On Wideband Deconvolution Using Wavelet Transforms

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***Abstract*—A discussion on the expression proposed in [1]–[3] for deconvolving the wideband *density function* is presented. We prove here that such an expression reduces to be proportional to the *wideband correlation receiver output,* or *continuous wavelet transform* of the received signal with respect to the transmitted one. Moreover, we show that the same result has been implicitly assumed in [1], when the deconvolution equation is derived. We stress the fact that the analyzed approach is just the orthogonal projection of the density function onto the image of the wavelet transform with respect to the transmitted signal. Consequently, the approach can be considered a good representation of the density function only under the prior knowledge that the density function belongs to such a subspace. The choice of the transmitted signal is thus crucial to this approach.**

I. Introduction

In [1]–[3], wideband processing of acoustic signals is applied to the problem of source localization, when there are several scatterers reflecting a transmitted signal $f(t)$. The wideband model for the received acoustic signal, $g(t)$, reflected by a single scatterer, is assumed to be $g(t) = (1/\sqrt{|s|})\, f\,[(t-\tau)/s]$, where τ is the time delay, and the scale s is related with the speed of the scatterer. In environments with multiple scatterers, the delay τ and scale s are assumed to be distributed according to the *density function* $S_W(s, \tau)$, so that the received signal, $g(t)$, is written as

$$g(t) = \int\int S_W(s,\tau) \frac{1}{\sqrt{|s|}}\, f\left(\frac{t-\tau}{s}\right) \frac{ds\, d\tau}{s^2}. \qquad (1)$$

By correlating this signal with hypothesized replicas of the transmitted signal, $f(t)$, the *wideband correlation receiver output*, $\mathrm{WC}_{gf}(s, \tau)$, is formed as

$$\mathrm{WC}_{gf}(s, \tau) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{\infty} g(t) f^*\left(\frac{t-\tau}{s}\right) dt \equiv W_f g(s, \tau) \qquad (2)$$

and is identical to the wavelet transform, $W_f g(s, \tau) = \langle g | f_{\tau, s}\rangle$, of the signal $g(t)$ with respect to the signal $f(t)$, which is assumed to be admissible as a *mother wavelet*. To simplify notation we denote $f_{\tau, s}(t) = (1/\sqrt{|s|})\, f\,[(t-\tau)/s]$.

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After some manipulation, the density function, $S_W(s, \tau)$, is derived in [1] and presented in [1]–[3] as

$$S_W(s', \tau') = \frac{1}{c_f} \int\int W_f g(s, \tau)\, W_f^* f\left(\frac{s}{s'}, \frac{\tau - \tau'}{s'}\right) \frac{ds\, d\tau}{s^2} \qquad (3)$$

where c_f is the *admissibility constant*, $c_f = \int_{-\infty}^{\infty} [|F(\omega)|^2/|\omega|]\, d\omega$, and $F(\omega)$ the Fourier transform of $f(t)$.

We shall show the following.

1) The c_f constant in (3) must be replaced by c_f^2.
2) Equation (3) (after changing c_f to c_f^2) is equivalent to
$$S_W(s', \tau') = \frac{W_f g(s', \tau')}{c_f}. \qquad (4)$$
3) The equivalence arises as a consequence of having implicitly assumed (4) in [1], when (3) is derived.
4) The approach being analyzed is just the orthogonal projection of the density function onto the image of the wavelet transform, with respect to the transmitted signal $f(t)$.
5) If the wavelet transform, $W_f g(s', \tau')$, is approximated by a function, say $F(s', \tau')$, then, provided that there is no function $g^o(t) \in L^2(R)$ such that $F(s', \tau')$ can be written as $F(s', \tau') = W_f g^o(s', \tau')$, (3) renders a better approximation to $W_f g(s', \tau')$ than $F(s', \tau')$ itself.

II. Discussion on the Deconvolution Expression (3)

Let us write the *resolution of the identity* property of the continuous wavelet transform, from which all the properties to be used here can be obtained. This property establishes that for $g(t), f(t) \in L^2(R)$

$$\langle g|f\rangle = \frac{1}{c_\phi} \int\int \langle g|\phi_{s,\tau}\rangle \langle \phi_{s,\tau}|f\rangle \frac{ds\, d\tau}{s^2} \qquad (5)$$

where $\phi(t)$ is any admissible *mother wavelet*, and $\phi_{s,\tau} = (1/\sqrt{|s|})\, \phi\,[(t-\tau)/s]$.

