Higher-Order Singular Systems and Polynomial Matrices

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Abstract.– There is a one-to-one correspondence between the set of quadruples of matrices defining singular linear time-invariant dynamical systems and a subset of the set of polynomial matrices. This correspondence preserves the equivalence relations introduced in both sets (feedback-similarity and strict equivalence): two quadruples of matrices are feedback-equivalent if, and only if, the polynomial matrices associated to them are also strictly equivalent.

Keywords: Singular systems, polynomial matrices.

1 Introduction

Linear singular systems (also called descriptor representations or DAEs, differential-algebraic equations) and their control have been widely studied from the end of the 1970s by many authors (see, for example, [1], [2], [3], [4], [7], [8], [11]). They arise naturally and frequently when mathematically modelling mechanical, electric, economic,... systems.

Minimal indices Kronecker's theory of singular matrix pencils has been widely used to obtain a canonical form of the matrices defining a system.

The matrix pencils $\begin{pmatrix} A & B \end{pmatrix} + \lambda \begin{pmatrix} I & 0 \end{pmatrix}$ are naturally associated to systems which can be represented in the form $\dot{x}(t) = Ax(t) + Bu(t)$. Block-equivalent pairs of matrices (equivalent pairs when considering basis changes, in the state and input spaces, and state feedback) are characterized as those whose associated matrix pencils are strictly equivalent. See [9], [10], [12], for example, for further details.

It is also known that equivalent triples of matrices defining systems of the form $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t)$, under the equivalence relation derived from: basis changes in the state, input and output spaces, state feedback and output injection, are those having strictly equivalent associated matrix pencils

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} + \lambda \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

(see [6], [9] and [10], for example).

Fist-order singular systems may be represented by

$$\left\{ \begin{array}{rl} E\dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{array} \right.$$

where $E, A \in \mathcal{M}_n(\mathbb{C})$, $B \in \mathcal{M}_{n\times m}(\mathbb{C})$, $C \in \mathcal{M}_{p\times n}(\mathbb{C})$ and $\operatorname{rk} E < n$. Singular systems are called regular when the matrix pencil $|sE - A|$ does not vanish identically. Regularity ensures the existence and uniqueness of a solution to the singular system ([2]).

In [5] a special type of polynomial matrices of degree two are associated to them and it is shown that there is a one-to-one correspondence between quadruples of matrices defining such singular systems under "feedback-similarity", the equivalence relation derived when applying one, or more, of the following elementary transformations:

(1) basis similarity for the state space,

(2) basis changes for the control space,

(3) basis changes for the output space,

(4) output injection,

(5) state feedback,

(6) derivative feedback

and degree two polynomial matrices of the form:

$$M(\lambda) = \begin{pmatrix} E & A & B \\ 0 & C & 0 \end{pmatrix} + \lambda \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

under "strict equivalence": $M_1(\lambda)$ and $M_2(\lambda)$ are said to be strictly equivalent when there exist invertible matrices L and R such that $M_2(\lambda) = LM_1(\lambda)R$.

In this work we generalize the result above to the case of higher-order singular systems. Though higher-order singular systems are often studied reducing them to first-order systems, this is not convenient in some cases (when the differentiability of $u(t)$ is limited) since such transformation of the system may lead to a non-equivalent problem: for instance, it is possible to find a continuous solution for the original system, but not for the first-order transformed one). This shows the necessity for directly treat higher-order systems.

2 Higher-order singular systems and polynomial matrices

Let us consider ℓth-order ($\ell \geq 1$) differential-algebraic equations with constant coefficients of the form

$$\left\{ \begin{array}{rl} E_\ell x^{(\ell)}(t) + E_{\ell-1}x^{(\ell-1)}(t) + \ldots + E_1\dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{array} \right.$$

where $E_1, \ldots, E_\ell, A \in \mathcal{M}_n(\mathbb{C})$, $B \in \mathcal{M}_{n\times m}(\mathbb{C})$, $C \in \mathcal{M}_{p\times n}(\mathbb{C})$ and $\operatorname{rk} E_\ell < n$.

First, we will generalize the equivalence relations considered in the Introduction to the set of $(\ell+3)$-tuples of matrices defining singular systems. This equivalence relation will be called *feedback-similarity*.

Then we will associate to each $(\ell+3)$-tuple of matrices describing a singular system a polynomial matrix and recall the notion of strict equivalence in [5], which generalizes strict equivalence for matrix pencils.

