# NEWTON POLYGONS OF HIGHER ORDER IN ALGEBRAIC NUMBER THEORY

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Abstract. We develop a theory of arithmetic Newton polygons of higher order, that provides the factorization of a separable polynomial over a $p$-adic field, together with relevant arithmetic information about the fields generated by the irreducible factors. This carries out a program suggested by Ø. Ore. As an application, we obtain fast algorithms to compute discriminants, prime ideal decomposition and integral bases of number fields.

## Introduction

R. Dedekind based the foundations of algebraic number theory on ideal theory, because the constructive attempts to find a rigorous general definition of the *ideal numbers* introduced by E. Kummer failed. This failure is due to the existence of inessential discriminant divisors; that is, there are number fields $K$ and prime numbers $p$, such that $p$ divides the index, $\mathrm{ind}(\theta) := (\mathbb{Z}_K \colon \mathbb{Z}[\theta])$, for any integral generator $\theta$ of $K$, where $\mathbb{Z}_K$ is the ring of integers. Dedekind gave a criterion to detect when $p \nmid \mathrm{ind}(\theta)$, and a procedure to construct the prime ideals of $K$ dividing $p$ in that case, in terms of the factorization of the minimal polynomial of $\theta$ modulo $p$ [Ded78].

M. Bauer introduced an arithmetic version of Newton polygons to construct prime ideals in cases where Dedekind's criterion failed [Bau07]. This theory was developed and extended by Ø. Ore in his 1923 thesis, and a series of papers that followed [Ore23, Ore24, Ore25, Ore26, Ore28]. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial that generates $K$. After K. Hensel's work, the prime ideals of $K$ lying above $p$ are in bijection with the irreducible factors of $f(x)$ over $\mathbb{Z}_p[x]$. Ore's work determines three successive factorizations of $f(x)$ in $\mathbb{Z}_p[x]$, known as the *three classical dissections* (cf. [Ber27], [Coh95]). The first dissection is determined by Hensel's lemma: $f(x)$ splits into the product of factors that are congruent to the power of an irreducible polynomial modulo $p$. The second dissection is a further splitting of each factor, according to the number of sides of certain Newton polygon. The third dissection is a further splitting of each of the late factors, according to the factorization of certain *residual polynomial* attached to each side of a polygon, which is a polynomial with coefficients in a finite field.

Unfortunately, the factors of $f(x)$ obtained after these three dissections are not always irreducible. Ore defined a polynomial to be *$p$-regular* when it satisfies a technical condition that ensures that the factorization of $f(x)$ is complete after the three dissections. Also, he proved the existence of a $p$-regular defining equation for every number field, but the proof is not constructive: it uses the Chinese remainder theorem with respect to the different prime ideals that one wants to construct. Ore

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himself suggested that it should be possible to introduce Newton polygons of higher order that continue the factorization process till all irreducible factors of $f(x)$ are achieved [Ore23, Ch.4,§8], [Ore28, §5].

Ore's program was carried out by the second author in his 1999 thesis [Mon99], under the supervision of the third author. For any natural number $r \geq 1$, Newton polygons of order $r$ were constructed, the case $r = 1$ corresponding to the Newton polygons introduced by Ore. Also, analogous to Ore's theorems were proved for polygons of order $r$, providing two more dissections of the factors of $f(x)$, for each order $r$. The whole process is controled by an invariant defined in terms of *higher order indices*, that ensures that the process finishes at most in $\mathrm{ind}(f) := v_p(\mathrm{ind}(\theta))$ steps, where $\theta$ is a root of $f(x)$. Once an irreducible factor of $f(x)$ is detected, the theory determines the ramification index and residual degree of the $p$-adic field generated by this factor, in terms of combinatorial data attached to the sides of the higher order polygons and the residual polynomials of higher order attached to each side. The process yields a computation of $\mathrm{ind}(f)$ as a by-product. An implementation in Mathematica of this factorization algorithm was worked out by the first author [Gua97].

We present these results for the first time in the form of a publication, after a thorough revision and some simplifications. In section 1 we review Ore's results, with proofs, which otherwise can be found only in the original papers by Ore in the language of "höhere Kongruenzen". In section 2 we develop the theory of Newton polygons of higher order, based in the concept of a *type* and its *representative*, which plays the analogous role in order $r$ to that played by an irreducible polynomial modulo $p$ in order one. In section 3 we prove analogous in order $r$ to Ore's Theorems of the polygon and of the residual polynomial (Theorems 3.1 and 3.7), that provide two more dissections for each order. In section 4 we introduce resultants and indices of higher order and we prove the Theorem of the index (Theorem 4.18), that relates $\mathrm{ind}(f)$ with the higher order indices constructed from the higher order polygons. This result guarantees that the factorization process ends after a finite number of steps.

Although the higher order Newton polygons are apparently involved and highly technical objects, they provide fast factorization algorithms, because all computations are mainly based on two reasonably fast operations: division with remainder of monic polynomials with *integer* coefficients, and factorization of polynomials over *finite* fields. Thus, from a modern perspective, the main application of these results is the design of fast algorithms to compute discriminants, prime ideal decomposition and integral bases of number fields. However, we present in this paper only the theoretical background of higher order Newton polygons. We shall describe the concrete design of the algorithms and discuss the relevant computational aspects elsewhere [GMN08a, GMN08b].

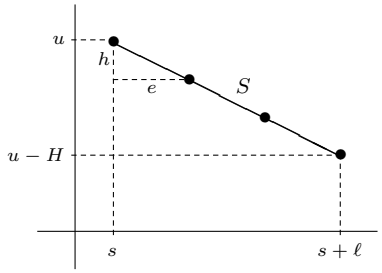
## 1. Newton polygons of the first order

1.1. **Abstract polygons.** Let $\lambda \in \mathbb{Q}^-$ be a negative rational number, expressed in lower terms as $\lambda = -h/e$, with $h, e$ positive coprime integers. We denote by $\mathcal{S}(\lambda)$ the set of segments of the Euclidian plane with slope $\lambda$ and end points having nonnegative integer coordinates. The points of $(\mathbb{Z}_{\geq 0})^2$ are also considered to be segments in $\mathcal{S}(\lambda)$, whose initial and final points coincide. The elements of $\mathcal{S}(\lambda)$ will be called *sides of slope* $\lambda$. For any side $S \in \mathcal{S}(\lambda)$, we define its *length*, $\ell := \ell(S)$,

and *height*, $H := H(S)$, to be the length of the respective projections of $S$ to the horizontal and vertical axis. We define the *degree* of $S$ to be

$$d := d(S) := \ell(S)/e = H(S)/h.$$

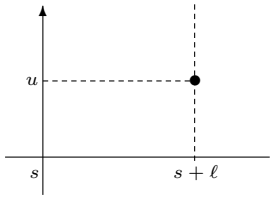
Note that any side $S$ of positive length is divided into $d$ segments by the points of integer coordinates that lie on $S$. A side $S \in \mathcal{S}(\lambda)$ is determined by the initial point $(s, u)$ and the length $\ell$, or equivalently, by the initial point and the degree $d$. The final point is $(s + \ell, u - H) = (s + de, u - dh)$. For instance, the next figure represents a side of slope $-1/2$, initial point $(s, u)$, and degree three.




The set $\mathcal{S}(\lambda)$ has the structure of an abelian semigroup with the following addition rule: given $S, T \in \mathcal{S}(\lambda)$, the sum $S + T$ is the side of length $\ell(S) + \ell(T)$ of $\mathcal{S}(\lambda)$, whose initial point is the sum of the initial points of $S$ and $T$. Thus, the addition is geometrically represented by the process of joining the two segments and choosing an apropriate initial point. The addition of a segment $S$ with a point $P$ is represented by the translation $P + S$ of $S$ by the vector represented by $P$. The neutral element is the point $(0, 0)$. The invariants $\ell(S)$, $H(S)$, $d(S)$ determine semigroup homomorphisms

$$\ell,\, H,\, d : \mathcal{S}(\lambda) \longrightarrow \mathbb{Z}_{\geq 0}.$$

For technical reasons we consider also a set of *sides of slope* $-\infty$, which is formally defined as $\mathcal{S}(-\infty) := \mathbb{Z}_{>0} \times (\mathbb{Z}_{\geq 0})^2$. If $S = (\ell, (s, u))$ is a side of slope minus infinity, we define $\ell(S) := \ell$, $H(S) := \infty$, $d(S) := 1$. Also, we take by convention $h = \infty$, $e = \ell$. This set has an obvious structure of an abelian monoid, and the length determines a monoid homomorphism, $\ell \colon \mathcal{S}(-\infty) \longrightarrow \mathbb{Z}_{>0}$. There is a geometric representation of such an $S$ as a side whose end points are $(s, \infty)$ and $(s + \ell, u)$.

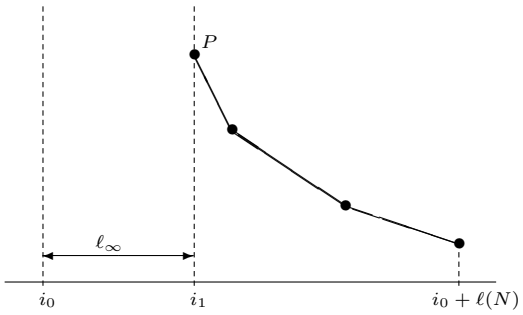



The *set of sides of negative slope* is defined as the formal disjoint union

$$\mathcal{S} := \mathcal{S}(-\infty) \coprod \left( \bigcup_{\lambda \in \mathbb{Q}^-} \mathcal{S}(\lambda) \right).$$

Note that the points of $(\mathbb{Z}_{\geq 0})^2$ belong formally to $\mathcal{S}(\lambda)$ for all finite $\lambda$, so that it is not possible (even in a formal sense) to attach a slope to them.

We have a natural geometric representation of a side. Let us introduce a geometric representation of a formal sum of sides as an open convex polygon of the plane. Let $N = S_1 + \cdots + S_t$ be a formal sum of sides of negative slope. Let $S_\infty = (\ell_\infty, P_\infty)$ be the sum of all sides of slope $-\infty$ among the $S_i$. Take $P_0$ to be the sum of all initial points of the $S_i$ that don't belong to $\mathcal{S}(-\infty)$ (the empty sum is considered to be $P_0 = (0,0)$). Let $P = P_\infty + (\ell_\infty, 0) + P_0$. Then, $N$ is represented as the polygon that starts at $P$ and is obtained by joining all sides of positive length and finite slope, ordered by increasing slopes. If $i_1$ is the abscissa of $P$, we have to think that the polygon starts at the abscisa $i_0 = i_1 - \ell_\infty$, that formally indicates the starting point (at infinity) of a side of slope $-\infty$. The typical shape of this polygon is




**Definition 1.1.** *The semigroup $\mathcal{PP}$ of principal polygons is defined to be the set of all these geometric configurations.*

By definition, every principal polygon represents a formal sum, $N = S_1 + \cdots + S_t$, of sides $S_i \in \mathcal{S}$. This expression is unique in any of the two following situations

(1) $N = S$, with $S \in (\mathbb{Z}_{\geq 0})^2$,
(2) $N = S_1 + \cdots + S_t$, with all $S_i$ of positive length and pairwise different slopes.

It is clear that any $N \in \mathcal{PP}$ can be expressed in one (and only one) of these canonical forms. Usually, when we speak of the *sides* of a principal polygon, we mean the sides of this canonical expression. If we need to emphasize this we shall use the term *canonical sides* of $N$. The finite end points of the canonical sides are called the *vertices* of the polygon.

The addition of polygons is defined in terms of the expression as a formal sum of sides (not necessarily the canonical ones). That is, if $N = S_1 + \cdots + S_r$ and $N' = S'_1 + \cdots + S'_s$, then $N + N'$ is the geometric representation of $S_1 + \cdots + S_r + S'_1 + \cdots + S'_s$. The reader may check easily that this is well-defined and $\mathcal{PP}$ has a structure of semigroup with neutral element $\{(0,0)\}$.

Also, it is clear that this addition is compatible with the sum operations that we had on all $\mathcal{S}(\lambda)$. Note that the addition of $N \in \mathcal{PP}$ with (the polygon represented by) a point $P \in (\mathbb{Z}_{\geq 0})^2$ is the translation $P + N$. The fact of adding to $N$ (the polygon represented by) a side of slope $-\infty$ is reflected by a horizontal shift of the finite part of $N$, without changing the starting abscissa $i_0$ of $N$.

**Definition 1.2.** *We define the length of a principal polygon $N = S_1 + \cdots + S_r$ to be $\ell(N) := \ell(S_1) + \cdots + \ell(S_r)$. Thus, the length determines a semigroup homomorphism, $\ell \colon \mathcal{PP} \longrightarrow \mathbb{Z}_{\geq 0}$.*

Let $N \in \mathcal{PP}$. Let $i_0$ be the abscissa where the polygon starts and $i_1$ the abscissa of the point $P$ where the finite part of $N$ starts. For any integer abscissa $i_0 \leq i \leq i_0 + \ell(N)$ we denote by

$$h_i = h_i(N) = \begin{cases} \infty, & \text{if } i_0 \leq i < i_1, \\ \text{the ordinate of the point of } N \text{ of abscissa } i, & \text{if } i_1 \leq i. \end{cases}$$

For $i \geq i_1$ these rational numbers form an strictly decreasing sequence.

**Definition 1.3.** *Let $P = (i, y)$ be a "point" of the plane, with $y \in \mathbb{R} \cup \{\infty\}$ and integer abscissa $i_0 \leq i \leq i_0 + \ell(N)$. We say that $P$* lies on *$N$ if $y = h_i$. We say that $P$* lies above *$N$ if $y \geq h_i$. We say that $P$* lies strictly above *$N$ if $y > h_i$.*

For any $i_1 < i \leq i_0 + \ell(N)$, let $\mu_i$ be the slope of the side of $N$ whose projection to the horizontal axis contains $i - 1/2$, or equivalently, the slope of the segment joining $(i-1, h_{i-1})$ and $(i, h_i)$. The sequence $\mu_{i_1+1} \leq \cdots \leq \mu_{i_0+\ell(N)}$ is an increasing sequence of negative rational numbers. We call these elements the *unit slopes* of $N$. Consider the multisets of unit slopes:

$$U_{i_1}(N) := \emptyset; \quad U_i(N) := \{\mu_{i_1+1}, \ldots, \mu_i\}, \quad \forall i_1 < i \leq i_0 + \ell(N).$$

Clearly, $h_i(N) = h_{i_1}(N) + \sum_{\mu \in U_i(N)} \mu$.

Let $N'$ be another principal polygon with starting abscissa $j_0$ and starting abscissa for the finite part $j_1$. Consider analogous multisets $U_j(N')$, for all $j_1 \leq j \leq j_0 + \ell(N')$. By the definition of the addition law of principal polygons, the multiset $U_k(N + N')$ contains the smallest $k - i_1 - j_1$ unit slopes of the multiset $U_{i_0+\ell(N)}(N) \cup U_{j_0+\ell(N')}(N')$ that contains all unit slopes of both polygons. Thus,

$$h_i(N) + h_j(N') \geq h_{i+j}(N + N'),$$

and equality holds if and only if $U_i(N) \cup U_j(N') = U_{i+j}(N + N')$.

**Lemma 1.4.** *Let $N, N' \in \mathcal{PP}$. Let $P = (i, u)$ be a point lying above the finite part of $N$ and $P' = (j, u')$ a point lying above the finite part of $N'$. Then $P + P'$ lies above the finite part of $N + N'$ and*

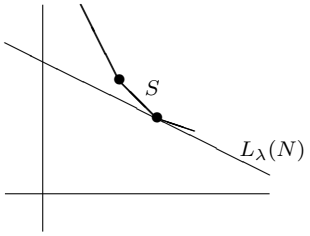
$$P + P' \in N + N' \iff P \in N,\ P' \in N',\ \text{and } U_i(N) \cup U_j(N') = U_{i+j}(N + N').$$

*Proof.* Clearly, $u + u' \geq h_i(N) + h_j(N') \geq h_{i+j}(N + N')$ and $P + P' \in N + N'$ if and only if both inequalities are equalities. □

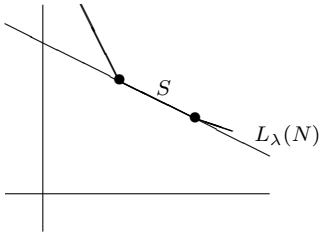
**Definition 1.5.** *Let $\lambda \in \mathbb{Q}^-$ and $N \in \mathcal{PP}$. Consider a line of slope $\lambda$ far below $N$ and let it move upwards till it touches $N$ for the first time. Denote by $L_\lambda(N)$ this line having first contact with $N$. We define the* $\lambda$-component *of $N$ to be $S_\lambda(N) := N \cap L_\lambda(N)$. We obtain in this way a map:*

$$S_\lambda \colon \mathcal{PP} \longrightarrow \mathcal{S}(\lambda).$$

If $N$ has a canonical side $S$ of positive length and finite slope $\lambda$, we have $S_\lambda(N) = S$, otherwise the $\lambda$-component $S_\lambda(N)$ reduces to a point.




$S_\lambda(N)$ = final point of $S$




$S_\lambda(N) = S$

Lemma 1.4 shows that $S_\lambda$ is a semigroup homomorphism:

$$S_\lambda(N+N') = S_\lambda(N) + S_\lambda(N'), \tag{1}$$

for all $N$, $N' \in \mathcal{PP}$ and all $\lambda \in \mathbb{Q}^-$.

1.2. **$\phi$-Newton polygon of a polynomial.** Let $p$ be a prime number and let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of the field $\mathbb{Q}_p$ of the $p$-adic numbers. For any finite extension, $\mathbb{Q}_p \subseteq L \subseteq \overline{\mathbb{Q}}_p$, of $\mathbb{Q}_p$ we denote by $v_L$ the $p$-adic valuation, $v_L\colon \overline{\mathbb{Q}}_p \longrightarrow \mathbb{Q}\cup\{\infty\}$, normalized by $v_L(L^*) = \mathbb{Z}$. Also, throughout the paper $\mathcal{O}_L$ will denote the ring of integers of $L$, $\mathfrak{m}_L$ its maximal ideal, and $\mathbb{F}_L$ the residue field. The canonical reduction map $\mathrm{red}_L\colon \mathcal{O}_L \longrightarrow \mathbb{F}_L$ will be usually indicated by a bar: $\overline{\alpha} := \mathrm{red}_L(\alpha)$.

We fix a finite extension $K$ of $\mathbb{Q}_p$ as a base field, and we denote $v := v_K$, $\mathcal{O} := \mathcal{O}_K$, $\mathfrak{m} := \mathfrak{m}_K$, $\mathbb{F} := \mathbb{F}_K$, $q := |\mathbb{F}|$. We fix also a prime element $\pi \in \mathcal{O}$.

We extend the valuation $v$ to polynomials with coefficients in $\mathcal{O}$ in a natural way:

$$v\colon \mathcal{O}[x] \longrightarrow \mathbb{Z}_{\geq 0}\cup\{\infty\}, \quad v(b_0+\cdots+b_rx^r) := \min\{v(b_j),\, 0\leq j\leq r\}.$$

Let $\phi(x) \in \mathcal{O}[x]$ be a monic polynomial of degree $m$ whose reduction modulo $\mathfrak{m}$ is irreducible. We denote by $\mathbb{F}_\phi$ the finite field $\mathcal{O}[x]/(\pi,\phi(x))$, and by $\mathrm{red}\colon \mathcal{O}[x] \longrightarrow \mathbb{F}_\phi$ the canonical homomorphism. We denote also by a bar the reduction of polynomials modulo $\mathfrak{m}$, $^-\colon \mathcal{O}[x] \longrightarrow \mathbb{F}[x]$.

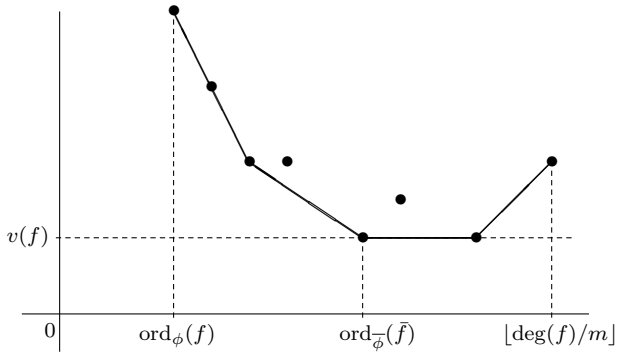
Any $f(x) \in \mathcal{O}[x]$ admits a unique $\phi$-adic development:

$$f(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_n(x)\phi(x)^n,$$

with $a_i(x) \in \mathcal{O}[x]$, $\deg a_i(x) < m$. For any coefficient $a_i(x)$ we let $u_i := v(a_i(x)) \in \mathbb{Z}\cup\{\infty\}$ and we attach to $a_i(x)$ the point $P_i = (i, u_i)$, which is a point of the plane if $u_i$ is finite, and it is thought to be the point at infinity of the vertical line with abscissa $i$, if $u_i = \infty$.

**Definition 1.6.** *The $\phi$-Newton polygon of a nonzero polynomial $f(x) \in \mathcal{O}[x]$ is the lower convex envelope of the set of points $P_i = (i, u_i)$, $u_i < \infty$, in the cartesian plane. We denote this polygon by $N_\phi(f)$.*

The abscissa of the last vertex of $N_\phi(f)$ is $n = \lfloor \deg(f)/m \rfloor$, and $\deg f(x) = mn + \deg a_n(x)$. The typical shape of this polygon is the following




**Remark 1.7.** *The $\phi$-Newton polygon of $f(x)$ is a side in $\mathcal{S}(-\infty)$ of length $\ell$ if and only if $f(x) = a(x)\phi(x)^\ell$, with* $\deg(a) < m$.

**Definition 1.8.** *The principal $\phi$-polygon of $f(x)$ is the element $N_\phi^-(f) \in \mathcal{PP}$ determined by the sides of negative slope of $N_\phi(f)$, including the side of slope $-\infty$ represented by the length $\operatorname{ord}_\phi(f)$. It always starts at the abscissa $i_0 = 0$ and has length $\operatorname{ord}_{\overline{\phi}}(\bar{f})$.*

*For any $\lambda \in \mathbb{Q}^-$ we shall denote by $S_\lambda(f) := S_\lambda(N_\phi^-(f))$ the $\lambda$-component of this polygon (cf. Definition 1.5)*

From now on, we denote $N = N_\phi^-(f)$ for simplicity. The principal polygon $N$ and the set of points $P_i = (i, u_i)$ that lie on $N$, contain the arithmetic information we are interested in. Note that, by construction, the points $P_i$ lie all above $N$.

We attach to any abscissa $\operatorname{ord}_\phi(f) \le i \le \ell(N)$ the following *residual coefficient* $c_i \in \mathbb{F}_\phi$:

$$c_i = \begin{cases} 0, & \text{if } (i,u_i) \text{ lies strictly above } N, \\ \operatorname{red}\left(\dfrac{a_i(x)}{\pi^{u_i}}\right), & \text{if } (i,u_i) \text{ lies on } N. \end{cases}$$

Note that $c_i$ is always nonzero in the latter case, because $\deg a_i(x) < m$.

Let $\lambda = -h/e$ be a negative rational number, with $h, e$ positive coprime integers. Let $S = S_\lambda(N)$ be the $\lambda$-component of $N$, $(s, u)$ the initial point of $S$, and $d := d(S)$ the degree of $S$. The points $(i, u_i)$ that lie on $S$ contain important arithmetic information that is kept in the form of two polynomials that are built with the coefficients of the $\phi$-adic development of $f(x)$ to whom these points are attached.

**Definition 1.9.** *We define the* virtual factor *of $f(x)$ attached to $S$ (or to $\lambda$) to be the polynomial*

$$f^S(x) := \pi^{-u}\phi(x)^{-s} f^0(x) \in K[x], \quad \textit{where} \quad f^0(x) := \sum_{(i,u_i)\in S} a_i(x)\phi(x)^i.$$

*We define the* residual polynomial *attached to $S$ (or to $\lambda$) to be the polynomial:*

$$R_\lambda(f)(y) := c_s + c_{s+e}\, y + \cdots + c_{s+(d-1)e}\, y^{d-1} + c_{s+de}\, y^d \in \mathbb{F}_\phi[y].$$

Note that only the points $(i, u_i)$ that lie on $S$ yield a nonzero coefficient of $R_\lambda(f)(y)$. In particular, $c_s$ and $c_{s+de}$ are always nonzero, so that $R_\lambda(f)(y)$ has degree $d$ and it is never divisible by $y$.

If $\pi' = \rho\pi$ is another prime element of $\mathcal{O}$, and $c = \bar{\rho} \in \mathbb{F}^*$, the residual coefficients of $N_\phi^-(f)$ with respect to $\pi'$ satisfy $c_i' = c_i c^{-u_i}$, so that the corresponding residual polynomial $R'_\lambda(f)(y)$ is equal to $c^{-u} R_\lambda(f)(c^h y)$.

We can define in a completely analogous way the residual polynomial of $f(x)$ with respect to a side $T$, which is not necessarily a $\lambda$-component of $N_\phi^-(f)$.

**Definition 1.10.** *Let $T \in \mathcal{S}(\lambda)$ be an arbitrary side of slope $\lambda$, with abscissas $s_0 \le s_1$ for the end points, and let $d' = d(T)$. We say that the polynomial $f(x)$* lies above *$T$ if all points of $N_\phi^-(f)$ with abscissa $s_0 \le i \le s_1$ lie above $T$; in this case we define*

$$R_\lambda(f,T)(y) := \tilde{c}_{s_0} + \tilde{c}_{s_0+e}\, y + \cdots + \tilde{c}_{s_0+(d'-1)e}\, y^{d'-1} + \tilde{c}_{s_0+d'e}\, y^{d'} \in \mathbb{F}_\phi[y],$$

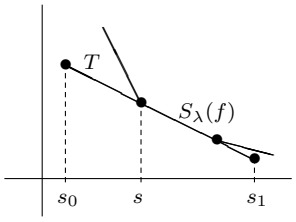
*where $\tilde{c}_i = c_i$ if $(i, u_i)$ lies on $T$ and $\tilde{c}_i = 0$ otherwise.*

Thus, if all points of $S_\lambda(f)$ lie strictly above $T$ we have $R_\lambda(f,T)(y) = 0$. Note that $\deg R_\lambda(f,T)(y) \le d'$ and equality holds if and only if the final point of $T$

belongs to $S_\lambda(f)$. Usually, $T$ will be an enlargement of $S_\lambda(f)$ and then,

$$T \supseteq S_\lambda(f) \implies R_\lambda(f,T)(y) = y^{(s-s_0)/e} R_\lambda(f)(y),$$

where $s$ is the abscissa of the initial point of $S_\lambda(f)$.




