Learning Engineering to Teach Mathematics

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Abstract – **The Bologna process is a good opportunity to bring together first-year mathematics courses of engineering degrees and technology courses offered in subsequent years. In fact, the Faculty Council has decided that 20% of the credits from basic courses must be related to technological applications. To this end, during the past academic year a mathematical engineering seminar was held with each session dealing with one technological discipline. The main goal of the seminar, which relied on the presence of speakers from both mathematics and engineering departments, was to identify the most commonly used mathematical tools. Furthermore, a set of exercises and some guidelines addressed to faculty lacking an engineering background were created. Here, we present some of this material: first, a summary of the collection of exercises illustrating the use of Linear Algebra in different engineering areas such as Mechanical Engineering, Control and Automation, and second, some exercises and the guideline for Electrical Engineering.**

Index Terms – applied exercises, Bologna process, engineering, mathematics

INTRODUCTION

One of the recurrent controversies at our Engineering Faculty concerns the orientation of first-year basic courses, particularly the subject area of mathematics, considering its role as an essential tool in technological disciplines.

The Bologna process is a good opportunity to delve into this debate. In fact, the Faculty Council has decided that 20% of the credits from basic courses must be related to technological applications. Thus, a mathematical engineering seminar was held during the past academic year at the Engineering Faculty of Barcelona. The organizers of this seminar were the authors belonging to Universitat Politècnica de Catalunya and the seminar was partially supported by AGAUR 2009MQD-00107. Sessions were each devoted to one technological discipline and aimed at identifying the most frequently used mathematical tools with the collaboration of guest speakers from mathematics and technology departments.

Moreover, the implementation of the Bologna process is presented as an excellent opportunity to substitute the traditional teaching-learning model by another one where students play a more active role. In this case, we can use the Problem-Based Learning (PBL) method. This environment is a really useful tool to increase student involvement as well as multidisciplinary. With PBL, before students increase their knowledge of the topic, they are given a real situation-based problem which will drive the learning process. Students will discover what they need to learn in order to solve the problem, either individually or in groups, using tools provided by the teacher or 'facilitator', or found by themselves. In contrast with conventional Lecture-Based Learning (LBL) methods, with the new technique students have more responsibility since they must decide how to approach their projects. They must identify their own learning requirements, find resources, analyze information obtained by research, and finally construct their own knowledge. Moreover, PBL helps students develop skills and competences such as group and self-assessment skills, which will allow them to keep up-to-date and continue to learn autonomously, or to become acquainted with the decision-making process, time scheduling and, last but not least, improve communication skills. Furthermore, theory and practice are integrated and motivation is enhanced, which results in increased academic performance.

A collection of exercises and problems was created to illustrate the applications identified in the seminar sessions. These exercises would be some of the real situation-based problems given for introducing the different mathematic topics. Two conditions were imposed: availability for first-year students and emphasis on the use of mathematical tools in technical subjects in later academic years. As additional material, guidelines for each technological area addressed to faculty without an engineering background were defined.

Here, we present some of this material: first, a summary of the collection of exercises showing how Linear Algebra can be useful in different engineering areas such as Mechanical Engineering, Control and Automation, and second, some exercises and the guideline for Electrical Engineering. As general references regarding Linear Algebra as well as referred engineering areas, see for example [1]-[5].

COLLECTION OF EXERCISES

The following exercises illustrate the use of Linear Algebra in different engineering areas. As said above, they are based on conclusions drawn at the above seminar.

- Set of complex numbers.
  Alternating current representation. Its use to calculate voltage drop and cancellation of reactive power.
- Matrices. Determinants. Rank.
  Controllability and observability of Control Linear System. Its controllability indices.
- Linear System Equations.
  Network flows. Leontief economic model.
- Vector Spaces. Bases. Coordinates.
  Color codes. Crystallography.
- Vector Subspaces.
  Reachable states of Control Linear Systems. Circuit analysis (mesh currents, node voltages).
- Linear maps.
  Controllability and observability matrices. Kalman decomposition.
- Diagonalization. Eigenvectors, eigenvalues.
  Strain and stress tensors. Circulant matrices.
- Non-diagonizable matrices;
  Control canonical form.
- Dynamic Discrete Systems.
  Leslie population model. Gould accessibility indices.

