceptions. Felsenthal and Machover [7, 8] consider games with abstention and provide for them extensions of the Banzhaf and Shapley–Shubik indices. Freixas and Zwicker [13] consider games with several levels of approval in the input and in the output and introduce weighted games for these structures. Freixas [10, 11] consider Banzhaf and Shapley extensions for these games. Extensions of the desirability relations for simple games have been studied in Tchantcho et. al. [22, 23, 19] and anonymous games have been considered in Freixas and Zwicker [14] and Zwicker [24].

In this paper we extend to ternary games the five power indices mentioned before and introduce three additional ones. Moreover, we emphasize the analogies and discrepancies between these power indices for games with abstention or with three levels of ordered approval and their versions for simple games. A unified probabilistic approach serves to regard these power indices as probabilities of certain natural events. Moreover, we provide a simple method to compute these measures.

Section 2 recalls the notions of (j,2) simple game, (j,2) weighted game and their restrictions to (3,2) games. In Section 3 we define several power indices for (3,2) games and remark some essential differences with their analogues for simple games. In section 4 we mainly consider uncertainty about voter's behaviour by means of a probability distribution. This more general approach permits, in Section 5, to interpret all power indices introduced as probabilities of relevant events. Section 6 is devoted to provide a method based on generating functions to compute these measures and to illustrate it by means of a real–world example.

2 The class of $(j, 2)$ simple games

The material on this section is essentially taken from Freixas and Zwicker [13]. Before the main notions are introduced we need some preliminary definitions. An *ordered j-partition* of the finite set N is a sequence $A = (A_1, ..., A_j)$ of mutually disjoint sets whose union is N. Any A_i is allowed to be empty, and we think of A_i as the set of those voters of N who vote approval level i for the issue at hand (where approval level 1 is the highest level of approval, 2 is the next highest, etc.), $|A_i|$ means de cardinality of A_i. Thus, an ordered j-partition is the analogue of a coalition for a standard simple game. Let j^N denote the set of all ordered j-partitions of N. For $A, B \in j^N$, we write $A \,{}^j\!\subseteq B$ to mean that either $A = B$ or A may be transformed into B by shifting 1 or more voters to higher levels of approval. This is the same as saying $A_i \uparrow \subseteq B_i\uparrow$ for each $i = 1, 2, ..., j$, where $A_i\uparrow$ denotes $A_1\cup\, A_2 \cup \ldots \cup\, A_i$ for each $i = 1, 2, .., j$; we write $A \,{}^j\!\subset B$ if $A \,{}^j\!\subseteq B$ and $A \neq B$. The ${}^j\!\subseteq$ order defined on j^N has minimum: the j-partition $\mathcal{N}$ such that $\mathcal{N}_j = N$, and maximum: the j-partition $\mathcal{M}$ such that $\mathcal{M}_1 = N$; *i.e.* for every j-partition A holds $\mathcal{N} \,{}^j\!\subseteq A \,{}^j\!\subseteq \mathcal{M}$.

Definition 2.1 *A (j,2) hypergraph $G = (N, V)$ consists of a finite set N together with a value function $V : j^N \to \{win, lose\}$; with win > lose.*

An equivalent way to uniquely define the hypergraph is by means the set of winning tripartitions $\mathcal{W}$, i.e. those tripartitions which are winning for V.

For ordered j-partitions A and B we write $A <_V B$ to mean $V(A) < V(B)$, and $A \leq_V B$ to mean $V(A) \leq V(B)$.

Definition 2.2 *A $(j, 2)$ simple game (henceforth $(j, 2)$ game) is a $(j, 2)$ hypergraph such that $V(\mathcal{N}) < V(\mathcal{M})$ and is monotonic: for all ordered j-partitions A and B, if $A \,{}^j\!\subseteq B$ then $A \leq_V B$.*

A $(j, 2)$ game can be called by equivalent denominations, as for example $(j, 2)$ voting rule. A $(j, 2)$ game is also defined by the set of winning tripartitions $\mathcal{W}$ such that $\mathcal{N} \notin \mathcal{W}$, $\mathcal{M} \in \mathcal{W}$ and with the monotonicity requirement: if $A \,{}^j\!\subseteq B$ and $A \in \mathcal{W}$ then $B \in \mathcal{W}$.

An ordinary simple game may be identified with a $(2, 2)$ game for which the value set is $\{win, lose\}$ with $win > lose$, and a simple game with abstention corresponds to a $(3, 2)$. In a $(j, 2)$ game can be extended the notions of winning and minimal winning coalitions: A is a *winning* j-partition whenever $V(A) = win$, A is a *minimal* winning j-partition whenever A is winning and B is a losing j-partition if $B \,{}^j\!\subset A$. The set of minimal winning tripartitions, denoted by $\mathcal{W}^m$, uniquely determines the (3,2) game.

Definition 2.3 *Let $G = (N, V)$ be a $(j, 2)$ game. A representation of G as a weighted $(j, 2)$ game consists of a sequence $w = (w_1, ..., w_j)$ of j weight functions, where $w_i : N \to \mathbb{R}$ for each i and the weight functions satisfy the additional weight-monotonicity requirement that for each $p \in N$, $w_1(p) \geq w_2(p) \geq \cdots \geq w_j(p)$, together with a real number quota q such that for every j-partition A and $V(A) = win$ if and only if $w(A) \geq q$, where $w(A)$ denotes*

$$\sum \Big\{ \sum \{ w_i(k) \ : \ k \in A_i \} \ : \ 1 \leq i \leq j \Big\}.$$

We say that $G = (N, V)$ is a weighted $(j, 2)$ game if it has such a representation.

As was observed in [13] in weighted voting with abstention, each 'yes' voter contributes the weight $w_{yes}(p)$ to the total weight T; each 'abstain' voter contributes $w_{abstain}(p)$ to T, and each 'no' voter contributes $w_{no}(p)$ to T, with the issue passing exactly if T meets or exceeds some preset quota r. That is, before any voting takes place each voter is pre-assigned three weights. We will also require that $w_{yes}(p) \geq w_{abstain}(p) \geq w_{no}(p)$ for each voter p, but will make no assumptions about the signs of $w_{yes}(p)$, $w_{abstain}(p)$ or $w_{no}(p)$. As occurs for simple games where two weights represent superfluous information, three weights represent superfluous information. If we renormalize by subtracting $w_{abstain}(p)$ from each of the weights $w_{yes}(p)$, $w_{abstain}(p)$ and $w_{no}(p)$ then the new triple of weights $w^Y(p) = w_{yes}(p) - w_{abstain}(p)$, 0 and $w^N(p) = w_{no}(p) - w_{abstain}(p)$ describes the same voting system, and satisfies $w^Y(p) \geq 0 \geq w^N(p)$.

Of course, it would be as easy to renormalize all 'no' weights to zero, or all 'yes' weights to zero, but the idea of abstention suggests that we renormalize at the middle level. This explains why it seems appropriate to allow weighted voting with abstention to assign positive weights to 'yes' votes, negative weights to 'no' votes, to require no other relationship between a player's 'yes' weight and their 'no' weight (except in special cases, discussed in what follows) and to count abstentions as zero.

Example 2.4 A resolution is carried in the Security Council if at least nine members support it and no permanent member is explicitely opposed, our formal description of the UNSC as a (3,2) game is as follows: let $P = \{1, 2, 3, 4, 5\}$ and $R = \{6, 7, ..., 15\}$ be respectively the set of permanent members and nonpermanent members, and

$$V(A) = V(A_1, A_2, A_3) = \begin{cases} \text{win} & \text{if } |A_1| \geq 9 \text{ and } A_3 \cap P = \emptyset \\ \text{lose} & \text{otherwise} \end{cases}$$

Here, we think of A_1 as the set of voters of N who vote 'yes,' A_2 as the set of those voters of N who abstain, and A_3 as the set of those voters of N who vote 'no.'

This voting system with abstention can be represented as

$$[\ 9;\ \underbrace{(1, -6), ..., (1, -6)}_{5}, \underbrace{(1, 0), ..., (1, 0)}_{10}\ ]$$

where 9 is the quota, $(w^Y(p), w^N(p)) = (1, -6)$ is the vector of weights for any arbitrary permanent member p and $(w^Y(r), w^N(r)) = (1, 0)$ for any nonpermanent member r.

Example 2.5 Let $\mathcal{W}$ be an arbitrary (3,2) game defined on N. The projection of $\mathcal{W}$ is the simple game $\mathcal{V}$ defined on N as:

$$S \in \mathcal{V} \quad \text{if and only if} \quad (S, \emptyset, N \setminus S) \in \mathcal{W}$$

i.e, tripartitions with abstainers are not taken into account in $\mathcal{W}$ to get its projection $\mathcal{V}$. Note that the coalition $S \subseteq N$ can also be identified with the bipartition $(S, N \setminus S)$.

We introduce here some necessary notation for tripartitions. Given (A_1, A_2, A_3) a tripartition such that $i \notin A_3$. We define:

$$A_{\downarrow i} = \begin{cases} (A_1 \setminus \{i\}, A_2 \cup \{i\}, A_3) & \text{if} \quad i \in A_1 \\ (A_1, A_2 \setminus \{i\}, A_3 \cup \{i\}) & \text{if} \quad i \in A_2 \end{cases}$$

and if $i \in A_1$

$$A_{\downarrow\downarrow i} = (A_1 \setminus \{i\}, A_2, A_3 \cup \{i\})$$

Analogous notation can be defined. Given (A_1, A_2, A_3) a tripartition such that $i \notin A_1$. We define:

$$A_{\uparrow i} = \begin{cases} (A_1 \cup \{i\}, A_2 \setminus \{i\}, A_3) & \text{if} \quad i \in A_2 \\ (A_1, A_2 \cup \{i\}, A_3 \setminus \{i\}) & \text{if} \quad i \in A_3 \end{cases}$$

and if $i \in A_3$

$$A_{\uparrow\uparrow i} = (A_1 \cup \{i\}, A_2, A_3 \setminus \{i\}).$$

Finally, we will denote $a_i = |A_i|$ for $i = 1, 2, 3$.