The motivation for this more general definition lies in the bad behaviour of the residual polynomial $R_\lambda(f)(y)$ with respect to sums. Nevertheless, if $T$ is a fixed side and $f(x)$, $g(x)$ lie both above $T$, it is clear that

$$R_\lambda(f+g,T)(y) = R_\lambda(f,T)(y) + R_\lambda(g,T)(y). \tag{2}$$

1.3. **Admissible $\phi$-developments and Theorem of the product.** Let

$$f(x) = \sum_{i\geq 0} a_i'(x)\phi(x)^i, \quad a_i'(x) \in \mathcal{O}[x], \tag{3}$$

be a $\phi$-development of $f(x)$, not necessarily the $\phi$-adic one. Take $u_i' = v(a_i'(x))$, and let $N'$ be the principal polygon of the set of points $(i, u_i')$. Let $i_1$ be the first abscissa with $a_{i_1}'(x) \neq 0$. To any $i_1 \leq i \leq \ell(N')$ we attach a residual coefficient as before:

$$c_i' = \begin{cases} 0, & \text{if } (i,u_i') \text{ lies strictly above } N', \\ \operatorname{red}\left(\dfrac{a_i'(x)}{\pi^{u_i'}}\right), & \text{if } (i,u_i') \text{ lies on } N' \end{cases}$$

For the points $(i, u_i')$ lying on $N'$ we can have now $c_i' = 0$; for instance, in the case $a_0'(x) = f(x)$, the Newton polygon has only one point $(0, v(f))$ and $c_0' = 0$ if $f(x)/\pi^{v(f)}$ is divisible by $\phi(x)$ modulo $\mathfrak{m}$.

Finally, for any negative rational number $\lambda$, we can define the *residual polynomial* attached to the $\lambda$-component $S' = S_\lambda(N')$ to be

$$R_\lambda'(f)(y) := c_{s'}' + c_{s'+e}'\, y + \cdots + c_{s'+(d'-1)e}'\, y^{d'-1} + c_{s'+d'e}'\, y^{d'} \in \mathbb{F}_\phi[y],$$

where $d' = d(S')$ and $s'$ is the abscissa of the initial point of $S'$.

**Definition 1.11.** *We say that the $\phi$-development (3) is admissible if for each abscissa $i$ of a vertex of $N'$ we have $c_i' \neq 0$.*

**Lemma 1.12.** *If a $\phi$-development is admissible, then $N' = N_\phi^-(f)$ and $c_i' = c_i$ for all abscissas $i$ of the finite part of $N'$. In particular, for any negative rational number $\lambda$ we have $R_\lambda'(f)(y) = R_\lambda(f)(y)$.*

*Proof.* Consider the $\phi$-adic developments of $f(x)$ and each $a_i'(x)$:

$$f(x) = \sum_{0\leq i} a_i(x)\phi(x)^i, \qquad a_i'(x) = \sum_{0\leq k} b_{i,k}(x)\phi(x)^k.$$

By the uniqueness of the $\phi$-adic development we have

$$a_i(x) = \sum_{0\le k\le i} b_{i-k,k}(x). \tag{4}$$

Clearly, $w_{i,k} := v(b_{i,k}) \ge u'_i$, for all $0 \le k$, $0 \le i \le \ell(N')$. In particular, all points $(i, u_i)$ lie above $N'$; in fact

$$u_i = v(a_i) \ge \min_{0\le k\le i}\{w_{i-k,k}\} = w_{i-k_0,k_0} \ge u'_{i-k_0} \ge h_{i-k_0}(N') \ge h_i(N'), \tag{5}$$

for some $0 \le k_0 \le i$. Also, for any abscissa $i$ of the finite part of $N'$,

$$w_{i-k,k} \ge u'_{i-k} \ge h_{i-k}(N') > h_i(N'),\ \forall k > 0. \tag{6}$$

Hence, for the abscissas with $u'_i = h_i(N')$ we have

$$c'_i = \operatorname{red}(a'_i(x)/\pi^{u'_i}) = \operatorname{red}(b_{i,0}(x)/\pi^{u'_i}). \tag{7}$$

Now, if $(i, u'_i)$ is a vertex of $N'$ we have $c'_i \ne 0$ by hypothesis, and from (7) we get $h_i(N') = u'_i = w_{i,0}$. By (6) and (4) we have $u_i = w_{i,0} = u'_i$. This shows that $N' = N^-_\phi(f)$. Let us denote this common polygon by $N$.

Finally, let us prove the equality of all residual coefficients. If $c_i \ne 0$, then $u_i = h_i(N)$, and from (5) we get $k_0 = 0$ and $u_i = w_{i,0} = u'_i$. By (6), (4) and (7), we get $c_i = \operatorname{red}(a_i(x)/\pi^{u_i}) = \operatorname{red}(b_{i,0}(x)/\pi^{u_i}) = c'_i$. If $c_i = 0$, then $u_i > h_i(N)$, and from (4) and (6) we get $w_{i,0} > h_i(N)$ too. By (7) we get $c'_i = 0$. □

The construction of the principal part of the $\phi$-Newton polygon of a polynomial can be interpreted as a mapping

$$N^-_\phi : \mathcal{O}[x] \setminus \{0\} \longrightarrow \mathcal{PP}, \qquad f(x) \mapsto N^-_\phi(f).$$

Also, for any negative rational number $\lambda$, the construction of the residual polynomial attached to $\lambda$ can be interpreted as a mapping

$$R_\lambda \colon \mathcal{O}[x] \setminus \{0\} \longrightarrow \mathbb{F}_\phi[y] \setminus \{0\}, \qquad f(x) \mapsto R_\lambda(f)(y).$$

The Theorem of the product says that both mappings are semigroup homomorphisms.

**Theorem 1.13** (Theorem of the product)**.** *For any $f(x), g(x) \in \mathcal{O}[x] \setminus \{0\}$ and any $\lambda \in \mathbb{Q}^-$ we have*

$$N^-_\phi(fg) = N^-_\phi(f) + N^-_\phi(g), \qquad R_\lambda(fg)(y) = R_\lambda(f)(y)R_\lambda(g)(y).$$

*Proof.* Consider the respective $\phi$-adic developments

$$f(x) = \sum_{0\le i} a_i(x)\phi(x)^i, \quad g(x) = \sum_{0\le j} b_j(x)\phi(x)^j,$$

and denote $u_i = v(a_i(x))$, $v_j = v(b_j(x))$, $N_f = N^-_\phi(f)$, $N_g = N^-_\phi(g)$. Then,

$$f(x)g(x) = \sum_{0\le k} A_k(x)\phi(x)^k, \qquad A_k(x) = \sum_{i+j=k} a_i(x)b_j(x). \tag{8}$$

Denote by $N'$ the principal part of the Newton polygon of $fg$, determined by this $\phi$-development.

We shall show that $N' = N_f + N_g$, that this $\phi$-development is admissible, and that $R'_\lambda(fg) = R_\lambda(f)R_\lambda(g)$ for all $\lambda$. The theorem will be then a consequence of Lemma 1.12.

Let $w_k := v(A_k(x))$ for all $0 \le k$. Lemma 1.4 shows that the point $(i,u_i)+(j,v_j)$ lies above $N_f+N_g$ for any $i,j \ge 0$. Since $w_k \ge \min\{u_i+v_j, i+j=k\}$, the points $(k,w_k)$ lie all above $N_f+N_g$. On the other hand, let $P_k=(k,h_k(N_f+N_g))$ be a vertex of $N_f+N_g$; that is, $P_k$ is the end point of $S_1+\cdots+S_r+T_1+\cdots+T_s$, for certain sides $S_i$ of $N_f$ and $T_j$ of $N_g$, ordered by increasing slopes among all sides of $N_f$ and $N_g$. By Lemma 1.4, for all pairs $(i,j)$ such that $i+j=k$, the point $(i,u_i)+(j,v_j)$ lies strictly above $N_f+N_g$ except for the pair $i_0=\operatorname{ord}_\phi(f)+\ell(S_1+\cdots+S_r)$, $j_0=\operatorname{ord}_\phi(g)+\ell(T_1+\cdots+T_s)$ that satisfies $(i_0,u_{i_0})+(j_0,v_{j_0})=P_k$. Thus, $(k,w_k)=P_k$ and

$$\operatorname{red}\left(\frac{A_k(x)}{\pi^{h_k(N_f+N_g)}}\right)=\operatorname{red}\left(\frac{a_{i_0}(x)b_{j_0}(x)}{\pi^{h_k(N_f+N_g)}}\right)=\operatorname{red}\left(\frac{a_{i_0}(x)}{\pi^{h_{i_0}(N_f)}}\right)\operatorname{red}\left(\frac{b_{j_0}(x)}{\pi^{h_{j_0}(N_g)}}\right)\neq 0.$$

This shows that $N'=N_f+N_g$ and that the $\phi$-development (8) is admisible.

Finally, by (1), the $\lambda$-components $S'=S_\lambda(N')$, $S_f=S_\lambda(N_f)$, $S_g=S_\lambda(N_g)$ are related by: $S'=S_f+S_g$. Let $(k,h_k(N'))$ be a point of integer coordinates lying on $S'$ (not necessarily a vertex). Denote by $I$ the set of the pairs $(i,j)$ such that $(i,u_i)$ lies on $S_f$, $(j,v_j)$ lies on $S_g$, and $i+j=k$. Take $P(x)=\sum_{(i,j)\in I}a_i(x)b_j(x)$. By Lemma 1.4, for all other pairs $(i,j)$ with $i+j=k$, the point $(i,u_i)+(j,v_j)$ lies strictly above $N'$. Therefore, $c'_k(fg)=\operatorname{red}(P(x)/\pi^{h_k(N')})=\sum_{(i,j)\in I}c_i(f)c_j(g)$. This shows that $R'_\lambda(fg)(y)=R_\lambda(f)(y)R_\lambda(g)(y)$. □

**Notation 1.14.** *Let $\mathcal{F}$ be a field and $\varphi(y)$, $\psi(y)\in\mathcal{F}[y]$ two polynomials. We write $\varphi(y)\approx\psi(y)$ to indicate that there exists a constant $c\in\mathcal{F}^*$ such that $\varphi(y)=c\psi(y)$.*

*We write $\varphi(y)\sim\psi(y)$ to indicate that there exist constants $a,b\in\mathcal{F}^*$ such that $\varphi(y)=a\psi(by)$.*

**Corollary 1.15.** *Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial. Let $\phi_1(x),\dots,\phi_r(x)$ be monic polyomials in $\mathcal{O}[x]$ such that their reductions modulo $\mathfrak{m}$ are pairwise different irreducible polynomials of $\mathbb{F}[x]$ and*

$$f(x)\equiv\phi_1(x)^{\ell_1}\cdots\phi_r(x)^{\ell_r}\pmod{\mathfrak{m}}.$$

*Let $f(x)=F_1(x)\cdots F_r(x)$ be the factorization into a product of monic polynomials of $\mathcal{O}[x]$ satisfying $F_i(X)\equiv\phi_i(x)^{\ell_i}\pmod{\mathfrak{m}}$, provided by Hensel's lemma. Then,*

$$N_{\phi_i}(F_i)=N^-_{\phi_i}(F_i)=N^-_{\phi_i}(f),\qquad R_\lambda(F_i)(y)\approx R_\lambda(f)(y),$$

*for all $1\le i\le r$ and all $\lambda\in\mathbb{Q}^-$.*

*Proof.* For any $1\le i\le r$, let $G_i(x)=\prod_{j\neq i}F_j(x)$. Since $\phi_i(x)$ does not divide $G_i(x)$ modulo $\mathfrak{m}$, the principal $\phi_i$-Newton polygon of $G_i(x)$ reduces to the point $(0,0)$. By the Theorem of the product, $N^-_{\phi_i}(f)=N_{\phi_i}(F_i)^-+N^-_{\phi_i}(G_i)=N^-_{\phi_i}(F_i)$. On the other hand, $N_{\phi_i}(F_i)=N^-_{\phi_i}(F_i)$ because both polygons have length $\ell_i$. Now, for any $\lambda\in\mathbb{Q}^-$, $S_\lambda(G_i)$ is a point and $R_\lambda(G_i)(y)$ is a nonzero constant. By the Theorem of the product, $R_\lambda(f)(y)=R_\lambda(F_i)(y)R_\lambda(G_i)(y)\approx R_\lambda(F_i)(y)$. □

1.4. **Theorems of the polygon and of the residual polynomial.** Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial divisible by $\phi(x)$ modulo $\mathfrak{m}$. By Hensel's lemma, $f(x)=f_\phi(x)G(x)$ in $\mathcal{O}[x]$, with monic polynomials $f_\phi(x)$, $G(x)$ such that $\operatorname{red}(G(x))\neq 0$ and $f_\phi(x)\equiv\phi(x)^\ell\pmod{\mathfrak{m}}$. The aim of this section is to obtain a further factorization of $f_\phi(x)$ and certain arithmetic data about the factors. Thanks to Corollary 1.15, we shall be able to read this information directly on $f(x)$; more precisely, on $N^-_\phi(f)=N_\phi(f_\phi)$ and $R_\lambda(f)(y)\approx R_\lambda(f_\phi)(y)$.

**Theorem 1.16** (Theorem of the polygon)**.** *Let $f(x) \in \mathcal{O}[x]$ be a monic polynomial divisible by $\phi(x)$ modulo $\mathfrak{m}$. Suppose that $N_\phi^-(f) = S_1 + \cdots + S_s$ has $s$ sides with pairwise different slopes $\lambda_1, \ldots, \lambda_s$. Then, $f_\phi(x)$ admits a factorization in $\mathcal{O}[x]$ into a product of monic polynomials*

$$f_\phi(x) = F_1(x) \cdot \cdots \cdot F_s(x),$$

*such that, for all $1 \leq i \leq s$,*

1. *$N_\phi(F_i) = S_i'$ is one-sided, and $S_i'$ is equal to $S_i$ up to a translation,*
2. *If $S_i$ has finite slope $\lambda_i$, then $R_{\lambda_i}(F_i)(y) \approx R_{\lambda_i}(f)(y)$,*
3. *For any root $\theta \in \overline{\mathbb{Q}}_p$ of $F_i(x)$, we have $v(\phi(\theta)) = |\lambda_i|$.*

*Proof.* By the Theorem of the product and Corollary 1.15, it is sufficient to show that if $F(x) := f_\phi(x)$ is irreducible, then $N_\phi(F) = S$ is one-sided and the roots $\theta \in \overline{\mathbb{Q}}_p$ have all $v(\phi(\theta))$ equal to minus the slope of $S$.

In fact, for all the roots $\theta \in \overline{\mathbb{Q}}_p$ of $F(x)$, the rational number $v(\phi(\theta))$ takes the same value because the $p$-adic valuation is invariant under the Galois action. Since $F(x)$ is congruent to a power of $\phi(x)$ modulo $\mathfrak{m}$ we have $\lambda := -v(\phi(\theta)) < 0$. We have $\lambda = -\infty$ if and only if $F(x) = \phi(x)$, and in this case the theorem is clear. Suppose $\lambda$ is finite.

Let $x^k + b_{k-1}x^{k-1} + \cdots + b_0 \in \mathcal{O}[x]$ be the minimal polynomial of $\phi(\theta)$ and let $g(x) = \phi(x)^k + b_{k-1}\phi(x)^{k-1} + \cdots + b_0$. We have $v(b_0) = k|\lambda|$ and $v(b_i) \geq (k-i)|\lambda|$ for all $i$; this implies that $N_\phi(g)$ is one-sided with slope $\lambda$. Since $g(\theta) = 0$, our polynomial $F(x)$ is an irreducible factor of $g(x)$ and by the Theorem of the product $N_\phi(F)$ is also one-sided with slope $\lambda$. □

We recall that the factor corresponding to a side $S_i$ of slope $-\infty$ is necessarily $F_i(x) = \phi(x)^{\mathrm{ord}_\phi(f)}$ (cf. Remark 1.7).

Let $\lambda = -h/e$, with $e, h$ coprime positive integers, be a negative rational number such that $S := S_\lambda(f)$ has positive length. Let $f_{\phi,\lambda}(x)$ be the factor of $f(x)$, corresponding to the pair $\phi, \lambda$ by the Theorem of the polygon. Choose a root $\theta \in \overline{\mathbb{Q}}_p$ of $f_{\phi,\lambda}(x)$ and let $L = K(\theta)$. Since $v(\phi(\theta)) > 0$, we can consider an embedding

$$\mathcal{O}[x]/(\pi, \phi(x)) = \mathbb{F}_\phi \hookrightarrow \mathbb{F}_L, \quad \mathrm{red}(x) \mapsto \overline{\theta}. \tag{9}$$

This embedding depends on the choice of $\theta$ (and not only on $L$). After this identification of $\mathbb{F}_\phi$ with a subfield of $\mathbb{F}_L$ we can think that all residual polynomials have coefficients in $\mathbb{F}_L$. The Theorem of the polygon yields certain arithmetic information on the field $L$.

**Corollary 1.17.** *The residual degree $f(L/K)$ is divisible by $m = \deg \phi(x)$, and the ramification index $e(L/K)$ is divisible by $e$. Moreover, the number of irreducible factors of $f_{\phi,\lambda}(x)$ is at most $d(S)$; in particular, if $d(S) = 1$ the polynomial $f_{\phi,\lambda}(x)$ is irreducible in $\mathcal{O}[x]$, and $f(L/K) = m$ and $e(L/K) = e$.*

*Proof.* The statement on the residual degree is a consequence of the embedding (9). By the theorem of the polygon, $v_L(\phi(\theta)) = e(L/K)h/e$. Since this is an integer and $h, e$ are coprime, necessarily $e$ divides $e(L/K)$. The upper bound for the number of irreducible factors is a consequence of the Theorem of the product. Finally, if $d(S) = 1$, we have $me = \deg(f_{\phi,\lambda}(x)) = f(L/K)e(L/K)$, and necessarily $f(L/K) = m$ and $e(L/K) = e$. □

**Notation 1.18.** $\gamma(x) := \phi(x)^e/\pi^h \in K[x]$. *Note that* $v(\gamma(\theta)) = 0$*; in particular,* $\gamma(\theta) \in \mathcal{O}_L$.

**Proposition 1.19.** *[Computation of* $v(P(\theta))$ *with the polygon] We keep the above notations for* $f(x)$, $\lambda = -h/e$, $\theta$, $L$, $\gamma$, *and the embedding* $\mathbb{F}_\phi \subseteq \mathbb{F}_L$ *of (9). Let* $P(x) \in \mathcal{O}[x]$ *be a nonzero polynomial,* $S = S_\lambda(P)$, $L_\lambda$ *the line of slope* $\lambda$ *that contains* $S$, *and* $H$ *the ordinate at the origin of this line. Then,*

1. $v(P^S(\theta)) \geq 0, \quad \overline{P^S(\theta)} = R_\lambda(P)(\overline{\gamma(\theta)})$,
2. $v(P(\theta) - P^0(\theta)) > H$.
3. $v(P(\theta)) \geq H$, *and equality holds if and only if* $R_\lambda(P)(\overline{\gamma(\theta)}) \neq 0$,
4. $R_\lambda(f)(\overline{\gamma(\theta)}) = 0$.
5. *If* $R_\lambda(f)(y) \approx \psi(y)^a$ *for an irreducible polynomial* $\psi(y) \in \mathbb{F}_\phi[y]$, *then* $v(P(\theta)) = H$ *if and only if* $\psi(y) \nmid R_\lambda(P)(y)$ *in* $\mathbb{F}_\phi[y]$.

*Proof.* Let $P(x) = \sum_{0\leq i} b_i(x)\phi(x)^i$ be the $\phi$-adic development of $P(x)$, and denote $u_i = v(b_i)$, $N = N_\phi^-(P)$. Recall that $P^S(x) = \phi(x)^{-s}\pi^{-u}P^0(x)$, where $(s,u)$ are the coordinates of the initial point of $S$, and $P^0(x) = \sum_{(i,u_i)\in S} b_i(x)\phi(x)^i$. Hence, for $d = d(S)$ we have

$$P^S(x) = \pi^{-u}\left(b_s(x) + b_{s+e}(x)\phi(x)^e + \cdots + b_{s+de}\phi(x)^{de}\right) \\ = \frac{b_s(x)}{\pi^u} + \frac{b_{s+e}(x)}{\pi^{u-h}}\gamma(x) + \cdots + \frac{b_{s+de}(x)}{\pi^{u-hd}}\gamma(x)^d.$$

Since $v(b_{s+ie}) \geq h_{s+ie}(N) = u - hi$ for all $1 \leq i \leq d$, the two statements of item 1 are clear.

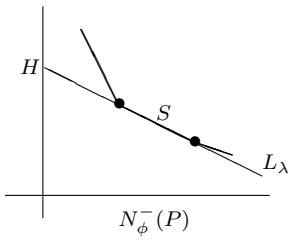
For any integer abscissa $i$

$$v(b_i(\theta)\phi(\theta)^i) = u_i + i\frac{h}{e} \geq h_i(N) + i\frac{h}{e} \geq H, \tag{10}$$

because all points of $N$ lie above the line $L_\lambda$. Also, equality holds in (10) if and only if $(i,u_i) \in S$. This proves item 2. Also, this shows that $v(P(\theta)) \geq H$. Since $v(\phi(\theta)^s\pi^u) = u + sh/e = H$, we have

$$v(P(\theta)) = H \iff v(P^0(\theta)) = H \iff v(P^S(\theta)) = 0 \iff R_\lambda(P)(\overline{\gamma(\theta)}) \neq 0,$$

the last equivalence by item 1. This proves item 3.




Since $f(\theta) = 0$, item 4 is a consequence of item 3 applied to $P(x) = f(x)$. Finally, if $R_\lambda(f)(y) \approx \psi(y)^a$, then $\psi(y)$ is the minimal polynomial of $\overline{\gamma(\theta)}$ over $\mathbb{F}_\phi$, by item 4. Hence, $R_\lambda(P)(\overline{\gamma(\theta)}) \neq 0$ is equivalent to $\psi(y) \nmid R_\lambda(P)(y)$ in $\mathbb{F}_\phi[y]$. □

We discuss now how Newton polygons and residual polynomials are affected by an extension of the base field by an unramified extension. We shall make an extensive use of Notation 1.14.

**Lemma 1.20.** *We keep the above notations for $f(x)$, $\lambda = -h/e$, $\theta$, $L$. Let $K'$ be the unramified extension of $K$ of degree $m$, and identify $\mathbb{F}_\phi = \mathbb{F}_{K'}$ through the embedding (9). Let $G(x) \in \mathcal{O}_{K'}[x]$ be the minimal polynomial of $\theta$ over $K'$. Let $\phi'(x) = x - \eta$, where $\eta \in K'$ is the unique root of $\phi(x)$ such that $G(x)$ is divisible by $x - \eta$ modulo $\mathfrak{m}_{K'}$. Then, for any nonzero polynomial $P(x) \in \mathcal{O}[x]$:*

$$N^-_{\phi'}(P) = N^-_\phi(P), \quad R'_\lambda(P)(y) \sim R_\lambda(P)(y),$$

*where $R'$ denotes the residual polynomial with respect to $\phi'(x)$ over $K'$.*

*Proof.* Consider the $\phi$-adic development of $P(x)$:

$$P(x) = \phi(x)^n + a_{n-1}(x)\phi(x)^{n-1} + \cdots + a_0(x) = \\ = \rho(x)^n\phi'(x)^n + a_{n-1}(x)\rho(x)^{n-1}\phi'(x)^{n-1} + \cdots + a_0(x),$$

where $\rho(x) = \phi(x)/\phi'(x) \in \mathbb{Z}_{K'}[x]$. Since $\phi(x)$ is irreducible modulo $\mathfrak{m}$, it is a separable polynomial modulo $\mathfrak{m}_{K'}$, so that the roots $\eta, \eta_2 \ldots, \eta_m$ of $\phi(x)$ are pairwise not congruent modulo $\mathfrak{m}_{K'}$. Thus, $v(\theta - \eta) > 0$ implies $v(\theta - \eta_i) = 0$ for all $i > 1$, so that $v(\rho(\theta)) = 0$. Therefore, the above $\phi'(x)$-development of $P(x)$ is admissible and $N^-_{\phi'}(P) = N^-_\phi(P)$. Moreover the residual coefficients of the two polygons are related by $c'_i = c_i\epsilon^i$, where $\epsilon = \overline{\rho(\theta)} \in \mathbb{F}^*_{K'}$, so that $R'_\lambda(P)(y) = \epsilon^s R_\lambda(P)(\epsilon^e y)$, where $s$ is the initial abscissa of $S_\lambda(P) = S'_\lambda(P)$. □

**Theorem 1.21** (Theorem of the residual polynomial)**.** *Let $f(x) \in \mathcal{O}[x]$ be a monic polynomial which is divisible by $\phi(x)$ modulo $\mathfrak{m}$. Let $S$ be a side of $N^-_\phi(f)$, of finite slope $\lambda$, and let*

$$R_\lambda(f)(y) = \psi_1(y)^{a_1} \cdots \psi_t(y)^{a_t}$$

*be the factorization of the residual polynomial of $f(x)$ into the product of powers of pairwise different irreducible polynomials in $\mathbb{F}_\phi[y]$. Then, the factor $f_{\phi,\lambda}(x)$ of $f(x)$, corresponding to $S$ by the Theorem of the polygon, admits a further factorization*

$$f_{\phi,\lambda}(x) = G_1(x) \cdots G_t(x)$$

*in $\mathcal{O}[x]$, such that all $N_\phi(G_i)$ are one-sided with slope $\lambda$, and $R_\lambda(G_i)(y) \approx \psi_i(y)^{a_i}$ in $\mathbb{F}_\phi[y]$, for all $1 \le i \le t$.*

*Proof.* We need only to prove that if $F(x) := f_{\phi,\lambda}(x)$ is irreducible, then $R_\lambda(F)(y)$ is the power of an irreducible polynomial of $\mathbb{F}_\phi[y]$. Let $\theta$, $L$, $K'$, $G(x)$ be as in Lemma 1.20, so that $F(x) = \prod_{\sigma \in \mathrm{Gal}(K'/K)} G^\sigma(x)$. Under the embedding $\mathbb{F}_\phi \longrightarrow \mathbb{F}_L$, the field $\mathbb{F}_\phi$ is identified with $\mathbb{F}_{K'}$. By Lemma 1.20, there is a polynomial of degree one, $\phi'(x) \in \mathcal{O}_{K'}[x]$, such that $R'_\lambda(F)(y) \sim R_\lambda(F)(y)$. By the construction of $\phi'(x)$, for any $\sigma \neq 1$, the polynomial $G^\sigma(x)$ is not divisible by $\phi'(x)$ modulo $\mathfrak{m}_{K'}$; thus, $N_{\phi'}(G^\sigma)$ is reduced to a point, and $R'_\lambda(G^\sigma)(y)$ is a constant. Therefore, by the Theorem of the product, $R'_\lambda(G)(y) \approx R'_\lambda(F)(y) \sim R_\lambda(F)(y)$, so that $R_\lambda(F)(y)$ is the power of an irreducible polynomial of $\mathbb{F}_\phi[y]$ if and only if $R'_\lambda(G)(y)$ has the same property over $\mathbb{F}_{K'}$. In conclusion, by extending the base field, we can suppose that $\deg \phi = m = 1$.

