SWITCHED SINGULAR LINEAR SYSTEMS AND REACHABILITY

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Abstract
We consider switched singular linear systems and conditions for such a system to be reachable/controllable in the cases where some hypotheses hold.

Key words
Switched linear system, controllability.

1 Introduction
Switched singular linear systems arise from various fields such as electrical and electronic engineering, aeronautical or automotive. [Sun and Ge, 2005] is a nice and complete survey on these systems. A very complete survey of the methods which are required to study singular systems (traditional approaches are not suitable for their study) is [Dai, 1989]. Switched singular linear systems have been studied by B. Meng and F.J. Zhang ([Meng and Zhang, 2006], [Meng and Zhang, 2007]), who provided necessary conditions and sufficient conditions for reachability.

Our goal is, under the assumption of some special conditions, the algebraic characterization of reachability.

2 Preliminaries
First, we recall the concept of switched singular linear systems.

Definition 2.1. A switched singular linear system \( \Sigma \) is a system which consists of several linear singular sub-systems and a piecewise constant map \( \sigma \) taking values into the index set \( M \) which determines the switching between them.

In the continuous case, such a system can be mathematically described by

\[
\begin{align*}
E_\sigma \dot{x}(t) &= A_\sigma x(t) + B_\sigma u(t) \\
y(t) &= C_\sigma x(t)
\end{align*}
\]

where \( E_\sigma, A_\sigma \in M_n(\mathbb{R}), B_\sigma \in M_{n \times m}(\mathbb{R}), C_\sigma \in M_{p \times n}(\mathbb{R}), \text{rk } E_\sigma < n \).

Definition 2.2. Given an initial time \( t_0 \), a switching path is a function of time \( \theta : [t_0, T) \rightarrow M \), with \( t_0 < T \leq \infty \) and the index set \( M = \{1, \ldots, \ell\} \),

\[\{ t \in [t_0, T) \mid \theta_1(t) \neq \theta_2(t)\}\]

is a set of isolated real numbers.

Definition 2.3. A switching path \( \theta \) is said to be well-defined on \([t_0, T)\) if it is defined on \([t_0, T)\) and for all \( t \in [t_0, T) \), both \( \lim_{s \rightarrow t^+} \theta(s) \) and \( \lim_{s \rightarrow t^-} \theta(s) \) exist and

\[\left\{ t \in [t_0, T) \mid \lim_{s \rightarrow t^+} \theta(s) \neq \lim_{s \rightarrow t^-} \theta(s) \right\}\]

is finite for any finite sub-interval of \([t_0, T)\) (in the case where \( t = t_0 \), we will consider \( \lim_{s \rightarrow t} \theta(s) = \theta(t_0) \)).

Time \( t \in (t_0, T) \) such that \( \lim_{s \rightarrow t^+} \theta(s) \neq \lim_{s \rightarrow t^-} \theta(s) \) is called a switching time. Let \( t_1, t_2, \ldots, t_\ell \) be the ordered switching times of \( \theta \). The sequence of ordered pairs

\[\{(t_0, \theta(t_0^+)), (t_1, \theta(t_1^+)), \ldots, (t_\ell, \theta(t_\ell^+))\}\]

is said to be the switching sequence of \( \theta \) over \([t_0, T)\).

Note that a switching sequence \( \{(t_i, k_i)\}_{i=0}^{\ell} \) uniquely determines a switching path (up to possibly re-arranging the value at the switching times) by the re-
3 Our set-up

We will assume from now that \( M = \{1, 2\} \) (an analogous reasoning might be applied to the case of more subsystems) and that the matrix pencils \( \lambda E_1 + A_1, \lambda E_2 + A_2 \) are regular and in the form

\[
E_1 = \begin{pmatrix} I_{n_1} & N_1 \end{pmatrix}, A_1 = \begin{pmatrix} G_1 & I_{n-n_1} \end{pmatrix} \\
E_2 = \begin{pmatrix} I_{n_2} & N_2 \end{pmatrix}, A_2 = \begin{pmatrix} G_2 & I_{n-n_2} \end{pmatrix}
\]

where \( N_1, N_2 \) are nilpotent matrices with nilpotent indices \( h_1, h_2 \). Let us denote by \( h \) the maximum of these nilpotent indices. We will finally assume that the function \( u(t) \) is a \( h \) times piecewise continuous differentiable function.

We will write

\[
B_1 = \begin{pmatrix} B_{1,1} \\ B_{1,2} \end{pmatrix}, B_2 = \begin{pmatrix} B_{2,1} \\ B_{2,2} \end{pmatrix}
\]

Then we introduce the following notation, for \( i = 1, 2 \):

\[
\overline{G}_i = \begin{pmatrix} G_i & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})
\]

Let us denote by \( \Phi(t, t_0, x_0, u, \sigma) \) the state trajectory at time \( t \) of the continuous-time switched linear system \( \Sigma \) starting from \( t_0 \) with initial value \( x_0 \), input \( u \) and switching well-defined path \( \sigma \).