3 Power indices for $(3, 2)$ simple games

In this section we extend some measures widely considered for simple games to $(3, 2)$ games, while some others are new. To define all these measures it is assumed that all the relevant information is encapsulated in the (3,2) game $\mathcal{W}$ so that all of them can be directly computed from it.

Rae's (3,2) extension We define the *Rae index* for (3,2) games (extension of the Rae index [20] for simple games) as

$$Rae_i(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\}|}{3^n} + \frac{|\{A : i \in A_3,\ A \notin \mathcal{W}\}|}{3^n}$$

The Rae index is regarded as a measure of the *success* of voter i in the (3,2) game $\mathcal{W}$. We understand that an abstainer is never satisfied whether the proposal passes or not, so abstainers do not contribute to the Rae index. Thus, the Rae index as defined is upper bounded by 2/3 since abstainers do not contribute to the index and both $|\{A : i \in A_1,\ A \in \mathcal{W}\}|$ and $|\{A : i \in A_3,\ A \notin \mathcal{W}\}|$ are bounded by 3^{n-1}. Note that the Rae index for simple games is upper bounded by 1 instead of 2/3.

An index that captures the opposite idea of the Rae index is the following

$$I_i(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \notin \mathcal{W}\}|}{3^n} + \frac{|\{A : i \in A_3,\ A \in \mathcal{W}\}|}{3^n}$$

This latter index measures how unsuccessful is voter i in the (3,2) game $\mathcal{W}$. The index I was not considered for simple games since it would follow the trivial equation: $Rae_i(\mathcal{W}) + I_i(\mathcal{W}) = 1$ for every simple game $\mathcal{W}$ and voter $i \in N$. However for every (3,2) game $\mathcal{W}$ and voter $i \in N$ it holds:

$$Rae_i(\mathcal{W}) + I_i(\mathcal{W}) = 2/3 \tag{1}$$

This is obvious since

$$Rae_i(\mathcal{W}) + I_i(\mathcal{W}) = \frac{|\{A : i \in A_1\}|}{3^n} + \frac{|\{A : i \in A_3\}|}{3^n} = \frac{3^{n-1}}{3^n} + \frac{3^{n-1}}{3^n} = \frac{2}{3}$$

Banzhaf's (3,2) extension Banzhaf's 'raw' index for a voter $i \in N$ in a (3,2) game $\mathcal{W}$ is defined in [8] and extended to (j,k) games in [11] as

$$\eta_i(\mathcal{W}) = |\{A : A \in \mathcal{W},\ A_{\downarrow i} \notin \mathcal{W}\}|$$

The number $\eta_i(\mathcal{W})$ or raw$Bz_i(\mathcal{W})$ counts the number of winning tripartitions in which i is positively decisive descending one single level of approval.

By observing that $|\{A : i \in A_2,\ A \in \mathcal{W},\ A_{\downarrow i} \notin \mathcal{W}\}| = |\ \{A : i \in A_1,\ A_{\downarrow i} \in \mathcal{W},\ A_{\downarrow\downarrow i} \notin \mathcal{W}\}\ |$, we see that $\eta_i(\mathcal{W})$ can also be expressed

$$\eta_i(\mathcal{W}) = |\{A :\ A \in \mathcal{W},\ A_{\downarrow\downarrow i} \notin \mathcal{W}\}|.$$

The *normalized Banzhaf index* for (3,2) simple games is the normalization of $\eta_i(\mathcal{W})$ for the sum of the raw indices:

$$\beta_i(\mathcal{W}) = \frac{\eta_i(\mathcal{W})}{\sum_{i \in N} \eta_i(\mathcal{W})}$$

The condition $\mathcal{M} \in \mathcal{W}$ and $\mathcal{N} \notin \mathcal{W}$ in Definition 2.2 guarantees that the denominator is not null. More interesting in probabilistic meaning is:

$$Bz_i(\mathcal{W}) = \frac{\eta_i(\mathcal{W})}{\text{total number of tripartitions with } i \in A_1} = \frac{\eta_i(\mathcal{W})}{3^{n-1}}$$

The Banzhaf index, $Bz_i(\mathcal{W})$ can be regarded as a measure of decisiveness. Here we consider a 'broader' type of decisiveness: 'A voter i is decisive in wide sense if he/she can alter the outcome by changing his/her vote'. Roughly speaking, a voter is *decisive* in a tripartition if the status of the tripartition (either winning or losing) changes by solely altering his/her vote to a different level of approval, it does not matter if this change is 'short' or 'long'. Note that if the voter votes 'yes' in the tripartition can alter his/her vote in two ways (one short change to abstention and one long change to 'no'), the same occurs if the voter votes 'no' in the tripartition he/she can make a short change or a long change in his/her vote, but if the voter is an abstainer in the tripartition only two short changes for his/her are feasible. This idea is fundamental for appropriate extensions of Banzhaf and Coleman indices to games with three levels of approval.

To make this precise consider $\eta'_i(\mathcal{W})$ as the sum of $\eta^*_i(\mathcal{W})$ and $\eta^{**}_i(\mathcal{W})$ where:

$$\begin{aligned}
\eta^*_i(\mathcal{W}) &= |\ \{A : i \in A_1,\ A \in \mathcal{W},\ A_{\downarrow\downarrow i} \notin \mathcal{W}\}\ | + |\ \{A : i \in A_2,\ A \in \mathcal{W},\ A_{\downarrow i} \notin \mathcal{W}\}|, \\
\eta^{**}_i(\mathcal{W}) &= |\ \{A : i \in A_3,\ A \notin \mathcal{W},\ A_{\uparrow\uparrow i} \in \mathcal{W}\}\ | + |\ \{A : i \in A_2,\ A \notin \mathcal{W},\ A_{\uparrow i} \in \mathcal{W}\}|,
\end{aligned}$$

that is, $\eta^*_i(\mathcal{W})$ counts the number of times that i is (positively) decisive altering his/her vote to a lower level of approval (either altering one single level or two levels). Conversely, $\eta^{**}_i(\mathcal{W})$ counts the number of times that i is (negatively) decisive altering his/her vote to an upper level of approval (either altering one single level or two levels). All this suggests the following 'alternative' version of the Banzhaf index for (3,2) games:

$$Bz'_i(\mathcal{W}) = \frac{\eta'_i(\mathcal{W})}{\text{number of tripartitions}} = \frac{\eta'_i(\mathcal{W})}{3^n}.$$

Proposition 3.1 *The two measures: $Bz_i(\mathcal{W}) = Bz'_i(\mathcal{W})$ coincide for all (3,2) game $\mathcal{W}$ and $i \in N$.*

Proof: It is enough to prove that $\eta'_i(\mathcal{W}) = 3\eta_i((\mathcal{W})$. We can write $\eta^*_i(\mathcal{W})$ and $\eta^{**}_i(\mathcal{W})$ as follows:

$$\begin{array}{rcl} \eta^*(\mathcal{W}) & = & \eta_i(\mathcal{W}) + |\{A : i \in A_2, A \in \mathcal{W}, A_{\downarrow i} \notin \mathcal{W}\}| \\ \eta^{**}(\mathcal{W}) & = & \eta_i(\mathcal{W}) + |\{A : i \in A_2, A \notin \mathcal{W}, A_{\uparrow i} \in \mathcal{W}\}| \end{array}$$

Thus, we need to prove that:

$$\eta_i(\mathcal{W}) = |\{A : i \in A_2, A \in \mathcal{W}, A_{\downarrow i} \notin \mathcal{W}| + |\{A : i \in A_2, A \notin \mathcal{W}, A_{\uparrow i} \in \mathcal{W}|$$

but the latter expression on the right hand–side coincides with

$$|\{A : i \in A_1, A_{\downarrow i} \in \mathcal{W}, A_{\downarrow\downarrow i} \notin \mathcal{W}| + |\{A : i \in A_1, A_{\downarrow i} \notin \mathcal{W}\}|$$

which is equal to

$$|\{A : i \in A_1, A \in \mathcal{W}, A_{\downarrow\downarrow i} \notin \mathcal{W}\}|$$

□

Coleman's (3,2) extensions We proceed to the extension of the different Coleman indices for simple games [5, 6] to (3,2) simple games. The *power of a collectivity to act* or *decisiveness* measures the ease of decision–making by means of a (3,2) game $\mathcal{W}$, and is given by

$$\mathcal{A}(\mathcal{W}) = \frac{|\{A\,:\,A \in \mathcal{W}\}|}{\text{number of tripartitions}} = \frac{|\mathcal{W}|}{3^n}$$

To extend the other two indices we should be careful since the numerators do not coincide with $\eta_i(\mathcal{W})$, weakening the analogy with simple games. Previously to introduce them it is valuable to observe the following identities:

$$\begin{array}{l} |\{A : i \in A_1,\ A \in \mathcal{W}\ ,\ A_{\downarrow i} \notin \mathcal{W}\}| = \\ |\{A : i \in A_3,\ A_{\uparrow i} \notin \mathcal{W}\ ,\ A_{\uparrow\uparrow i} \in \mathcal{W}\}| = \\ |\{A : i \in A_2,\ A \notin \mathcal{W}\ ,\ A_{\uparrow i} \in \mathcal{W}\}| \end{array} \tag{2}$$

and

$$\begin{array}{l} |\{A : i \in A_1,\ A_{\downarrow i} \in \mathcal{W}\ , A_{\downarrow\downarrow i} \notin \mathcal{W}\}| = \\ |\{A : i \in A_3,\ A \notin \mathcal{W}\ ,\ A_{\uparrow i} \in \mathcal{W}\}| = \\ |\{A : i \in A_2,\ A \in \mathcal{W}\ ,\ A_{\downarrow i} \notin \mathcal{W}\}| \end{array} \tag{3}$$

Voter i's *Coleman index to prevent* action (Col^P_i) measures the i's capacity to convert winning tripartitions into losing tripartitions by changing his/her vote into a lower level of approval.

$$Col^P_i(\mathcal{W}) = \frac{|\{A : i \in A_1, A \in \mathcal{W}, A_{\downarrow\downarrow i} \notin \mathcal{W}\}| + |\{A : i \in A_2, A \in \mathcal{W}, A_{\downarrow i} \notin \mathcal{W}\}|}{|\mathcal{W}|} = \frac{\eta^*_i(\mathcal{W})}{|\mathcal{W}|}$$