Consider now the minimal polynomial $h(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0 \in K[x]$ of $\gamma(\theta) = \phi(\theta)^e/\pi^h$ over $K$. Since $v(\gamma(\theta)) = 0$, we have $v(b_0) = 0$. Thus, the polynomial

$$g(x) = \phi(x)^{ek} + \pi^h b_{k-1}\phi(x)^{e(k-1)} + \cdots + \pi^{kh}b_0,$$

has one-sided $N_\phi^-(g)$ of slope $\lambda$, and $R_\lambda(g)(y)$ is the reduction of $h(y)$ modulo $\mathfrak{m}$, which is the power of an irreducible polynomial because $h(x)$ is irreducible in $K[x]$. Since $g(\theta)=0$, $F(x)$ divides $g(x)$, and it has the same properties by the Theorem of the product. □

**Corollary 1.22.** *With the above notations, let $\theta \in \overline{\mathbb{Q}}_p$ be a root of $G_i(x)$, and $L=K(\theta)$. Then, $f(L/K)$ is divisible by $m\deg\psi_i$. Moreover, if $a_i=1$ then $G_i(x)$ is irreducible in $\mathcal{O}[x]$, and $f(L/K)=m\deg\psi_i$, $e(L/K)=e$.*

*Proof.* The statement about $f(L/K)$ is a consequence of the embedding of the finite field $\mathbb{F}_\phi[y]/(\psi_i(y))$ into $\mathbb{F}_L$ determined by $\mathrm{red}(x)\mapsto\overline{\theta}$, $y\mapsto\overline{\gamma(\theta)}$. This embedding is well-defined by item 4 of Proposition 1.19. The statement about $a_i=1$ is a consequence of the Theorem of the product and Corollary 1.17. □

1.5. **Types of order one.** Starting with a monic and separable polynomial $f(x)\in\mathcal{O}[x]$, the Newton polygon techniques provide partial information on the factorization of $f(x)$ in $\mathcal{O}[x]$, obtained after three classical *dissections* (cf. [Ber27]). In the first dissection we obtain as many factors of $f(x)$ as pairwise different irreducible factors modulo $\mathfrak{m}$ (by Hensel's lemma). In the second dissection, each of these factors splits into the product of as many factors as sides of certain Newton polygon of $f(x)$ (by the Theorem of the polygon). In the third dissection, the factor that corresponds to a side of finite slope splits into the product of as many factors as irreducible factors of the residual polynomial of $f(x)$ attached to this side (by the Theorem of the residual polynomial).

The final list of factors of $f(x)$ obtained by this procedure can be parameterized by certain data, which we call *types of order zero and of order one*.

**Definition 1.23.** *A type of order zero is a monic irreducible polynomial $\mathbf{t}=\psi_0(y)\in\mathbb{F}[y]$. We attach to any type of order zero the map*

$$\omega_{\mathbf{t}}\colon \mathcal{O}[x]\setminus\{0\}\longrightarrow\mathbb{Z}_{\geq 0},\quad P(x)\mapsto \mathrm{ord}_{\psi_0}(\overline{P(x)/\pi^{v(P)}}).$$

*Let $f(x)\in\mathcal{O}[x]$ be a monic and separable polynomial. We say that the type $\psi_0(y)$ is $f$-complete if $\omega_{\mathbf{t}}(f)=1$. In this case, we denote by $f_{\mathbf{t}}(x)\in\mathcal{O}[x]$ the monic irreducible factor of $f(x)$ determined by $\psi_0(y)=f_{\mathbf{t}}(y)\pmod{\mathfrak{m}}$.*

**Definition 1.24.** *A type of order one is a triple $\mathbf{t}=(\phi(x);\lambda,\psi(y))$, where*

1. *$\phi(x)\in\mathbb{Z}[x]$ is a monic polynomial which is irreducible modulo $\mathfrak{m}$,*
2. *$\lambda=-h/e\in\mathbb{Q}^-$, with $h,e$ positive coprime integers.*
3. *$\psi(y)\in\mathbb{F}_\phi[y]$ is a monic irreducible polynomial, $\psi(y)\neq y$.*

*By truncation of $\mathbf{t}$ we obtain the type of order zero $\mathbf{t}_0:=\phi(y)\pmod{\mathfrak{m}}$.*

We denote by $\mathbf{t}_1(f)$ the set of all types of order one obtained from $f(x)$ along the process of applying the three classical dissections: $\phi(x)$ is a monic lift to $\mathcal{O}[x]$ of an irreducible factor $\mathbf{t}_0$ of $f(x)$ modulo $\mathfrak{m}$ which is not $f$-complete, $\lambda$ is the finite slope of a side of positive length of $N_\phi^-(f)$, and $\psi(y)$ is an irreducible factor of the residual polynomial $R_\lambda(f)(y)\in\mathbb{F}_\phi[y]$. These types are not intrinsical objects of $f(x)$. There is a non-canonical choice of the lifts $\phi(x)\in\mathcal{O}[x]$, and the data $\lambda$, $\psi(y)$ depend on this choice.

We denote by $\mathbf{T}_1(f)$ the union of $\mathbf{t}_1(f)$ and the set of all $f$-complete types of order zero. By the previous results we have a factorization in $\mathcal{O}[x]$

$$f(x) = f_\infty(x) \prod_{\mathbf{t}\in\mathbf{T}_1(f)} f_{\mathbf{t}}(x),$$

where $f_\infty(x)$ is the product of the different $\phi(x)$ that divide $f(x)$ in $\mathcal{O}[x]$, and, if $\mathbf{t}$ has order one, $f_{\mathbf{t}}(x)$ is the unique monic divisor of $f(x)$ in $\mathcal{O}[x]$ satisfying the following properties:

$$\begin{aligned}
&f_{\mathbf{t}}(x) \equiv \phi(x)^{fea} \pmod{\mathfrak{m}}, \quad \text{where } a = \operatorname{ord}_\psi(R_\lambda(f)),\\
&N_\phi(f_{\mathbf{t}}) \text{ is one-sided with slope } \lambda,\\
&R_\lambda(f_{\mathbf{t}})(y) \approx \psi(y)^a \text{ in } \mathbb{F}_1[y].
\end{aligned}$$

The factor $f_\infty(x)$ is already expressed as a product of irreducible polynomials in $\mathcal{O}[x]$. Also, if $a = 1$, the Theorem of the residual polynomial shows that $f_{\mathbf{t}}(x)$ is irreducible too. Thus, the remaining task is to obtain the further factorization of $f_{\mathbf{t}}(x)$, for the types $\mathbf{t} \in \mathbf{t}_1(f)$ with $a > 1$. The factors of $f_{\mathbf{t}}(x)$ will bear a reminiscence of $\mathbf{t}$ as a birth mark (cf. Lemma 2.5).

Once a type of order one $\mathbf{t} = (\phi(x); \lambda, \psi(y))$ is fixed, we change the notation of several objects that depend on the data of the type. We omit the data from the notation but we include the subscript "1" to emphasize that they are objects of the first order. From now on, for any nonzero polynomial $P(x) \in \mathcal{O}[x]$, any principal polygon $N \in \mathcal{PP}$, and any $T \in \mathcal{S}(\lambda)$, we shall denote

$$\begin{aligned}
&v_1(P) := v(P), && \omega_1(P) := \omega_\phi^-(P) = \operatorname{ord}_\phi^-(\overline{P(x)/\pi^{v(P)}}),\\
&N_1(P) := N_\phi(P), && N_1^-(P) := N_\phi^-(P),\\
&S_1(N) := S_\lambda(N), && S_1(P) := S_\lambda(P) = S_\lambda(N_1^-(P)),\\
&R_1(P)(y) := R_\lambda(P)(y), && R_1(P,T)(y) := R_\lambda(P,T)(y).
\end{aligned}$$

Note that $\omega_1(P), N_1(P)$ depend only on $\phi(x)$, wheras $S_1(P)$, $R_1(P)$, $R_1(P,T)$ depend on the pair $\phi(x), \lambda$.

The aim of the next two sections is to introduce Newton polygons of higher order and prove similar theorems, yielding information on a further factorization of each $f_{\mathbf{t}}(x)$. As before, we shall obtain arithmetic information about the factors of $f_{\mathbf{t}}(x)$ just by a direct manipulation of $f(x)$, without actually computing a $p$-adic approximation to these factors. This fact is crucial to ensure that the whole process has a low complexity. However, once an irreducible factor of $f(x)$ is "detected", the theory provides a reasonable approximation of this factor as a by-product (cf. Proposition 3.12).

## 2. Newton polygons of higher order

Throughout this section, $r$ is an integer, $r \geq 2$. We shall construct Newton polygons of order $r$ and prove their basic properties and the Theorem of the product in order $r$, under the assumption that analogous results have been already obtained in orders $1, 2, \ldots, r-1$. The case $r = 1$ has been already considered in section 1.

2.1. **Types of order $r-1$.** A *type of order $r-1$* is a sequence of data

$$\mathbf{t} = (\phi_1(x); \lambda_1, \phi_2(x); \cdots ; \lambda_{r-2}, \phi_{r-1}(x); \lambda_{r-1}, \psi_{r-1}(y)),$$

where $\phi_i(x)$ are monic polynomials in $\mathcal{O}[x]$, $\lambda_i$ are negative rational numbers and $\psi_{r-1}(y)$ is a monic polynomial over certain finite field (to be specified below), that satisfy the following recursive properties:

(1) $\phi_1(x)$ is irreducible modulo $\mathfrak{m}$. We denote by $\psi_0(y) \in \mathbb{F}[y]$ the polynomial obtained by reduction of $\phi_1(y)$ modulo $\mathfrak{m}$. We define $\mathbb{F}_1 := \mathbb{F}[y]/(\psi_0(y))$.
(2) For all $1 \le i < r-1$, the Newton polygon of $i$-th order, $N_i(\phi_{i+1})$, is one-sided, with positive length and slope $\lambda_i$.
(3) For all $1 \le i < r-1$, the residual polynomial of $i$-th order, $R_i(\phi_{i+1})(y)$, is an irreducible polynomial in $\mathbb{F}_i[y]$. We denote by $\psi_i(y) \in \mathbb{F}_i[y]$ the monic polynomial determined by $R_i(\phi_{i+1})(y) \approx \psi_i(y)$. We define $\mathbb{F}_{i+1} := \mathbb{F}_i[y]/(\psi_i(y))$.
(4) $\psi_{r-1}(y) \in \mathbb{F}_{r-1}[y]$ is a monic irreducible polynomial, $\psi_{r-1}(y) \neq y$. We define $\mathbb{F}_r := \mathbb{F}_{r-1}[y]/(\psi_{r-1}(y))$.

The type determines a tower $\mathbb{F} =: \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \cdots \subseteq \mathbb{F}_r$ of finite fields. The field $\mathbb{F}_i$ should not be confused with the finite field with $i$ elements.

By the Theorem of the product in orders $1, \dots, r-1$, the polynomials $\phi_i(x)$ are all irreducible over $\mathcal{O}[x]$.

Let us be more precise about the meaning of $N_i(-)$, $R_i(-)$, used in item 2.

**Notation 2.1.** *For all $1 \le i < r$, we obtain by truncation of $\mathbf{t}$ a type of order $i$, and a* reduced type *of order $i$, defined respectively as:*

$$\mathbf{t}_i := (\phi_1(x); \lambda_1, \phi_2(x); \cdots ; \lambda_{i-1}, \phi_i(x); \lambda_i, \psi_i(y)),$$
$$\mathbf{t}_i^0 := (\phi_1(x); \lambda_1, \phi_2(x); \cdots ; \lambda_{i-1}, \phi_i(x); \lambda_i).$$

*For $1 \le i < r-1$, we define the* extended *type of order $i-1$ to be*

$$\tilde{\mathbf{t}}_{i-1} := (\phi_1(x); \lambda_1, \phi_2(x); \cdots ; \lambda_{i-1}, \phi_i(x)).$$

*We have semigroup homomorphisms:*

$$N_i^- \colon \mathcal{O}[x]\backslash\{0\} \to \mathcal{PP}, \quad S_i \colon \mathcal{O}[x]\backslash\{0\} \to \mathcal{S}(\lambda_i), \quad R_i \colon \mathcal{O}[x]\backslash\{0\} \to \mathbb{F}_i[y].$$

*For any nonzero polynomial $P(x) \in \mathcal{O}[x]$, $N_i(P)$ is the $i$-th order Newton polygon with respect to the extended type $\tilde{\mathbf{t}}_{i-1}$, $S_i(P)$ is the $\lambda_i$-component of $N_i^-(P)$, and $R_i(P)(y)$ is the residual polynomial of $i$-th order. Both $S_i$ and $R_i$ depend on the reduced type $\mathbf{t}_i^0$.*

*Finally, we denote by $s_i(P)$ the initial abscissa of $S_i(P)$.*

Other data attached to the type $\mathbf{t}$ deserve an specific notation. For all $1 \le i < r$:

- $\lambda_i = -h_i/e_i$, with $e_i$, $h_i$ positive coprime integers,
- $f_i := \deg \psi_i(y)$, $f_0 := \deg \psi_0(y) = \deg \phi_1(x)$,
- $m_i := \deg \phi_i(x)$, and $m_r := m_{r-1} e_{r-1} f_{r-1}$. Note that $m_{i+1} = m_i e_i f_i = m_1 e_1 f_1 \cdots e_i f_i$,
- $\ell_i$, $\ell_i' \in \mathbb{Z}$ are fixed integers such that $\ell_i h_i - \ell_i' e_i = 1$,
- $z_i := y \pmod{\psi_i(y)} \in \mathbb{F}_{i+1}^*$, $z_0 := y \pmod{\psi_0(y)} \in \mathbb{F}_1$. Note that $\mathbb{F}_{i+1} = \mathbb{F}_i(z_i)$,

Also, for all $0 \leq i < r$ we have semigroup homomorphisms

$$\omega_{i+1}\colon \mathcal{O}[x] \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0}, \qquad P(x) \mapsto \operatorname{ord}_{\psi_i}(R_i(P)),$$

where, by convention: $R_0(P) = \overline{P(y)/\pi^{v(P)}} \in \mathbb{F}[y]$.

**Definition 2.2.** *We say that a monic polynomial $P(x) \in \mathcal{O}[x]$ has type $\mathbf{t}$ if,*

(1) $P(x) \equiv \phi_1(x)^{a_0} \pmod{\mathfrak{m}}$, *for some positive integer $a_0$,*
(2) *For all $1 \leq i < r$, the Newton polygon $N_i(P)$ is one-sided, of slope $\lambda_i$, and $R_i(P)(y) \approx \psi_i(y)^{a_i}$ in $\mathbb{F}_i[y]$, for some positive integer $a_i$.*

**Lemma 2.3.** *For any nonzero polynomial $P(x) \in \mathcal{O}[x]$ we have*

$$\omega_1(P) \geq e_1 f_1 \omega_2(P) \geq \cdots \geq e_1 f_1 \cdots e_{r-1} f_{r-1} \omega_r(P).$$

*If $P(x)$ has type $\mathbf{t}$ then all these inequalities are equalities, and*

$$\deg P(x) = m_r \omega_r(P) = m_{r-1}\omega_{r-1}(P) = \cdots = m_1 \omega_1(P).$$

*Proof.* $e_i f_i \omega_{i+1}(P) \leq e_i \deg R_i(P) = e_i d(S_i(P)) = \ell(S_i(P)) \leq \ell(N_i^-(P)) = \omega_i(P)$, the last equality by Lemma 2.18 in order $i$. If $P(x)$ has type $\mathbf{t}$, we have $\deg P = m_1 a_0 = m_1 \omega_1(P)$, and the two inequalities above are equalities. □

**Definition 2.4.** *Let $P(x) \in \mathcal{O}[x]$ be a monic polynomial with $\omega_r(P) > 0$. We denote by $P_{\mathbf{t}}(x)$ the monic factor of $P(x)$ of greatest degree that has type $\mathbf{t}$. By the Theorem of the residual polynomial in order $r-1$,*

$$\omega_r(P_{\mathbf{t}}) = \omega_r(P), \quad \deg P_{\mathbf{t}} = m_r \omega_r(P). \tag{11}$$

**Lemma 2.5.** *Let $P(x)$, $Q(x) \in \mathcal{O}[x]$ be monic polynomials of positive degree.*

(1) $\deg P < m_r \implies \omega_r(P) = 0$,
(2) *$P(x)$ is of type $\mathbf{t}$ if and only if $\deg P = m_r \omega_r(P) > 0$,*
(3) *$P(x)Q(x)$ has type $\mathbf{t}$ if and only if $P(x)$ and $Q(x)$ have both type $\mathbf{t}$.*

*Proof.* Items 1 and 2 are an immediate consequence of (11). Item 3 follows from the Theorem of the product in orders $1, \ldots, r-1$. □

We fix a type $\mathbf{t}$ of order $r-1$ for the rest of section 2.

2.2. **The $p$-adic valuation of $r$-th order.** In this section we shall attach to $\mathbf{t}$ a discrete valuation $v_r\colon K(x)^* \longrightarrow \mathbb{Z}$, that restricted to $K$ extends $v$ with index $e_1 \cdots e_{r-1}$. We need only to define $v_r$ on $\mathcal{O}[x]$. Consider the mapping

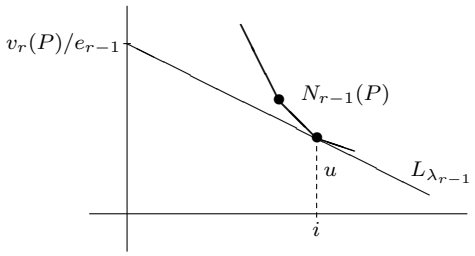
$$H_{r-1}\colon \mathcal{S}(\lambda_{r-1}) \longrightarrow \mathbb{Z}_{\geq 0},$$

that assigns to each side $S \in \mathcal{S}(\lambda_{r-1})$ the non-negative integer obtained as $e_{r-1}$ times the ordinate at the origin of the line $L_{\lambda_{r-1}}$ of slope $\lambda_{r-1}$ that contains $S$. If $(i,u)$ is any point of integer coordinates lying on $S$, then $H_{r-1}(S) = h_{r-1}i + e_{r-1}u$; thus, $H_{r-1}$ is a semigroup homomorphism.

**Definition 2.6.** *For any polynomial $P(x) \in \mathcal{O}[x]$, $P(x) \neq 0$, we define*

$$v_r(P) := H_{r-1}(S_{r-1}(P)).$$

*Note that $v_r$ depends only on the reduced type $\mathbf{t}^0$.*




**Proposition 2.7.** *The natural extension of $v_r$ to $K(x)^*$ is a discrete valuation, whose restriction to $K^*$ extends $v$ with index $e_1 \cdots e_{r-1}$.*

*Proof.* The mapping $v_r$ restricted to $\mathcal{O}[x] \setminus \{0\}$ is a semigroup homomorphism, because it is the composition of two semigroup homomorphisms; hence, $v_r \colon K(x)^* \longrightarrow \mathbb{Z}$ is a group homomorphism.

Let $P(x)$, $Q(x) \in \mathcal{O}[x]$ be two nonzero polynomials and denote $N_P = N_{r-1}^-(P)$, $N_Q = N_{r-1}^-(Q)$, $L_P = L_{\lambda_{r-1}}(N_P)$, $L_Q = L_{\lambda_{r-1}}(N_Q)$ (cf. Definition 1.5). All points of $N_P$ lie above the line $L_P$ and all points of $N_Q$ lie above the line $L_Q$. If $v_r(P) \leq v_r(Q)$, all points of both polygons lie above the line $L_P$. Thus, all points of $N_{r-1}^-(P+Q)$ lie above this line too, and this shows that $v_r(P+Q) \geq v_r(P)$.

Finally, for any $a \in \mathcal{O}$, we have $v_r(a) = e_{r-1}v_{r-1}(a)$ by definition, since the $(r-1)$-th order Newton polygon of $a$ is the single point $(0, v_{r-1}(a))$. □

This valuation was introduced by S. MacLane without using Newton polygons [McL36a], [McL36b]. In [Mon99, Ch.2,§2], J. Montes computed explicit generators of the residue field of $v_r$ as a transcendental extension of a finite field. These results lead to a more conceptual and elegant definition of residual polynomials in higher order, as the reductions modulo $v_r$ of the virtual factors. However, we shall not follow this approach, in order not to burden the paper with even more technicalities.

The main properties we need of this discrete valuation are gathered in the next proposition.

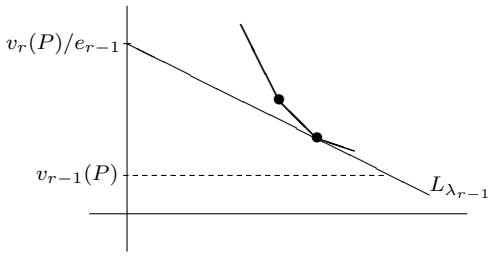
**Proposition 2.8.** *Let $P(x)$, $Q(x) \in \mathcal{O}[x]$ be nonzero polynomials.*

1. *$v_r(P) \geq e_{r-1}v_{r-1}(P)$ and equality holds if and only if $\omega_{r-1}(P) = 0$.*
2. *$v_r(P) = 0$ if and only if $v_2(P) = 0$ if and only if $\operatorname{red}(P) \neq 0$.*
3. *$v_r(\phi_{r-1}) = e_{r-1}v_{r-1}(\phi_{r-1}) + h_{r-1}$.*
4. *If $P(x) = \sum_{0\leq i} a_i(x)\phi_{r-1}(x)^i$ is the $\phi_{r-1}$-adic development of $P(x)$, then*

$$v_r(P) = \min_{0\leq i}\{v_r(a_i(x)\phi_{r-1}(x)^i)\} = e_{r-1}\min_{0\leq i}\{v_{r-1}(a_i(x) + i(v_{r-1}(\phi_{r-1}) + |\lambda_{r-1}|)\}.$$

5. *$v_r(P-Q) > v_r(P)$ if and only if $S_{r-1}(P) = S_{r-1}(Q)$ and $R_{r-1}(P) = R_{r-1}(Q)$. In this case, $\omega_r(P) = \omega_r(Q)$.*

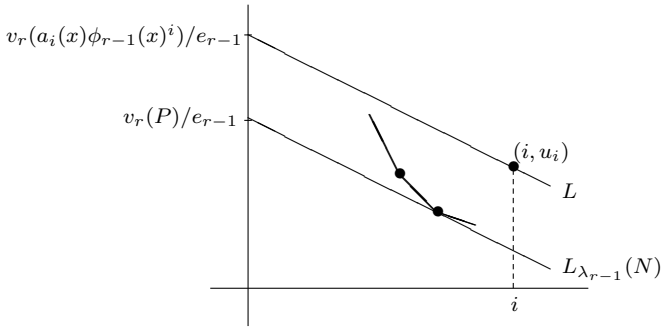
*Proof.* We denote throughout the proof, $N = N_{r-1}^-(P)$, $N' = N_{r-1}^-(Q)$. By (1) of Lemma 2.18 in order $r-1$, all points of $N$ lie above the horizontal line with ordinate $v_{r-1}(P)$. Hence, $v_r(P) \geq e_{r-1}v_{r-1}(P)$. Equality holds if and only if the first point of $N$ is $(0, v_{r-1}(P))$; this is equivalent to $\omega_{r-1}(P) = 0$, by (2) of Lemma 2.18 in order $r-1$. This proves item 1.




By a recurrent aplication of item 1, $v_r(P) = 0$ is equivalent to $v_1(P) = 0$ and $\omega_1(P) = \cdots = \omega_{r-1}(P) = 0$. By Lemma 2.3 this is equivalent to $v_1(P) = 0$ and $\omega_1(P) = 0$, which is equivalent to $v_2(P) = 0$, and also to $P(x) \notin (\pi, \phi_1(x))$. This proves item 2.

Item 3 is immediate from the definition. The polygon $N_{r-1}(\phi_{r-1})$ has only two points $(0, \infty)$, $(1, v_{r-1}(\phi_{r-1}))$; hence it has only one side of length one and slope $-\infty$. The line $L_{\lambda_{r-1}}$ touches the polygon at the point $(1, v_{r-1}(\phi_{r-1}))$, so that $v_r(\phi_{r-1}) = h_{r-1} \cdot 1 + e_{r-1} \cdot v_{r-1}(\phi_{r-1})$.

By definition, $v_r(a_i(x)\phi_{r-1}(x)^i)$ is $e_{r-1}$ times the ordinate at the origin of the line $L$ that has slope $\lambda_{r-1}$ and passes through $(i, v_{r-1}(a_i(x)\phi_{r-1}(x)^i))$. Since all points of $N$ lie above the line $L_{\lambda_{r-1}}(N)$, the line $L$ lies above $L_{\lambda_{r-1}}(N)$ too, and $v_r(a_i(x)\phi_{r-1}(x)^i) \geq v_r(P)$. On the other hand, for the points lying on $S_{r-1}(P)$ we have $L = L_{\lambda_{r-1}}(N)$ and $v_r(a_i(x)\phi_{r-1}(x)^i) = v_r(P)$. This proves item 4.