First, we show that

$$\int\int W_f g(s, \tau) W_f^* f\left(\frac{s}{s'}, \frac{\tau-\tau'}{s'}\right) \frac{ds\, d\tau}{s^2} = c_f W_f g(s', \tau'). \qquad (6)$$

Since

$$W_f^* f\left(\frac{s}{s'}, \frac{\tau-\tau'}{s'}\right) = \int_{-\infty}^{\infty} f(t)^* \sqrt{\frac{|s'|}{|s|}}\, f\left[\left(t - \frac{\tau-\tau'}{s'}\right)\frac{s'}{s}\right] dt \qquad (7)$$

by performing the change of variables $t = (t - \tau')/s'$ in the integral, we obtain

$$W_f^* f\left(\frac{s}{s'}, \frac{\tau - \tau'}{s'}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{|ss'|}} f\left(\frac{t - \tau}{s}\right) \cdot f^*\left(\frac{t - \tau'}{s'}\right) dt = \langle f_{\tau,\,s} | f_{\tau',\,s'} \rangle. \quad (8)$$

Thus, the left hand side in (6) can be recast as

$$\int\int W_f g(s,\,\tau) W_f^* f\left(\frac{s}{s'}, \frac{\tau - \tau'}{s'}\right) \frac{ds\,d\tau}{s^2} = \int\int \langle g | f_{s,\,\tau} \rangle \langle f_{\tau,\,s} | f_{\tau',\,s'} \rangle \frac{ds\,d\tau}{s^2} \quad (9)$$

and replacing $\phi_{s,\,\tau}$ by $f_{s,\,\tau}$ and f by $f_{\tau',\,s'}$ in (5), it is proved that

$$\int\int W_f g(s,\,\tau) W_f^* f\left(\frac{s}{s'}, \frac{\tau - \tau'}{s'}\right) \frac{ds\,d\tau}{s^2} = c_f \langle g | f_{s',\,\tau'} \rangle = c_f W_f g(s',\,\tau') \quad (10)$$

so, the integral in (3) is reduced to $c_f W_f g(s',\,\tau')$.

Now we discuss the derivation of (3) given in [1], and we show that such a derivation assumes $S_W(s',\,\tau') = W_f g(s',\,\tau')/c_f$.

In that publication, the property that $W_f^* f[\hat{s}/s', (\hat{\tau} - \tau')/s']$ is a reproducing kernel in a subspace of the Hilbert space $\mathcal{L} = L^2(R; c_f^{-1} s^{-2} ds\,d\tau)$ is applied for deconvolving $S_W(s',\,\tau')$. That $W_f^* f[\hat{s}/s', (\hat{\tau} - \tau')/s']$ is a reproducing kernel in a subspace of $\mathcal{L}$ readily follows because $W_f^*\, f\,[\hat{s}/s', (\hat{\tau} - \tau')/s'] = \langle f_{\hat{\tau},\,\hat{s}} | f_{\tau',\,s'} \rangle$ [see (8)] and from (5), with $f = f_{\tau',\,s'}$ and $\phi_{s,\,\tau} = f_{\hat{\tau},\,\hat{s}}$, we have

$$\langle g | f_{\tau',\,s'} \rangle = \frac{1}{c_f} \int\int \langle g | f_{\hat{\tau},\,\hat{s}} \rangle \langle f_{\hat{\tau},\,\hat{s}} | f_{\tau',\,s'} \rangle \frac{d\hat{\tau}\,d\hat{s}}{\hat{s}^2}. \quad (11)$$

However, (11) does not hold for all the Hilbert space, but only for the subspace generated by $W_f[L^2(R)]$ [4]. Let us call such a subspace $\mathcal{W}$, i.e., $\mathcal{W} = \{W_f g(s,\,\tau) = \langle g | f_{\tau,\,s} \rangle;\ g(t) \in L^2(R)\}$. Equation (11) provides the necessary and sufficient condition for a function $F(s,\,\tau)$ to belong to $\mathcal{W}$. This means that if a function $F(s,\,\tau)$ satisfies

$$F(\tau',\,s') = \frac{1}{c_f} \int\int F(\hat{\tau},\,\hat{s}) \langle f_{\hat{\tau},\,\hat{s}} | f_{\tau',\,s'} \rangle \frac{d\hat{\tau}\,d\hat{s}}{\hat{s}^2} \quad (12)$$