Alternatively, $Col_i^P(\mathcal{W})$ can be expressed as a function of the number of tripartitions where i votes 'yes' and the number of winning tripartitions. As a direct consequence of applying equations (2) and (3) we obtain the equivalent expression:

$$Col_i^P(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\ ,\ A_{\downarrow i} \notin \mathcal{W}\}| + 2|\{A : i \in A_1,\ A_{\downarrow i} \in \mathcal{W}\ , A_{\downarrow\downarrow i} \notin \mathcal{W}\}|}{|\mathcal{W}|}$$

Voter i's *Coleman index to initiate* action (Col_i^I) measures the i's capacity to convert losing tripartitions into winning tripartitions by changing his/her vote into an upper level.

$$Col_i^I(\mathcal{W}) = \frac{|\{A : i \in A_3,\ A \notin \mathcal{W}\ ,\ A_{\uparrow\uparrow i} \in \mathcal{W}\}| + |\{A : i \in A_2,\ A \notin \mathcal{W}\ ,\ A_{\uparrow i} \in \mathcal{W}\}|}{|N \setminus \mathcal{W}|} =$$

$$= \frac{\eta_i^{**}(\mathcal{W})}{|N \setminus \mathcal{W}|} = \frac{\eta_i'(\mathcal{W}) - \eta_i^*(\mathcal{W})}{3^n - |\mathcal{W}|} = \frac{3\eta_i(\mathcal{W}) - \eta_i^*(\mathcal{W})}{3^n - |\mathcal{W}|}$$

or, equivalently

$$Col_i^I(\mathcal{W}) = \frac{2|\{A : i \in A_1,\ A \in \mathcal{W}\ ,\ A_{\downarrow i} \notin \mathcal{W}\}| + |\{A : i \in A_1,\ A_{\downarrow i} \in \mathcal{W}\ , A_{\downarrow\downarrow i} \notin \mathcal{W}\}|}{3^n - |\mathcal{W}|}$$

For simple games the *normalizations* of these three indices coincide, giving rise to the so-called 'Banzhaf–Coleman' index. In formula, we have the following relation for any simple game $\mathcal{W}$:

$$\frac{Bz_i(\mathcal{W})}{\sum_{j \in N} Bz_j(\mathcal{W})} = \frac{Col_i^P(\mathcal{W})}{\sum_{j \in N} Col_j^P(\mathcal{W})} = \frac{Col_i^I(\mathcal{W})}{\sum_{j \in N} Col_j^I(\mathcal{W})} \tag{4}$$

However, this relation is false for (3,2) simple games. The next example illustrates this.

Example 3.2 Consider the $(3,2)$ game $\mathcal{W}$ defined on $N = \{1,2,3\}$ by its set of minimal winning tripartitions $\mathcal{W}^m = \{(12, \emptyset, 3), (1, 23, \emptyset), (23, 1, \emptyset)\}$ (keys and commas omitted for tripartitions).

Consider $\eta_i(\mathcal{W})$ for each player $i \in N$ and let $\eta(\mathcal{W}) = (\eta_1(\mathcal{W}), \eta_2(\mathcal{W}), \eta_3(\mathcal{W}))$ and analogous notation for $\eta^*(\mathcal{W})$, $\eta^{**}(\mathcal{W})$, $Bz(\mathcal{W})$, $Col^P(\mathcal{W})$ and $Col^I(\mathcal{W})$. To calculate $Bz(\mathcal{W})$, $Col^P(\mathcal{W})$ and $Col^I(\mathcal{W})$ we previously need to compute the following numbers:

$$\begin{array}{rcl} \eta(\mathcal{W}) &=& (4,2,1) + (1,2,1) &=& (5,4,2) \\ \eta^*(\mathcal{W}) &=& (4,2,1) + 2(1,2,1) &=& (6,6,3) \\ \eta^{**}(\mathcal{W}) &=& 2(4,2,1) + (1,2,1) &=& (9,6,3) \\ |\mathcal{W}| &=& 6 \\ |N \setminus \mathcal{W}| &=& 21 \end{array}$$

Note that $\eta^{**}(\mathcal{W})$ also follows from $3\eta(\mathcal{W}) - \eta^*(\mathcal{W})$. From these figures we obtain:

$$\begin{array}{rcl} Bz(\mathcal{W}) &=& \frac{1}{9}(5,4,2) \\ Col^P(\mathcal{W}) &=& \frac{1}{6}(6,6,3) \\ Col^I(\mathcal{W}) &=& \frac{1}{21}(9,6,3) \end{array}$$

The relation for simple games given in (4) is not true for (3,2) games as is illustrated for $\mathcal{W}$. Indeed,

$$\frac{Bz(\mathcal{W})}{\sum_{j\in N} Bz_j(\mathcal{W})} = \frac{1}{11}(5,4,2)$$

$$\frac{Col^P(\mathcal{W})}{\sum_{j\in N} Col_j^P(\mathcal{W})} = \frac{1}{15}(6,6,3)$$

$$\frac{Col^I(\mathcal{W})}{\sum_{j\in N} Col_j^I(\mathcal{W})} = \frac{1}{18}(9,6,3)$$

Concerning the Rae index and $I(\mathcal{W})$ it is not difficult to check that

$$Rae(\mathcal{W}) = \frac{1}{27}(14,13,11) \qquad \text{while} \qquad I(\mathcal{W}) = \frac{1}{27}(4,5,7)$$

The lack of proportionality among these three indices for (3,2) games tells us that these are three independent concepts. This difference with simple games is very important. For instance, the proportionality for simple games implies that these three indices are *ordinally equivalent* (see [4] and [12] for ordinal equivalence of power indices for simple games), whereas for (3,2) games they do not necessarily need to be ordinally equivalent, and hence they can rank voters in different orders.

There exists a linear relationship between the Rae and Banzhaf indices for simple games which is

$$Rae_i(\mathcal{W}) = \frac{1}{2} + \frac{1}{2} Bz_i(\mathcal{W}) \tag{5}$$

but the next proposition shows that a different linear relationship holds for (3,2) games:

Proposition 3.3 *For each (3,2) game $\mathcal{W}$ and $i \in N$ it holds*

$$Rae_i(\mathcal{W}) = \frac{1}{3} + \frac{1}{3} Bz_i(\mathcal{W}). \tag{6}$$

Proof: It is enough to prove that $|\{A : i \in A_1,\ A \in \mathcal{W}\}| + |\{A : i \in A_3,\ A \notin \mathcal{W}\}| = 3^{n-1} + \eta_i(\mathcal{W})$ and dividing by 3^n we get the equality. To see this, consider any tripartition of those counted in $\eta_i(\mathcal{W})$, i.e. with i being positively crucial, and change (if necessary) his/her vote to the abstention. Let A one of these tripartitions for which i abstains; as i is crucial in it his/her vote after the swap will coincide with the output in both $A_{\uparrow i}$ and in $A_{\downarrow i}$. Hence, i will be successful in $2\eta_i(\mathcal{W})$ tripartitions. Consider now one of the remaining $3^{n-1} - \eta_i(\mathcal{W})$ tripartitions for which i abstains. In such a situation, i will be successful in only one of the two next: $A_{\uparrow i}$ or $A_{\downarrow i}$ tripartitions, since his/her vote is indifferent to the final result. In summary, i will be successful in $2 \cdot \eta_i(\mathcal{W}) + (3^{n-1} - \eta_i(\mathcal{W})) = 3^{n-1} + \eta_i(\mathcal{W})$. $\square$

Relationship (6) for (3,2) games is different from relationship (5) for simple games. We can give an interpretation of Equation (6) for (3,2) games as that Penrose [18] did for equation (5) for simple games. The power of the individual vote can be measured by the amount by which her chance of being on the winning side exceeds one third. the power, thus defined, is the same as one third the likelihood of a situation in which an individual vote can be decisive. Penrose's measure of power for (3,2) games would correspond to $(1/3)Bz_i(\mathcal{W})$.

As $0 \leq Bz_i(\mathcal{W}) \leq 1$ for all $i \in N$ and (3,2) game $\mathcal{W}$, it trivially follows

$$\frac{1}{3} \leq Rae_i(\mathcal{W}) \leq \frac{2}{3}. \tag{7}$$

From Equations (1) and (6) also follow:

$$I_i(\mathcal{W}) = \frac{1}{3} - \frac{1}{3} Bz_i(\mathcal{W})$$

and therefore

$$0 \leq I_i(\mathcal{W}) \leq \frac{1}{3} \tag{8}$$

for all $i \in N$ and (3,2) game $\mathcal{W}$. In contrast to Equations (7) and (8) for simple games the bounds are:

$$\frac{1}{2} \leq Rae_i(\mathcal{W}) \leq 1 \quad \text{and} \quad 0 \leq I_i(\mathcal{W}) \leq \frac{1}{2}.$$

K¨onig and Br¨auninger's (3,2) extension The extension of the K¨onig and Br¨auninger voter i's *inclusiveness* for simple games [15] to (3,2) games is direct if we take into account that its meaning is to measure the success proportion of a player with respect to the collective success:

$$KB_i(\mathcal{W}) = \frac{\text{number of winning tripartitions with } i \text{ voting 'yes'}}{\text{total number of winning tripartitions}}$$

or, equivalently

$$KB_i(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\}|}{|\mathcal{W}|}$$

Note that if voter $i \in N$ has *yes–veto*, i.e. $A \in \mathcal{W}$ implies $i \in A_1$ then $KB_i(\mathcal{W}) = 1$. If voter $i \in N$ is null, i.e. $A \in \mathcal{W}^m$ implies $i \in A_3$ then $KB_i(\mathcal{W}) = 1/2$. However, $KB_i(\mathcal{W}) = 1/2$ is not exclusive for null voters in (3,2) games. Thus,

$$\frac{1}{2} \leq KB_i(\mathcal{W}) \leq 1$$

for every (3,2) game and each $i \in N$. These bounds are the same for simple games, but there $KB_i(\mathcal{W}) = 1/2$ iff i is null in the simple game $\mathcal{W}$.