Let us prove finally item 5. If $v_r(P) < v_r(Q)$ the parallel lines $L_{\lambda_{r-1}}(N)$, $L_{\lambda_{r-1}}(N')$ are different and $S_{r-1}(P) \neq S_{r-1}(Q)$. If $v_r(P) = v_r(Q)$, the above parallel lines coincide and we can consider the shortest segment $S$ of $L_{\lambda_{r-1}}(N)$ that contains $S_{r-1}(P)$ and $S_{r-1}(Q)$. By (16) in order $r-1$, the condition $S_{r-1}(P) = S_{r-1}(Q)$, $R_{r-1}(P) = R_{r-1}(Q)$, is equivalent to $R_{r-1}(P, S) = R_{r-1}(Q, S)$, which is equivalent to $R_{r-1}(P - Q, S) = 0$, by Lemma 2.24 in order $r-1$. This is equivalent to $N^-_{r-1}(P - Q)$ lying strictly above $L_{\lambda_{r-1}}(N)$, which is equivalent in turn to $v_r(P - Q) > v_r(P)$. □

In a natural way, $\omega_r$ induces a group homomorphism from $K(x)^*$ to $\mathbb{Z}$, but it is not a discrete valuation of this field. For instance, for $\mathbf{t} = (x; -1, y + 1)$ and $P(x) = x + p$, $Q(x) = x + p + p^2$, we have

$$R_1(P) = y + 1, \qquad R_1(Q) = y + 1, \qquad R_1(P - Q) = 1,$$
$$\omega_2(P) = 1, \qquad \omega_2(Q) = 1, \qquad \omega_2(P - Q) = 0.$$

The following proposition stablishes a very particular relationship between $v_r$ and $\omega_r$. We shall say that $\omega_r$ is a *pseudo-valuation with respect to* $v_r$.

**Proposition 2.9.** *Let $P(x), Q(x) \in \mathcal{O}[x]$ be two nonzero polynomials such that $v_r(P) = v_r(Q)$ and $\omega_r(P) \neq \omega_r(Q)$. Then,*

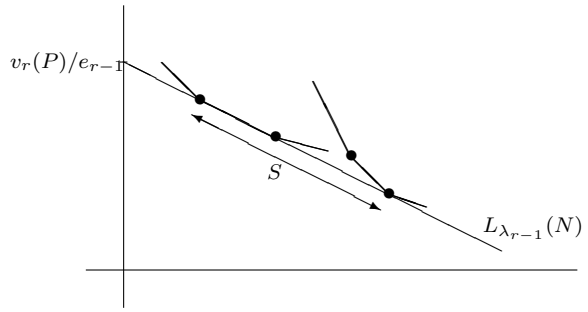
$$v_r(P+Q) = v_r(P) = v_r(Q) \quad \text{and} \quad \omega_r(P+Q) = \min\{\omega_r(P), \omega_r(Q)\}.$$

*Proof.* Suppose $\omega_r(P) < \omega_r(Q)$. Since $\omega_r(Q) = \omega_r(-Q)$, item 5 of the last proposition shows that $v_r(P-(-Q)) = v_r(P)$.

Let $N = N_{r-1}^-(P)$ and let $S$ be the shortest segment of $L_{\lambda_{r-1}}(N)$ that contains $S_{r-1}(P)$ and $S_{r-1}(Q)$. By Lemma 2.24 in order $r-1$ we have $R_{r-1}(P+Q,S) = R_{r-1}(P,S) + R_{r-1}(Q,S)$, and by (16) in order $r-1$ this translates into

$$y^a R_{r-1}(P+Q)(y) = y^b R_{r-1}(P)(y) + y^c R_{r-1}(Q)(y),$$

for certain nonnegative integers $a, b, c$. Since the residual polynomials are never divisible by $y$, and $\psi_{r-1}(y) \neq y$, from $\operatorname{ord}_{\psi_{r-1}}(R_{r-1}(P)) < \operatorname{ord}_{\psi_{r-1}}(R_{r-1}(Q))$ we deduce $\operatorname{ord}_{\psi_{r-1}}(R_{r-1}(P+Q)) = \operatorname{ord}_{\psi_{r-1}}(R_{r-1}(P))$. □




We can reinterpret the computation of $v(P(\theta))$ given in item 5 of Proposition 3.5 in order $r-1$ (cf. Proposition 1.19 for $r=2$), in terms of the pair $v_r$, $\omega_r$.

**Proposition 2.10.** *Let $\theta \in \overline{\mathbb{Q}}_p$ be a root of a polynomial in $\mathcal{O}[x]$ of type $\mathbf{t}$. Then, for any nonzero polynomial $P(x) \in \mathcal{O}[x]$,*

$$v(P(\theta)) \geq v_r(P(x))/e_1 \cdots e_{r-1},$$

*and equality holds if and only if $\omega_r(P) = 0$.*

2.3. **Construction of a representative of t.** By Lemma 2.3, a nonconstant polynomial of type $\mathbf{t}$ has degree at least $m_r$. In this section we shall show how to construct in a effective (and recursive) way a polynomial $\phi_r(x)$ of type $\mathbf{t}$ and minimal degree $m_r$.

We first show how to construct a polynomial with prescribed residual polynomial.

**Proposition 2.11.** *Let $V$ be an integer, $V \geq e_{r-1}f_{r-1}v_r(\phi_{r-1})$. Let $\varphi(y) \in \mathbb{F}_{r-1}[y]$ be a nonzero polynomial of degree less than $f_{r-1}$, and let $\nu = \operatorname{ord}_y(\varphi)$. Then, we can construct in an effective way a polynomial $P(x) \in \mathcal{O}[x]$ satisfying the following properties*

$$\deg P(x) < m_r, \qquad v_r(P) = V, \qquad y^\nu R_{r-1}(P)(y) = \varphi(y).$$

*Proof.* Let $L$ be the line of slope $\lambda_{r-1}$ with ordinate $V/e_{r-1}$ at the origin. Let $T$ be the greatest side contained in $L$, whose end points have nonnegative integer coordinates. Let $(s,u)$ be the initial point of $T$ and denote $u_j := u - jh_{r-1}$, for all $0 \le j < f_{r-1}$, so that $(s + je_{r-1}, u_j)$ lies on $L$. Clearly, $s < e_{r-1}$ and, for all $j$,

$$j < f_{r-1},\, s < e_{r-1} \implies s + je_{r-1} < e_{r-1}f_{r-1}. \tag{12}$$

Let us check that $u_j \ge 0$, so that $(s + je_{r-1}, u_j)$ actually lies on $T$. In fact, let us prove a stronger inequality; denote $V_j := u_j - (s + je_{r-1})v_{r-1}(\phi_{r-1})$. Since $u = (V - sh_{r-1})/e_{r-1}$, we get

$$\begin{aligned}
V_j =& \frac{1}{e_{r-1}}\,(V - (s + je_{r-1})(e_{r-1}v_{r-1}(\phi_{r-1}) + h_{r-1}) \text{ ( by item 3 of Prop. 2.8 )}\\
=& \frac{1}{e_{r-1}}\,(V - (s + je_{r-1})v_r(\phi_{r-1})) \ge \text{ ( by (12) )}\\
\ge& \frac{1}{e_{r-1}}\,(V - (e_{r-1}f_{r-1} - 1)v_r(\phi_{r-1})) \ge \text{ ( by hypothesis )}\\
\ge& \frac{1}{e_{r-1}}v_r(\phi_{r-1}) = v_{r-1}(\phi_{r-1}) + \frac{h_{r-1}}{e_{r-1}} > v_{r-1}(\phi_{r-1}) = e_{r-2}f_{r-2}v_{r-1}(\phi_{r-2}),
\end{aligned}$$

the last equality by (13) below, in order $r-1$..

Let $\varphi(y) = \sum_{0\le j<f_{r-1}} c_j y^j$, with $c_j \in \mathbb{F}_{r-1}$. Select polynomials $c_j(y) \in \mathbb{F}_{r-2}[y]$ of degree less than $f_{r-2}$, such that $c_j$ is the class of $c_j(y)$ modulo $\psi_{r-2}(y)$, or equivalently, $c_j(z_{r-2}) = c_j$.

We proceed by induction on $r \ge 2$. For $r = 2$ the polynomials $c_j(y)$ belong to $\mathbb{F}[y]$; we abuse of language and denote by $c_j(x) \in \mathcal{O}[x]$ the polynomials obtained by choosing arbitrary lifts to $\mathcal{O}$ of the nonzero coefficients of $c_j(y)$. The polynomial $P(x) = \sum_{0\le j<f_{r-1}} \pi^{u-jh_1}c_j(x)\phi_1(x)^{s+je_1}$ satisfies the required properties. In fact, by (12),

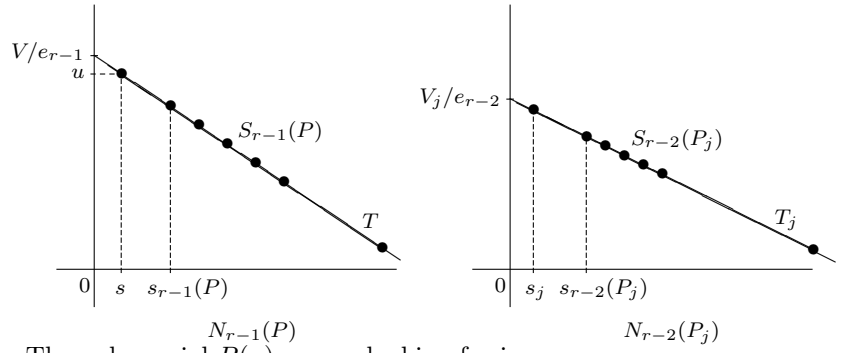
$$\deg(c_j(x)\phi_1(x)^{s+je_1}) < m_1 + (e_1f_1 - 1)m_1 = m_2,$$

for all $j$. For the coefficients $c_j = 0$ we take $c_j(x) = 0$. For the coefficients $c_j \ne 0$, we have $c_j(y) \ne 0$ and $v(c_j(x)) = 0$; hence, $v(\pi^{u-jh_1}c_j(x)) = u - jh_1 = u_j$. Thus, the coefficient $\pi^{u-jh_1}c_j(x)$ determines a point of $N_1^-(P)$ lying on $T$, and $v_2(P) = V$. Finally, it is clear by construction that $\nu = (s_1(P) - s)/e_1$ and $y^\nu R_1(P)(y) = R_1(P,T)(y) = \varphi(y)$.

Suppose now that the proposition has been proved for orders $2,\dots,r-1$. For any $0 \le j < f_{r-1}$ we have seen above that $V_j > e_{r-2}f_{r-2}v_{r-1}(\phi_{r-2})$. Let $L_j$ be the line of slope $\lambda_{r-2}$ with ordinate at the origin $V_j/e_{r-2}$. Let $T_j$ be the greatest side contained in $L_j$, whose end points have nonnegative integer coordinates. Let $s_j$ be the initial abscissa of $T_j$. Consider the unique polynomial $\varphi_j(y) \in \mathbb{F}_{r-2}[y]$, of degree less than $f_{r-2}$, such that

$$\varphi_j(y) \equiv y^{(\ell_{r-2}u_j - s_j)/e_{r-2}}c_j(y) \pmod{\psi_{r-2}(y)},$$

and let $\nu_j = \operatorname{ord}_y(\varphi_j)$. By induction hypothesis, we are able to construct a polynomial $P_j(x)$ of degree less than $m_{r-1}$, with $v_{r-1}(P_j) = V_j$, $\nu_j = (s_{r-2}(P_j) - s_j)/e_{r-2}$, and $y^{\nu_j}R_{r-2}(P_j)(y) = \varphi_j(y)$ in $\mathbb{F}_{r-2}[y]$.




The polynomial $P(x)$ we are looking for is:

$$P(x) = \sum_{0\le j<f_{r-1}} P_j(x)\phi_{r-1}(x)^{s+je_{r-1}} \in \mathcal{O}[x].$$

In fact, by (12), $\deg(P_j(x)\phi_{r-1}(x)^{s+je_{r-1}}) < m_{r-1}+(e_{r-1}f_{r-1}-1)m_1 = m_r$, for all $j$. If $P_j(x)\neq 0$, then $v_{r-1}(P_j(x)\phi_{r-1}(x)^{s+je_{r-1}}) = V_j+(s+je_{r-1})v_{r-1}(\phi_{r-1}) = u_j$, so that all of these coefficients determine points of $N^-_{r-1}(P)$ lying on $T$; this shows that $v_r(P) = V$. For $c_j = 0$ we take $P_j(x) = 0$; hence, $\nu = (s_{r-1}(P)-s)/e_{r-1}$, and by the definition of the residual polynomial in order $r-1$ (cf. Definition 2.21):

$$y^\nu R_{r-1}(P)(y) = \sum_{P_j(x)\neq 0} (z_{r-2})^{t_{r-2}(j)} R_{r-2}(P_j)(z_{r-2})y^j = R_{r-1}(P,T)(y),$$

where $t_{r-2}(j) = (s_{r-2}(P_j) - \ell_{r-2}u_j)/e_{r-2}$. Finally,

$$\begin{aligned}(z_{r-2})^{t_{r-2}(j)} R_{r-2}(P_j)(z_{r-2}) &= (z_{r-2})^{t_{r-2}(j)-\nu_j}\varphi_j(z_{r-2})\\ &= (z_{r-2})^{t_{r-2}(j)-\nu_j+\frac{\ell_{r-2}u_j-s_j}{e_{r-2}}} c_j(z_{r-2}) = c_j,\end{aligned}$$

so that $y^\nu R_{r-1}(P)(y) = \varphi(y)$. □

**Theorem 2.12.** *We can effectively construct a monic polynomial $\phi_r(x)$ of type $\mathbf{t}$ such that $R_{r-1}(\phi_r)(y) = \psi_{r-1}(y)$. This polynomial is irreducible over $\mathcal{O}[x]$ and it satisfies*

$$\deg\phi_r = m_r, \quad \omega_r(\phi_r) = 1, \quad v_r(\phi_r) = e_{r-1}f_{r-1}v_r(\phi_{r-1}). \tag{13}$$

*Proof.* The polynomial $\varphi(y) := \psi_{r-1}(y) - y^{f_{r-1}}$ has degree less than $f_{r-1}$, and $\nu = \operatorname{ord}_y(\varphi) = 0$. Let $P(x)$ be the polynomial attached by Proposition 2.11 to $\varphi(y)$ and $V = e_{r-1}f_{r-1}v_r(\phi_{r-1})$. Since $\deg(P(x)) < m_r$, the polynomial $\phi_r(x) := \phi_{r-1}(x)^{e_{r-1}f_{r-1}} + P(x)$ is monic and it has degree $m_r$. Let $T$ be the auxiliary side used in the construction of $P(x)$; we saw along the proof of Proposition 2.11 that $R_{r-1}(P)(y) = \varphi(y) = R_{r-1}(P,T)(y)$. By (16), $S_{r-1}(P)$ has the same initial point than $T$ and $R_{r-1}(\phi_r)(y) = R_{r-1}(\phi_r,T)(y)$ too. Finally,

$$R_{r-1}(\phi_r,T)(y) = R_{r-1}(\phi_{r-1}^{e_{r-1}f_{r-1}},T)(y)+R_{r-1}(P,T)(y) = y^{f_{r-1}}+\varphi(y) = \psi_{r-1}(y),$$

and $\omega_r(\phi_r) = 1$. The polynomial $\phi_r(x)$ is irreducible over $\mathcal{O}[x]$ by the Theorem of the product in order $r-1$. Finally, it has $v_r(\phi_r) = V$ because all points of $N_{r-1}(\phi_r)$ lie on $T$. □

**Definition 2.13.** *A* representative *of the type $\mathbf{t}$ is a monic polynomial $\phi_r(x)$ of type $\mathbf{t}$ such that $R_{r-1}(\phi_r)(y) \approx \psi_{r-1}(y)$. This object plays the analogous role in order $r-1$ to that of an irreducible polynomial modulo $\mathfrak{m}$ in order one.*

By Theorem 2.12, we can always find a representative $\phi_r(x)$ of $\mathbf{t}$ such that $R_{r-1}(\phi_r)(y) = \psi_{r-1}(y)$; however, we do not impose this condition in the definition of a representative, because in some instances we need to work with representatives satisfying some extra conditions that are incompatible with such an equality (cf. Proposition 3.6).

From now on, we fix a representative $\phi_r(x)$ of $\mathbf{t}$, without necessarily assuming that it has been constructed by the method of Propositon 2.11.

2.4. **Certain rational functions.** We introduce in a recursive way several rational functions in $K(x)$.

**Definition 2.14.** *We define* $\phi_0(x) = x$, $\pi_0(x) = 1$, $\pi_1(x) = \pi$, *and*

$$\Phi_i(x) = \frac{\phi_i(x)}{\pi_{i-1}(x)^{f_{i-1}v_i(\phi_{i-1})}}, \qquad \gamma_i(x) = \frac{\Phi_i(x)^{e_i}}{\pi_i(x)^{h_i}}, \qquad \pi_{i+1}(x) = \frac{\Phi_i(x)^{\ell_i}}{\pi_i(x)^{\ell'_i}},$$

*for all* $1 \le i \le r$.

Each of these rational functions can be written as $\pi^{n_0}\phi_1(x)^{n_1}\cdots\phi_r(x)^{n_r}$, for adequate (positive or negative) integers $n_i$. Also,

$$\Phi_i(x) = \cdots\phi_i(x), \quad \gamma_i(x) = \cdots\phi_i(x)^{e_i}, \quad \pi_{i+1}(x) = \cdots\phi_i(x)^{\ell_i}, \tag{14}$$

where the dots indicate a product of integral powers of $\pi$ and $\phi_j(x)$, with $j < i$. We want to compute the value of $v_r$ on all these functions.

**Lemma 2.15.** *Let* $1 \le i < j \le r$.

(1) $\omega_j(\phi_i) = 0$,
(2) *If* $m_i = m_j$ *then* $v_j(\phi_i) = v_j(\phi_i - \phi_j) \le v_j(\phi_j)$.

*Proof.* Since $N_i(\phi_i)$ is a side of slope $-\infty$, we have $\omega_{i+1}(\phi_i) = 0$ because $S_i(\phi_i)$ reduces to a point. By Lemma 2.3, $\omega_j(\phi_i) = 0$ for all $j > i$.

By (13), $\omega_j(\phi_j) = 1$; hence, $v_j(\phi_i - \phi_j) = \min\{v_j(\phi_i), v_j(\phi_j)\}$ by item 5 of Proposition 2.8. Now, $m_i = m_j$ implies $\deg(\phi_i - \phi_j) < m_j$, and $\omega_j(\phi_i - \phi_j) = 0$ by Lemma 2.5. Again by item 5 of Proposition 2.8, $v_j(\phi_i) = \min\{v_j(\phi_i - \phi_j), v_j(\phi_j)\}$. This proves item 2. □

**Proposition 2.16.** *For all* $1 \le i < r$ *we have*

(1) $v_r(\phi_i) = \sum_{j=1}^{i} (e_{j+1}\cdots e_{r-1})\,(e_j f_j \cdots e_{i-1}f_{i-1})\,h_j$,
(2) $v_r(\Phi_i) = e_{i+1}\cdots e_{r-1}h_i$,
(3) $v_r(\pi_{i+1}) = e_{i+1}\cdots e_{r-1}$,
(4) $v_r(\gamma_i) = 0$.
(5) $\omega_r(\phi_i) = \omega_r(\Phi_i) = \omega_r(\gamma_i) = \omega_r(\pi_{i+1}) = 0$.

*Moreover,* $v_r(\phi_r) = \sum_{j=1}^{r-1} (e_{j+1}\cdots e_{r-1})\,(e_j f_j \cdots e_{i-1}f_{i-1})\,h_j$ *and* $v_r(\Phi_r) = 0$.

*Proof.* We proceed by induction on $r$. For $r = 2$ all formulas are easily deduced from $v_2(\phi_1) = h_1$, that was proved in Proposition 2.8. Suppose $r \ge 3$ and all statements true for $r - 1$.

Let us start with item 1. By Proposition 2.8 and (13),

$$v_r(\phi_{r-1}) = h_{r-1} + e_{r-1}v_{r-1}(\phi_{r-1}), \qquad v_{r-1}(\phi_{r-1}) = e_{r-2}f_{r-2}v_{r-1}(\phi_{r-2}).$$

Hence, the formula for $i = r - 1$ follows from the induction hypothesis. Suppose from now on $i < r - 1$. If $m_i < m_{r-1}$, then $N_{r-1}(\phi_i) = (0, v_{r-1}(\phi_i))$, so that $v_r(\phi_i) = e_{r-1}v_{r-1}(\phi_i)$ and the formula follows by induction. Finally, if $m_i = m_{r-1}$,

then $\phi_i = (\phi_i - \phi_{r-1}) + \phi_{r-1}$ is the $\phi_{r-1}$-adic development of $\phi_i$, and $N_{r-1}(\phi_i)$ has two points $(0, v_{r-1}(\phi_i - \phi_{r-1}))$ and $(1, v_{r-1}(\phi_{r-1}))$. By the above lemma, $v_{r-1}(\phi_i) = v_{r-1}(\phi_i - \phi_{r-1}) \le v_{r-1}(\phi_{r-1})$; thus, $N^-_{r-1}(\phi_i) = (0, v_{r-1}(\phi_i - \phi_{r-1})) = (0, v_{r-1}(\phi_i))$, and $v_r(\phi_i) = e_{r-1}v_{r-1}(\phi_i)$. The formula follows by induction as well.

Let us prove now simultaneously items 2 and 3 by induction on $i$. For $i = 1$ we have by item 1,

$$v_r(\Phi_1) = v_r(\phi_1) = e_2 \cdots e_{r-1}h_1,$$
$$v_r(\pi_2) = \ell_1 v_r(\Phi_1) - \ell'_1 v_r(\pi) = (\ell_1 h_1 - \ell'_1 e_1)e_2 \cdots e_{r-1} = e_2 \cdots e_{r-1}.$$

Suppose now $i > 1$ and the formulas hold for $1, \dots, i-1$.

$$v_r(\Phi_i) = v_r(\phi_i) - f_{i-1}v_i(\phi_{i-1})e_{i-1} \cdots e_{r-1} = e_{i+1} \cdots e_{r-1}h_i,$$
$$v_r(\pi_{i+1}) = \ell_i v_r(\Phi_i) - \ell'_i v_r(\pi_i) = (\ell_i h_i - \ell'_i e_i)e_{i+1} \cdots e_{r-1} = e_{i+1} \cdots e_{r-1}.$$

Item 4 is easily deduced from the previous formulas, and item 5 is an immediate consequence of (14) and item 1 of Lemma 2.15. The last statements follow from (13) and the previous formulas. □

**Lemma 2.17.** *For $\mathbf{n} = (n_0, \dots, n_{r-1}) \in \mathbb{Z}^r$, consider the rational function $\Phi(\mathbf{n}) = \pi^{n_0}\phi_1(x)^{n_1} \cdots \phi_{r-1}(x)^{n_{r-1}} \in K(x)$. Then, if $v_r(\Phi(\mathbf{n})) = 0$, there exists a unique sequence $i_1, \dots, i_{r-1}$ of integers such that $\Phi(\mathbf{n}) = \gamma_1(x)^{i_1} \cdots \gamma_{r-1}(x)^{i_{r-1}}$. Moreover, $i_s$ depends only on $n_s, \dots, n_{r-1}$, for all $1 \le s < r$.*

*Proof.* Since the polynomials $\phi_s(x)$ are irreducible and pairwise different, we have $\Phi(\mathbf{n}) = \Phi(\mathbf{n}')$ if and only if $\mathbf{n} = \mathbf{n}'$. By (14), any product $\gamma_1(x)^{i_1} \cdots \gamma_{r-1}(x)^{i_{r-1}}$ can be expressed as $\Phi(\mathbf{j})$, for a suitable $\mathbf{j} = (j_0, \dots, j_{r-2}, e_{r-1}i_{r-1})$. Thus, if $\gamma_1(x)^{i_1} \cdots \gamma_{r-1}(x)^{i_{r-1}} = 1$ we have necessarily $i_{r-1} = 0$, and recursively, $i_1 = \cdots = i_{r-2} = 0$. This proves the unicity of the expression of any $\Phi(\mathbf{n})$ as a product of powers of gammas.

Let us prove the existence of such an expression by induction on $r \ge 1$. For $r = 1$, let $\mathbf{n} = (n_0)$; the condition $v_r(\pi^{n_0}) = 0$ implies $n_0 = 0$ and $\Phi(\mathbf{n}) = 1$. Suppose $r \ge 2$ and the lemma proven for all $\mathbf{n}' \in \mathbb{Z}^{r-1}$. By item 1 of the last lemma, $v_r(\Phi(\mathbf{n})) \equiv n_{r-1}h_{r-1} \pmod{e_{r-1}}$; hence, if $v_r(\Phi(\mathbf{n})) = 0$ we have necessarily $n_{r-1} = e_{r-1}i_{r-1}$ for some integer $i_{r-1}$ that depends only on $n_{r-1}$. By (14), $\gamma_{r-1}(x)^{i_{r-1}} = \Phi(\mathbf{j})$, for some $\mathbf{j} = (j_0, \dots, j_{r-2}, e_{r-1}i_{r-1})$; hence, $\Phi(\mathbf{n})\gamma_{r-1}(x)^{-i_{r-1}} = \Phi(\mathbf{n}')$, with $\mathbf{n}' = (n'_0, \dots, n'_{r-2}, 0)$, and each $n'_s$ depends only on $n_s$ and $n_{r-1}$. By item 4 of the last lemma, we have still $v_r(\Phi(\mathbf{n}')) = 0$, and by induction hypothesis we get the desired expression of $\Phi(\mathbf{n})$ as a product of powers of gammas. □

2.5. **Newton polygon and residual polynomials of $r$-th order.** Let $f(x) \in \mathcal{O}[x]$ be a nonzero polynomial, and consider its unique $\phi_r$-adic development

$$f(x) = \sum_{0 \le i \le \lfloor \deg(f)/m_r \rfloor} a_i(x)\phi_r(x)^i, \quad \deg a_i(x) < m_r. \tag{15}$$

We define the Newton polygon $N_r(f)$ of $f(x)$, with respect to the extended type

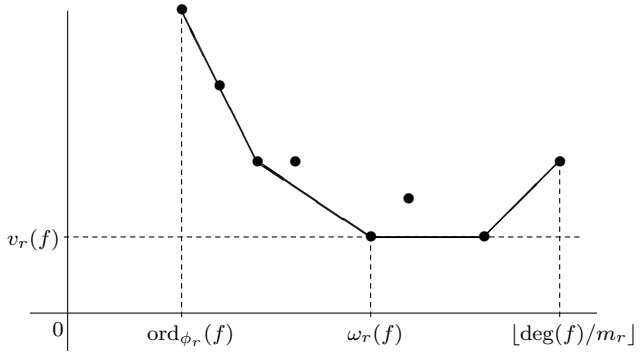
$$\tilde{\mathbf{t}} := (\phi_1(x); \lambda_1, \phi_2(x); \cdots; \lambda_{r-2}, \phi_{r-1}(x); \lambda_{r-1}, \phi_r(x)),$$

to be the lower convex envelope of the set of points $(i, u_i)$, where

$$u_i := v_r(a_i(x)\phi_r(x)^i) = v_r(a_i(x)) + iv_r(\phi_r(x)).$$

Note that we consider the $v_r$-value of the whole monomial $a_i(x)\phi_r(x)^i$. Actually, we did the same for the Newton polygons of first order, but in that case $v_1(a_i(x)\phi_1(x)^i) = v_1(a_i(x))$, because $v_1(\phi_1(x)) = 0$.