then there exists a function $g(t) \in L^2(R)$ such that $F(\hat{s},\,\hat{\tau}) = \langle g | f_{\hat{\tau},\,\hat{s}} \rangle$. Nevertheless, not every function $F(\hat{\tau},\,\hat{s}) \in \mathcal{L}$ satisfies (12), i.e., not every $F(\hat{\tau},\,\hat{s}) \in \mathcal{L}$ can be expressed as $\langle g | f_{\hat{\tau},\,\hat{s}} \rangle$ for some $g(t) \in L^2(R)$. Therefore, to assume, as in [1], that the density function satisfies (12) is equivalent to assuming that $c_f S_W(\tau,\,s) = \langle g | f_{\tau,\,s} \rangle = W_f g(\tau,\,s)$. The constant c_f arises due to the fact that it is not present in the reproducing kernel equation (17) in [1]. Thus, in the deconvolution equation (18) of [1], written here as (3), c_f should be replaced by c_f^2 and, according to (10), the equation reduces to $S_W(\tau,\,s) = W_f g(\tau,\,s)/c_f$ (which is consistent with the way it has been derived).

Certainly, there is no reason, *per se*, why the density function should satisfy (12). Denoting $\mathcal{W}^{\perp}$ to be the orthogonal complement of $\mathcal{W}$ in $\mathcal{L}$, we have that any function $S^{\perp}(\tau,\,s) \in \mathcal{W}^{\perp}$ fulfills $\int\int S^{\perp}(\tau,\,s) f_{s,\,\tau}(t)(ds\,d\tau/s^2) = 0$, because $\int\int S^{\perp}(\tau,\,s) \langle f_{s,\,\tau} | f_{s',\,\tau'} \rangle (ds\,d\tau/s^2) = 0\, \forall\, f_{s',\,\tau'}(t)$, due to the fact that $\langle f_{s,\,\tau} | f_{s',\,\tau'} \rangle \in \mathcal{W}$ and $S^{\perp}(\tau,\,s)$ is orthogonal to every function in $\mathcal{W}$. Consequently, the most general solution of the inverse problem (1) can be expressed as $S_W(\tau,\,s) = W_f g(\tau,\,s)/c_f + S^{\perp}(\tau,\,s)$, which implies that the determination of the density function has no unique solution, if only one transmitted signal $f(t)$ is used. The hypothesis that $S_W(\tau,\,s)$ satisfies (12) is tantamount to setting $S^{\perp}(\tau,\,s) = 0$. Otherwise, in general, the right-hand side of (12) gives rise to the orthogonal projection of $S_W(\tau,\,s)$ onto $\mathcal{W}$. It is pertinent to stress then, that $W_f g(\tau,\,s)/c_f$ can be accepted as a good representation of the density function only in those exceptional cases in which one has the prior knowledge that the density function must lie in $\mathcal{W}$, or be close to it. As $\mathcal{W}$ is generated by the transmitted signal, $f(t)$, it is obvious that the selection of $f(t)$, far from being arbitrary, is of crucial importance.

In order to discuss the effect of noise in the approach being analyzed, let us assume that the received signal, $g(t) \in L^2(R)$, is affected by additive random noise $\epsilon(t)$ and one is able to obtain only $g^o(t) = g(t) + \epsilon(t)$. If the random noise $\epsilon(t) \in L^2(R)$, then $\langle g^o | f_{\tau,\,s} \rangle$can be evaluated and the deconvolution equation (3) is equivalent to writing $S_W^o(\tau,\,s) = \langle g^o | f_{\tau,\,s} \rangle / c_f$. In practice, however, the exact computation of $\langle g^o | f_{\tau,\,s} \rangle$ (or $\langle g | f_{\tau,\,s} \rangle$ in the noiseless case) may not be possible but rather an approximation of $\langle g^o | f_{\tau,\,s} \rangle$ is available. If $F^o(\tau,\,s)$ is the realizable approximation of $\langle g^o | f_{\tau,\,s} \rangle$ there is no longer any guarantee that $F^o(\tau,\,s) \in \mathcal{W}$, and so, when $F^o(\tau,\,s) \notin \mathcal{W}$ a better approximation to $\langle g^o | f_{\tau,\,s} \rangle$ than $F^o(\tau,\,s)$ itself is, at least in theory, possible. Indeed, using $F^o(\tau,\,s)$ we can build an artificial signal $g'(t)$ as

$$g'(t) = \frac{1}{c_f} \int\int F^o(\tau,\,s) \frac{1}{\sqrt{|s|}}\, f\left(\frac{t - \tau}{s}\right) \frac{ds\,d\tau}{s^2} \quad (13)$$

and, taking the inner product of $g'(t)$ with $(1/\sqrt{|s'|})\, f\,[(t - \tau')/s']$, we have

$$\langle g' | f_{s',\,\tau'} \rangle = \frac{1}{c_f} \int\int F^o(\tau,\,s) \langle f_{s,\,\tau} | f_{s',\,\tau'} \rangle \frac{ds\,d\tau}{s^2} \equiv \frac{1}{c_f} \int\int F^o(s,\,\tau) W_f^*\, f\left(\frac{s}{s'}, \frac{\tau - \tau'}{s'}\right) \frac{ds\,d\tau}{s^2}. \quad (14)$$

