A note on the degree sequences of graphs with restrictions

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Abstract

Degree sequences of some types of graphs will be studied and characterized in this paper.

Key words: Graph, degree sequence, simple graph, $1-$graph, digraph, graphic sequence, 1-graphic sequence, digraphic sequences, bigraphic sequences.

1 Introduction.

For the undefined notation in this paper, we follow [3]. For a simple graph G, that is to say a graph in which we do not allow neither loops nor multiple edges, the degree sequence of G is the list of vertex degrees of G, usually written in a non increasing order, as $D : d_1 \geq d_2 \geq \cdots \geq d_n$.

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A graphic sequence is a list of nonnegative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence D 'realizes' D. The same concepts also extend to graphs in general in the obvious way, where loops increment the degree of a vertex by two.

Havel [2] in 1955 and Hakimi [1] in 1962 introduced independently the following result that characterizes degree sequences for simple graphs.

Theorem 1.1 *For $n \geq 1$ the nonnegative integer list D of size D is graphic if and only if D' is graphic, where D' is the list of size $n-1$ obtained from D by deleting its largest element Δ and subtracting 1 from its Δ next largest elements.*

Let $P : p_1 \geq p_2 \geq \cdots \geq p_m$ and $Q : q_1 \geq q_2 \geq \cdots \geq q_n$ be sequences of nonnegative integers. The pair (P, Q) is bigraphic if and only if there is a simple bipartite graph in which $p_1 \geq p_2 \geq \cdots \geq p_m$ are the degrees of the vertices of one partite set and $q_1 \geq q_2 \geq \cdots \geq q_n$ are the degrees of the vertices of the other partite set. (See [3]).

The next theorem is a version of Havel-Hakimi's theorem for bipartite sequences (See [3]).

Theorem 1.2 *The pair (P, Q) is bigraphic if and only if the pair (P', Q') is bigraphic, where (P', Q') is obtained from (P, Q) by deleting the largest element p_1 of P and subtracting one from each of the p_1 largest elements of Q.*

If we do not restrict ourselves to simple graphs, then we have the following result (See [3]).

Theorem 1.3 *The nonnegative integers $d_1, \ldots, d_n$ are the vertex degrees of some graph if and only if $\sum_{i=1}^{n} d_i$ is even.*

Let $\vec{G}$ be any digraph of order n. The sequences $\begin{cases} F : o_1; o_2; \ldots; o_n \\ G : i_1; i_2; \ldots; i_n \end{cases}$

are called the sequences of out degrees and in degrees of $\vec{G}$ if we can rename the vertices of $\vec{G}$ as $V(\vec{G}) = \{v_1, v_2, \ldots, v_n\}$ in such away that $out(v_k) = o_k$ and $in(v_k) = i_k$ for every $k \in \{1, 2, \ldots, n\}$.

Two sequences P and Q are said to be digraphic if there is a digraph $\vec{G}$ such that the sequences P and Q are the sets of out degrees and of in degrees of $\vec{G}$ respectively. In this case $\vec{G}$ is said to realize the sequences P and Q.

In light of theorems 1.1 and 1.3, it is clear that it is known how to determine the sequences of integers which are graphic, for graphs in general and for simple graphs.

Therefore in this work we deal with graphs which are not simple, but for which the number of loops and multiple edges is restricted. In particular, we consider graphs in the set **G** defined as follows:

G = {simple graphs} $\cup$ {graphs with at most one loop attached at each vertex and at most two edges joining every pair of vertices}

We call the elements of **G** $1-$graphs.

For a $1-$graph G, the degree sequence of G is the list of vertex degrees of G, usually written in a non increasing order as $d_1 \geq d_2 \geq \cdots \geq d_n$.

A $1-$graphic sequence is a list of nonnegative numbers that is the degree sequence of some $1-$graph. A $1-$graph with degree sequence D 'realizes' D.

In this paper we characterize $1-$graphic sequences in terms of bigraphic sequences. However, in order to do this we need to use the following theorem which is well known in the graph theory literature.

Theorem 1.4 *Let G be any graph with all vertices of even degree. Then G admits a decomposition into edge disjoint cycles.*

2 Characterization of $1-$graphic sequences.

Now we are ready to state and prove the following characterization.

Theorem 2.1 *Let $D : d_1 \geq d_2 \geq \cdots \geq d_n$ be a sequence of non negative integers. Then D is $1-$graphic if and only if there exist two sequences F and G*

$$\begin{cases} F : f_1; f_2; \ldots; f_n \\ G : g_1; g_2; \ldots; g_n \end{cases}$$

such that

1. *if d_i is even then $f_i = g_i = \frac{d_i}{2}$.*
2. *if d_i is odd then $f_i + g_i = d_i$ and $|f_i - g_i| = 1$.*
3. *The pair (F, G) is bigraphic.*

Proof.
($\Longrightarrow$) Let $D : d_1 \geq d_2 \geq \cdots \geq d_n$ be a $1-$graphic sequence. Then there is a $1-$graph G that realizes D. Let the vertex of G with degree d_i be denoted by v_i, and construct a new graph H as follows:

$$\begin{cases} V(H) = V(G) \cup \{u\} \\ E(H) = E(G) \cup \{uv_i \ : \ d_i \text{ is odd}\} \end{cases}$$