The principal part $N_r^-(f)$ is the part of all sides of negative slope, including the side of slope $-\infty$ if $f(x)$ is divisible by $\phi_r(x)$ in $\mathcal{O}[x]$. The typical shape of the polygon is the following




**Lemma 2.18.** (1) $\min_{0\le i\le \ell}\{u_i\} = v_r(f)$,
(2) *The length of* $N_r^-(f)$ *is* $\omega_r(f)$,
(3) *The side of slope* $-\infty$ *of* $N_r^-(f)$ *has length* $ord_{\phi_r}(f)$.

*Proof.* The third item is obvious. Let us prove items 1, 2. Let $u := \min_{0\le i\le \ell}\{u_i\}$, and consider the polynomial

$$g(x) := \sum_{u_i = u} a_i(x)\phi_r(x)^i.$$

All monomials of $g(x)$ have the same $v_r$-value and a different $\omega_r$-value:

$$\omega_r(a_i(x)\phi_r(x)^i) = \omega_r(a_i(x)) + \omega_r(\phi_r(x)^i) = i,$$

because $\omega_r(a_i) = 0$ by Lemma 2.5. By Proposition 2.9, $v_r(g) = u$ and $\omega_r(g) = i_0$, the least abscissa with $u_{i_0} = u$. Since, $v_r(f-g) > u$, we have $v_r(f) = v_r(g) = u$, and this proves item 1. On the other hand, item 5 of Propositon 2.8 shows that $R_1(f) = R_1(g)$; in particular, $\omega_r(f) = \omega_r(g) = i_0$, and this proves item 2. □

The following observation is a consequence of Lemmas 2.5 and 2.18.

**Corollary 2.19.** *If* $f(x)$ *has type* **t** *then* $N_r(f) = N_r^-(f)$.

From now on let $N = N_r^-(f)$. As we did in order one, we attach to any abscissa $i$ of $N$ a *residual coefficient* $c_i \in \mathbb{F}_r$. The natural idea is to consider $c_i = R_{r-1}(a_i)(z_{r-1})$ for the points lying on $N$. However, this does not lead to the right concept of residual polynomial attached to a side; it is necessary to twist these coefficients by certain powers of $z_{r-1}$.

**Definition 2.20.** *For any nonzero* $a(x) \in \mathcal{O}[x]$ *and any integer* $i \ge 0$, *we denote*

$$t_{r-1}(a)_i := \frac{s_{r-1}(a) - \ell_{r-1}v_r(a\phi_r^i)}{e_{r-1}}.$$

*For any nonzero $f(x)\in\mathcal{O}[x]$ with $\phi_r$-adic development (15), we denote*

$$t_{r-1}(i):=t_{r-1}(i,f):=t_{r-1}(a_i)_i=\frac{s_{r-1}(a_i)-\ell_{r-1}u_i}{e_{r-1}}.$$

This number $t_{r-1}(a)_i$ is always an integer. In fact, if $u_{r-1}(a)$ denotes the ordinate of the initial point of $S_{r-1}(a)$,

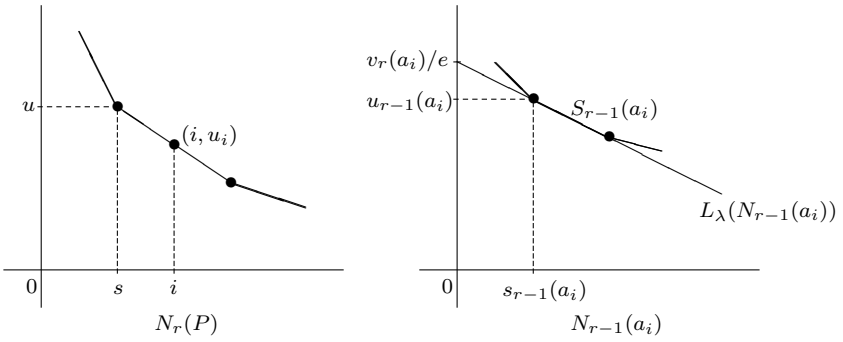
$$\begin{aligned}v_r(a\phi_r^i)=v_r(a)+iv_r(\phi_r)\equiv v_r(a)=h_{r-1}s_{r-1}(a)+e_{r-1}u_{r-1}(a)\pmod{e_{r-1}}\\ \equiv h_{r-1}s_{r-1}(a)\pmod{e_{r-1}},\end{aligned}$$

the first congruence by (13). Hence, $\ell_{r-1}v_r(a\phi_r^i)\equiv s_{r-1}(a)\pmod{e_{r-1}}$.

The correct definition of the residual coefficient is:

$$c_i:=c_i(f):=\begin{cases}0, & \text{if } (i,u_i) \text{ lies strictly above } N,\\ z_{r-1}^{t_{r-1}(i)}R_{r-1}(a_i)(z_{r-1})\in\mathbb{F}_r, & \text{if } (i,u_i) \text{ lies on } N\end{cases}$$

Note that $c_i\neq 0$ if $(i,u_i)$ lies on $N$ because $\omega_r(a_i)=0$ and $\psi_{r-1}(y)$ is the minimal polynomial of $z_{r-1}$ over $\mathbb{F}_{r-1}$.




**Definition 2.21.** *Let $\lambda_r=-h_r/e_r$ be a negative rational number, with $h_r,e_r$ positive coprime integers. Let $S=S_{\lambda_r}(N)$ be the $\lambda_r$-component of $N$, $d=d(S)$ the degree, and $(s,u)$ the initial point of $S$.*

*We define the* virtual factor *of $f(x)$ attached to $S$ (or to $\lambda_r$) to be the rational function*

$$f^S(x):=\Phi_r(x)^{-s}\pi_r(x)^{-u}f^0(x)\in K(x),\quad f^0(x):=\sum_{(i,u_i)\in S}a_i(x)\phi_r(x)^i,$$

*where $\Phi_r(x)$, $\pi_r(x)$ are the rational functions introduced in Definition 2.14.*

*We define the* residual polynomial *attached to $S$ (or to $\lambda_r$) to be the polynomial:*

$$R_{\lambda_r}(f)(y):=c_s+c_{s+e_r}\,y+\cdots+c_{s+(d-1)e_r}\,y^{d-1}+c_{s+de_r}\,y^d\in\mathbb{F}_r[y].$$

Only the points $(i,u_i)$ that lie on $S$ yield a non-zero coefficient of $R_{\lambda_r}(f)(y)$. In particular, $c_s$ and $c_{s+de}$ are always nonzero, so that $R_{\lambda_r}(f)(y)$ has degree $d$ and it is never divisible by $y$. We emphasize that $R_{\lambda_r}(f)(y)$ does not depend only on $\lambda_r$; as all other objects in Sect.2, it depends on $\mathbf{t}$ too.

We define in an analogous way the residual polynomial of $f(x)$ with respect to a side $T$ that is not necessarily a $\lambda_r$-component of $N$. Let $T\in\mathcal{S}(\lambda_r)$ be an arbitrary side of slope $\lambda_r$, with abscissas $s_0\le s_{r-1}$ for the end points. Let $d'=d(T)$. We say

that $f(x)$ *lies above* $T$ *in order* $r$ if all points of $N_r^-(f)$ with abscissa $s_0 \le i \le s_{r-1}$ lie above $T$. In this case we define

$$R_{\lambda_r}(f,T)(y) := \tilde{c}_{s_0} + \tilde{c}_{s_0+e_r}\, y + \cdots + \tilde{c}_{s_0+(d'-1)e_r}\, y^{d'-1} + \tilde{c}_{s_0+d'e_r}\, y^{d'} \in \mathbb{F}_r[y],$$

where $\tilde{c}_i := \tilde{c}_i(f) := c_i$ if $(i,u_i)$ lies on $T$ and $\tilde{c}_i = 0$ otherwise.

Note that $\deg R_{\lambda_r}(f,T)(y) \le d'$ and equality holds if and only if the final point of $T$ belongs to $S_{\lambda_r}(f)$. Usually, $T$ will be an enlargement of $S_{\lambda_r}(f)$ and then,

$$T \supseteq S_{\lambda_r}(f) \implies R_{\lambda_r}(f,T)(y) = y^{(s-s_0)/e_r} R_{\lambda_r}(f)(y), \tag{16}$$

where $s$ is the abscissa of the initial point of $S_{\lambda_r}(f)$.

For technical reasons, we express $c_i$ in terms of a residual polynomial attached to certain side.

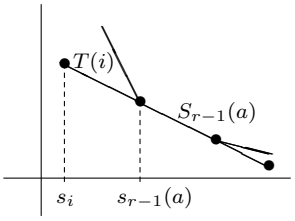
**Notation 2.22.** *Let $N \in \mathcal{PP}$ be a principal polygon, and $(i, h_i(N))$ a (finite) point lying on $N$, with integer abscissa $i$. Let $V = h_i(N) - iv_r(\phi_r)$, and let $L_{\lambda_{r-1}}$ be the line of slope $\lambda_{r-1}$ that has ordinate $V/e_{r-1}$ at the origin. We denote by $T(i)$ the greatest side contained in $L_{\lambda_{r-1}}$, whose end points have nonnegative integer coordinates. We denote by $s_i$ the abscissa of the initial point of $T(i)$.*

**Lemma 2.23.** *Let $N \in \mathcal{PP}$ be a principal polygon, and $(i, h_i(N))$ a point lying on $N$, with integer abscissa $i$. Let $a(x) \in \mathcal{O}[x]$ be a nonzero polynomial such that $u_i := v_r(a\phi_r^i) \ge h_i(N)$. Then,*

$$y^{(s_i - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(a, T(i))(y) = \begin{cases} 0, & \text{if } u_i > h_i(N), \\ y^{t_{r-1}(a)_i} R_{r-1}(a)(y), & \text{if } u_i = h_i(N). \end{cases}$$

*In particular, $c_i = z_{r-1}^{(s_i - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(a, T(i))(z_{r-1})$.*

*Proof.* If $v_r(a\phi_r^i) = h_i(N)$, we have $v_r(a) = V$ and $S_{r-1}(a) \subseteq T(i)$. Then, the lemma follows from (16) in order $r-1$. If $v_r(a\phi_r^i) > h_i(N)$ then $S_{r-1}(a)$ lies strictly above $T(i)$ and $R_{r-1}(a, T(i))(y) = 0$. □




**Lemma 2.24.** *let $T \in \mathcal{S}(\lambda_r)$ be a side of slope $\lambda_r$ and let $f(x), g(x) \in \mathcal{O}[x]$. If $f(x)$ and $g(x)$ lie above $T$ in order $r$, then $(f+g)(x)$ lies above $T$ in order $r$ and*

$$R_{\lambda_r}(f+g, T) = R_{\lambda_r}(f, T) + R_{\lambda_r}(g, T).$$

*Proof.* Let $s_0 \le s_{r-1}$ be the abscissas of the end points of $T$. We want to check that, for all $s_0 \le i \le s_{r-1}$,

$$\tilde{c}_i(f+g) = \tilde{c}_i(f) + \tilde{c}_i(g). \tag{17}$$

Let $a_i(x)$, $b_i(x)$, be the respective $i$-th coefficients of the $\phi_r$-adic development of $f(x)$, $g(x)$; then, $a_i(x) + b_i(x)$ is the $i$-th coefficient of the $\phi_r$-adic development of $f(x) + g(x)$. By Lemma 2.23 applied to the point $(i, h_i(T))$ of $T$,

$$\tilde{c}_i(f) = z_{r-1}^{(s_i - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(a_i, T(i))(z_{r-1}).$$

Analogous equalities hold for $g(x)$ and $(f+g)(x)$, and (17) follows from Lemma 2.24 itself, in order $r-1$ (cf. (2) for $r=1$). □

2.6. **Admissible $\phi_r$-developments and Theorem of the product in order $r$.** Let

$$f(x)=\sum_{i\ge 0} a'_i(x)\phi_r(x)^i,\quad a'_i(x)\in\mathcal{O}[x], \tag{18}$$

be a $\phi_r$-development of $f(x)$, not necessarily the $\phi_r$-adic one. Let $N'$ be the principal polygon of the set of points $(i,u'_i)$, with $u'_i=v_r(a'_i(x)\phi_r(x)^i)$. Let $i_1$ be the first abscissa with $a'_{i_1}(x)\neq 0$. As we did in order one, to each abscisa $i_1\le i\le \ell(N')$ we attach a residual coefficient

$$c'_i=\begin{cases} 0, & \text{if } (i,u'_i) \text{ lies strictly above } N',\\ z_{r-1}^{t'_{r-1}(i)}R_{r-1}(a'_i)(z_{r-1})\in\mathbb{F}_r, & \text{if } (i,u'_i) \text{ lies on } N'\end{cases}$$

where $t'_{r-1}(i):=t_{r-1}(a'_i)_i$. For the points $(i,u'_i)$ lying on $N'$ we may have now $c'_i=0$; for instance in the case $a'_0(x)=f(x)$ the Newton polygon has only one point $(0,v_r(f))$ and $c'_0=0$ if $\omega_r(f)>0$.

Finally, for any negative rational number $\lambda_r=-h_r/e_r$, with $h_r,e_r$ positive coprime integers, we define the residual polynomial attached to the $\lambda_r$-component $S'=S_{\lambda_r}(N')$ to be

$$R'_{\lambda_r}(f)(y):=c'_{s'}+c'_{s'+e_r}\,y+\cdots+c'_{s'+(d'-1)e_r}\,y^{d'-1}+c'_{s'+d'e_r}\,y^{d'}\in\mathbb{F}_r[y],$$

where $d'=d(S')$ and $s'$ is the initial abscissa of $S'$.

**Definition 2.25.** *We say that the $\phi_r$-development (18) is admissible if for each abscissa $i$ of a vertex of $N'$ we have $c'_i\neq 0$, or equivalently, $\omega_r(a'_i)=0$.*

**Lemma 2.26.** *If a $\phi_r$-development is admissible then $N'=N_r^-(f)$ and $c'_i=c_i$ for all abscissas $i$ of the finite part of $N'$. In particular, for any negative rational number $\lambda_r$ we have $R'_{\lambda_r}(f)(y)=R_{\lambda_r}(f)(y)$.*

*Proof.* Consider the $\phi_r$-adic developments of $f(x)$ and each $a'_i(x)$:

$$f(x)=\sum_{0\le i} a_i(x)\phi_r(x)^i,\qquad a'_i(x)=\sum_{0\le k} b_{i,k}(x)\phi_r(x)^k.$$

By the uniqueness of the $\phi_r$-adic development we have

$$a_i(x)=\sum_{0\le k\le i} b_{i-k,k}(x). \tag{19}$$

Let us denote $w_{i,k}:=v_r(b_{i,k})$, $w:=v_r(\phi_r)$. By item 1 of Lemma 2.18, $u'_i=v_r(a'_i)+iw=\min_{0\le k}\{w_{i,k}+(k+i)w\}$. Hence, for all $0\le k$ and all $0\le i\le\ell(N')$:

$$w_{i,k}+(k+i)w\ge u'_i\ge h_i(N'). \tag{20}$$

Therefore, by (19) and (20), all points $(i,u_i)$ lie above $N'$; in fact

$$\begin{aligned} u_i=v_r(a_i)+iw\ge \min_{0\le k\le i}\{w_{i-k,k}+iw\}&=w_{i-k_0,k_0}+iw\\ &\ge u'_{i-k_0}\ge h_{i-k_0}(N')\ge h_i(N'),\end{aligned} \tag{21}$$

for some $0 \le k_0 \le i$. On the other hand, for any abscissa $i$ of the finite part of $N'$ and for all $k > 0$ we have by (20)

$$w_{i-k,k} \ge u'_{i-k} - iw \ge h_{i-k}(N') - iw > h_i(N') - iw. \tag{22}$$

The following claim ends the proof of the lemma:

**Claim.** Let $i$ be an abscissa of the finite part of $N'$ such that $(i, u'_i) \in N'$. Then, $u_i = u'_i$ if and only if $c'_i \ne 0$; and in this case $c'_i = c_i$.

In fact, suppose $c'_i \ne 0$, or equivalently, $\omega_r(a'_i) = 0$. We decompose

$$a'_i(x) = b_{i,0}(x) + B(x), \quad B(x) = \sum_{0<k} b_{i,k}(x)\phi_r(x)^k.$$

By (20) we have $v_r(b_{i,0}) = w_{i,0} \ge u'_i - iw = v_r(a'_i)$. Since $\omega_r(a'_i) = 0$ and $\omega_r(B) > 0$ (because $\phi_r(x)|B(x)$), item 5 of Proposition 2.8 shows that $v_r(b_{i,0}) = v_r(a'_i)$. By (19) and (22) we get $u_i - iw = v_r(a_i) = w_{i,0} = u'_i - iw$, so that $u_i = u'_i$. Let $T(i)$ be the side attached to the point $(i, u'_i) \in N'$ in Notation 2.22. Since $R_{r-1}(B)(z_{r-1}) = 0$, we have $R_{r-1}(B, T(i))(z_{r-1}) = 0$. Hence, $R_{r-1}(a'_i, T(i)) = R_{r-1}(b_{i,0}, T(i))$, by Lemma 2.24 in order $r-1$. Lemma 2.23 shows that

$$\begin{aligned} c'_i &= (z_{r-1})^{(s_i - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(a'_i, T(i)) \\ &= (z_{r-1})^{(s_i - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(b_{i,0}, T(i)) \\ &= (z_{r-1})^{(s_{r-1}(b_{i,0}) - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(b_{i,0})(z_{r-1}) \\ &= (z_{r-1})^{(s_{r-1}(a_i) - \ell_{r-1}u_i)/e_{r-1}} R_{r-1}(a_i)(z_{r-1}) = c_i, \end{aligned}$$

the last but one equality because $S_{r-1}(a_i) = S_{r-1}(b_{i,0})$, $R_{r-1}(a_i) = R_{r-1}(b_{i,0})$, by (22) and item 5 of Proposition 2.8.

Conversely, if $u_i = u'_i = h_i(N')$ we have necessarily $k_0 = 0$ in (21) and all inequalities of (21) are equalities. Hence, $w_{i,0} + iw = u'_i$, or equivalently, $v_r(a'_i) = v_r(b_{i,0})$. Since $\omega_r(b_{i,0}) = 0$ and $\omega_r(B) > 0$, Proposition 2.9 shows that $\omega_r(a'_i) = 0$. This ends the proof of the claim. □

**Theorem 2.27** (Theorem of the product in order $r$)**.** *For any nonzero $f(x), g(x) \in \mathcal{O}[x]$ and any negative rational number $\lambda_r$ we have*

$$N_r^-(fg) = N_r^-(f) + N_r^-(g), \qquad R_{\lambda_r}(fg)(y) = R_{\lambda_r}(f)(y)R_{\lambda_r}(g)(y).$$

*Proof.* Consider the respective $\phi_r$-adic developments

$$f(x) = \sum_{0\le i} a_i(x)\phi_r(x)^i, \quad g(x) = \sum_{0\le j} b_j(x)\phi_r(x)^j,$$

and denote $u_i = v_r(a_i\phi_r^i)$, $v_j = v_r(b_j\phi_r^j)$, $N_f = N_r^-(f)$, $N_g = N_r^-(g)$. Take

$$f(x)g(x) = \sum_{0\le k} A_k(x)\phi_r(x)^k, \qquad A_k(x) = \sum_{i+j=k} a_i(x)b_j(x), \tag{23}$$

and denote by $N'$ the principal part of the Newton polygon of order $r$ of $fg$, determined by this $\phi_r$-development.

We shall show that $N' = N_f + N_g$, that this $\phi_r$-development is admissible, and that $R'_{\lambda_r}(fg) = R_{\lambda_r}(f)R_{\lambda_r}(g)$ for all negative $\lambda_r$. The theorem will be then a consequence of Lemma 2.26.

Let $w_k := v_r(A_k\phi_r^k)$ for all $0 \le k$. Lemma 1.4 shows that the point $(i, u_i)+(j, v_j)$ lies above $N_f + N_g$ for any $i, j \ge 0$. Since $w_k \ge \min\{u_i + v_j, i + j = k\}$, the points $(k, w_k)$ lie all above $N_f + N_g$ too. On the other hand, let $P_k = (k, h_k(N_f + N_g))$ be a

vertex of $N_f+N_g$; that is, $P_k$ is the end point of $S_1+\cdots+S_r+T_1+\cdots+T_s$, for certain sides $S_i$ of $N_f$ and $T_j$ of $N_g$, ordered by increasing slopes among all sides of $N_f$ and $N_g$. By Lemma 1.4, for all pairs $(i,j)$ such that $i+j=k$, the point $(i,u_i)+(j,v_j)$ lies strictly above $N_f+N_g$ except for the pair $i_0=\mathrm{ord}_{\phi_r}(f)+\ell(S_{r-1}+\cdots+S_r)$, $j_0=\mathrm{ord}_{\phi_r}(g)+\ell(T_{r-1}+\cdots+T_s)$ that satisfies $(i_0,u_{i_0})+(j_0,v_{j_0})=P_k$. Thus, $(k,w_k)=P_k$. This shows that $N'=N_f+N_g$.

Moreover, for all $(i,j)\neq(i_0,j_0)$ we have

$$v_r(A_k\phi_r^k)=v_r(a_{i_0}b_{j_0}\phi_r^k)<v_r(a_ib_j\phi_r^k),$$

so that $v_r(A_k)=v_r(a_{i_0}b_{j_0})<v_r(a_ib_j)$. By item 5 of Proposition 2.8, $\omega_r(A_k)=\omega_r(a_ib_j)=\omega_r(a_i)+\omega_r(b_j)=0$, and the $\phi_r$-development (23) is admissible.

Finally, by (1), the $\lambda_r$-components $S'=S_{\lambda_r}(N')$, $S_f=S_{\lambda_r}(N_f)$, $S_g=S_{\lambda_r}(N_g)$ are related by: $S'=S_f+S_g$. Let $(k,h_k(N'))$ be a point of integer coordinates lying on $S'$ (not necessarily a vertex), and let $T(k)$ be the corresponding side of slope $\lambda_{r-1}$ given in Notation 2.22, with starting abscissa $s_k$. Denote by $I$ the set of the pairs $(i,j)$ such that $(i,u_i)$ lies on $S_f$, $(j,v_j)$ lies on $S_g$, and $i+j=k$. Take $P(x)=\sum_{(i,j)\in I}a_i(x)b_j(x)$. By Lemma 1.4, for all other pairs $(i,j)$ with $i+j=k$, the point $(i,u_i)+(j,v_j)$ lies strictly above $N'$. By Lemma 2.24,

$$R_{\lambda_r}(A_k,T(k))=R_{\lambda_r}(P,T(k))=\sum_{(i,j)\in I}R_{\lambda_r}(a_ib_j,T(k)).$$

Lemma 2.23 and the Theorem of the product in order $r-1$ show that

$$\begin{aligned}
c'_k(fg)&=(z_{r-1})^{\frac{s_k-\ell w_k}{e}}R_{\lambda_r}(A_k,T(k))(z_{r-1})\\
&=(z_{r-1})^{\frac{s_k-\ell w_k}{e}}\sum_{(i,j)\in I}R_{\lambda_r}(a_ib_j,T(k))(z_{r-1})\\
&=\sum_{(i,j)\in I}(z_{r-1})^{\frac{s_{r-1}(a_ib_j)-\ell w_k}{e}}R_{r-1}(a_ib_j)(z_{r-1})\\
&=\sum_{(i,j)\in I}(z_{r-1})^{t_{r-1}(i,f)}R_{r-1}(a_i)(z_{r-1})(z_{r-1})^{t_{r-1}(j,g)}R_{r-1}(b_j)(z_{r-1})\\
&=\sum_{(i,j)\in I}c_i(f)c_j(g).
\end{aligned}$$

This shows that the residual polynomial attached to $S'$ with respect to the $\phi_r$-development (23) is $R_{\lambda_r}(f)R_{\lambda_r}(g)$. □

**Corollary 2.28.** *Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial with $\omega_r(f)>0$, and let $f_{\mathbf{t}}(x)$ be the monic factor of $f(x)$ determined by $\mathbf{t}$ (cf. Definition 2.4). Then $N_r(f_{\mathbf{t}})$ is equal to $N_r^-(f)$ up to a vertical shift, and $R_{\lambda_r}(f)\approx R_{\lambda_r}(f_{\mathbf{t}})$ for any negative rational number $\lambda_r$.*

*Proof.* Let $f(x)=f_{\mathbf{t}}(x)g(x)$. By (11), $\omega_r(g)=0$. By the Theorem of the product, $N_r^-(f)=N_r^-(f_{\mathbf{t}})+N_r^-(g)$ and $R_{\lambda_r}(f)=R_{\lambda_r}(f_{\mathbf{t}})R_{\lambda_r}(g)$. Since $N_r^-(g)$ reduces to a point with abscissa 0 (cf. Lemma 2.18), the polygon $N_r^-(f)$ is a vertical shift of $N_r^-(f_{\mathbf{t}})$ and $R_{\lambda_r}(g)$ is a constant. □

## 3. Dissections in order $r$

In this section we extend to order $r$ the Theorems of the polygon and of the residual polynomial. As before, we fix throughout a type $\mathbf{t}$ of order $r-1$. We proceed by induction and we asume that all results of this section have been proved already in orders $1,\dots,r-1$. The case $r=1$ was considered in section 1.