Following Coleman's idea one may consider (even for simple games) an alternative index $J(\mathcal{W})$ that measures the (negative) success proportion of a player with respect to the collective failure:

$$J_i(\mathcal{W}) = \frac{\text{number of losing tripartitions with } i \text{ voting 'no'}}{\text{total number of losing tripartitions}}$$

or, equivalently

$$J_i(\mathcal{W}) = \frac{|\{A : i \in A_3,\ A \notin \mathcal{W}\}|}{|N \setminus \mathcal{W}|}$$

The two indices above can be regarded as measures of success, now we can define analogous notions for unsuccess. Let

$$L_i(\mathcal{W}) = \frac{\text{number of winning tripartitions with } i \text{ voting 'no'}}{\text{total number of winning tripartitions}}$$

and

$$M_i(\mathcal{W}) = \frac{\text{number of losing tripartitions with } i \text{ voting 'yes'}}{\text{total number of losing tripartitions}}$$

Consider now Example 3.2. It is not difficult to check that:

$$\begin{aligned} KB(\mathcal{W}) &= \frac{1}{6}(5,4,3) \\ J(\mathcal{W}) &= \frac{1}{21}(9,9,8) \\ L(\mathcal{W}) &= \frac{1}{6}(1,2,3) \\ M(\mathcal{W}) &= \frac{1}{21}(5,4,3) \end{aligned}$$

4 Success, unsuccess and decisiveness in $(3,2)$ games

In this section we extend the approach followed by Laruelle and Valenciano in [17] for simple games to (3,2) games.

4.1 Ex post model

First let's consider the situation ex post, that is, once a committee voted on a given proposal, a tripartition has emerged, and this tripartition has prescribed the final outcome, i.e. passage or rejection of the proposal. A distinction can be made between voters. If the proposal is accepted (rejected), those voters who voted in favour (against) are satisfied with the result, while the others are not. Extending Barry's contribution for simple games [1], we will say that they have been successful. Thus, being *successful* means obtaining the outcome—acceptance or rejection— that one voted for. We also say that a successful voter has been *decisive* in a vote if his/her vote was crucial: i.e., if he/she had changed his/her vote the outcome would have been different. Note that the ex post notions below are binary definitions for each player.

Success ex post (3,2) extension

Definition 4.1 *After a decision is made according to a (3,2) game $\mathcal{W}$, if the resulting configuration of votes is tripartition A,* (*i*) *Voter i is said to have been* successful *if the decision coincides with voter i's vote, i.e. iff*

$$(i \in A_1, A \in \mathcal{W}) \quad or \quad (i \in A_3, A \notin \mathcal{W})$$

(*ii*) *Voter i is said to have been* unsuccessful *if the decision is contrary with voter i's vote, i.e. iff*

$$(i \in A_1, A \notin \mathcal{W}) \quad or \quad (i \in A_3, A \in \mathcal{W})$$

(*iii*) *Voter i is said to have been* neutral *if voter i abstains, i.e. iff*

$$i \in A_2$$

Note that for simple games there are not neutral voters.

Decisiveness ex post (3,2) extension

Definition 4.2 *After a decision is made according to a (3,2) game $\mathcal{W}$, if the resulting configuration of votes is tripartition A,*

(*i*) *Voter i is said to have been* successfully decisive*, if he/she was successful and his/her vote was critical to that success, i.e. iff*

$$(i \in A_1,\, A \in \mathcal{W},\, A_{\downarrow\downarrow i} \notin \mathcal{W}) \quad or \quad (i \in A_3,\, A \notin \mathcal{W},\, A_{\uparrow\uparrow i} \in \mathcal{W})$$

(*ii*) *Voter i is said to have been* neutrally decisive*, if he/she was neutral and his/her vote was critical to the outcome, i.e. iff*

$$(i \in A_2,\, A \in \mathcal{W},\, A_{\downarrow i} \notin \mathcal{W}) \quad or \quad (i \in A_2,\, A \notin \mathcal{W},\, A_{\uparrow i} \in \mathcal{W})$$

Neutrally decisiveness is a new concept with respect to the ex post model for simple games.

Luckiness ex post (3,2) extension

Definition 4.3 *After a decision is made according to a (3,2) game $\mathcal{W}$, if the resulting configuration of votes is tripartition A,*

(*i*) *Voter i is said to have been* lucky*, i.e. iff*

$$(i \in A_1,\, A \in \mathcal{W},\, A_{\downarrow\downarrow i} \in \mathcal{W}) \quad or \quad (i \in A_3,\, A \notin \mathcal{W},\, A_{\uparrow\uparrow i} \notin \mathcal{W})$$

(*ii*) *Voter i is said to have been* neutrally indecisive*, if he/she was neutral and his/her vote was not critical to the outcome, i.e. iff*

$$(i \in A_2,\, A \in \mathcal{W},\, A_{\downarrow i} \in \mathcal{W}) \quad or \quad (i \in A_2,\, A \notin \mathcal{W},\, A_{\uparrow i} \notin \mathcal{W})$$

Neutrally indecisiveness is also a new concept with respect to the ex post model for simple games.

The three previous definitions suggest the consideration of the following sets and the existent relations.

$$
\begin{aligned}
\text{SUC} &= \{i \in N : i \text{ is successful}\} \\
\text{UNS} &= \{i \in N : i \text{ is unsuccessful}\} \\
\text{NEU} &= \{i \in N : i \text{ is neutral}\} \\
\text{DEC} &= \{i \in N : i \text{ is decisive}\} \\
\text{SDE} &= \{i \in N : i \text{ is successfully decisive}\} \\
\text{NDE} &= \{i \in N : i \text{ is neutrally decisive}\} \\
\text{LUC} &= \{i \in N : i \text{ is lucky}\} \\
\text{NID} &= \{i \in N : i \text{ is neutrally indecisive}\}
\end{aligned}
$$

We have the following obvious relationships:

$$i \in SUC \;\veebar\; i \in UNS \;\veebar\; i \in NEU$$

$$
\begin{aligned}
i \in SUC &\iff i \in SDE \;\veebar\; i \in LUC \\
i \in NEU &\iff i \in NDE \;\veebar\; i \in NID \\
i \in DEC &\iff i \in SDE \;\veebar\; i \in NDE
\end{aligned}
$$

where '$\veebar$' stands for an exclusive 'or'.

4.2 Uncertainty about voters' behaviour

We assume that we know —or at least have an estimate of— the probability of occurrence of any vote configuration that may arise. In other words, the elementary events are the vote configurations in 3^N. As their number is finite (3^n), we can represent any such probability by a map $p_N : 3^N \to \mathbb{R}$, that associates with each vote configuration A its probability of occurrence $p_N(A)$, i.e. $p_N(A)$ gives the probability that voters in A_1 will vote 'yes', those in A_2 will abstain, and those in A_3 will vote 'no'. To keep notation as simple as below, when N is clear from the context we will write p. Of course, $0 \leq p(A) \leq 1$ for any $A \in 3^N$, and $\sum\limits_{A \in 3^N} p(A) = 1$.

The probability of i voting 'yes' is denoted by $\gamma_i^{yes}(p)$ that is

$$\gamma_i^{yes}(p) = Prob\,(i \text{ votes 'yes'}) = \sum_{A: i \in A_1} p(A)$$

Analogously,

$$\gamma_i^{abs}(p) = Prob\,(i \text{ abstains}) = \sum_{A: i \in A_2} p(A),$$

$$\gamma_i^{no}(p) = Prob\,(i \text{ votes 'no'}) = \sum_{A: i \in A_3} p(A),$$

Let $\mathbf{P}_N$ denote the set of all distributions of probability over 3^N. This set can be interpreted as the set of all conceivable voting behaviours or preference profiles of n voters within the present probabilistic setting.

The following special distributions of probability will be considered. A distribution is *anonymous* if the probability of a vote configuration depends only on the number of voters in each level of approval, that is $p(A) = p(B)$ whenever $|A_1| = |B_1|$ and $|A_2| = |B_2|$. A distribution is *independent* if each voter i independently votes 'yes' with a probability t_i^{yes}, abstention with a probability t_i^{abs} and 'no' with a probability $1 - t_i^{yes} - t_i^{abs}$. The probability of the configuration A is then given by

$$p(A) = \prod_{i \in A_1} t_i^{yes} \cdot \prod_{i \in A_2} t_i^{abs} \cdot \prod_{i \in A_3} (1 - t_i^{yes} - t_i^{abs})$$

The next result illustrates that these two conditions are independent.

Proposition 4.4 *Independent and anonymous probability distributions are not related conditions.*

Proof: It is enough to prove the existence of probability distributions of one type but not of the other.

For all tripartition $A = (A_1, A_2, A_3)$ consider the anonymous probability distribution:

$$p(A) = \begin{cases} 1/3 & \text{if} \quad |A_1| = n \text{ or } |A_2| = n \text{ or } |A_3| = n \\ 0 & \quad \text{otherwise} \end{cases}$$

Assume, moreover that this probability distribution was independent. Then:
From $p(N, \emptyset, \emptyset) = 1/3$ it would be $t_i^{yes} \neq 0$ for all $i \in N$.
From $p(\emptyset, N, \emptyset) = 1/3$ it would be $t_i^{abs} \neq 0$ for all $i \in N$.
From $p(\emptyset, \emptyset, N) = 1/3$ it would be $t_i^{no} \neq 0$ for all $i \in N$.

However, $p(A) = 0$ for any other tripartition A, but

$$p(A) = \prod_{j \in A_1} t_j^{yes} \prod_{j \in A_2} t_j^{abs} \prod_{j \in A_3} t_j^{no} \neq 0$$

which is a contradiction.