Finally, we will prove that there is a one-to-one correspondence between the set of quadruples of matrices and a subset of the set of polynomial matrices of degree two which preserves the considered equivalence relations: *feedback-similarity* and *strict equivalence*. That is to say, two quadruples of matrices are *feedback-similar* if, and only if, the associated matrix polynomials are *strictly equivalent*.

Given a linear time-invariant ℓth-order ($\ell \geq 1$) differential-algebraic equations with constant coefficients, we will consider the following elementary transformations:

(1) basis similarity for the state space:

$(E_\ell, \ldots, E_1, A, B, C) \longrightarrow (P^{-1}E_\ell P, \ldots, P^{-1}E_1P, P^{-1}AP, P^{-1}B, CP)$;

(2) basis changes for the control space:

$(E_\ell, \ldots, E_1, A, B, C) \longrightarrow (E_\ell, \ldots, E_1, A, BR, C)$;

(3) basis changes for the output space:

$(E_\ell, \ldots, E_1, A, B, C) \longrightarrow (E_\ell, \ldots, E_1, A, B, SC)$;

(4) output injection:

$(E_\ell, \ldots, E_1, A, B, C) \longrightarrow (E_\ell, \ldots, E_1, A + WC, B, C)$;

(5) state feedback:

$(E_\ell, \ldots, E_1, A, B, C) \longrightarrow (E_\ell, \ldots, E_1, A + BU, B, C)$;

(6) derivative feedback of order $i \geq 1$:

$(E_\ell, \ldots, E_i, \ldots, E_1, A, B, C) \longrightarrow (E_\ell, \ldots, E_i + BV_i, \ldots, E_1, A, B, C)$

for some matrices $P \in Gl_n(\mathbb{C})$, $R \in Gl_m(\mathbb{C})$, $S \in Gl_p(\mathbb{C})$, $U, V_1, \ldots, V_r \in M_{m\times n}(\mathbb{C})$ and $W \in M_{n\times p}(\mathbb{C})$. These transformations lead to the definition of the following equivalence relation.

Definition 1. $(E_\ell, \dots, E_1, A, B, C)$, $({E'}_\ell, \dots, {E'}_1, A', B', C')$ are *feedback-similar* if, and only if, there exist matrices $P \in Gl_n(\mathbb{C})$, $R \in Gl_m(\mathbb{C})$, $S \in Gl_p(\mathbb{C})$, $U, V_1, \dots, V_\ell \in M_{m\times n}(\mathbb{C})$ and $W \in M_{n\times p}(\mathbb{C})$ such that

$$\begin{array}{rl} {E'}_\ell & = P^{-1}E_\ell P + P^{-1}BV_\ell, \dots, {E'}_1 = P^{-1}E_1P + P^{-1}BV_1, \\ A' & = P^{-1}AP + WCP + P^{-1}BU, B' = P^{-1}BR, C' = SCP \end{array}$$

That is to say, two $(\ell+3)$-tuples of matrices are said to be feedback-similar when one can be obtained from the other by means of one, or more, of the elementary transformations (1) - (6) above.

Let us associate to the $(\ell+3)$-tuple $(E_\ell, \dots, E_1, A, B, C)$ the polynomial matrix

$$M(\lambda) = \begin{pmatrix} E_\ell & \dots & E_1 & A & B \\ 0 & \dots & 0 & C & 0 \end{pmatrix} + \lambda M_1(\lambda) + \ldots + \lambda^\ell M_\ell(\lambda) + \lambda^{\ell+1} M_{\ell+1}(\lambda)$$

$$M_1(\lambda) = \begin{pmatrix} I_n & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$M_2(\lambda) = \begin{pmatrix} 0 & I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\dots$$

$$M_\ell(\lambda) = \begin{pmatrix} 0 & \dots & 0 & I_n & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{\ell+1}(\lambda) = \begin{pmatrix} 0 & \dots & 0 & 0 & I_n & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strict equivalence in the set of matrix pencils may be generalized to the set of polynomial matrices.

Definition 2. We will say that two polynomial matrices $M(\lambda)$ and $N(\lambda)$ are *strictly equivalent* when there exist constant regular matrices L and R such that $LM(\lambda)R = N(\lambda)$.

We can state now the relationship between high-order singular systems under feedback-similarity and polynomial matrices under strict equivalence.