3.1. **Theorem of the polygon in order $r$.** Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial such that $\omega_r(f)>0$. The aim of this section is to obtain a factorization of $f_{\mathbf{t}}(x)$ and certain arithmetic data about the factors. Thanks to Corollary 2.28, we shall be able to read this information directly on $N_r^-(f)$, and the different residual polynomials $R_{\lambda_r}(f)(y)$.

**Theorem 3.1** (Theorem of the polygon in order $r$)**.** *Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial such that $\omega_r(f)>0$. Suppose that $N_r^-(f)=S_{r-1}+\cdots+S_s$ has $s$ sides with pairwise different slopes $\lambda_{r,1},\dots,\lambda_{r,s}$. Then, $f_{\mathbf{t}}(x)$ admits a factorization*

$$f_{\mathbf{t}}(x)=F_1(x)\cdots F_s(x),$$

*as a product of monic polynomials of $\mathcal{O}[x]$ satisfying the following properties:*

1. *$N_r(F_i)$ is equal to $S_i$ up to a translation,*
2. *If $S_i$ has finite slope, then $R_{\lambda_{r,i}}(F_i)(y)\approx R_{\lambda_{r,i}}(f)(y)$*
3. *For any root $\theta\in\overline{\mathbb{Q}}_p$ of $F_i(x)$, $v(\phi_r(\theta))=(v_r(\phi_r)+|\lambda_{r,i}|)/e_1\cdots e_{r-1}$.*

*Proof.* Let us denote $e=e_1\cdots e_{r-1}$. We deal first with the case $f_{\mathbf{t}}(x)$ irreducible. Note that $\deg f_{\mathbf{t}}=m_r\omega_r(f)>0$, by Lemma 2.5, and $N_r(f_{\mathbf{t}})=N_r^-(f_{\mathbf{t}})$ by Corollary 2.19. Since $f_{\mathbf{t}}(x)$ is irreducible, $\rho:=v(\phi_r(\theta))$ is constant among all roots $\theta\in\overline{\mathbb{Q}}_p$ of $f_{\mathbf{t}}(x)$, and $0\le v_r(\phi_r)/e<\rho$, by Proposition 2.10. We have $\rho=\infty$ if and only if $f_{\mathbf{t}}(x)=\phi_r(x)$, and in this case the theorem is clear. Suppose $\rho$ is finite.

Let $P(x)=\sum_{0\le i\le k}b_ix^i\in\mathcal{O}[x]$ be the minimal polynomial of $\phi_r(\theta)$, and let $g(x)=P(\phi_r(x))=\sum_{0\le i\le k}b_i\phi_r(x)^i$. By the Theorem of the polygon in order one, the $x$-polygon of $P$ has only one side and it has slope $-\rho$. The end points of $N_r(P)$ are $(0,ek\rho)$ and $(k,kv_r(\phi_r))$. Now, for all $0\le i\le k$,

$$\frac{v_r(b_i\phi_r^i)-kv_r(\phi_r)}{k-i}=\frac{ev(b_i)+iv_r(\phi_r)-kv_r(\phi_r)}{k-i}\ge e\rho-v_r(\phi_r).$$

This implies that $N_r(g)$ has only one side and it has slope $\lambda_r:=-(e\rho-v_r(\phi_r))$. Since $g(\theta)=0$, $f_{\mathbf{t}}(x)$ divides $g(x)$ and the Theorem of the product shows that $N_r(f_{\mathbf{t}})$ is one-sided, with the same slope. Also, $R_{\lambda_r}(f_{\mathbf{t}})\approx R_{\lambda_r}(f)$ by Corollary 2.28. This ends the proof of the theorem when $f_{\mathbf{t}}(x)$ is irreducible.

If $f_{\mathbf{t}}(x)$ is not necessarily irreducible, we consider its decomposition $f_{\mathbf{t}}(x)=\prod_j P_j(x)$ into a product of monic irreducible factors in $\mathcal{O}[x]$. By Lemma 2.5, each $P_j(x)$ has type $\mathbf{t}$ and by the proof in the irreducible case, each $P_j(x)$ has a one-sided $N_r(P_j)$. The Theorem of the product shows that the slope of $N_r(P_j)$ is $\lambda_{r,i}$ for some $1\le i\le s$. If we group these factors according to the slope, we get the desired factorization. By the Theorem of the product, $R_{\lambda_{r,i}}(F_i)\approx R_{\lambda_{r,i}}(f_{\mathbf{t}})$, because $R_{\lambda_{r,i}}(F_j)$ is a constant for all $j\ne i$. Finally, $R_{\lambda_{r,i}}(f_{\mathbf{t}})\approx R_{\lambda_{r,i}}(f)$ by Corollary 2.28. The statement about $v(\phi_r(\theta))$ is obvious because $P_j(\theta)=0$ for some $j$, and we have already proved the formula for an irreducible polynomial. □

We recall that the factor corresponding to a side $S_i$ of slope $-\infty$ is necessarily $F_i(x)=\phi_r(x)^{\mathrm{ord}_{\phi_r}(f)}$ (cf. Remark 1.7).

Let $\lambda_r = -h_r/e_r$, with $h_r, e_r$ positive coprime integers, be a negative rational number such that $S := S_{\lambda_r}(f)$ has positive length. Let $f_{\mathbf{t},\lambda_r}(x)$ be the factor of $f(x)$, corresponding to the pair $\tilde{\mathbf{t}}, \lambda_r$ by the Theorem of the polygon. Choose a root $\theta \in \overline{\mathbb{Q}}_p$ of $f_{\mathbf{t},\lambda_r}(x)$, and let $L = K(\theta)$. By item 4 of Proposition 3.5 in orders $1, \dots, r-1$, there is a well-defined embedding $\mathbb{F}_r \longrightarrow \mathbb{F}_L$, determined by

$$\mathbb{F}_r \hookrightarrow \mathbb{F}_L,\ z_0 \mapsto \overline{\theta},\ z_{r-1} \mapsto \overline{\gamma_1(\theta)},\ \dots,\ z_{r-1} \mapsto \overline{\gamma_{r-1}(\theta)}. \tag{24}$$

This embedding depends on the choice of $\theta$. After this identification of $\mathbb{F}_r$ with a subfield of $\mathbb{F}_L$ we can think that all residual polynomials of $r$-th order have coefficients in $\mathbb{F}_L$.

**Corollary 3.2.** *The residual degree $f(L/K)$ is divisible by $f_0 \cdots f_{r-1}$, and the ramification index $e(L/K)$ is divisible by $e_1 \cdots e_r$. Moreover, the number of irreducible factors of $f_{\mathbf{t},\lambda_r}(x)$ is at most $d(S)$; in particular, if $d(S) = 1$ the polynomial $f_{\mathbf{t},\lambda_r}(x)$ is irreducible in $\mathcal{O}[x]$, and $f(L/K) = f_0 \cdots f_{r-1}$, $e(L/K) = e_1 \cdots e_r$.*

*Proof.* The statement on the residual degree is a consequence of the embedding (24). Let $e_{\mathfrak{m}} = e(L/K)$, and denote $e = e_1 \cdots e_{r-1}$, $f = f_0 \cdots f_{r-1}$. By the same result in order $r-1$, $e_{\mathfrak{m}}$ is divisible by $e$. Now, by the theorem of the polygon, $v_L(\phi_r(\theta)) = (e_{\mathfrak{m}}/e)v_r(\phi_r) + (e_{\mathfrak{m}}/e)(h_r/e_r)$. Since this is an integer and $h_r, e_r$ are coprime, necessarily $e_r$ divides $e_{\mathfrak{m}}/e$.

The upper bound for the number of irreducible factors is a consequence of the Theorem of the product. Finally, if $d(S) = 1$, we have $ef = \deg(f_{\mathbf{t},\lambda_r}) = f(L/K)e(L/K)$, and necessarily $f(L/K) = f$ and $e(L/K) = e$. □

Let us prove now an identity between the rational functions of Definition 2.14, that plays an essential role in what follows.

**Lemma 3.3.** *Let $P = \sum_{0 \le i} a_i(x)\phi_r(x)^i$ be the $\phi_r$-adic development of a nonzero polynomial in $\mathcal{O}[x]$. Let $\lambda_r = -h_r/e_r$ be a negative rational number, where $h_r$, $e_r$ are coprime positive integers. Let $S = S_{\lambda_r}(P)$ be the $\lambda_r$-component of $N_r^-(P)$, let $(s,u)$ be the initial point of $S$ and $(i,u_i)$ any point lying on $S$. Let $(s(a_i), u(a_i))$ be the initial point of the side $S_{r-1}(a_i)$. Then, the following identity holds in $K(x)$:*

$$\phi_r(x)^i \frac{\Phi_{r-1}(x)^{s(a_i)}\pi_{r-1}(x)^{u(a_i)}}{\Phi_r(x)^s \pi_r(x)^u} = \gamma_{r-1}(x)^{t_{r-1}(i)}\gamma_r(x)^{\frac{i-s}{e_r}}. \tag{25}$$

*Proof.* If we substitute $u = u_i + (i-s)\frac{h_r}{e_r}$ and $\gamma_r = \Phi_r^{e_r}/\pi_r^{h_r}$ in (25), we see that the identity is equivalent to:

$$\phi_r(x)^i \frac{\Phi_{r-1}(x)^{s(a_i)}\pi_{r-1}(x)^{u(a_i)}}{\pi_r(x)^{u_i}} = \gamma_{r-1}(x)^{t_{r-1}(i)}\Phi_r(x)^i.$$

If we substitute now $\Phi_r$, $\pi_r$ and $\gamma_{r-1}$ by its defining values and we use $e_{r-1}t_{r-1}(i) = s(a_i) - \ell_{r-1}u_i$, we get an equation involving only $\pi_{r-1}$, which is equivalent to:

$$u(a_i) + \ell'_{r-1}u_i + h_{r-1}t_{r-1}(i) + if_{r-1}v_r(\phi_{r-1}) = 0.$$

This equality is easy to check by using $u(a_i) + s(a_i)\frac{h_{r-1}}{e_{r-1}} = v_r(a_r) = u_i - iv_r(\phi_r)$, $v_r(\phi_r) = e_{r-1}f_{r-1}v_r(\phi_{r-1})$, and $\ell_{r-1}h_{r-1} - \ell'_{r-1}e_{r-1} = 1$. □

**Lemma 3.4.** *The rational function $\gamma_r(x) \in K(x)$ satisfies: $v(\gamma_r(\theta)) = 0$.*

*Proof.* It is sufficient to check that

$$v(\pi_r(\theta)) = 1/(e_1 \cdots e_{r-1}), \quad v(\Phi_r(\theta)) = h_r/(e_1 \cdots e_r). \tag{26}$$

The first equality follows from Proposition 2.10, because $v_r(\pi_r) = 1$, $\omega_r(\pi_r) = 0$ by Proposition 2.16. The second equality of (26) follows from the Theorem of the polygon and the first equality for $v(\pi_{r-1})(\theta)$. □

**Proposition 3.5.** *We keep the above notations for $f(x)$, $\lambda_r = -h_r/e_r$, $\theta$, $L$, and the embedding (24). Let $P(x) \in \mathcal{O}[x]$ be a nonzero polynomial, $S = S_{\lambda_r}(P)$, $L_{\lambda_r}$ the line of slope $\lambda_r$ that contains $S$, and $H$ the ordinate at the origin of this line. Denote $e = e_1 \cdots e_{r-1}$. Then,*

1. $v(P^S(\theta)) \geq 0, \qquad \overline{P^S(\theta)} = R_{\lambda_r}(P)(\overline{\gamma_r(\theta)})$,
2. $v(P(\theta) - P^0(\theta)) > H/e$.
3. $v(P(\theta)) \geq H/e$, *and equality holds if and only if* $R_{\lambda_r}(P)(\overline{\gamma_r(\theta)}) \neq 0$,
4. $R_{\lambda_r}(f)(\overline{\gamma_r(\theta)}) = 0$.
5. *If* $R_{\lambda_r}(f)(y) \approx \psi_r(y)^a$ *for some irreducible* $\psi_r(y) \in \mathbb{F}_r[y]$ *then* $v(P(\theta)) = H/e$ *if and only if* $R_{\lambda_r}(P)(y)$ *is not divisible by* $\psi_r(y)$ *in* $\mathbb{F}_r[y]$.

*Proof.* Let $P(x) = \sum_{0 \leq i} a_i(x)\phi_r(x)^i$ be the $\phi_r$-adic development of $P(x)$, and denote $u_i = v_r(a_i\phi_r^i)$, $N = N_r^-(P)$, Recall that

$$P^S(x) = \Phi_r(x)^{-s}\pi_r(x)^{-u}P^0(x), \quad P^0(x) = \sum_{(i,u_i)\in S} a_i(x)\phi_r(x)^i,$$

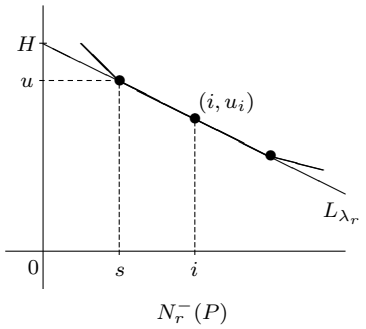
where $(s, u)$ are the coordinates of the initial point of $S$. By (26),

$$v(\Phi_r(\theta)^s\pi_r(\theta)^u) = \frac{1}{e}\left(s\frac{h_r}{e_r} + u\right) = \frac{H}{e}. \tag{27}$$

On the other hand, by the Theorem of the polygon, for all $i$:

$$v(a_i(\theta)\phi_r(\theta)^i) = \frac{v_r(a_i)}{e} + \frac{i}{e}\left(v_r(\phi_r) + \frac{h_r}{e_r}\right) = \frac{1}{e}\left(u_i + i\frac{h_r}{e_r}\right) \geq \frac{H}{e}, \tag{28}$$

with equality if and only if $(i, u_i) \in S$. This proves item 2.




Also, (28) shows that $v(P^S(\theta)) \geq 0$, so that $P^S(\theta)$ belongs to $\mathcal{O}_L$. Denote for simplicity $z_r = \overline{\gamma_r(\theta)}$. In order to prove the equality $\overline{P^S(\theta)} = R_{\lambda_r}(P)(z_r)$, we need to show that for every abscissa $i$ in the projection of $S$ to the horizontal axis:

$$\mathrm{red}_L\left(\frac{a_i(\theta)\phi_r(\theta)^i}{\Phi_r(\theta)^s\pi_r(\theta)^u}\right) = (z_{r-1})^{t_{r-1}(i)}R_{r-1}(a_i)(z_{r-1})(z_r)^{(i-s)/e_r}, \tag{29}$$

if $(i,u_i)\in S$, and $\operatorname{red}_L\left((a_i(\theta)\phi_r(\theta)^i)/(\Phi_r(\theta)^s\pi_r(\theta)^u)\right)=0$, if $(i,u_i)\notin S$. The latter equality is a consequence of (27) and (28). Suppose now $(i,u_i)\in S$. By items 1,2 of the proposition in order $r-1$, applied to the polynomial $a_i(x)$,

$$\overline{(a_i)^{S_{r-1}(a_i)}(\theta)}=R_{r-1}(a_i)(z_{r-1}),$$
$$a_i(\theta)\equiv\Phi_{r-1}(\theta)^{s(a_i)}\pi_{r-1}(\theta)^{u(a_i)}(a_i)^{S_{r-1}(a_i)}(\theta)\ (\mathrm{mod}\ \mathfrak{m}_L^{v_r(a_i/e)}),$$

where $(s(a_i),u(a_i))$ is the initial point of $S_{r-1}(a_i)$. Thus, it is sufficient to check the following identity in $L$,

$$\phi_r(\theta)^i\frac{\Phi_{r-1}(\theta)^{s(a_i)}\pi_{r-1}(\theta)^{u(a_i)}}{\Phi_r(\theta)^s\pi_r(\theta)^u}=\gamma_{r-1}(\theta)^{t_{r-1}(i)}\gamma_r(\theta)^{\frac{i-s}{e_r}},$$

which is a consequence of Lemma 3.3. This ends the proof of item 1.

Also, (28) shows that $v(P(\theta))\geq H/e$, and

$$v(P(\theta))=H/e\iff v(P^0(\theta))=H/e\overset{(27)}{\iff}v(P^S(\theta))=0\iff R_{\lambda_r}(P)(z_r)\neq 0,$$

the last equivalence by item 1. This proves item 3. The last two items are proved by similar arguments to that of the proof of Proposition 1.19. □

3.2. **Theorem of the residual polynomial in order $r$.** We discuss now how Newton polygons and residual polynomials are affected by an extension of the base field by an unramified extension. We keep the above notations for $f(x)$, $\lambda_r=-h_r/e_r$, $\theta$, $L$ and the embedding (24).

**Proposition 3.6.** *Let $K'$ be the unramified extension of $K$ of degree $f_0\dots f_{r-1}$, and identify $\mathbb{F}_r=\mathbb{F}_{K'}$ through the embedding (24). Let $G(x)\in\mathcal{O}_{K'}[x]$ be the minimal polynomial of $\theta$ over $K'$ Then, there exist a type of order $r-1$ over $K'$, $\mathbf{t}'=(\phi_1'(x);\lambda_1,\phi_2'(x);\cdots;\lambda_{r-1},\psi_{r-1}'(y))$, and a representative $\phi_r'(x)$ of $\mathbf{t}'$, with the following properties (where the superscript $'$ indicates that the objects are taken with respect to $\mathbf{t}'$):*

1. *$f_0'=\cdots=f_{r-1}'=1$,*
2. *$G(x)$ is of tytpe $\mathbf{t}'$,*
3. *For any nonzero polynomial $P(x)\in\mathcal{O}[x]$,*

$$(N')_r^-(P)=N_r^-(P),\quad R'_{\lambda_r}(P)(y)=\sigma_r^s\tau_r^uR_{\lambda_r}(P)(\mu_r y),$$

*where $(s,u)$ is the initial point of $S_{\lambda_r}(P)$ and $\sigma_r$, $\tau_r$, $\mu_r\in\mathbb{F}_{K'}^*$ are constants that depend only on $\mathbf{t}$ and $\theta$.*

*Proof.* We proceed by induction on $r$. The case $r=1$ is considered in Lemma 1.20; for the constant $\epsilon$ defined there, we can take $\sigma_1=\epsilon$, $\tau_1=1$, and $\mu_1=\epsilon^{e_1}$. Let $r\geq 2$ and suppose we have already constructed $\mathbf{t}'_{r-2}$ and a representative $\phi'_{r-1}(x)$ satisfying these properties. Let $\eta_1,\dots,\eta_{f_{r-1}}\in\mathbb{F}_{K'}$ be the roots of $\psi_{r-1}(y)$. We have,

$$R'_{r-1}(\phi_r)(y)\approx R_{r-1}(\phi_r)(\mu_{r-1}y)\approx\psi_{r-1}(\mu_{r-1}y)=\textstyle\prod_{i=1}^{f_{r-1}}(\mu_{r-1}y-\eta_i),$$
$$R'_{r-1}(F)(y)\approx R_{r-1}(F)(\mu_{r-1}y)\approx\psi_{r-1}(\mu_{r-1}y)^{a_{r-1}}=\textstyle\prod_{i=1}^{f_{r-1}}(\mu_{r-1}y-\eta_i)^{a_{r-1}}.$$

Since $G(x)$ is of type $\mathbf{t}'_{r-2}$, Lemma 2.5 shows that $\deg G=m'_{r-1}\omega'_{r-1}(G)$. Since $(N')_{r-1}^-(F)=N_{r-1}^-(F)$, the Theorem of the product shows that $(N')_{r-1}^-(G)$ is one-sided, with slope $\lambda_{r-1}$ and positive length $\omega'_{r-1}(G)$. By the Theorem of the

residual polynomial, $R'_{r-1}(G) \approx (\mu_{r-1}y - \eta)^a$, for some root $\eta \in \mathbb{F}_{K'}$ of $\psi'_{r-1}(y)$ and some positive integer $a$. We take

$$\mathbf{t}' = (\phi'_1(x); \lambda_1, \phi'_2(x); \cdots ; \lambda_{r-2}, \phi'_{r-1}(y); \lambda_{r-1}, y - \mu_{r-1}^{-1}\eta),$$

so that $f'_{r-1} = 1$. We have $\deg G = m'_{r-1}\omega'_{r-1}(G) = m'_{r-1}e_{r-1}a = m'_r a$, and $a = \omega'_r(G)$. Thus, $G(x)$ is of type $\mathbf{t}'$, again by Lemma 2.5.

The same argument shows that there is a unique irreducible factor $\phi'_r(x)$ of $\phi_r(x)$ in $\mathcal{O}_{K'}[x]$ such that $R'_{r-1}(\phi'_r(x)) \approx (\mu_{r-1}y - \eta)$. We choose $\phi'_r(x)$ as a representative of $\mathbf{t}'$. Let $\rho_r(x) = \phi_r(x)/\phi'_r(x) \in \mathcal{O}_{K'}[x]$. By construction, $\omega'_r(\rho_r) = 0$, because $R'_{r-1}(\rho_r) \approx \psi_{r-1}(\mu_{r-1}y)/(\mu_{r-1}y - \eta)$.

Let $P(x) \in \mathcal{O}[x]$ be a nonzero polynomial. Clearly,

$$(N')^-_{r-1}(P) = N^-_{r-1}(P) \implies v'_r(P) = v_r(P),$$
$$R'_{r-1}(P)(y) \approx R_{r-1}(P)(\mu_{r-1}y) \implies \omega'_r(P) = \omega_r(P).$$

Consider the $\phi_r$-adic development of $P(x)$:

$$P(x) = \phi_r(x)^n + a_{n-1}(x)\phi_r(x)^{n-1} + \cdots + a_0(x) = \\ = \rho_r(x)^n\phi'_r(x)^n + a_{n-1}(x)\rho_r(x)^{n-1}\phi'_r(x)^{n-1} + \cdots + a_0(x).$$

Since $\omega'_r(\rho_r) = 0$, this $\phi'_r$-adic development of $P(x)$ is admissible. On the other hand, the tautology

$$v_r(a_i(x)\phi_r(x)^i) = v'_r(a_i(x)\phi_r(x)^i) = v'_r(a_i(x)\rho_r(x)^i(\phi'_r)(x)^i),$$

shows that $(N')^-_r(P) = N^-_r(P)$.

In order to prove the relationship between $R'_{\lambda_r}(P)(y)$ and $R_{\lambda_r}(P)(y)$, we introduce some elements in $\mathbb{F}^*_{K'}$, constructed in terms of the rational functions of Definition 2.14. By Lemma 3.4, $v(\gamma_r(\theta)) = 0 = v(\gamma'_r(\theta))$. By (26), $v(\pi_r(\theta)) = (e_1 \cdots e_{r-1})^{-1} = v(\pi'_r(\theta))$, and

$$v(\rho_r(\theta)) = (v_r(\phi_r) - v'_r(\phi'_r))/(e_1 \cdots e_{r-1}) = v(\pi'_{r-1}(\theta))(v_r(\phi_r) - v'_r(\phi'_r))/e_{r-1}.$$

We introduce the following elements of $\mathbb{F}^*_{K'}$:

$$\mu_r := \overline{\gamma_r(\theta)/\gamma'_r(\theta)}, \qquad \tau_r := \overline{\pi_r(\theta)/\pi'_r(\theta)},$$
$$\sigma_r := \overline{\Phi_r(\theta)/\Phi'_r(\theta)}, \qquad \epsilon_r := \overline{\rho_r(\theta)/\pi'_{r-1}(\theta)^{(v_r(\phi_r)-v'_r(\phi'_r))/e_{r-1}}},$$

where we used $f_{r-1}v_r(\phi_{r-1}) = v_r(\phi_r)/e_{r-1}$ in the last equality. By the recursive definition of the functions of Definition 2.14, we get the following identities:

$$\sigma_r = \epsilon_r/(\tau_{r-1})^{v_r(\phi_r)/e_{r-1}}, \qquad \tau_r = (\sigma_{r-1})^{\ell_{r-1}}/(\tau_{r-1})^{\ell'_{r-1}}. \tag{30}$$

We need still another interpretation of $\epsilon_r$. Since $(N')^-_{r-1}(\phi_r) = N^-_{r-1}(\phi_r)$, the Theorem of the product shows that $(N')_{r-1}(\rho_r)$ is one-sided with slope $\lambda_{r-1}$; hence, the initial point $(s'_{r-1}(\rho_r), u'_{r-1}(\rho_r))$ of $S := S'_{r-1}(\rho_r)$ is given by $s'_{r-1}(\rho_r) = 0$ and

$$u'_{r-1}(\rho_r) = v'_r(\rho_r)/e_{r-1} = (v'_r(\phi_r) - v'_r(\phi'_r))/e_{r-1} = (v_r(\phi_r) - v'_r(\phi'_r))/e_{r-1}. \tag{31}$$

Recall that the virtual factor $\rho^S_r(x)$ is by definition $\rho_r(x)/(\pi')^{u'_{r-1}(\rho_r)}_{r-1}$; therefore, item 1 of Proposition 3.5 shows that, for $r \geq 2$:

$$\epsilon_r = R'_{r-1}(\rho_r)(z'_{r-1}). \tag{32}$$

We have seen above that for each integer abscissa $i$, the $i$-th terms of the $\phi_r$ and $\phi'_r$-developments of $P(x)$ determine the same point $(i, u_i)$ of the plane. Let $i =$

$s+je_r$ be an abscissa such that $(i,u_i)$ lies on $S_{\lambda_r}(P)=S'_{\lambda_r}(P)$; the corresponding residual coefficients at this abscissa are respectively

$$c_i=(z_{r-1})^{t_{r-1}(i)}R_{r-1}(a_i)(z_{r-1}),\quad c'_i=(z'_{r-1})^{t'_{r-1}(i)}R'_{r-1}(a_i\rho_r^i)(z'_{r-1}),$$

and $R_{\lambda_r}(P)(y)=\sum_{0\le j\le d}c_iy^j$, $R'_{\lambda_r}(P)(y)=\sum_{0\le j\le d}c'_iy^j$. Hence, the last equality of item 3 is equivalent to $c'_i=c_i\sigma_r^s\tau_r^u\mu_r^j$, for all such $i$.