On the other hand, let $n \geq 2$ and consider the following independent distribution of probabilities:
$(t_1^{yes}, t_1^{abs}, 1 - t_1^{yes} - t_1^{abs}) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(t_j^{yes}, t_j^{abs}, 1 - t_j^{yes} - t_j^{abs}\right) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ if $j \neq 1$.
Let $A = (1, N \setminus \{1,2\}, 2)$ and $B = (2, N \setminus \{1,2\}, 1)$. Then $p(1, N \setminus \{1,2\}, 2) \neq 0$ and $p(2, N \setminus \{1,2\}, 1) = 0$ but $|A_1| = |B_1|$ and $|A_2| = |B_2|$. Hence, the distribution of probabilities p is not anonymous. □

Proposition 4.5 *If a distribution is both anonymous and independent then each voter votes 'yes' with a probability* t^{yes}*, abstains with probability* t^{abs} *and votes no with probability* $1 - t^{yes} - t^{abs}$*, and the probability of the configuration* A *is then given by*

$$p(A) = (t^{yes})^{a_1}(t^{abs})^{a_2}(1 - t^{yes} - t^{abs})^{n-a_1-a_2}$$

Proof: Let $n \geq 2$ Consider the following three pairs of tripartitions for any pair of voters $i, j \in N$:

$$(i, j, N \setminus \{i, j\}), (j, i, N \setminus \{i, j\}),$$
$$(i, N \setminus \{i, j\}, j), (j, N \setminus \{i, j\}, i),$$
$$(N \setminus \{i, j\}, i, j), (N \setminus \{i, j\}, j, i).$$

From anonymity and independence these three pairs lead to the respective equations:

$$\frac{t_i^{yes}}{t_j^{yes}} = \frac{t_i^{abs}}{t_j^{abs}}, \quad \frac{t_i^{yes}}{t_j^{yes}} = \frac{t_i^{no}}{t_j^{no}}, \quad \frac{t_i^{abs}}{t_j^{abs}} = \frac{t_i^{no}}{t_j^{no}}$$

Let

$$\frac{t_i^{yes}}{t_j^{yes}} = \frac{t_i^{abs}}{t_j^{abs}} = \frac{t_i^{no}}{t_j^{no}} = M$$

If $M \neq 1$, then either $t_i^{yes} + t_i^{abs} + t_i^{no} \neq 1$ or $t_j^{yes} + t_j^{abs} + t_j^{no} \neq 1$ which would be a contradiction. Hence, $M = 1$ and $t_i^{yes} = t_j^{yes}$, $t_i^{abs} = t_j^{abs}$ and $t_i^{no} = t_j^{no}$ for each pair of voters i and j. □

An anonymous and independent distribution is, in fact, a probability distribution, i.e.

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n-a_1} \binom{n}{a_1}\binom{n-a_1}{a_2}(t^{yes})^{a_1} \cdot (t^{abs})^{a_2} \cdot (1 - t^{yes} - t^{abs})^{n-a_1-a_2} = 1$$

because the expression of the left hand–side can be written as

$$[t^{yes} + t^{abs} + (1 - t^{yes} - t^{abs})]^n$$

Three special cases of anonymous and independent distributions that accumulate some symmetry deserve to be mentioned.

1. When all vote configuration have the same probability, denoted by

   $$p^*(A) = \frac{1}{3^n}$$

   for all tripartition A. This is equivalent to assuming that each voter, independently of the others, votes 'yes' with probability 1/3, abstains with probability 1/3, and votes 'no' with probability 1/3. Thus this probability accumulates all symmetries: it is anonymous and independent, with equal inclination towards 'yes', 'abstention' and 'no'.

2. Another significant vote configuration is when all voters are active, without abstainers, and with equal inclination towards 'yes' and 'no'. This is equivalent to assuming that each voter, independently of the others, votes 'yes' with probability 1/2, abstains with probability 0 and votes 'no' with probability 1/2.

$$p^{**}(A) = \begin{cases} \frac{1}{2^n}, & \text{if} \quad a_2 = 0; \\ 0, & \text{if} \quad a_2 > 0. \end{cases}$$

   for all configurations $A \in 3^N$ and $|A_2| = a_2$.

3. As Felsenthal and Machover [8] suggest, a more cautious approach would be to take the probability $1 - 2p$ of abstention as an undetermined parameter in the general theory, leaving $2p$ to be shared equally between 'no' and 'yes'. In any application of the theory, the value $1 - 2p$ might be determined on the basis of specific arguments, including perhaps empirical data. This would lead us to the family of vote configurations with equal inclination towards 'yes', and 'no', which is an extension of vote configuration p^*. It presupposes the knowledge of the independent probability that a player will actively participate in the election. Assume that this probability to actively vote is $2p$ with $p < 1/2$ and it is equally distributed between 'yes' and 'no'. This is equivalent to assuming that each voter, independently of the others, votes 'yes' with probability p, abstains with probability $1 - 2p$, and votes 'no' with probability p.

$$p^{***}(A) = p^{n-a_2}(1-2p)^{a_2} \qquad \text{if} \quad p < 1/2$$

   for all configurations $A \in 3^N$ and $|A_2| = a_2$. Braham and Steffen [2] argue that abstention is a totally different action and voters first decide whether they want to form an opinion on the issue and if the answer is yes, then they vote. This would suggest a fourth probability distribution with the actively vote $2p$ distributed between 'yes' and 'no' in a non–necessarily symmetric way.

4.3 Ex ante model

Assume that a probability distribution over vote configurations p enters the picture as a second input besides $\mathcal{W}$ that governs decisions in a (3,2) game. In a voting situation thus described by pair $(\mathcal{W}, p)$ the ease of passing proposals or probability of acceptance is given by

$$\alpha(\mathcal{W}, p) = Prob\,(\text{acceptance}) = \sum_{A:A \in \mathcal{W}} p(A)$$

Furthermore, success, decisiveness and luckiness can also be defined ex ante. It suffices to replace the sure configuration A by the random vote configuration specified by p in ex post Definitions 4.1, 4.2, 4.3. This yields the following extension of these concepts.

Success ex ante (3,2) extension

Definition 4.6 *Let* $(\mathcal{W}, p)$ *be an* N *voting situation, where* $\mathcal{W}$ *is the (3,2) game and* $p \in \mathbf{P}_N$ *is the probability distribution over vote configurations, and let* $i \in N$:

(*i*) *Voter i's (ex ante)* success *is the probability that i is successful:*

$$\Omega_i^{SUC}(\mathcal{W}, p) = Prob\,(i \text{ is successful}) = \sum_{\substack{A: i\in A_1\\ A\in\mathcal{W}}} p(A) + \sum_{\substack{A: i\in A_3\\ A\notin\mathcal{W}}} p(A)$$

(*ii*) *Voter i's (ex ante)* failure *is the probability that i is unsuccessful*

$$\Omega_i^{UNS}(\mathcal{W}, p) = Prob\,(i \text{ is unsuccessful}) = \sum_{\substack{A: i\in A_1\\ A\notin\mathcal{W}}} p(A) + \sum_{\substack{A: i\in A_3\\ A\in\mathcal{W}}} p(A)$$

(*iii*) *Voter i's (ex ante)* neutrality *is the probability that i is neutral*

$$\Omega_i^{NEU}(\mathcal{W}, p) = Prob\,(i \text{ is neutral}) = \sum_{A: i\in A_2} p(A)$$

Decisiveness ex ante (3,2) extension

Definition 4.7 *Let* $(\mathcal{W}, p)$ *be an* N *voting situation, where* $\mathcal{W}$ *is the (3,2) simple game and* $p \in \mathbf{P}_N$ *is the probability distribution over vote configurations, and let* $i \in N$:

(*i*) *Voter i's (ex ante)* success decisiveness *is the probability that i is successfully decisive:*

$$\Phi_i^{SDE}(\mathcal{W}, p) = Prob\,(i \text{ is successfully decisive}) = \sum_{\substack{A\,:\, i\in A_1\\ A\in\mathcal{W}\\ A_{\downarrow\downarrow i}\notin\mathcal{W}}} p(A) + \sum_{\substack{A\,:\, i\in A_3\\ A\notin\mathcal{W}\\ A_{\uparrow\uparrow i}\in\mathcal{W}}} p(A) \tag{9}$$

$$= \left(\overbrace{\sum_{\substack{A: i\in A_1\\ A\in\mathcal{W}\\ A_{\downarrow i}\notin\mathcal{W}}} p(A)}^{a} + \overbrace{\sum_{\substack{A: i\in A_1\\ A_{\downarrow i}\in\mathcal{W}\\ A_{\downarrow\downarrow i}\notin\mathcal{W}}} p(A)}^{b} \right) + \left(\overbrace{\sum_{\substack{A: i\in A_3\\ A\notin\mathcal{W}\\ A_{\uparrow i}\in\mathcal{W}}} p(A)}^{c} + \overbrace{\sum_{\substack{A: i\in A_3\\ A_{\uparrow i}\notin\mathcal{W}\\ A_{\uparrow\uparrow i}\in\mathcal{W}}} p(A)}^{d} \right)$$

(*ii*) *Voter i's (ex ante)* neutral decisiveness, *is the probability that i is neutrally decisive*

$$\Phi_i^{NDE}(\mathcal{W}, p) = Prob(i \text{ is neutrally decisive}) = \overbrace{\sum_{\substack{A\,:\, i\in A_2\\ A\in\mathcal{W}\\ A_{\downarrow i}\notin\mathcal{W}}} p(A)}^{e} + \overbrace{\sum_{\substack{A\,:\, i\in A_2\\ A\notin\mathcal{W}\\ A_{\uparrow i}\in\mathcal{W}}} p(A)}^{f} \tag{10}$$

(*iii*) *Voter i's (ex ante)* decisiveness, *is the probability that i is successfully decisive and neutrally decisive, i.e.*

$$\Phi_i^{DEC}(\mathcal{W}, p) = \Phi_i^{SDE}(\mathcal{W}, p) + \Phi_i^{NDE}(\mathcal{W}, p) \tag{11}$$

Obviously $\Phi_i^{SDE}(\mathcal{W},p) \leq \Phi_i^{DEC}(\mathcal{W},p)$.

Note that the parameters a, b, c, d, e and f that respectively appear in (9) and (10) are defined for each voter and are functions of $\mathcal{W}$ and p, however in some places we will simply write a instead of the more precise expression $a_i(\mathcal{W},p)$ and the same holds for the other five parameters considered: b, c, d, e and f.