GUIDELINE FOR ELECTRICAL ENGINEERING

The guideline for Electrical Engineering is included here as an example of the guidelines defined for faculty lacking an engineering background.

I. Circuit Analysis

Study of the equation set derived from Ohm's and Kirchoff's laws.

Approach

Considering a network having *N* nodes, *B* branches and *M* meshes, we obtain (Euler)

$$N\text{-}B+M=1. \quad (1)$$

In order to simplify the calculation, we will consider a direct current, so that every magnitude is represented by a real number (the same theory is also valid for alternating current if we represent its magnitudes by the complex numbers).

The objective is to determine the *currents and voltages* distribution (i_k, u_k), $1 \le k \le B$, with

- OHM'S LAW: $u_k = R_k i_k + e_k$ in each branch, where e_k is the generated voltage (if it exists).
- KIRCHOFF'S CURRENT LAW (KCL): $\sum i_k = 0$ at each node.
- KIRCHOFF'S VOLTAGE LAW (KVL): $\sum u_k = 0$ in each mesh.

Note that for certain points it is useful to consider

- MESH CURRENTS**:** fictitious; then any branch current is the difference between the surrounding mesh currents (usually the positive sign is assigned to the counterclockwise one).
- NODE VOLTAGES**:** absolute voltage at each node. A node with a preset voltage must be previously selected as a reference (usually null, considering this node bonded to ground).

Theoretical Analysis

From a theoretical point of view, Ohm's law establishes an *isomorphism between currents and voltages*, so we just need to refer to one of both variables, for example, the currents

$$\text{(KCL)} \qquad \sum i_k = 0 \text{ at each node.} \qquad (2)$$

$$\text{(KVL)} \qquad \sum R_k i_k = -\sum e_k \text{ in each mesh.} \qquad (3)$$

It is obvious that the sum of all equations in (2) will be null (each current appears at two nodes with opposite sign), so *one node equation is redundant*. Therefore, we have *N+M-1=B* equations with *B* unknowns. It is also evident that this is a *compatible system with a unique solution* because the homogeneous associate is so (if $e_k=0$, there is no energy contribution, meaning that the currents must be null).

Then,

dim *{solutions KCL}=B-(N-1)=M*

dim *{solutions KVL}=B-M=N-1*

Easily, we can confirm (also by "practical" reasoning) that *the mesh currents are linearly independent*, implying that they are a basis of the first subspace.

To summarize, in this basis I_k, $1 \le k \le M$, the analysis reduces to solving a compatible system with a unique solution:

$$\sum R_k \left(I_{k(k^+)} - I_{k(k^-)} \right) = -e_k \text{ in each mesh.} \qquad (4)$$

Alternatively, the *node voltages* could be taken as a basis of the second subspace, and the analysis is then reduced to a KCL.

The Modified Nodal Analysis method (MNA)

In practice, the first equations to be considered are the "structural" ones (KCL and KVL), which do not depend on the consumption (which is generally variable) but on the network structure only.

The KCL is $A_T I = 0$, where A_T is the (complete) *incidence matrix* of the network. Obviously, each column must contain a 1, a -1 and the rest of the numbers must be 0, so the row addition is null. Therefore, $rank A_T \le N-1$. Because the mesh currents are linearly independent, we obtain $rank A_T = N-1$. Deleting one equation (usually the one corresponding to the reference node), the KCL provides a system of maximum rank:

$$\text{(KCL)} \qquad AI = 0,\ A \in M_{(N-1)\times B} \qquad (5)$$

where A is the (reduced) incidence matrix and I the branch current vector.

As the node voltages are linearly independent solutions of KVL, this is a system of maximum rank and it becomes

$$\text{(KVL)} \qquad U = A^t V \qquad (6)$$

where V is the node voltage vector (except for the reference one) and U the branch voltage vector.