Note that $t_{r-1}(i)=(s_{r-1}(a_i)-\ell_{r-1}u_i)/e_{r-1}=t'_{r-1}(i)$, because

$$s'_{r-1}(a_i\rho_r^i)=s'_{r-1}(a_i)+is'_{r-1}(\rho_r)=s'_{r-1}(a_i)=s_{r-1}(a_i),$$

the last equality because $N^-_{r-1}(a_i)=(N')^-_{r-1}(a_i)$. For simplicity we denote by $(s(a_i),u(a_i))$ the initial point of $S_{r-1}(a_i)$. By (31), the initial point of $S'_{r-1}(a_i\rho_r^i)$ is $(s(a_i),u(a_i)+i(v_r(\phi_r)-v'_r(\phi'_r))/e_{r-1})$. Now, by induction, the Theorem of the product, and (32), we have

$$\begin{aligned}c'_i&=(z'_{r-1})^{t_{r-1}(i)}R'_{r-1}(a_i)(z'_{r-1})\epsilon_r^i\\&=(z'_{r-1})^{t_{r-1}(i)}(\sigma_{r-1})^{s(a_i)}(\tau_{r-1})^{u(a_i)}R_{r-1}(a_i)(z_{r-1})\epsilon_r^i\\&=c_i(\mu_{r-1})^{-t_{r-1}(i)}(\sigma_{r-1})^{s(a_i)}(\tau_{r-1})^{u(a_i)}\epsilon_r^i\\&=c_i\mu_r^j\left(\mu_r^{-j}(\mu_{r-1})^{-t_{r-1}(i)}\right)(\sigma_{r-1})^{s(a_i)}(\tau_{r-1})^{u(a_i)}\epsilon_r^i\end{aligned}$$

By Lemma 3.3,

$$\begin{aligned}\gamma_r(\theta)^j\gamma_{r-1}(\theta)^{t_{r-1}(i)}&=\phi_r(\theta)^i\Phi_{r-1}(\theta)^{s(a_i)}\pi_{r-1}(\theta)^{u(a_i)}\Phi_r(\theta)^{-s}\pi_r(\theta)^{-u}\\&=\phi_r(\theta)^i\Phi_{r-1}(\theta)^{s(a_i)-\ell_{r-1}u}\pi_{r-1}(\theta)^{u(a_i)+\ell'_{r-1}u}\Phi_r(\theta)^{-s}.\end{aligned}$$

We get an analogous expression for $\gamma'_r(\theta)^j\gamma'_{r-1}(\theta)^{t_{r-1}(i)}$, just by putting $'$ everywhere and by substituting $u(a_i)$ by $u(a_i\rho_r^i)=u(a_i)+i(v_r(\phi_r)-v'_r(\phi'_r))/e_{r-1}$. By taking the quotient of both expressions and taking classes modulo $\mathfrak{m}_{K'}$ we get

$$\mu_r^j(\mu_{r-1})^{t_{r-1}(i)}=\epsilon_r^i(\sigma_{r-1})^{s(a_i)-\ell_{r-1}u}(\tau_{r-1})^{u(a_i)+\ell'_{r-1}u}\sigma_r^{-s}.$$

Therefore, $c'_i=c_i\mu_r^j(\sigma_{r-1})^{\ell_{r-1}u}(\tau_{r-1})^{-\ell'_{r-1}u}\sigma_r^s=c_i\mu_r^j\tau_r^u\sigma_r^s$, by (30). □

**Theorem 3.7** (Theorem of the residual polynomial in order $r$)**.** *Let $f(x)\in\mathcal{O}[x]$ be a monic polynomial with $\omega_r(f)>0$, and let $S$ be a side of $N_r^-(f)$, of finite slope $\lambda_r$. Consider the factorization*

$$R_{\lambda_r}(f)(y)=\psi_{r,1}(y)^{a_1}\cdots\psi_{r,t}(y)^{a_t},$$

*of the residual polynomial of $f(x)$ into the product of powers of pairwise different irreducible polynomials in $\mathbb{F}_r[y]$. Then, the factor $f_{\mathbf{t},\lambda_r}(x)$ of $f_{\mathbf{t}}(x)$, corresponding to $S$ by the Theorem of the polygon, admits a factorization in $\mathcal{O}[x]$,*

$$f_{\mathbf{t},\lambda_r}(x)=G_1(x)\cdots G_t(x),$$

*with all $N_r(G_i)$ one-sided of slope $\lambda_r$, and $R_{\lambda_r}(G_i)(y)\approx\psi_{r,i}(y)^{a_i}$ in $\mathbb{F}_r[y]$.*

*Proof.* Let us deal first with the case $F(x):=f_{\mathbf{t},\lambda_r}(x)$ irreducible. We need only to prove that $R_{\lambda_r}(F)(y)$ is the power of an irreducible polynomial of $\mathbb{F}_r[y]$. Let $\theta\in\overline{\mathbb{Q}}_p$ be a root of $F(x)$, let $L=K(\theta)$, and fix the embedding $\mathbb{F}_r\longrightarrow\mathbb{F}_L$ as in (24). Let $K'$ be the unramified extension of $K$ of degree $f_0\cdots f_{r-1}$, and let $G(x)\in\mathcal{O}_{K'}[x]$ be the minimal polynomial of $\theta$ over $K'$, so that $F(x)=\prod_{\sigma\in\mathrm{Gal}(K'/K)}G^\sigma(x)$. Under the embedding $\mathbb{F}_r\longrightarrow\mathbb{F}_L$, the field $\mathbb{F}_r$ is identified with $\mathbb{F}_{K'}$. By Proposition 3.6, we can construct a type $\mathbf{t}'$ of order $r-1$ over $K'$ such that $R'_{\lambda_r}(F)(y)\sim R_{\lambda_r}(F)(y)$.

By the construction of $\mathbf{t}'$, for any $\sigma \neq 1$, the polynomial $G^\sigma(x)$ is not divisible by $\phi_1'(x)$ modulo $\mathfrak{m}_{K'}$; thus, $\omega_r(G^\sigma) \leq \omega_1(G^\sigma) = 0$, and $R'_{\lambda_r}(G^\sigma)(y)$ is a constant. Therefore, by the Theorem of the product, $R'_{\lambda_r}(G)(y) \approx R'_{\lambda_r}(F)(y) \sim R_{\lambda_r}(F)(y)$, so that $R_{\lambda_r}(F)(y)$ is the power of an irreducible polynomial of $\mathbb{F}_r[y]$ if and only if $R'_{\lambda_r}(G)(y)$ has the same property over $\mathbb{F}_{K'}$. In conclusion, by extending the base field, we can suppose that $f_0 = \cdots f_{r-1} = 1$.

Let $P(x) = \sum_{j=0}^k b_j x^j \in \mathcal{O}[x]$ be the minimal polynomial of $\gamma_r(\theta)$. Let

$$\Pi(x) := \gamma_r(x)/\phi_r(x)^{e_r} = \pi_{r-1}(x)^{-e_r f_{r-1} v_r(\phi_{r-1})} \pi_r(x)^{-h_r}.$$

By (14), $\Pi(x)$ admits an expression $\Pi(x) = \pi^{n_0'} \phi_1(x)^{n_1'} \cdots \phi_{r-1}(x)^{n_{r-1}'}$ for some integers $n_1', \ldots, n_r'$. Take $\Phi(x) := \pi^{n_0} \phi_1(x)^{n_1} \cdots \phi_{r-1}(x)^{n_{r-1}}$ with sufficiently large positive integers $n_i$ so that $\Pi(x)^k \Phi(x)$ is a polynomial in $\mathcal{O}[x]$. Then, the following rational function is actually a polynomial in $\mathcal{O}[x]$:

$$g(x) := \Phi(x) P(\gamma_r(x)) = \sum_{j=0}^k B_{je_r}(x) \phi_r(x)^{je_r}, \quad B_{je_r}(x) = \Phi(x) \Pi(x)^j b_j.$$

Moreover, by Proposition 2.16, $\omega_r(B_{je_r}) = 0$ for all $j$ such that $B_{je_r} \neq 0$, so that this $\phi_r$-development of $g(x)$ is admissible.

Our aim is to show that $N_r(g)$ is one-sided with slope $\lambda_r$, and $R_{\lambda_r}(g)(y)$ is equal to $P(y)$ modulo $\mathfrak{m}$, which is the power of an irreducible polynomial of $\mathbb{F}_r[y]$ because $P(x)$ is irreducible. Since $g(\theta) = 0$, $F(x)$ is a divisor of $g(x)$ and by the Theorem of the product the residual polynomial of $F(x)$ will be the power of an irreducible polynomial too. This will end the proof of the theorem in the irreducible case.

Let us bound by below all $v_r(B_{je_r} \phi_r^{je_r})$. Denote $u := v_r(\Phi)$. By Proposition 2.16 and (13), we get: $v_r(\pi_{r-1}) = e_{r-1}$, $v_r(\pi_r) = 1$, and $v_r(\Pi) = -e_r v_r(\phi_r) - h_r$. Therefore,

$$u_{je_r} := v_r(B_{je_r} \phi_r^{je_r}) = v_r(b_j) + u - j(e_r v_r(\phi_r) + h_r) + je_r v_r(\phi_r) \geq u - jh_r. \tag{33}$$

For $j = 0, k$ we have $v(b_0) = 0$ (because $v(\gamma_r(\theta)) = 0$) and $v(b_k) = 0$ (because $b_k = 1$). Hence, equality holds in (33) for these two abscissas. This proves that $N_r(g)$ has only one side $T$, with end points $(0, u)$, $(ke_r, u - kh_r)$, and slope $\lambda_r$.

Let $R_{\lambda_r}(g)(y) = \sum_{j=0}^k c_{je_r} y^j$. We want to show that $c_{je_r} = c\bar{b}_j$ for certain constant $c \in \mathbb{F}_r^*$ independent of $j$. Recall that $c_{je_r} = 0$ if and only if $(je_r, u_{je_r}) \notin T$, and by (33), this is equivalent to $\bar{b}_j = 0$. Suppose now $(je_r, u_{je_r}) \in T$; by item 1 of Proposition 3.5 (cf. (29))

$$\mathrm{red}_L \left( \frac{B_{je_r}(\theta) \phi_r(\theta)^{je_r}}{\pi_r(\theta)^u} \right) = c_{je_r} \overline{\gamma_r(\theta)}^j.$$

Hence, we want to check that for all $j$

$$\mathrm{red}_L \left( \frac{B_{je_r}(\theta) \phi_r(\theta)^{je_r}}{\pi_r(\theta)^u \gamma_r(\theta)^j} \right) = c\bar{b}_j,$$

for some nonzero constant $c$. Now, by substitution of the defining value of $\gamma_r$ it is easily checked that the left hand side is equal to $c\bar{b}_j$, for $c = \mathrm{red}_L(\Phi(\theta)/\pi_r(\theta)^u)$. This ends the proof of the theorem in the irreducible case.

In the general case, consider the decomposition, $F(x) = \prod_j P_j(x)$, into a product of monic irreducible factors in $\mathcal{O}[x]$. By Lemma 2.5, each $P_j(x)$ has type $\mathbf{t}$, so that $\omega_r(P_j) > 0$. By the Theorem of the product, $N_r(P_j)$ is one-sided, of ositive

length and slope $\lambda_r$. By the proof in the irreducible case, the residual polynomial $R_{\lambda_r}(P_j)(y)$ is the positive power of an irreducible polynomial, and by the Theorem of the product it must be $R_{\lambda_r}(P_j)(y) \approx \psi_{r,i}(y)^{b_j}$ for some $1 \le i \le t$. If we group these factors according to the irreducible factor of the residual polynomial, we get the desired factorization. □

**Corollary 3.8.** *With the above notations, let $\theta \in \overline{\mathbb{Q}}_p$ be a root of $G_i(x)$, and $L = K(\theta)$. Let $f_r = \deg \psi_{r,i}(y)$, $e_r = e_{r,i}$. Then, $f(L/K)$ is divisible by $f_0 f_1 \cdots f_r$. Moreover, if $a_i = 1$ then $G_i(x)$ is irreducible in $\mathcal{O}[x]$ and*

$$f(L/K) = f_0 f_1 \cdots f_r, \quad e(L/K) = e_1 \cdots e_{r-1} e_r.$$

*Proof.* The statement about $f(L/K)$ is a consequence of the extension of the embeding (24) to an embedding

$$\mathbb{F}_r[y]/\psi_{r,i}(y) \hookrightarrow \mathbb{F}_L, \qquad y \mapsto \overline{\gamma_r(\theta)}, \tag{34}$$

which is well-defined by item 4 of Proposition 3.5. The irreducibility of $G_i(x)$ when $a_i = 1$ is a consequence of the Theorem of the product. The computation of $f(L/K)$ and $e(L/K)$ follows from

$$f(L/K)e(L/K) = \deg G_i = f_0 f_1 \cdots f_r e_1 \cdots e_{r-1} e_r,$$

and the fact that $f(L/K)$ is divisible by $f_0 \cdots f_r$ and $e(L/K)$ is divisible by $e_1 \cdots e_r$ (cf. Corollary 3.2). □

3.3. **Types of order $r$.** Let $\mathbf{t}$ be a type of order $r-1$, and let $f(x) \in \mathcal{O}[x]$ be a monic separable polynomial.

**Definition 3.9.** *We say that $\mathbf{t}$ is $f$-*complete*, if $\omega_r(f) = 1$. In this case, $f_{\mathbf{t}}(x)$ is irreducible and the ramification index and residual degree of the extension of $K$ determined by $f_{\mathbf{t}}(x)$ can be computed in terms of some data of $\mathbf{t}$, by applying Corollary 3.8 in order $r-1$.*

If $\mathbf{t}$ is a type of order $r-1$ and $\omega_r(f) > 1$, the results of Sect. 3 can be interpreted as the addition of *two more dissections*, for each order $2, \dots, r$, to the three classical ones, in the process of factorization of $f(x)$. The factor $f_{\mathbf{t}}(x)$ has experimented further factorizations at two levels: first $f_{\mathbf{t}}(x)$ factorizes in as many factors as sides of $N_r^-(f)$, and then, the factors corresponding to finite slopes split into the product of as many factors of pairwise different irreducible factors of the residual polynomial.

We can think that the type $\mathbf{t}$ has sprouted to produce several types of order $r$,

$$\mathbf{t}' = (\tilde{\mathbf{t}}; \lambda_r, \psi_r(y)),$$

each of them distinguished by the choice of a slope $\lambda_r$ of a side of $N_r^-(f)$, and an irreducible factor $\psi_r(y)$ of $R_{\lambda_r}(f)(y)$ in $\mathbb{F}_r[y]$.

**Definition 3.10.** *In Sect. 1.5, we defined two sets $\mathbf{t}_0(f)$, $\mathbf{t}_1(f)$. We recursively define $\mathbf{t}_r(f)$ to be the set of all types of order $r$ constructed as above, $\mathbf{t}' = (\tilde{\mathbf{t}}; \lambda_r, \psi_r(y))$, from those $\mathbf{t} \in \mathbf{t}_{r-1}(f)$ that are not $f$-complete. This set is not an intrinsic invariant of $f(x)$ because it depends on the choices of the representatives $\phi_1(x), \dots, \phi_r(x)$ of the truncations of $\mathbf{t}$.*

*We denote by* $\mathbf{t}_r(f)^{\mathrm{compl}}$ *the subset of the* $f$*-complete types of* $\mathbf{t}_r(f)$*. We define*

$$\mathbf{T}_r(f) := \mathbf{t}_r(f) \cup \left( \bigcup_{0 \le s < r} \mathbf{t}_s(f)^{\mathrm{compl}} \right).$$

Hensel's lemma and the theorems of the polygon and of the residual polynomial in orders $1, \dots, r$ determine a factorization

$$f(x) = f_{r,\infty}(x) \prod_{\mathbf{t} \in \mathbf{T}_r(f)} f_{\mathbf{t}}(x), \tag{35}$$

where $f_{r,\infty}(x)$ is the product of the different representatives $\phi_i(x)$, of the different types, that divide $f(x)$ in $\mathcal{O}[x]$.

The following remark is an immediate consequence of the definitions.

**Lemma 3.11.** *The following conditions are equivalent:*

1. $\mathbf{t}_{r+1}(f) = \emptyset$,
2. $\mathbf{t}_r(f)^{\mathrm{compl}} = \mathbf{t}_r(f)$,
3. *For all* $\mathbf{t} \in \mathbf{t}_{r-1}(f)$ *and all* $\lambda_r \in \mathbb{Q}^-$*, the residual polynomial of* $r$*-th order,* $R_{\lambda_r}(f)(y)$ *is separable.*

If these conditions are satisfied, then (35) is a factorization of $f(x)$ into the product of monic irreducible polynomials in $\mathcal{O}[x]$, and we get arithmetic information about each factor by Corollary 3.8. As long as there is some $\mathbf{t} \in \mathbf{t}_r(f)$ which is not $f$-complete, we must apply the results of this section in order $r+1$ to get further factorizations of $f_{\mathbf{t}}(x)$, or to detect that it is irreducible. We need some invariant to control the whole process and ensure that after a finite number of steps we shall have $\mathbf{t}_r(f)^{\mathrm{compl}} = \mathbf{t}_r(f)$. This is the aim of the next section.

We end with a remark about $p$-adic approximations to the irreducible factors of $f(x)$, that is an immediate consequence of Lemma 2.3, the Theorem of the polygon and Proposition 2.16.

**Proposition 3.12.** *Let* $\mathbf{t}$ *be a type of order* $r$ *of* $f(x)$ *that is* $f$*-complete. Let* $\phi_{r+1}(x) \in \mathcal{O}[x]$ *be a representative of* $\mathbf{t}$*. Then,* $\deg \phi_{r+1}(x) = \deg f_{\mathbf{t}}(x)$*, and* $\phi_{r+1}(x)$ *is a* $p$*-adic approximation to* $f_{\mathbf{t}}(x)$ *satisfying*

$$v(\phi_{r+1}(\theta)) = (v_{r+1}(\phi_{r+1}) + h_{r+1})/e(L/K) = \sum_{i=1}^{r+1} e_i f_i \cdots e_r f_r \, \frac{h_i}{e_1 \dots e_i},$$

*where* $\theta \in \overline{\mathbb{Q}}_p$ *is a root of* $f_{\mathbf{t}}(x)$*.*

## 4. Indices and resultants of higher order

### 4.1. Computation of resultants with Newton polygons.

**Definition 4.1.** *Let* $r \ge 1$ *be a natural number. Let* $\mathbf{t}$ *be a type of order* $r-1$ *and let* $\phi_r(x) \in \mathcal{O}[x]$ *be a representative of* $\mathbf{t}$*. For any pair of monic polynomials* $P(x), Q(x) \in \mathcal{O}[x]$ *we define*

$$\mathrm{Res}_{\mathbf{t}}(P,Q) := f_0 \cdots f_{r-1} \left( \sum_{i,j} \min\{E_i H'_j, E'_j H_i\} \right),$$

*where* $E_i = \ell(S_i)$*,* $H_i = H(S_i)$ *are the lengths and heights of the sides of* $N_r^-(P)$*, and* $E'_j = \ell(S'_j)$*,* $H'_j = H(S'_j)$ *are the lengths and heights of the sides of* $N_r^-(Q)$*.*

We recall that for a side $S$ of slope $-\infty$ we took $H(S)=\infty$ by convention. Thus, the part of $\mathrm{Res}_{\mathbf{t}}(P,Q)$ that involves sides of slope $-\infty$ is always

$$f_0\cdots f_{r-1}(\mathrm{ord}_{\phi_r}(P)H(Q)+\mathrm{ord}_{\phi_r}(Q)H(P)), \tag{36}$$

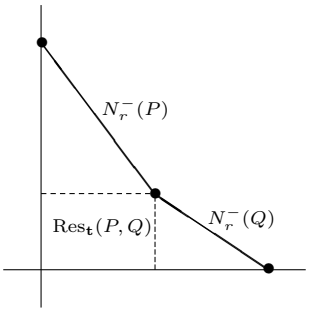
where $H(P)$, $H(Q)$ are the total heights of the finite parts respectively of $N_r^-(P)$, $N_r^-(Q)$.

**Lemma 4.2.** *Let $P(x),P'(x),Q(x)\in\mathcal{O}[x]$ be monic polynomials.*

1. $\mathrm{Res}_{\mathbf{t}}(P,Q)=0$ *if and only if* $\omega_r(P)\omega_r(Q)=0$,
2. $\mathrm{Res}_{\mathbf{t}}(P,Q)<\infty$ *if and only if* $\mathrm{ord}_{\phi_r}(P)\,\mathrm{ord}_{\phi_r}(Q)=0$,
3. $\mathrm{Res}_{\mathbf{t}}(P,Q)=\mathrm{Res}_{\mathbf{t}}(Q,P)$,
4. $\mathrm{Res}_{\mathbf{t}}(PP',Q)=\mathrm{Res}_{\mathbf{t}}(P,Q)+\mathrm{Res}_{\mathbf{t}}(P',Q)$.

*Proof.* The three first items are an immediate consequence of the definition. Item 4 follows from $N_r^-(PP')=N_r^-(P)+N_r^-(P')$. □

In the simplest case when $N_r^-(P)$ and $N_r^-(Q)$ are both one-sided, $\mathrm{Res}_{\mathbf{t}}(P,Q)$ represents the area of the rectangle joining the two triangles determined by the sides, if they are ordered by increasing slope. The reader may figure out a similar geometrical interpretation of $\mathrm{Res}_{\mathbf{t}}(P,Q)$ in the general case, as the area of a union of rectangles below the Newton polygon $N_r^-(PQ)=N_r^-(P)+N_r^-(Q)$.




Our aim is to compute $v(\mathrm{Res}(P,Q))$ as a sum of several $\mathrm{Res}_{\mathbf{t}}(P,Q)$ for an adequate choice of the types $\mathbf{t}$. To this end, we want to compare types attached to $P$ and $Q$, and this is uneasy because in the definition of the sets $\mathbf{t}_r(P)$, $\mathbf{t}_r(Q)$, we had freedom in the choices of the different representatives $\phi_i(x)$. For commodity in the exposition, we shall assume in this section that these polynomials are universally fixed.

**Convention.** *We fix from now on a monic lift $\phi_1(x)\in\mathcal{O}[x]$ of every monic irreducible polynomial $\psi_0(y)\in\mathbb{F}[y]$. We proceed now recursively: for any $1\le i<r$ and any type of order $i$*

$$\mathbf{t}=(\phi_1(x);\lambda_1,\phi_2(x);\cdots;\lambda_{i-1},\phi_i(x);\lambda_i,\psi_i(y)),$$

*with $\phi_1(x),\dots,\phi_i(x)$ belonging to the (infinite) family of previously chosen polynomials, we fix a representative $\phi_{i+1}(x)$ of $\mathbf{t}$ such that $R_i(\phi_{i+1})=\psi_i(y)$.*

*Also, we assume from now on that all types are made up only with our chosen polynomials $\phi_i(x)$.*

Once these choices are made, the set $\mathbf{t}_r(P)$ is uniquely determined by $r$ and $P(x)$. More precisely, $\mathbf{t}_r(P)$ is the set of all types of order $r$ such that $\omega^{\mathbf{t}}_{r+1}(P) > 0$ and the truncation $\mathbf{t}_{r-1}$ is not $P$-complete; in other words,

$$\mathbf{t}_r(P) = \{\mathbf{t} \text{ type of order } r \text{ such that } \omega^{\mathbf{t}}_{r+1}(P) > 0,\, \omega^{\mathbf{t}}_{r}(P) > 1\}.$$

However, in view of the computation of resultants, we need a broader concept of "type attached to a polynomial".

**Definition 4.3.** *For any monic polynomial $P(x) \in \mathcal{O}[x]$, and any natural number $r \geq 1$, we define*

$$\hat{\mathbf{t}}_r(P) := \{\mathbf{t} \textit{ type of order } r \textit{ such that } \omega^{\mathbf{t}}_{r+1}(P) > 0\} \supseteq \mathbf{t}_r(P).$$

The following observation is a consequence of the fact that $\omega^{\mathbf{t}}_{r+1}$ is a semigroup homomorphism for every type $\mathbf{t}$ of order $r$.

**Lemma 4.4.** *For any two monic polynomials $P(x), Q(x) \in \mathcal{O}[x]$, we have $\hat{\mathbf{t}}_r(PQ) = \hat{\mathbf{t}}_r(P) \cup \hat{\mathbf{t}}_r(Q)$.*

Note that the analogous statement for the sets $\mathbf{t}_r(P)$ is false. For instance, let $P(x), Q(x)$ be two monic polynomials congruent to the same irreducible polynomial $\psi(y)$ modulo $\mathfrak{m}$. We have $\mathbf{t}_0(P) = \mathbf{t}_0(Q) = \{\psi(y)\} = \mathbf{t}_0(PQ)$, and the type of order zero $\psi(y)$ is $P$-complete and $Q$-complete; thus, $\mathbf{t}_1(P) = \emptyset = \mathbf{t}_1(Q)$. However, $\psi(y)$ is not $PQ$-complete, and $\mathbf{t}_1(PQ) \neq \emptyset$.

We could also build the set $\hat{\mathbf{t}}_r(P)$ in a constructive way analogous to that used in the last section to construct $\mathbf{t}_r(P)$. The only difference is that the $P$-complete types are expanded to produce types of order $r+1$ as well. Thanks to our above convention about fixing a universal family of representatives of the types, these expansions are unique.