Theorem 1. *$({E'}_\ell, \dots, {E'}_1, A', B', C')$ and $(E_\ell, \dots, E_1, A, B, C)$ are feedback-similar if, and only if, the associated polynomial matrices $M(\lambda)$ and $M'(\lambda)$ are strictly equivalent.*

Proof. Let us assume $(E_\ell, \dots, E_1, A, B, C)$ and $({E'}_\ell, \dots, {E'}_1, A', B', C')$ feedback-similar. Then there exist $P \in Gl_n(\mathbb{C})$, $R \in Gl_m(\mathbb{C})$, $S \in Gl_p(\mathbb{C})$, $U, V_1, \dots, V_\ell \in M_{m\times n}(\mathbb{C})$ and $W \in M_{n\times p}(\mathbb{C})$ such that

$$\begin{array}{rl} {E'}_\ell & = P^{-1}E_\ell P + P^{-1}BV_\ell, \dots, {E'}_1 = P^{-1}E_1P + P^{-1}BV_1, \\ A' & = P^{-1}AP + WCP + P^{-1}BU, B' = P^{-1}BR, C' = SCP \end{array}$$

Equivalently,

$$\begin{pmatrix} P & W \\ 0 & Q \end{pmatrix} \begin{pmatrix} E_\ell & \dots & E_1 & A & B \\ 0 & \dots & 0 & C & 0 \end{pmatrix} \begin{pmatrix} P^{-1} & \dots & 0 & 0 & 0 \\ \dots & & \dots & \dots & \dots \\ 0 & \dots & P^{-1} & 0 & 0 \\ 0 & \dots & 0 & P^{-1} & 0 \\ V_\ell & \dots & V_1 & U & S \end{pmatrix} = \begin{pmatrix} E'_\ell & \dots & E'_1 & A' & B' \\ 0 & \dots & 0 & C' & 0 \end{pmatrix}$$

It suffices to take

$$L = \begin{pmatrix} P & W \\ 0 & Q \end{pmatrix}, \ R = \begin{pmatrix} P^{-1} & \dots & 0 & 0 & 0 \\ \dots & & \dots & \dots & \dots \\ 0 & \dots & P^{-1} & 0 & 0 \\ 0 & \dots & 0 & P^{-1} & 0 \\ V_\ell & \dots & V_1 & U & S \end{pmatrix}$$

It is straightforward that

$$LM_i(\lambda)R = M'_i(\lambda) \quad 1 \le i \le \ell+1$$

and therefore

$$LM(\lambda)R = M'(\lambda)$$

Conversely. We assume that

$$LM(\lambda)R = M'(\lambda)$$

where

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}, \ R = \begin{pmatrix} R_1^1 & \dots & R_{\ell+2}^1 \\ \dots & & \dots \\ R_1^{\ell+2} & \dots & R_{\ell+2}^{\ell+2} \end{pmatrix}$$

Equivalently,

$$L \begin{pmatrix} E_\ell & \dots & E_1 & A & B \\ 0 & \dots & 0 & C & 0 \end{pmatrix} R = \begin{pmatrix} E'_\ell & \dots & E'_1 & A' & B' \\ 0 & \dots & 0 & C' & 0 \end{pmatrix}$$

$$LM_i(\lambda)R = M'_i(\lambda) \quad 1 \le i \le \ell+1$$

From these equalities we conclude that L_1 is regular, $R_1^1 = {L_1}^{-1}$, $L_3 = 0$, $R_j^i = 0$ for $1 \le i,j \le \ell+1$ $(i \ne j)$ and thus

$$\begin{pmatrix} L_1 & L_2 \\ 0 & L_4 \end{pmatrix} \begin{pmatrix} E_\ell & \dots & E_1 & A & B \\ 0 & C & 0 & & \end{pmatrix} \begin{pmatrix} {L_1}^{-1} & \dots & 0 & 0 & 0 \\ \dots & & \dots & \dots & \dots \\ 0 & \dots & {L_1}^{-1} & 0 & 0 \\ 0 & \dots & 0 & L_1^{-1} & 0 \\ R_1^{\ell+2} & \dots & R_\ell^{\ell+2} & R_{\ell+1}^{\ell+2} & R_{\ell+2}^{\ell+2} \end{pmatrix} =$$

$$\begin{pmatrix} E'_\ell & \dots & E'_1 & A' & B' \\ 0 & \dots & 0 & C' & 0 \end{pmatrix}$$

Therefore $(E_\ell, \dots, E_1, A, B, C)$ and $(E'_\ell, \dots, E'_1, A', B', C')$ are feedback-similar. $\diamond$

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