Assuming

$$\{I\} = \mathrm{Ker}A,\ \{U\} = \mathrm{Im}A^t = (\mathrm{Ker}A)^{\perp}, \quad (7)$$

We obtain *Tellegen's theorem:*

$$\sum_{k=1}^{B} u_k i_k = 0 \quad (8)$$

Returning to the previous equations, we can begin from the mesh currents or the node voltages, and apply the other Kirchoff's law and Ohm's law. Nowadays the second option is preferred as it has a better physical meaning and may be generalized to non-planar networks.

The *MNA starts from node voltages V***.** It is applied directly to "Norton branches" (where i_k depends on u_k) .Later, "Tevenin branches" (where u_k depends on i_k) will be included. Specifically, in $AI = 0$ we replace

$$i_k = Y_k u_k + J_k \quad (9)$$

where Y_k are the branches admittances (=resistance inverse, or more generally, the impedance) and J_k the current generators. Replacing $U = A^t V$ and writing $A = (A_1 \dots A_B)$ and $J = \sum A_k J_k$ (the so-called *node injection*), we obtain

$$YV = J\ ,\ Y = \sum A_k Y_k A_k^t \in Sim_{N-1} \quad (10)$$

It is the *Fundamental equation of the MNA*, which allows the determination of the V node voltages. As we have already seen, the branch voltages U and the branch currents I are easily derivable.

Note that the matrix Y, called *node admittance matrix*, is easily calculated with the appropriate algorithm and often by direct inspection of the network, for example when new branches are included.

II. Complex representation of electric magnitudes for alternating currents

The treatment of direct currents can be generalized to alternating currents if their electric magnitudes are represented by complex numbers instead of real numbers. More specifically, the electric magnitudes in alternating currents have cosinusoidal form:

$$m(t) = \sqrt{2} M \cos(\omega t + \alpha) \quad (11)$$

where M is the *root mean square* value (rms), α the *phase angle* and ω the *frequency* (constant). Each magnitude is associated with its complex *phasor*:

$$\underline{M} = Me^{i\alpha}$$

As it is a linear correspondence, Kirchoff's laws become

(KCL) $\sum \underline{I_k} = 0$ at each node, (12)

(KVL) $\sum \underline{U_k} = 0$ in each mesh, (13)

now with the algebraic operations on complex numbers. As this set of numbers has a field structure, Ohm's law also generalizes to

(Ohm) $\underline{U_k} / \underline{I_k} = Z_k$ in each branch,

(14)

where Z_k is the *impedance* including not only the resistance case:

$$u(t) = Ri(t), \qquad \underline{U} = R\underline{I} \quad (15)$$

but thanks to

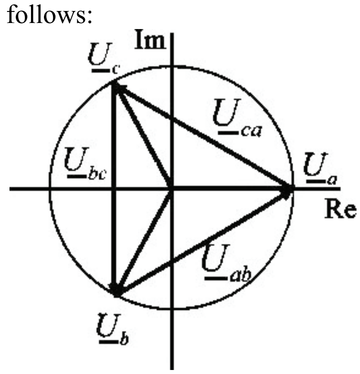
$$m'(t) = j\omega m(t) \quad (16)$$

the condenser and coil case as well:

$$i(t) = Cu'(t), \qquad \underline{I} = j\omega C\underline{U} \quad (17)$$

$$u(t) = Li'(t), \qquad \underline{U} = j\omega L\underline{I} \quad (18)$$

where C and L are the *capacitance* and the *inductance*, respectively. For example, the phase voltage phasor (phase/neutral) of a three-phase current is distributed as follows:



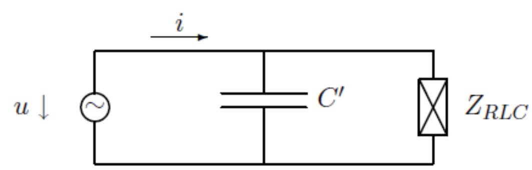
The complex algebraic operations allow us to easily prove that

$$U_A + U_B + U_C = 0 \quad (19)$$

or calculate the composed voltages (phase/phase):

$$U_{AB} = U_A - U_B, \dots \quad (20)$$

Equally, we can calculate the capacitance C, which turns the resultant impedance of the following circuit:



into real (condition by which the *reactive power is cancelled,* and therefore the *active power is maximized*).