Note that strictly speaking i's decisiveness depends only on the other voter's behaviour, not on his/her own. To see this, voter i's decisiveness can be rewritten as

$$\Phi_i^{DEC}(\mathcal{W},p) = \sum_{\substack{A:i\in A_1\\ A\in\mathcal{W}\\ A_{\downarrow i}\notin\mathcal{W}}} [p(A)+p(A_{\downarrow i})+p(A_{\downarrow\downarrow i})] + \sum_{\substack{A:i\in A_1\\ A_{\downarrow i}\in\mathcal{W}\\ A_{\downarrow\downarrow i}\notin\mathcal{W}}} [p(A)+p(A_{\downarrow i})+p(A_{\downarrow\downarrow i})]$$

Such decomposition is held after reordering the six addends of Φ_i^{DEC}: first (a), (f) and (d), second (b), (c) and (e). Observe that for each A, $[p(A)+p(A_{\downarrow i})+p(A_{\downarrow\downarrow i})]$ is the probability of all voters in $A_1 \setminus \{i\}$ voting 'yes', those in A_2 abstain and those in $N \setminus (A_1 \cup A_2)$ voting 'no'. In this case, whatever voter i's vote, he/she is decisive, while the measures of success depend on all voters' behaviour.

Luckiness ex ante (3,2) extension

Definition 4.8 *Let $(\mathcal{W},p)$ be an N voting situation, where $\mathcal{W}$ is the (3,2) game and $p \in \mathbf{P}_N$ is the probability distribution over vote configurations, and let $i \in N$:*

(*i*) *Voter i's (ex ante)* luckiness *is the probability that i is successfully lucky:*

$$\Lambda_i^{LUC}(\mathcal{W},p) = Prob\,(i \text{ is lucky}) = \sum_{\substack{A\,:\,i\,\in\,A_1\\ A_{\downarrow\downarrow i}\,\in\,\mathcal{W}}} p(A) + \sum_{\substack{A\,:\,i\,\in\,A_3\\ A_{\uparrow\uparrow i}\,\notin\,\mathcal{W}}} p(A)$$

(*ii*) *Voter i's (ex ante)* neutrally indecisiveness *or the probability that i is 'neutrally indecisive'*

$$\Lambda_i^{NIND}(\mathcal{W},p) = \sum_{\substack{A\,:\,i\,\in\,A_2\\ A_{\downarrow i}\,\in\,\mathcal{W}}} p(A) + \sum_{\substack{A\,:\,i\,\in\,A_2\\ A_{\uparrow i}\,\notin\,\mathcal{W}}} p(A)$$

Barry's equation and some other equations From the three previous definitions it trivially follows the next two equations. The first equation concerns the ex ante voter i's satisfaction, while the second equation concerns decisiveness.

$$\Omega_i^{SUC}(\mathcal{W},p) + \Omega_i^{UNS}(\mathcal{W},p) + \Omega_i^{NEU}(\mathcal{W},p) = 1$$

$$\Phi_i^{DEC}(\mathcal{W},p) = \Phi_i^{SDE}(\mathcal{W},p) + \Phi_i^{NDE}(\mathcal{W},p)$$

The Barry's equation for simple games for all probability distribution p on 2^N:

$$\Omega_i(\mathcal{W},p) = \Phi_i(\mathcal{W},p) + \Lambda_i(\mathcal{W},p)$$

is still true for (3,2) simple games but it can be decomposed into two separate identities. The first identity is the direct extension of Barry's (success) equation for simple games, while the second is a new version for neutrality.

$$\Omega_i^{SUC}(\mathcal{W},p) = \Phi_i^{SDE}(\mathcal{W},p) + \Lambda_i^{LUC}(\mathcal{W},p)$$

$$\Omega_i^{NEU}(\mathcal{W},p) = \Phi_i^{NDE}(\mathcal{W},p) + \Lambda_i^{NIND}(\mathcal{W},p)$$

4.4 Conditional probabilities

A little of notation is necessary. We will superindex the measures. It is interesting to separate the notions of success and decisiveness as a function of the vote of the player. To this end, we define

$$\Omega_i^{S+}(\mathcal{W},p) = Prob\ (i \text{ is successful } \&\ i \text{ votes } yes) = \sum_{\substack{A:i\in A_1\\ A\in\mathcal{W}}} p(A)$$

$$\Omega_i^{S-}(\mathcal{W},p) = Prob\ (i \text{ is successful } \&\ i \text{ votes } no) = \sum_{\substack{A:i\in A_3\\ A\notin\mathcal{W}}} p(A)$$

so that $\Omega_i^{S}(\mathcal{W},p) = \Omega_i^{S+}(\mathcal{W},p) + \Omega_i^{S-}(\mathcal{W},p)$, as it occurs for simple games, ($\Omega_i^{S0}(\mathcal{W},p) = Prob\,(i \text{ is successful } \&\ i \text{ abstains}) = 0$ since it is not feasible to have success when abstaining). Analogously we would reach to $\Omega_i^{U+}(\mathcal{W},p)$ and $\Omega_i^{U-}(\mathcal{W},p)$.

For decisiveness measures we have:

$$\Phi_i^{+}(\mathcal{W},p) = Prob\ (i \text{ is decisive } \&\ i \text{ votes } yes) = \sum_{\substack{A:i\in A_1\\ A\in\mathcal{W}\\ A_{\downarrow\downarrow i}\notin\mathcal{W}}} p(A) = (a) + (b)$$

$$\Phi_i^{-}(\mathcal{W},p) = Prob\ (i \text{ is decisive } \&\ i \text{ votes } no) = \sum_{\substack{A:i\in A_3\\ A\notin\mathcal{W}\\ A_{\uparrow\uparrow i}\in\mathcal{W}}} p(A) = (c) + (d)$$

$$\Phi_i^{0}(\mathcal{W},p) = Prob\ (i \text{ is decisive } \&\ i \text{ abstains}) = \Phi_i^{NDE}(\mathcal{W},p) = (e) + (f)$$

so that, $\Phi_i^{DEC}(\mathcal{W},p) = \Phi_i^{+}(\mathcal{W},p) + \Phi_i^{0}(\mathcal{W},p) + \Phi_i^{-}(\mathcal{W},p)$.

The probabilistic measures that will lead us to the extension of the power indices defined in Section 2 are the conditional probabilities. Let us consider several examples:

$$\Omega_i^{Si-}(\mathcal{W},p) = Prob\ (i \text{ is successful} \mid i \text{ votes } no) = \frac{1}{\gamma_i^{-}(p)} \sum_{\substack{A:i\in A_3\\ A\notin\mathcal{W}}} p(A)$$

$$\Phi_i^{i+}(\mathcal{W},p) = Prob\ (i \text{ is decisive} \mid i \text{ votes } yes) = \frac{1}{\gamma_i^+(p)} \sum_{\substack{A: i \in A_1 \\ A \in \mathcal{W} \\ A_{\downarrow\downarrow i} \notin \mathcal{W}}} p(A) = \frac{1}{\gamma_i^+(p)} \; [(a)+(b)]$$

$$\Phi_i^{i0}(\mathcal{W},p) = Prob\ (i \text{ is decisive} \mid i \text{ abstains}) = \frac{1}{1-\gamma_i^+(p)-\gamma_i^-(p)} \; [(e)+(f)]$$

$$\Omega_i^{SAcc}(\mathcal{W},p) = Prob\ (i \text{ is successful} \mid \text{acceptance}) = \frac{1}{\alpha(\mathcal{W},p)} \sum_{\substack{A: i \in A_1 \\ A \in \mathcal{W}}} p(A)$$

$$\Omega_i^{URej}(\mathcal{W},p) = Prob\ (i \text{ is unsuccessful} \mid \text{rejection}) = \frac{1}{1-\alpha(\mathcal{W},p)} \sum_{\substack{A: i \in A_1 \\ A \notin \mathcal{W}}} p(A)$$

$$\Phi_i^{Acc}(\mathcal{W},p) = Prob\ (i \text{ is decisive} \mid \text{acceptance}) = \frac{1}{\alpha(\mathcal{W},p)} \; [(a)+(b)+(e)]$$

$$\Phi_i^{Rej}(\mathcal{W},p) = Prob\ (i \text{ is decisive} \mid \text{rejection}) = \frac{1}{1-\alpha(\mathcal{W},p)} \; [(c)+(d)+(f)]$$

Next table summarizes all the considered probabilities:

Condition:	-	i votes *yes*	i abstains	i votes *no*	acceptance	rejection
Success	Ω_i^{SUC}	Ω_i^{Si+}	$-$	Ω_i^{Si-}	Ω_i^{SAcc}	Ω_i^{SRej}
Unsuccess	Ω_i^{UNS}	Ω_i^{Ui+}	$-$	Ω_i^{Ui-}	Ω_i^{UAcc}	Ω_i^{URej}
Decisiveness	Φ_i^{DEC}	Φ_i^{i+}	Φ_i^{i0}	Φ_i^{i-}	Φ_i^{Acc}	Φ_i^{Rej}

Table 1: Sixteen different unconditional and conditional probabilities of success, unsuccess and decisiveness.

5 When all vote configurations are equally probable a priori

Consider now the special case of anonymous and independent distribution where $p^*(A) = \frac{1}{3^n}$ for all $A \in 3^N$. Now we are going to establish relationships between the power indices considered in Section 3 and the probabilities of some events.