If $F^o(\tau,\,s) \notin \mathcal{W}$, it will not satisfy the reproducing kernel equation (12), and from (14) we have: $F^o(\tau,\,s) = \langle g' | f_{s,\,\tau} \rangle + F^{\perp}(\tau,\,s)$, with $F^{\perp}(\tau,\,s) \in \mathcal{W}^{\perp}$. Therefore, as it follows from the inequality, $||\langle g^o | f_{s,\,\tau} \rangle - F^o(\tau,\,s)||^2 \geq ||\langle g^o | f_{s,\,\tau} \rangle - \langle g' | f_{s,\,\tau} \rangle||^2$, $\langle g' | f_{s,\,\tau} \rangle$ is a better approximation to $\langle g^o | f_{s,\,\tau} \rangle$ than $F^o(\tau,\,s)$. The proof is immediate: Let us denote $\Delta = \langle g^o | f_{s,\,\tau} \rangle - \langle g' | f_{s,\,\tau} \rangle$, then $||\langle g^o | f_{s,\,\tau} \rangle - F^o(\tau,\,s)||^2 = ||\Delta - F^{\perp}(\tau,\,s)||^2 = ||\Delta||^2 + ||F^{\perp}(\tau,\,s)||^2 \geq ||\Delta||^2$. [The previous equality holds for Δ being orthogonal to $F^{\perp}(\tau,\,s)$.]

By comparing (3) and (14) we can conclude that when the wavelet transform, $W_f g(s,\,\tau)$, is approximated by a function $F^o(\tau,\,s) \notin \mathcal{W}$, the deconvolution equation (3) gives a better approximation to $W_f g(s, \tau)$ than $F^o(\tau, s)$ itself. Notice that, in practice, the statement may not be true if the computation of (3) introduces large errors.

III. Conclusions

We have discussed the deconvolution equation for the wideband density function given in [1]–[3]. We have proved that this equation is equivalent to $S_W(s, \tau) = W_f g(s, \tau)/c_f$. We have shown that this equality has been used implicitly in the derivation of the deconvolution equation and that is why the equation is reduced to it.

We have remarked that, in fact, what the analyzed approach provides is the orthogonal projection of the density function onto the subspace generated by the image of the wavelet transform, with respect to the transmitted signal $f(t)$. Thus, the approach makes sense only in those cases for which one has the prior knowledge that the density function belongs to such a subspace. As a consequence, the selection of the transmitted signal can not be arbitrary.

Finally, we have proved that when the wavelet transform $W_f g(s, \tau)$ is approximated by a function $F^o(\tau, s)$ that is not in the image of the wavelet transform, i.e., when no signal $g^o(t) \in L^2(R)$ such that $F^o(\tau, s) = W_f g^o(s, \tau)$ exists, then what the analyzed deconvolution equation yields is a better approximation to $W_f g(s, \tau)$ than $F^o(\tau, s)$ itself.

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References

[1] L. G. Weiss, R. K. Young, and L. H. Sibul, “Wide-band processing of acoustic-signals using wavelet transforms 1: Theory,” *J. Acoust. Soc. Amer.,* vol. 96, pp. 850–856, 1994.

[2] L. G. Weiss, “Wide-band processing of acoustic-signals using wavelet transforms 2: Efficient implementation and examples,” *J. Acoust. Soc. Amer.,* vol. 96, pp. 857–866, 1994.

[3] ——, “Wavelets and wideband correlation processing,” *IEEE Signal Processing Mag.,* vol. 11, pp. 13–32, 1994.

[4] I. Daubechies, “Ten lectures on wavelets,” in *CBMS-NSF,* SIAM, Philadelphia, 1992.

[5] T. L. Dixon and L. H. Sibul, “A parameterized Hough transform approach for estimating the support of the wideband spreading function of a distributed object,” *Multidimen. Syst. Signal Processing,* vol. 7, pp. 75–86, 1996.

[6] ——, “Wideband imaging and parameter estimation of distributed objects using the continuous wavelet transform,” *Signal Process.,* vol. 54, pp. 207–223, 1996.