III. Circulant matrices

Among the various types of matrices appearing in the subject of Electrical Engineering, we highlight coupling matrices in different types of induction machinery having an *inductance operator* as follows:

$$Z = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix} \in M_3(C) \quad (21)$$

which are a particular case of the so-called *circulant matrices*.

Proposition – The following statements are equivalent:

- Z is a circulant matrix
- Z diagonalizes by the orthogonal transformation

$$F = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix},\ a = e^{j\frac{2\pi}{3}} \tag{22}$$

We point out that

1. The addition and the product of circulant matrices are also circulant matrices.
2. If (λ_i) are the eigenvalues of Z, and (λ_i') are the eigenvalues of Z', then $(\lambda_i + \lambda_i')$ are the eigenvalues of $Z + Z'$, and $(\lambda_i \lambda_i')$ are the eigenvalues of ZZ'.

Remark

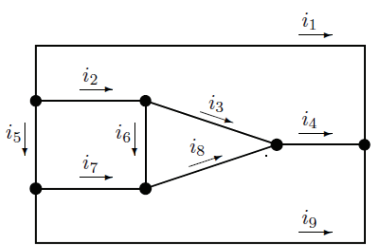
- The above circulant matrices also appear in other contexts, like nested polygons, coupled oscillators and the discrete Fourier transform.
- The previous properties are generalized to any dimension and to circulant matrices by blocks.

EXERCISES IN ELECTRICAL ENGINEERING

We now present some exercises in Electrical Engineering.

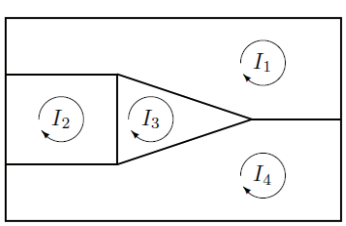
Exercise 1

Consider the following mesh:



Being E the set of possible current distribution, find the subset $F \subset E$ verifying Kirchoff's law; that is, at each of the nodes the sum of input currents must be equal to the sum of output currents.

In practice, the used currents are not the above currents but the so-called *mesh currents*:



To justify this use,

1) Prove that E is a vector space of dimension 9 and that F is a subspace of E of dimension 4.

2) Determine a basis of F so that (I_1, I_2, I_3, I_4) are its coordinates.

3) Prove that one of Kirchoff's equations is redundant; that is, if it is verified at 5 nodes, it must also be verified at the sixth node.

Exercise 2

The set of complex numbers allows us to work with alternating current in a similar way to how the set of real numbers enables us to work with direct current. In this context, complex numbers are expressed as $a + jb$ (the reason for this change is the fact that i is used to symbolize the current). Several electric magnitudes (voltage, current...) are cosinuidal functions like

$$m(t) = \sqrt{2}M\cos(\omega t + \alpha) \tag{23}$$

where M and α are called its rms and its phase, respectively. The key fact is the representation of *m(t)* by a complex number $\underline{M}$, called its phasor and defined by

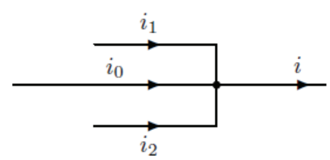
$$phasor(m(t)) \equiv \underline{M} = Me^{j\alpha} \tag{24}$$

Kirchoff's laws are valid for phasors; that is, the sum of input current phasors at a node is equal to the sum of output current phasors, and the sum of voltage phasors in a mesh is null.

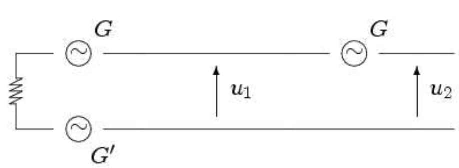
- Let us use these laws to calculate the current resulting from adding to $i_0(t) = \sqrt{2}I_0\cos\omega t$ two other currents of 75% and 50% rms and phase-shifted 120º and 90º, respectively:

$$i_1(t) = 0'75\sqrt{2}I_0\cos\left(\omega t + \frac{\pi}{3}\right) \tag{25}$$

$$i_2(t) = 0'50\sqrt{2}I_0\cos\left(\omega t + \frac{\pi}{4}\right) \tag{26}$$



- Let us now use these laws in the schema



where G and G' are the generated voltages, respectively,

$$u(t) = \sqrt{2}U\cos\omega t \tag{27}$$

$$u'(t) = \sqrt{2}U\cos\left(\omega t + \frac{\pi}{4}\right) \tag{28}$$

Calculate the root mean square values of voltages u_1 and u_2 . Is their difference equal to the root mean square value generated by *G*?