Success and Rae (3,2) measures

$$Rae_i(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\}|}{3^n} + \frac{|\{A : i \in A_3,\ A \notin \mathcal{W}\}|}{3^n} = \Omega_i^{SUC}(\mathcal{W}, p^*).$$

Decisiveness, Banzhaf and Coleman (3,2) measures To link the Banzhaf measure for (3,2) games with decisiveness in the context $(\mathcal{W}, p^*)$ we need to relate $\eta_i(\mathcal{W})$ with the considered decisiveness measures:

$$Bz_i(\mathcal{W}) = \frac{1}{1/3} \cdot \frac{\eta_i(\mathcal{W})}{3^n} = \frac{1}{\gamma_i^+(p^*)} \sum_{\substack{A:i\in A_1 \\ A\in\mathcal{W} \\ A_{\downarrow\downarrow i}\notin\mathcal{W}}} p^*(A) = \Phi_i^{i+}(\mathcal{W}, p^*)$$

it holds: $\Phi_i^{i+}(\mathcal{W}, p^*) = \Phi_i^{i-}(\mathcal{W}, p^*) = \Phi_i^{i0}(\mathcal{W}, p^*) = \Phi_i^{DEC}(\mathcal{W}, p^*)$; therefore, $Bz_i(\mathcal{W})$ is equivalent to four probabilistic decisiveness measures. This is because $\gamma_i^+(p^*) = \gamma_i^-(p^*) = \gamma_i^0(p^*) = 1/3$, and that $(a) + (b) = (c) + (d) = (e) + (f)$ when $p = p^*$, since the number of tripartitions that take part in each pair of addends is the same. Those addends that appear in (a), (d) and (f) are identical, in which we pass from ones to others by switching voter i by its different voting options (the same for (b), (c) and (e)). As immediate corollaries, we have that $\Phi_i^{SDE}(\mathcal{W}, p^*) = 2 \cdot \Phi_i^{NDE}(\mathcal{W}, p^*)$ and

$$Bz_i(\mathcal{W}) = 3[a(\mathcal{W}, p^*) + b(\mathcal{W}, p^*)]$$

As Coleman index to prevent action (Col_i^P) is equal to $\frac{\eta_i^*(\mathcal{W})}{|\mathcal{W}|}$, it is sufficient to relate $\eta_i^*(\mathcal{W})$ with probability measures to get a probabilistic interpretations for Col_i^P. Analogously, as Coleman index to initiate action (Col_i^I) is equal to $\frac{\eta_i^*(\mathcal{W})}{|\mathcal{W}|}$, it is sufficient to relate $\eta_i^{**}(\mathcal{W})$ with probability measures to get a probabilistic interpretations for Col_i^I.

As $|\mathcal{W}| = \alpha(\mathcal{W}, p^*)$, $\eta_i^*(\mathcal{W}) = [(a) + (b) + (e)]$ and $\eta_i^{**}(\mathcal{W}) = [(c) + (d) + (f)]$ it follows:

$$Col_i^P(\mathcal{W}) = \frac{1}{\alpha(\mathcal{W}, p^*)}\ [(a) + (b) + (e)] = \Phi_i^{Acc}(\mathcal{W}, p^*)$$

$$Col_i^I(\mathcal{W}) = \frac{1}{1 - \alpha(\mathcal{W}, p^*)}\ [(c) + (d) + (f)] = \Phi_i^{Rej}(\mathcal{W}, p^*)$$

Alternative expressions of Coleman's measures are:

$$Col_i^P(\mathcal{W}) = \frac{a(\mathcal{W}, p^*) + 2b(\mathcal{W}, p^*)}{\mathcal{A}(\mathcal{W})}$$

$$Col_i^I(\mathcal{W}) = \frac{2a(\mathcal{W}, p^*) + b(\mathcal{W}, p^*)}{1 - \mathcal{A}(\mathcal{W})}$$

König and Bräuninger and other probabilistic (3,2) measures The nature of the *inclusiveness index* is to measure the rate of success of a player with respect to the collective success, that is:

$$KB_i(\mathcal{W}) = \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\}|}{|\mathcal{W}|}$$

By using the same argument that in the Coleman's measures we obtain:

$$KB_i(\mathcal{W}) = \frac{1}{\alpha(\mathcal{W}, p^*)} \frac{|\{A : i \in A_1,\ A \in \mathcal{W}\}|}{3^n} = \Omega_i^{SAcc}(\mathcal{W}, p^*)$$

For the other three indices J, L and M we have analogous relationships.

Next table summarizes the relations studied in this Section for $p = p^*$. Note that the complete family of (3,2) measures introduced in Section 3 appear in the table: Rae, Bz, Col^P, Col^I, KB, J, L, M. Thus, all these eight measures have interpretation as probabilities of some events in the context $(\mathcal{W}, p^*)$.

Condition:	none	i votes *yes*	i abstains	i votes *no*	acceptance	rejection
Success	$Rae_i(\mathcal{W})$	Ω_i^{Si+}	–	Ω_i^{Si-}	$KB_i(\mathcal{W})$	$J_i(\mathcal{W})$
Unsuccess	$I_i(\mathcal{W})$	Ω_i^{Ui+}	–	Ω_i^{Ui-}	$L_i(\mathcal{W})$	$M_i(\mathcal{W})$
Decisiveness	$Bz_i(\mathcal{W})$	$Bz_i(\mathcal{W})$	$Bz_i(\mathcal{W})$	$Bz_i(\mathcal{W})$	$Col_i^P(\mathcal{W})$	$Col_i^I(\mathcal{W})$

Table 2: Relationships between power indices for (3,2) games and the current probabilistic model.

For simple games Laruelle and Valenciano [16] prove that the relation

$$\Omega_i(\mathcal{W}, p) = \frac{1}{2} + \frac{1}{2}\Phi_i(\mathcal{W}, p)$$

holds for all simple game $\mathcal{W}$ and probability distribution p on 2^N *if and only if* $p = p^*$, i.e. under the assumption that *all vote configurations are equally probable.* Thus, for each $p \neq p^*$ there is a simple game $\mathcal{W}$ such that

$$\Omega_i(\mathcal{W}, p) \neq \frac{1}{2} + \frac{1}{2}\Phi_i(\mathcal{W}, p)$$

Hence, for simple games success and decisiveness are not only conceptually different but also analytically independent as there is no general way to derive one concept from the other if $p \neq p^*$.

But, what happen for (3,2) games? A natural but slightly different extension of Laruelle and Valenciano's result for simple games holds for (3,2) games. Indeed, part (*i*) of Proposition 5.1 follows the same guidelines of their proof for simple games, part (*ii*) is derived by applying their result for simple games, while part (iii) follows from part (i) and by considering $p \neq p^*$ and three (or more voting rules) with the same number of voters giving rise to an incompatible system with unknowns k_1 and k_2.

Proposition 5.1 *For all $i \in N$ and real numbers k_1 and k_2 it yields:*

i) $\Omega_i^{SUC}(\mathcal{W}, p) = \dfrac{1}{3} + \dfrac{1}{3}\Phi_i^{DEC}(\mathcal{W}, p)$ *for all (3,2) game* $\mathcal{W}$ if and only if $p = p^*$.

ii) $\Omega_i(\mathcal{V}, p) = \dfrac{1}{2} + \dfrac{1}{2}\Phi_i(\mathcal{V}, p)$ *for all projection* $\mathcal{V}$ *of an arbitrary (3,2) game* $\mathcal{W}$ if and only if $p = (1/2, 1/2)$.

iii) $\Omega_i^{SUC}(\mathcal{W}, p) = k_1 + k_2\Phi_i^{DEC}(\mathcal{W}, p)$ *for all (3,2) game* $\mathcal{W}$ if and only if $p = p^*$.

Example 5.2 Consider again Example 3.2, i.e. the $(3, 2)$ game $\mathcal{W}$ defined on $N = \{1, 2, 3\}$ by its set of minimal winning tripartitions $\mathcal{W}^m = \{(12, \emptyset, 3), (1, 23, \emptyset), (23, 1, \emptyset)\}$ (keys and commas omitted for tripartitions).

Consider decisiveness for each player, i.e.

$$\Phi^{DEC}(\mathcal{W}, p) = (\Phi_1^{DEC}(\mathcal{W}, p), \Phi_2^{DEC}(\mathcal{W}, p), \Phi_3^{DEC}(\mathcal{W}, p))$$

As we have seen in (9), (10) and (11): $\Phi_i^{DEC}(\mathcal{W}, p) = a_i + b_i + c_i + d_i + e_i + f_i$. Let $a = (a_1, a_2, a_3)$ and analogous notation for b, c, d, e and f.

For voting configuration p^* then one may easily check that: $a = d = f$ and $b = c = e$ independently of $\mathcal{W}$. Moreover, for the particular $\mathcal{W}$ of this example we have:

$$a = d = f = \tfrac{1}{3^3}(4, 2, 1) = \left(\tfrac{4}{27}, \tfrac{2}{27}, \tfrac{1}{27}\right)$$

$$b = c = e = \tfrac{1}{3^3}(1, 2, 1) = \left(\tfrac{1}{27}, \tfrac{2}{27}, \tfrac{1}{27}\right)$$

so that:

$$\Phi^{DEC}(\mathcal{W}, p^*) = (\Phi_1^{DEC}(\mathcal{W}, p^*), \Phi_2^{DEC}(\mathcal{W}, p^*), \Phi_3^{DEC}(\mathcal{W}, p^*)) = \left(\frac{15}{27}, \frac{12}{27}, \frac{6}{27}\right)$$

while for success we obtain the expected result according to Proposition 5.1-(i)

$$\Omega^{SUC}(\mathcal{W}, p^*) = \left(\frac{14}{27}, \frac{13}{27}, \frac{11}{27}\right).$$

Consider now p^{**} then one may easily check that:

$$\Phi^{DEC}(\mathcal{W}, p^{**}) = \left(\frac{4}{8}, \frac{4}{8}, \frac{0}{8}\right) \qquad \text{and} \qquad \Omega^{SUC}(\mathcal{W}, p^{**}) = \left(\frac{6}{8}, \frac{6}{8}, \frac{4}{8}\right)$$

so that one might suspect that the relationship:

$$\Omega_i(\mathcal{W}, p^{**}) = \frac{1}{2} + \frac{1}{2}\Phi_i(\mathcal{W}, p^{**}) \tag{12}$$

is true, but this is not the case as the following example illustrates. Let
$\mathcal{W}^m = \{(1, 2, 3), (13, \emptyset, 2), (2, 13, \emptyset)\}$ then:

$$\Phi^{DEC}(\mathcal{W}, p^{**}) = \left(\frac{6}{8}, \frac{2}{8}, \frac{1}{8}\right) \qquad \text{and} \qquad \Omega^{SUC}(\mathcal{W}, p^{**}) = \left(\frac{7}{8}, \frac{5}{8}, \frac{5}{8}\right)$$

Thus, Equation (12) is not true for this (3,2) game.