4 Reachable states

Let us remember the notion of reachability.

**Definition 4.1.** System \( \Sigma \) is (completely) reachable if for any given initial time \( t_0 \in \mathbb{R} \) and state \( x_f \in \mathbb{R}^n \), there exists a real number \( t_f > t_0 \), a switching well-defined path \( \sigma : [t_0, t_f] \rightarrow M = \{1, 2\} \) and an input \( u : [t_0, t_f] \rightarrow \mathbb{R}^n \), such that:

1. \( (I_{n_i} \sigma)(t_f) = \lim_{s \rightarrow t_f} \Phi(t, t_0, x_f, u, \sigma), i \in \{1, 2\} \)

2. \( (0)I_{n-n_i}x_f = \lim_{s \rightarrow t_f} (N_i^j B_{i,2} u(\sigma)(t_f), i \in \{1, 2\} \)

5 A previous result

In [Clotet, Ferrer and Magret, 2009], the authors determined the space of controllable and the space of reachable states and characterized (completely) controllable and (completely) reachable switched singular systems satisfying the “equisingularity condition” \((n_1 = n_2)\).

**Theorem.** (see [Clotet, Ferrer and Magret, 2009]) For system \( \Sigma \), the following conditions are equivalent:

1. \( \Sigma \) is (completely) controllable.
2. \( \Sigma \) is (completely) reachable.
3. \( \mathbb{R}^n = \mathbb{R}^2 \setminus \{1\} | B_{1,2} > \) or \( \mathbb{R}^n = \mathbb{R}^2 \setminus \{2\} | B_{2,2} > \).

where \( \mathbb{R} = \sum_{p=1}^{n_1} \mathbb{R}, \) being

\[
\begin{align*}
\mathbb{R}_1 &= Im | B_{1,1} | B_{1,2} > \\
\mathbb{R}_2 &= \mathbb{R}_1 + G_1 \mathbb{R}_1 + G_2 \mathbb{R}_1 + \ldots + G_{n_1} - 1 \mathbb{R}_1 \\
&= \ldots + G_{n_1} \mathbb{R}_p + G_{n_2} \mathbb{R}_p + \ldots \mathbb{R}_p + 1 \\
&= \ldots + G_{n_1} \mathbb{R}_p + G_{n_2} \mathbb{R}_p + \ldots \mathbb{R}_p + 1
\end{align*}
\]

6 Characterization of reachable states

Let us define

\[
\begin{align*}
V_1 &= (\overline{G}_1 | B_1 > \oplus ([0] \times \{1\} | B_{1,2} > )) + \\
&= (\overline{G}_2 | B_2 > \oplus ([0] \times \{2\} | B_{2,2} > ))
\end{align*}
\]

where \( \overline{G}_i | B_i > (i = 1, 2) \) is the vector subspace spanned by

\[
\left( \begin{array}{cccc}
I_n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
G_1 & 0 & 0 & 0 \\
G_2 & 0 & 0 & 0 \\
\end{array} \right) \mathbb{R}_i, \ldots
\]

and \( \mathbb{R}_i | B_i > (i = 1, 2) \) is the vector subspace spanned by \( B_{1,2} \mathbb{R}_i, B_{1,2} \mathbb{R}_i, \ldots, B_{1,2} \mathbb{R}_i \).

Similarly, for \( k > 1 \),

\[
\begin{align*}
V_k &= (\overline{G}_1 | V_{k-1} > \oplus ([0] \times \{1\} | B_{1,2} > )) + \\
&= (\overline{G}_2 | V_{k-1} > \oplus ([0] \times \{2\} | B_{2,2} > ))
\end{align*}
\]

where \( \overline{G}_i | V_{k-1} > (i = 1, 2) \) is the vector subspace spanned by

\[
\left\{ \left( \begin{array}{cccc}
I_n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
G_1 & 0 & 0 & 0 \\
G_2 & 0 & 0 & 0 \\
\end{array} \right) v, \ldots, v \in V_{k-1} \right\}
\]

Note that \( V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n_0} = V_{n_0+1} = \ldots \)

where \( n_0 = \max \{n_1, n_2\} \).

B. Meng and F.J. Zhang found necessary and sufficient conditions for a switched singular linear system to be (completely) controllable / (completely) reachable. Concretely, they obtained (adapted to our case) the following results.

**Theorem.** ([Meng and Zhang, 2007]) For system \( \Sigma \),

1. if \( \Sigma \) is (completely) controllable then \( V_n = \mathbb{R}^n \).
2. if \( \Sigma \) is (completely) reachable, then \( V_n = \mathbb{R}^n \).