Exercise 3

In direct current, Ohm's law establishes that the *u(t)/i(t)* quotient is constant and is called *resistance*, but in alternating currents it is not so when capacitors or inductances are present. Let us see that the problem is easily solved by considering phasors and the elementary operations between complex numbers.

The laws governing the three alternate circuits corresponding to a resistance *R*, a capacitor of capacitance *C* and an inductance *L,* are, respectively,

$$u(t) = Ri(t) \qquad (29)$$

$$i(t) = CD_t u(t) \qquad (30)$$

$$u(t) = LD_t i(t) \qquad (31)$$

- Prove that, in general, for a cosinusoidal magnitude $m(t) = \sqrt{2}M\cos(\omega + \alpha)$, the following expression is verified:

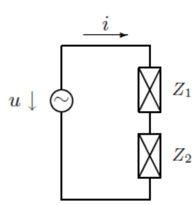
$$phasor(D_t m(t)) \equiv j\omega \underline{M} \qquad (32)$$

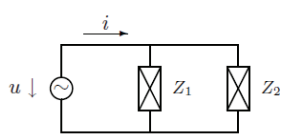
- Prove that the relation between the phasors $\underline{U}$ and $\underline{I}$ can be expressed by $\underline{U} = z\underline{I}$, where

$$Z_R = R\,,\;\; Z_C = -j\frac{1}{\omega C}\,,\;\; Z_L = j\omega L\,.$$

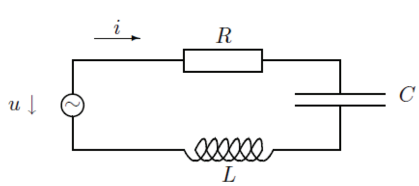
- In general, the *impedance z* of a circuit is defined as the complex quotient $\underline{U}/\underline{I}$, fulfilling *Ohm's law* between phasors.
  Prove that similarly to direct current, the *equivalent impedance* in serial connexion and parallel connexion is given by

$$Z_s = Z_1 + Z_2\,,\quad \frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2}\,.$$



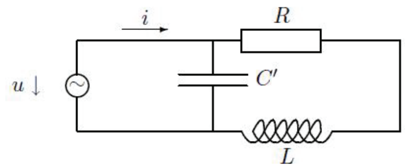


- Prove that for the *R-L-C circuit*



the impedance is $Z = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$. Note that in particular the impedance is a real number if and only if $\sqrt{LC} = \dfrac{1}{\omega}$, a situation called *serial resonance*.

- Prove that for the following circuit



the *impedance* Z is a real number if and only if $\omega L = \sqrt{\dfrac{L}{C} - R^2}$, and then $Z = \dfrac{L}{RC}$. This situation is called *parallel resonance*.

Exercise 4

The magnetic coupling matrices in induction machinery have the following *impedance operator*:

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \qquad a,b,c \in C$$

which is a particular case of the *circulant matrices.*

- Prove that any matrix of this type diagonalizes with

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}$$

  where $\alpha^3 = 1,\ \alpha \neq 1$. Calculate the diagonal form.
- Prove that with *A* and *B* being circulant matrices, *A*+*B* and *AB* are also circulant matrices and their eigenvalues are the sum and the product of *A* and *B*, respectively.

CONCLUSIONS

This paper confirms the possibility of illustrating and motivating concepts and basic results of Linear Algebra in Engineering degrees through application exercises of technological disciplines such as Electrical Engineering. The exercises are accessible to early-year students since they are self-contained in terms of technological requirements and only basic knowledge of Linear Algebra is necessary. Furthermore, they can be implemented by means of PBL methodology.

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