6 Computation of power indices for (3,2) games

In this section, we show how to compute for weighted (3,2) games, in an efficient way, the indices we have been working with along the previous sections. To this end, we extend to weighted (3,2) games the idea by Brams and Affuso [3], which consists of using generating functions to obtain $\eta(\mathcal{W})$ as the sum of some of its coefficients. We want to note that the procedure below can be easily extended to the class of (j,k) games, considered in Freixas and Zwicker [13], and similar indices to those considered in Section 3 of this paper.

Let's consider a weighted (3,2) simple game as described in Definition 2.3 with quota q and weights $w^Y(p)$ and $w^N(p)$ for each player $p \in N$. We build the *global generating function* $f(x)$ and the *individual generating functions* $f_i(x)$ in the following way:

$$f(x) = \prod_{p \in N} (x^{w^N(p)} + 1 + x^{w^Y(p)}) = \sum_{j=\bar{W}^N}^{\bar{W}^Y} \alpha_j x^j$$

$$f_i(x) = \prod_{p \neq i} (x^{w^N(p)} + 1 + x^{w^Y(p)}) = \sum_{j=\bar{W}^N - w^N(i)}^{\bar{W}^Y - w^Y(i)} \alpha_j x^j$$

where $\bar{W}^Y = \sum_{p \in N} w^Y(p)$, $\bar{W}^N = \sum_{p \in N} w^N(p)$.

Once the individual generating function is computed, we have to sum their *relevant* coefficients to get the value of $\eta_i(\mathcal{W})$. Note that each α_j indicates the number of tripartitions with total weight j that can be set up using any of the N players apart from i, so that the relevant coefficients are those referred to those tripartitions which change their condition (from winners to losers) when player i descends one single level of approval:

$$\sum_{j=q-w^Y(i)}^{q-w^N(i)-1} \alpha_j = (\alpha_{q-w^Y(i)} + ... + \alpha_{q-1}) + (\alpha_q + ... + \alpha_{q-w^N(i)-1}) =$$

$$= |\{A : A \in \mathcal{W},\ A_{\downarrow i} \notin \mathcal{W},\ i \in A_1\}| + |\{A : A \in \mathcal{W},\ A_{\downarrow i} \notin \mathcal{W},\ i \in A_2\}| = \eta_i(\mathcal{W})$$

Recall that here $w^Y(i) \geq 0$ and $w^N(i) \leq 0$ by the convention to normalize in the middle level, but this convention is not compulsory to compute indices with generating functions.

A small modification in the previous formula serves to obtain $\eta_i^*(\mathcal{W})$ and $\eta_i^{**}(\mathcal{W})$:

$$\eta_i^*(\mathcal{W}) = (\alpha_{q-w^Y(i)} + ... + \alpha_{q-1}) + 2 \cdot (\alpha_q + ... + \alpha_{q-w^N(i)-1})$$
$$\eta_i^{**}(\mathcal{W}) = 2 \cdot (\alpha_{q-w^Y(i)} + ... + \alpha_{q-1}) + (\alpha_q + ... + \alpha_{q-w^N(i)-1})$$

as it is obvious using the same reasoning which was applied to the Coleman indices' numerators in Section 3.

Rae, Banzhaf and Coleman indices are immediately obtained from these *raw* ones. The numerator of the König-Bräuninger index can also be computed in an analogous way, now considering as relevant the following coefficients:

$$\sum_{j=q-w^Y(i)}^{\bar{W}^Y - w^Y(i)} \alpha_j = (\alpha_{q-w^Y(i)} + ... + \alpha_{\bar{W}^Y - w^Y(i)}) = |\{A : i \in A_1,\ A \in \mathcal{W}\}|$$

Note that the sum starts at the same first coefficient as before but ends at the last one, as we are considering all the winning tripartitions in which voter i votes 'yes'. To compute the König and Bräuninger index denominator (and Coleman's *decisiveness* index, as well), we need to use the *global GF*:

$$|\mathcal{W}| = \sum_{j=q}^{\bar{W}^Y} \alpha_j = \alpha_q + \alpha_{q+1} ... + \alpha_{\bar{W}^Y}$$

The other success and unsuccess notions derived from the König and Bräuninger index ($J(\mathcal{W})$, $L(\mathcal{W})$ and $M(\mathcal{W})$) would be computed in similar ways. Let's check in the next example that all these procedures work:

Example 6.1 Consider again the game $\mathcal{W}$ described in the Example 3.2, in which $\mathcal{W}^m = \{(12, \emptyset, 3), (1, 23, \emptyset), (23, 1, \emptyset)\}$. This (3,2) simple game can also be represented as a weighted (3,2) simple game by $[\ 2\ ;\ (2, -1), (1, -2), (1, -1)]$ (where $q = 2$ and each player's 'yes' and 'no' votes appear in brackets). First of all, let's compute the individual generating function:

$$f_1(x) = (x^{-2} + 1 + x)(x^{-1} + 1 + x) = x^{-3} + x^{-2} + 2x^{-1} + 2 + 2x + x^2$$
$$f_2(x) = (x^{-1} + 1 + x^2)(x^{-1} + 1 + x) = x^{-2} + 2x^{-1} + 2 + 2x + x^2 + x^3$$
$$f_3(x) = (x^{-1} + 1 + x^2)(x^{-2} + 1 + x) = x^{-3} + x^{-2} + x^{-1} + 3 + x + x^2 + x^3$$

The next step is obtaining the different *raw* indices using the procedure explained above:

$$\left.\begin{array}{l|l} q - w^Y(1) = 0 & q = 2 \\ q - 1 = 1 & q - w^N(1) - 1 = 2 \end{array}\right\} \rightarrow \eta_1(\mathcal{W}) = (\alpha_0 + \alpha_1) + a_2 = 4 + 1 = 5$$

$$\left.\begin{array}{l|l} q - w^Y(2) = 1 & q = 2 \\ q - 1 = 1 & q - w^N(2) - 1 = 3 \end{array}\right\} \rightarrow \eta_2(\mathcal{W}) = \alpha_1 + (\alpha_2 + \alpha_3) = 2 + 2 = 4$$

$$\left.\begin{array}{l|l} q - w^Y(3) = 1 & q = 2 \\ q - 1 = 1 & q - w^N(3) - 1 = 2 \end{array}\right\} \rightarrow \eta_3(\mathcal{W}) = \alpha_1 + \alpha_2 = 1 + 1 = 2$$

Indeed, these values coincide with the ones given in Example 3.2. Now we can easily compute $\eta^*(\mathcal{W})$ and $\eta^{**}(\mathcal{W})$:

$$\eta^*(\mathcal{W}) = (4 + 2 \cdot 1, 2 + 2 \cdot 2, 1 + 2 \cdot 1) = (6, 6, 3)$$

$$\eta^{**}(\mathcal{W}) = (2 \cdot 4 + 1, 2 \cdot 2 + 2, 2 \cdot 1 + 1) = (9, 6, 3)$$

Finally, we deal with the KB index. Its numerators for each player are:

$$|\{A : 1 \in A_1,\ A \in \mathcal{W}\}| = \alpha_0 + ... = 2 + 2 + 1 = 5$$

$$|\{A : 2 \in A_1,\ A \in \mathcal{W}\}| = \alpha_1 + ... = 2 + 1 + 1 = 4$$

$$|\{A : 3 \in A_1,\ A \in \mathcal{W}\}| = \alpha_1 + ... = 1 + 1 + 1 = 3$$

while the denominator is obtained from the global generating function:

$$f(x) = (x^{-1}+1+x^2)(x^{-2}+1+x)(x^{-1}+1+x) = x^{-4}+2x^{-3}+3x^{-2}+5x^{-1}+5+5x+3x^2+2x^3+x^4$$

$$|\mathcal{W}| = \alpha_2 + ... = 3 + 2 + 1 = 6$$

Again, this method provides the same results which were given in the Example considered.

For large values of $n = |N|$, these calculations can become really tedious. That is why some mathe-matical software may help. The indices we are showing in the next example have been computed by an algorithm ran with Maple.

Example 6.2 Let's return to the UNSC voting system. In several works abstention is not considered, so that this system can be modelled as a weighted simple game:

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

As it was shown in Example 2.4, when abstention is taken into account, the voting system admits a representation as a weighted (3,2) game:

$$[\ 9;\ \underbrace{(1, -6), ..., (1, -6)}_{5}, \underbrace{(1, 0), ..., (1, 0)}_{10}\ ]$$

The next table shows the differences between both cases for the most important indices among those we have been studying along this paper:

Indices	Simple game		(3,2) game	
	Permanent M.	*Nonpermanent M.*	*Permanent M.*	*Nonpermanent M.*
$Rae(\mathcal{W})$	0, 5259	0, 5026	0, 3409	0, 337
$Bz(\mathcal{W})$	0, 0518	0, 0051	0, 0227	0, 0111
$Col^P(\mathcal{W})$	1	0, 0991	1	0, 3621
$Col^I(\mathcal{W})$	0, 0266	0, 0026	0, 0125	0, 0075
$KB(\mathcal{W})$	1	0, 5495	0, 738	0, 5747

One thing we can easily check at first sight is that the possibility of abstention shortens the relative power differences between permanent and nonpermanent members, specially when talking about decisiveness. Permanent members lose cruciality because there are tri-partitions which do not need their 'yes' votes to be winners, while that was simply impossible in the simple game model. Simultaneously, this fact enlarges the power of nonpermanent members, as the number of associations in which they become crucial increases.

Another fact we can observe is the loss of proporcionality between the Banzhaf and Coleman indices, in the way it was announced in Section 3 (see Example 3.2): while the proportion of power between permanent and nonpermanent members is exactly the same ($\simeq 10 : 1$) for each of the three indices in the simple game model, in the (3,2) game all of them are considerably different: ($Bz \simeq (2 : 1)$, $(2 : 1) \leq Col^P \leq (3 : 1)$, $Col^I \leq (2 : 1)$). This is another evidence of the independence of these three concepts when abstention takes part as an input.

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