**Theorem.** ([Meng and Zhang, 2007]) For system \( \Sigma \),
(a) if \( V_n = \mathbb{R}^n \) and \( < N_i | B_{1,2} > = \mathbb{R}^{n-n_i} \) for all \( i \in M \), then \( \Sigma \) is (completely) controllable.

(b) if \( V_n = \mathbb{R}^n \) and \( < N_i | B_{1,2} > = \mathbb{R}^{n-n_o} \) for all \( i \in M \), then \( \Sigma \) is (completely) reachable.

The main result is the following one.

**Theorem 6.1.** Let us assume that \( V_1 = \mathbb{R}^n \) and there exists \( i_0 \in M \) such that \( < N_{i_0} | B_{1,2} > = \mathbb{R}^{n-n_o} \). Then the switched singular linear system \( \Sigma \) is (completely) reachable.

**Proof.** For a given switching sequence

\[
\sigma = \{ t_i, i+1 \}_{i=0}^1, \quad t_0 < t_1 < t_2
\]

we consider

\[
\mathcal{R}_i = \{ x = \Phi(t_i, t_0, 0, u, \sigma) | u : [t_0, t_2] \rightarrow \mathbb{R}^m \},
\]

\( 1 \leq i \leq 2 \).

Let \( \ell_i = t_i - t_{i-1}, \ i = 1, 2 \). Then

\[
\mathcal{R}_1 = < G_1 | B_{1,1} > \oplus < N_i | B_{1,2} >
\]

according to [Dai, 1989]. On the other hand,

\[
\mathcal{R}_2 = \left( e^{G_2 \ell_2} (I_{n_2} | 0) \mathcal{R}_1 + < G_2 | B_{2,1} > < N_2 | B_{2,2} > \right)
\]

because \( \mathcal{R}_2 \) is the set

\[
\begin{cases}
  x = \left( x_1(u, \sigma) \\ x_2(u) \right),
  u : [t_0, t_2] \rightarrow \mathbb{R}^m
\end{cases}
\]

where \( x_1(u, \sigma) = e^{G_2 \ell_2} (I_{n_2} | 0) \Phi(t_1, t_0, 0, u, \sigma) + \int_{t_1}^{t_2} e^{G_2 (t_2 - \tau)} B_{1,2} u(\tau) d\tau, \) and

\[
x_2(u) = -\sum_{j=0}^{t_1} N_2 B_{2,2} u(j)(t_2). \]

Then \( \mathcal{R}_2 \) is equal to

\[
\left( e^{G_2 \ell_2} (I_{n_2} | 0) \mathcal{R}_1 + < G_2 | B_{2,1} > \oplus < N_2 | B_{2,2} > \right) = \left( I_{n_2} | 0 \right) e^{G_2 \ell_2} (I_{n_2} | 0) \mathcal{R}_1 + \left( I_{n_2} | 0 \right) < G_2 | B_{2,1} >
\]

\[
+ \left( 0 | I_{n-n_2} \right) < N_2 | B_{2,2} >
\]

Using a Lemma by Meng-Zhang, the dimension of \( \mathcal{R}_2 \) is greater or equal than

\[
\dim \left( \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{n-n_2} \end{bmatrix} \right) < G_2 | B_{2,1} > + \left( 0 | I_{n-n_2} \right) < N_2 | B_{2,2} >
\]

\[
= \dim \left( \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{n-n_2} \end{bmatrix} \right) < G_2 | B_{2,1} > \oplus < N_2 | B_{2,2} >
\]

\[
= \dim \left( \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{n-n_2} \end{bmatrix} \right) < G_2 | B_{2,1} > \oplus < N_2 | B_{2,2} >
\]

Since \( \mathcal{R}_1 \subseteq ((I_{n_2} \oplus N_2 | B_{2,2} >) \) (Meng-Zhang),

\[
\dim \mathcal{R}_2 \geq \dim (\mathcal{R}_1 + ( < G_2 | B_{2,1} > \oplus < N_2 | B_{2,2} > ) = \dim ( ( < G_1 | B_{1,1} > \oplus < N_1 | B_{1,2} > ) + ( < G_2 | B_{2,1} > \oplus < N_2 | B_{2,2} > )
\]

\[
= \dim \mathcal{Y}_1 = n
\]

Therefore, \( \mathcal{R}_2 = \mathbb{R}^n \) and \( \Sigma \) is (completely) reachable. \( \square \)

**7 Conclusion**

In this paper a similar characterization of reachable switched singular linear systems to that in [Clotet, Ferrer and Magret, 2009] is obtained. Note that the hypothesis of “equisingularity” in [Clotet, Ferrer and Magret, 2009] is no longer in the statement. The controls are not assumed to be in the set of admissible controls, as in [Meng and Zhang, 2006].

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**References**