Then, since every graph contains an even number of vertices of odd degree, it follows that all vertices in H have even degree. Therefore, by theorem 1.4, we can decompose H into edge disjoint cycles. We orient each cycle in the decomposition cyclically obtaining a new digraph that we call $\vec{H}$. The digraph $\vec{H}$ has the property that $out_{\vec{H}} = in_{\vec{H}}$ for every vertex $x \in V(\vec{H})$. At this point we define a new digraph $\vec{G}$ to be the digraph obtained from $\vec{H}$ by deleting vertex u. That is to say $\vec{G} = \vec{H} \setminus u$. Hence, it is clear that $out_{\vec{G}} + in_{\vec{G}} = d_i$ and that the vertices of $\vec{G}$ have the following property:

1. if d_i is even then $out_{\vec{G}} = in_{\vec{G}} = \frac{d_i}{2}$
2. if d_i is odd then $|out_{\vec{G}} - in_{\vec{G}}| = 1$

Consider the adjacency matrix of $\vec{G}$, namely $A(\vec{G}) = (a_{ij})$, and construct the bipartite digraph $\vec{K}$ with bipartite set $V_1 = \{u_i \ : \ 1 \leq i \leq n\}$ and $V_2 = \{v_i \ : \ 1 \leq i \leq n\}$ and with an arc u_iv_j if and only if $a_{ij} = 1$. The sequences of out degrees and of in degrees of $\vec{K}$ satisfy the theorem.

($\Longleftarrow$) Let K be a bipartite graph with bigraphic pair (F, G) and bipartite sets $V_1 = \{u_i \ : \ 1 \leq i \leq n\}$ and $V_2 = \{v_i \ : \ 1 \leq i \leq n\}$ and let (b_{ij}) the $n \times n$ matrix with $b_{ij} = 1$ if and only if (u_iv_j) is an arc of $\vec{K}$ and $b_{ij} = 0$ otherwise. Let $\vec{G}$ a digraph with $V(G) = \{w_i \ : \ 1 \leq i \leq n\}$ and adjacency matrix (b_{ij}); as $in_{\vec{G}} + out_{\vec{G}} = f_i + g_i = d_i$, und$(\vec{G})$ is $1-$graphic and realizes D. □

Now, with the previous results in hand, we obtain the following immediate corollaries:

Corollary 2.1 *Let $D : d_1 \geq d_2 \geq \cdots \geq d_n$ be a sequence of non negative integers. Then D is $1-$graphic if and only if the sequences*

$$H_1 = H_2 = \frac{d_1}{2} \geq \frac{d_2}{2} \geq \cdots \geq \frac{d_n}{2}$$

are bigraphic

Corollary 2.2 *Let $D : d_1 \geq d_2 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{k+l} \geq d_{k+l+1} \geq d_{k+l+2} \geq \cdots \geq d_n$ be a sequence of non negative integers such that*

1. *d_i is even for all $i \in \{1, 2, \ldots, k-1, k+l+1, k+l+2, \ldots, n\}$*
2. *d_i is odd for all $i \in \{k, k+1, k+2, \ldots, k+l\}$*

Then D is $1-$graphic if and only if the sequences

$$H_1 = H_2 = \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_{k-1}}{2}, \lceil \frac{d_k}{2} \rceil, \lceil \frac{d_{k+1}}{2} \rceil, \ldots, \lceil \frac{d_{k+\frac{l-1}{2}}}{2} \rceil, \lfloor \frac{d_{k+\frac{l}{2}}}{2} \rfloor,$$

$$\lfloor \frac{d_{k+\frac{l+1}{2}}}{2} \rfloor, \ldots, \lfloor \frac{d_{k+l}}{2} \rfloor, \frac{d_{k+l+1}}{2}, \frac{d_{k+l+2}}{2}, \ldots, \frac{d_n}{2}$$

are bigraphic.

Corollary 2.3 *Let $D : d_1 \geq d_2 \geq \cdots \geq d_n$ be a sequence of non negative integers with the property that for some k, l such that $1 \leq k < l \leq n$, we have that d_k and d_l are odd integers, and for $i \in \{1, 2, \ldots, n\} \setminus \{k, l\}$ d_i is even. Then, D is $1-$graphic if and only if the sequences*

$$H_1 : \frac{d_1}{2}, \ldots, \frac{d_{k-1}}{2}, \lceil \frac{d_k}{2} \rceil, \frac{d_{k+1}}{2}, \ldots, \frac{d_{l-1}}{2}, \lfloor \frac{d_l}{2} \rfloor, \frac{d_{l+1}}{2}, \frac{d_n}{2}$$

$$H_2 : \frac{d_1}{2}, \ldots, \frac{d_{k-1}}{2}, \lfloor \frac{d_k}{2} \rfloor, \frac{d_{k+1}}{2}, \ldots, \frac{d_{l-1}}{2}, \lceil \frac{d_l}{2} \rceil, \frac{d_{l+1}}{2}, \frac{d_n}{2}$$

are bigraphic.

3 Conclutions.

We have found necessary and sufficient conditions for a sequence of non negative integers to determine whether or not such sequence is $1-$graphic. Furthermore, when the sequence contains an small number of odd integers, then we can see that algorithm suggested by the characterization is fast. However this is not the case when the sequence has many odd integers. This leads us to formulate the following question:

Question 1: Given a sequence of non negative integers, is it possible to find an efficient algorithm that will allow us to determine whether the sequence is $1-$graphic or it is not, when we have many odd integers in the sequence?

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