**Lemma 4.5.** *Let $r \geq 1$ be a natural number. Let $P(x) \in \mathcal{O}[x]$ be a monic polynomial, $P(x) \neq \phi_r(x)$. Let $\mathbf{t}$ be a $P$-complete type of order $r-1$. Then, $N_r(P)$ is one-sided of length one, and $R_{\lambda_r}(P)(y)$ has degree one, where $\lambda_r \in \mathbb{Q}^-$ is the slope of $N_r(P)$. Moreover, let $\psi_r(y)$ be the monic polynomial determined by $R_{\lambda_r}(P)(y) \approx \psi_r(y)$. Then, the type $\mathbf{t}' = (\tilde{\mathbf{t}}; \lambda_r, \psi_r(y))$ is $P$-complete, and it is the unique type of order $r$ such that $\mathbf{t}'_{r-1} = \mathbf{t}$ and $\omega^{\mathbf{t}'}_{r+1}(P) > 0$.*

*Proof.* By Lemma 2.18 and Corollary 2.19, $N_r(P)$ is one-sided of length one, and $\deg R_{\lambda_r}(P)(y) = d(N_r(P)) = 1$. Clearly, $\omega^{\mathbf{t}'}_{r+1}(P) = 1$. Finally, $\omega^{\mathbf{t}''}_{r+1}(P) = 0$ for any $\mathbf{t}'' = (\tilde{\mathbf{t}}; \lambda'_r, \psi'_r(y)) \neq \mathbf{t}'$. In fact, if $\lambda'_r \neq \lambda_r$, then $R_{\lambda'_r}(P)$ is a constant; if $\lambda'_r = \lambda_r$, but $\psi_r(y) \neq \psi'_r(y)$ then $\psi'_r(y)$ cannot divide $R_{\lambda_r}(P)(y)$. □

**Corollary 4.6.** *Let $P(x) \in \mathcal{O}[x]$ be a monic polynomial of positive degree. Then, $\hat{\mathbf{t}}_r(P) = \emptyset$ if and only if all irreducible factors of $P(x)$ belong to $\{\phi_1(x), \ldots, \phi_r(x)\}$.*

*Proof.* By Lemma 4.4, we can assume that $P(x)$ is irreducible. Since $\hat{\mathbf{t}}_0(P) \neq \emptyset$, by the above lemma, $\hat{\mathbf{t}}_r(P) \neq \emptyset$ as long as $P(x) \neq \phi_i(x)$ for $i = 1, \ldots, r$. On the other hand, $N_r(\phi_r)$ is one-sided of slope $-\infty$; hence, $R_{\lambda_r}(\phi_r)$ is a constant for every $\lambda_r \in \mathbb{Q}^-$, and $\omega^{\mathbf{t}'}_{r+1}(\phi_r) = 0$, for every type $\mathbf{t}'$ of order $\geq r$. □

By the Theorems of the polygon and of the residual polynomial, if $P(x)$ is irreducible and $P(x) \neq \phi_i(x)$ for $i = 1, \ldots, r$, then $|\hat{\mathbf{t}}_r(P)| = 1$.

**Definition 4.7.** *For any pair of monic polynomials $P(x), Q(x) \in \mathcal{O}[x]$, and any natural number $r \geq 1$, we define*

$$\mathrm{Res}_r(P,Q) := \sum_{\mathbf{t} \in \hat{\mathbf{t}}_{r-1}(P) \cap \hat{\mathbf{t}}_{r-1}(Q)} \mathrm{Res}_{\mathbf{t}}(P,Q).$$

The following observation is an immediate consequence of Lemma 4.2.

**Lemma 4.8.** *The following conditions are equivalent:*

1. $\mathrm{Res}_{r+1}(P,Q) = 0$,
2. $\hat{\mathbf{t}}_r(P) \cap \hat{\mathbf{t}}_r(Q) = \emptyset$,
3. *For all $\mathbf{t} \in \hat{\mathbf{t}}_{r-1}(P) \cap \hat{\mathbf{t}}_{r-1}(Q)$ and all $\lambda_r \in \mathbb{Q}^-$, the residual polynomials of $r$-th order, $R_{\lambda_r}(P)(y)$, $R_{\lambda_r}(Q)(y)$, do not have a common factor in $\mathbb{F}_r[y]$.*

The following result is an immediate consequence of Lemmas 4.2 and 4.4.

**Lemma 4.9.** *For any three monic polynomials $P(x), P'(x), Q(x) \in \mathcal{O}[x]$, and any natural number $r \geq 1$, we have $\mathrm{Res}_r(PP', Q) = \mathrm{Res}_r(P,Q) + \mathrm{Res}_r(P',Q)$.*

**Theorem 4.10.** *Let $P(x), Q(x) \in \mathcal{O}[x]$ be two monic polynomials having no common factors, and let $r \geq 1$ be natural number. Then,*

1. $v(\mathrm{Res}(P,Q)) \geq \mathrm{Res}_1(P,Q) + \cdots + \mathrm{Res}_r(P,Q)$,
2. *Equality holds if and only if* $\mathrm{Res}_{r+1}(P,Q) = 0$.

*Proof.* Let us deal first with the case where $P(x)$, $Q(x)$ are both irreducible and $\hat{\mathbf{t}}_{r-1}(P) = \hat{\mathbf{t}}_{r-1}(Q) = \{\mathbf{t}\}$, for some type $\mathbf{t}$ of order $r-1$. For $0 \leq i \leq r$, let $E_i, H_i$ be the length and heigth of the unique side of $N_i(P)$, and $E_i', H_i'$ be the length and heigth of the unique side of $N_i(Q)$. Since $P$ and $Q$ have both type $\mathbf{t}$, we have $H_i/E_i = H_i'/E_i'$, for all $1 \leq i < r$, with $E_iE_i' > 0$, $0 < H_iH_i' < \infty$ (because $P$, $Q$ cannot be both equal to $\phi_i(x)$). Suppose that $H_r/E_r \leq H_r'/E_r'$; in particular, $H_r < \infty$, so that $P(x) \neq \phi_r(x)$ in $\mathcal{O}[x]$.

Let $\tilde{\mathbf{t}} = (\phi_1(x); \cdots; \lambda_{r-1}, \phi_r(x))$ be the extended type of $\mathbf{t}$, $\lambda_r' = H_r'/E_r'$, and $R_{\lambda_r'}(Q)(y)$ the $r$-th order residual polynomial of $Q(x)$ with respect to $(\tilde{\mathbf{t}}; \lambda_r')$. By the Theorem of the residual polynomial, $R_{\lambda_r'}(Q)(y)$ is irreducible; let $\psi_r'(y) \in \mathbb{F}_r[y]$ be the monic irreducible polynomial determined by $R_{\lambda_r'}(Q)(y) \approx \psi_r'(y)$, and consider the type of order $r$, $\mathbf{t}' = (\tilde{\mathbf{t}}; \lambda_r', \psi_r'(y))$.

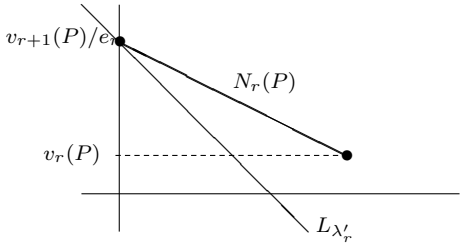
It is well-known that

$$\mathrm{Res}(P,Q) = \pm \prod_{Q(\theta)=0} P(\theta).$$

By applying Proposition 2.10 to the type $\mathbf{t}'$, we get

$$v(P(\theta)) \geq v_{r+1}(P)/e_1 \cdots e_r = (v_r(P) + H_r)/e_1 \cdots e_{r-1}, \tag{37}$$

the last equality by the definition of $v_{r+1}$. Also, equality holds in (37) if and only if $\omega_{r+1}'(P) = 0$, where $\omega_{r+1}'$ is the pseudo-valuation of order $r+1$ attached to $\mathbf{t}'$.



By Lemma 2.3, $\deg Q = m_r\omega_r(Q) = f_0e_1f_1\cdots e_{r-1}f_{r-1}E'_r$. If we apply recursively $v_{s+1}(P) = e_s(v_s(P_s) + H_s)$, $E'_{s+1} = (e_sf_s)^{-1}E'_s$, for all $1 \le s < r$, and $v_1(P) = 0$, we get

$$\begin{aligned} v(\mathrm{Res}(P,Q)) &= \deg(Q)v(P(\theta)) \ge \deg Q\,\frac{v_r(P)+H_r}{e_1\cdots e_{r-1}} \\ &= f_0\cdots f_{r-1}E'_r(v_r(P)+H_r) \\ &= \sum_{s=1}^{r} f_0\cdots f_{s-1}E'_sH_s = \sum_{s=1}^{r}\mathrm{Res}_s(P,Q), \end{aligned}$$

and equality holds if and only if $\omega'_{r+1}(P) = 0$, i.e. if and only if $R_{\lambda'_r}(P)(y)$ is not divisible by $\psi'_r(y)$. This condition is equivalent to (3) of Lemma 4.8 because $R_{\lambda_r}(P)(y)$ is irreducible and $R_{\lambda''_r}(P)(y)$ is a constant for any negative rational number $\lambda''_r \ne \lambda_r$. This ends the proof of the theorem in this case.

Let us still assume that $P(x)$, $Q(x)$ are both irreducible, but now $\hat{\mathbf{t}}_{r-1}(P) \cap \hat{\mathbf{t}}_{r-1}(Q) = \emptyset$. If $\hat{\mathbf{t}}_0(P) \cap \hat{\mathbf{t}}_0(Q) = \emptyset$, then $\mathrm{Res}_1(P,Q) = \cdots = \mathrm{Res}_r(P,Q) = \mathrm{Res}_{r+1}(P,Q) = 0$, by definition. On the other hand, $v(\mathrm{Res}(P,Q)) = 0$, because $P(x)$ and $Q(x)$ have no common factors modulo $\mathfrak{m}$; hence, the theorem is proven in this case. Assume $\hat{\mathbf{t}}_0(P) \cap \hat{\mathbf{t}}_0(Q) \ne \emptyset$, and let $1 \le s < r$ be maximal with the property $\hat{\mathbf{t}}_{s-1}(P) \cap \hat{\mathbf{t}}_{s-1}(Q) \ne \emptyset$. Clearly, $\mathrm{Res}_r(P,Q) = 0$ for all $r > s$, so that the condition of item 2 is satisfied for all $r \ge s$; thus, we want to show that $v(\mathrm{Res}(P,Q)) = \mathrm{Res}_1(P,Q) + \cdots + \mathrm{Res}_s(P,Q)$. This follows from the proof of the previous case for $r = s$.

Let now $P(x) = P_1(x)\cdots P_g(x)$, $Q(x) = Q_1(x)\cdots Q_{g'}(x)$ be the factorizations of $P(x)$, $Q(x)$ into a product of monic irreducible polynomials in $\mathcal{O}[x]$. We have proved above that $v(\mathrm{Res}(P_i,Q_j)) \ge \mathrm{Res}_1(P_i,Q_j)+\cdots+\mathrm{Res}_r(P_i,Q_j)$ for all $i,j$; hence, item 1 follows from Lemma 4.9 and the bilineality of resultants. Also, equality in item 1 holds for the pair $P$, $Q$ if and only if it holds for each pair $P_i$, $Q_j$; that is, if and only if $\mathrm{Res}_{r+1}(P_i,Q_j) = 0$, for all $i,j$. This is equivalent to $\mathrm{Res}_{r+1}(P,Q) = 0$, again by Lemma 4.9. □

We end this section with an example that illustrates the necessity to introduce the sets $\hat{\mathbf{t}}_r(P)$. Let $\mathcal{O} = \mathbb{Z}_p$, $P(x) = x+p$, $Q(x) = x+p+p^{100}$, and let $\mathbf{t}_0 = y \in \mathbb{F}[y]$. Clearly, $\mathbf{t}_0(P) = \{\mathbf{t}_0\} = \mathbf{t}_0(Q)$, and $\mathbf{t}_0$ is both $P$-complete and $Q$-complete, so that $\mathbf{t}_1(P) = \emptyset = \mathbf{t}_1(Q)$. On the other hand, $\mathrm{Res}_1(P,Q) = \mathrm{Res}_{\mathbf{t}_0}(P,Q) = 1$, whereas $v(\mathrm{Res}(P,Q)) = 100$. Thus, we need to consider the expansions of $\mathbf{t}_0$ to types of higher order in order to reach the right value of $v(\mathrm{Res}(P,Q))$. The number of expansions to consider depends on the choices of the representatives $\phi_i(x)$; for instance, if we take $\phi_1(x) = x + p$ we have already $\mathrm{Res}_2(P,Q) = 99$.

Nevertheless, the sets $\hat{\mathbf{t}}_r(P)$ were introduced only as an auxiliary tool to prove Theorem 4.10. In practice, the factorization algorithm computes only the sets $\mathbf{t}_r(P)$, as we shall show in the next section.

4.2. **Index of a polynomial and index of a polygon.** All types that we consider are still assumed to be made up with polynomials $\phi_i(x)$ belonging to a universally fixed family, as indicated in the last section.

Let $F(x) \in \mathcal{O}[x]$ be a monic irreducible polynomial, $\theta \in \overline{\mathbb{Q}}_p$ a root of $F(x)$, and $L = K(\theta)$. It is well-known that $(\mathcal{O}_L\colon \mathcal{O}[\theta]) = q^{\mathrm{ind}(F)}$, for some natural number

$\mathrm{ind}(F)$ that will be called the *index* of $F(x)$. Note that

$$\mathrm{ind}(F) = v(\mathcal{O}_L : \mathcal{O}[\theta])/[K : \mathbb{Q}_p].$$

Recall the well-known relationship, $v(\mathrm{disc}(F)) = 2\,\mathrm{ind}(F) + v(\mathrm{disc}(L/K))$, linking $\mathrm{ind}(F)$ with the discriminant of $F(x)$ and the discriminant of $L/K$.

**Definition 4.11.** *Let $f(x) \in \mathcal{O}[x]$ be a monic separable polynomial and $f(x) = F_1(x)\cdots F_k(x)$ its decomposition into the product of monic irreducible polynomials in $\mathcal{O}[x]$. We define the* index *of $f(x)$ by the formula*

$$\mathrm{ind}(f) := \sum_{i=1}^{k} \mathrm{ind}(F_i) + \sum_{1\le i<j\le k} v(\mathrm{Res}(F_i, F_j)).$$

**Definition 4.12.** *Let $S$ be a side of negative slope, and denote $E = \ell(S)$, $H = H(S)$, $d = d(S)$. We define*

$$\mathrm{ind}(S) := \begin{cases} \frac{1}{2}(EH - E - H + d), & \text{if } S \text{ has finite slope,} \\ 0, & \text{otherwise.} \end{cases}$$

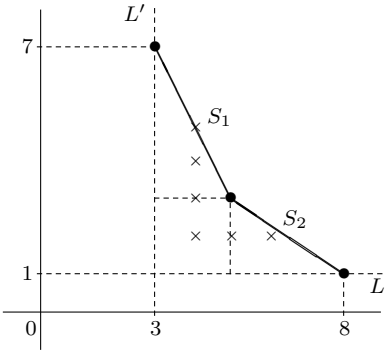
*Let $N = S_1 + \cdots + S_g$ be a principal polygon, with sides ordered by increasing slopes $\lambda_1 < \cdots < \lambda_g$. We define*

$$\mathrm{ind}(N) := \sum_{i=1}^{g} \mathrm{ind}(S_i) + \sum_{1\le i<j\le g} E_i H_j.$$

If $N$ has a side of slope $-\infty$ and length $E_\infty$, it contributes with $E_\infty H(N)$ to $\mathrm{ind}(N)$, where $H(N)$ is the total height of the finite part of $N$.

**Remark 4.13.** *The contribution of the sides of finite slope to* $\mathrm{ind}(N)$ *is the number of points with integer coordinates that lie below the finite part of $N$, strictly above the horizontal line $L$ that passes through the last point of $N$, and strictly beyond the vertical line $L'$ that passes through the initial point of the finite part of $N$.*

For instance, the polygon below has index 25, the infinite side contributes with 18 (the area of the rectangle $3 \times 6$) and the finite part has index 7, corresponding to the marked seven points of integers coordinates, distributed into $\mathrm{ind}(S_1) = 2$, $\mathrm{ind}(S_2) = 1$, $E_1H_2 = 4$.




**Remark 4.14.** *Note that* $\mathrm{ind}(N) = 0$ *if and only if either $N$ is reduced to a point, or $N$ is one-sided with slope $-\infty$, or $N$ is one-sided with $E = 1$ or $H = 1$.*

Let $i_1 \leq i_2$ be the respective abscissas of the starting point and the last point of the finite part $N_{\text{fin}}$ of $N$. For any integer abscissa $i_1 \leq i \leq i_2$, let $\nu_i$ be the distance of the point of $N$ of abscissa $i$ to the line $L$. Clearly, we can count the points of integer coordinates below $N_{\text{fin}}$, above $L$ and beyond $L'$, as the sum of the points with given abscissa:

$$\text{ind}(N_{\text{fin}}) = \lfloor \nu_{i_1+1} \rfloor + \cdots + \lfloor \nu_{i_2-1} \rfloor. \tag{38}$$

For instance, in the above figure we have $\nu_4 = 4$, $\nu_5 = 2$, $\nu_6 = 1$ and $\nu_7 = 0$.

**Definition 4.15.** *Let $P(x) \in \mathcal{O}[x]$ be a monic and separable polynomial. For any type $\mathbf{t}$ of order $r-1$ we define*

$$\text{ind}_{\mathbf{t}}(P) := f_0 \cdots f_{r-1} \, \text{ind}(N_r^-(P)),$$

*where $N_r(P)$ is the Newton polygon of $r$-th order with respect to $\tilde{\mathbf{t}}$.*

*For any natural number $r \geq 1$ we define*

$$\text{ind}_r(P) := \sum_{\mathbf{t} \in \mathbf{t}_{r-1}(P)} \text{ind}_{\mathbf{t}}(P).$$

**Lemma 4.16.** *Let $P(x) \in \mathcal{O}[x]$ be a monic and separable polynomial.*

1. *If $\mathbf{t}$ a type of order $r-1$, and $\mathbf{t} \notin \mathbf{t}_{r-1}(P)$, then $\text{ind}_{\mathbf{t}}(P) = 0$,*
2. *If $\text{ind}_r(P) = 0$, then $\mathbf{t}_{r+1}(P) = \emptyset$.*

*Proof.* If $\mathbf{t} \notin \mathbf{t}_{r-1}(P)$, then either $\omega_r(P) = 0$ or $\omega_{r-1}(P) = 1$. By Lemma 2.18, in both cases $N_r^-(P)$ has length less than or equal to one, and $\text{ind}_{\mathbf{t}}(P) = 0$ by Remark 4.14. This proves item 1.

If $\text{ind}_r(P) = 0$, we have $\text{ind}_{\mathbf{t}}(P) = 0$ for all $\mathbf{t} \in \mathbf{t}_{r-1}(P)$. For any such $\mathbf{t}$ we have $\omega_r(P) > 0$, so that $N_r^-(P)$ is not reduced to a point. By Remark 4.14, either $P$ is equal to the representative $\phi_r(x)$ of $\mathbf{t}$, or $N_r^-(P)$ is one-sided and the side has either length one or height one. If $P(x) = \phi_r(x)$, then $\mathbf{t}$ is $P$-complete, and $\mathbf{t}_r(P) = \emptyset$. If $N_r^-(P)$ is one-sided with slope $\lambda_r$, and the side has degree one, then the residual polynomial $R_{\lambda_r}(P)(y)$ has degree one; thus, the unique expansion $\mathbf{t}'$ of $\mathbf{t}$ to a type of order $r$ is $P$-complete. Since this occurs for all $\mathbf{t} \in \mathbf{t}_{r-1}(P)$, the set $\mathbf{t}_{r+1}(P)$ is empty. □

**Lemma 4.17.** *Let $P(x), Q(x) \in \mathcal{O}[x]$ be two monic and separable polynomials, without common factors. Let $r \geq 1$ be a natural number and $\mathbf{t}$ a type of order $r-1$. Then,*

$$\text{ind}_{\mathbf{t}}(PQ) = \text{ind}_{\mathbf{t}}(P) + \text{ind}_{\mathbf{t}}(Q) + \text{Res}_{\mathbf{t}}(P, Q),$$

$$\text{ind}_r(PQ) = \text{ind}_r(P) + \text{ind}_r(Q) + \text{Res}_r(P, Q).$$

*Proof.* For commodity, in the discussion we omit the weight $f_0 \cdots f_{r-1}$ that multiplies all terms in the identities.

All terms involved in the first identity are the sum of a finite part and an infinite part. If $P(x)Q(x)$ is not divisible by $\phi_r(x)$, all infinite parts are zero. If $\phi_r(x)$ divides (say) $P(x)$, then the infinite part of $\text{ind}_{\mathbf{t}}(PQ)$ is $\text{ord}_{\phi_r}(P)(H(P) + H(Q))$, the infinite part of $\text{ind}_{\mathbf{t}}(P)$ is $\text{ord}_{\phi_r}(P)H(P)$, the infinite part of $\text{ind}_{\mathbf{t}}(Q)$ is zero, and the infinite part of $\text{Res}_{\mathbf{t}}(P, Q)$ is $\text{ord}_{\phi_r}(P)H(Q)$, by (36). Thus, the first identity is correct, in what the infinite parts concerns.

The finite part of the first identity follows from $N_r^-(PQ) = N_r^-(P) + N_r^-(Q)$ and Remark 4.13. Let $N = N_r^-(PQ)$ and let $\mathcal{R}$ be the region of the plane that lies

below $N$, above the line $L$ and beyond the line $L'$, as indicated in Remark 4.13. The number $\mathrm{ind}_{\mathbf{t}}(PQ)$ counts the total number of points of integer coordinates in $\mathcal{R}$, the number $\mathrm{ind}_{\mathbf{t}}(P)+\mathrm{ind}_{\mathbf{t}}(Q)$ counts the number of points of integer cordinates in the regions determined by the right triangles whose hypotenuses are the sides of $N_r^-(P)$ and $N_r^-(Q)$. The region of $\mathcal{R}$ not covered by these triangles is a union of rectangles and $\mathrm{Res}_{\mathbf{t}}(P,Q)$ is precisely the number of points of integer coordinates of this region.

In order to prove the second identity, we note first that for any monic polynomial $R(x)\in\mathcal{O}[x]$,

$$\mathrm{ind}_r(R):=\sum_{\mathbf{t}\in\hat{\mathbf{t}}_{r-1}(R)}\mathrm{ind}_{\mathbf{t}}(R),$$

by (1) of Lemma 4.16. Now, if we apply this to $R=P,Q,PQ$, the identity follows from the first one and Lemma 4.4, having in mind that $\mathrm{ind}_{\mathbf{t}}(Q)=0=\mathrm{Res}_{\mathbf{t}}(P,Q)$ if $\mathbf{t}\notin\hat{\mathbf{t}}_{r-1}(Q)$, because $N_r^-(Q)$ reduces to a point. □

**Theorem 4.18** (Theorem of the index)**.** *Let $f(x)\in\mathcal{O}[x]$ be a monic and separable polynomial, and $r\geq 1$ a natural number. Then,*

(1) $\mathrm{ind}(f)\geq\mathrm{ind}_1(f)+\cdots+\mathrm{ind}_r(f)$,
(2) *Equality holds if and only if* $\mathrm{ind}_{r+1}(f)=0$.

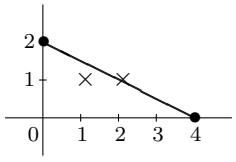
Therefore, the factorization algorithm, based in the computation of the sets $\mathbf{t}_r(f)$ and the higher indices $\mathrm{ind}_r(f)$, ends after a finite number of steps. Also, when the process finishes we obtain as a by-product a computation of $\mathrm{ind}(f)$.

**Corollary 4.19.** *Let $f(x)\in\mathcal{O}[x]$ be a monic and separable polynomial. There exists $r\geq 0$ such that all types in $\mathbf{t}_r(f)$ are $f$-complete, or equivalently, such that $\mathbf{t}_{r+1}(f)=\emptyset$.*

*Proof.* By the Theorem of the index, there exists $r\geq 1$ such that $\mathrm{ind}_r(f)=0$, and by (2) of Lemma 4.16 this implies $\mathbf{t}_{r+1}(f)=\emptyset$. □

In the next section we exhibit an example where the factorization is achieved in order three. More examples, and a more accurate discussion of the computational aspects can be found in [GMN08a].

4.3. **An example.** Take $p=2$, and $f(x)=x^4+ax^2+bx+c\in\mathbb{Z}[x]$, with $v(a)\geq 2$, $v(b)=3$, $v(c)=2$. Since $f(x)\equiv x^4 \pmod 2$, all types we are going to consider will start with $\phi_1(x)=x$. The Newton polygon $N_1(f)$ has slope $\lambda_1=-1/2$




and the residual polynomial of $f(x)$ with respect to $\lambda_1$ is $R_1(f)(y)=y^2+1=(y+1)^2\in\mathbb{F}$, where $\mathbb{F}$ is the field ot two elements. Hence, $\mathbf{t}_1(f)=\{\mathbf{t}\}$, where $\mathbf{t}:=(x;-1/2,y+1)$. We have $h_1=1,e_1=2,f_0=f_1=1$ and $\omega_2(f)=2$, so that $\mathbf{t}$ is not $f$-complete. The partial information we get in order one is $\mathrm{ind}_1(f)=2$, and the fact that all irreducible factors of $f(x)$ will generate extensions $L/\mathbb{Q}_2$ with even ramification number, because $e_1=2$.

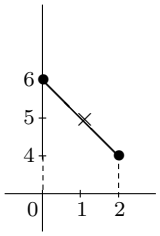
Take $\phi_2(x) = x^2 - 2$ as a representative of $\mathbf{t}$. The $\phi_2$-adic development of $f(x)$ is

$$f(x) = \phi_2(x)^2 + (a+4)\phi_2(x) + (bx + c + 2a + 4).$$

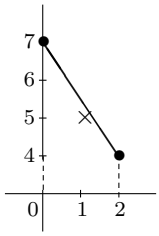
By Proposition 2.8 and (13), we have

$$v_2(x) = 1,\ v_2(\phi_2) = 2,\ v_2(a+4) \geq 4,\ v_2(bx) = 7,\ v_2(c+2a+4) \geq 6.$$

Hence, according to $v(c+2a+4) = 3$ or $v(c+2a+4) \geq 4$, the Newton polygon of second order, $N_2(f)$, is:




$v(c+2a+4) = 3$




$v(c+2a+4) \geq 4$

If $v(c+2a+4) \geq 4$, $N_2(f)$ is one-sided with slope $\lambda_2 = -3/2$, and $R_2(f)(y) = y+1$. The type $\mathbf{t}' := (x; -1/2, x^2 - 2; -3/2, y+1)$ is $f$-complete and $\mathbf{t}_2(f) = \{\mathbf{t}'\}$. We have $h_2 = 3, e_2 = f_2 = 1$. Thus, $f(x)$ is irreducible over $\mathbb{Z}_2[x]$, and it generates an extension $L/\mathbb{Q}_2$ with $e(L/\mathbb{Q}_2) = e_1e_2 = 4$, $f(L/\mathbb{Q}_2) = f_0f_1f_2 = 1$. Moreover, $\mathrm{ind}_2(f) = 1$, so that $\mathrm{ind}(f) = \mathrm{ind}_1(f) + \mathrm{ind}_2(f) = 3$.

If $v(c+2a+4) = 3$, $N_2(f)$ is one-sided with slope $\lambda_2 = -1$, and $R_2(f)(y) = y^2 + 1 = (y+1)^2$. The type $\mathbf{t}' := (x; -1/2, x^2 - 2; -1, y+1)$ is not $f$-complete, $\mathbf{t}_2(f) = \{\mathbf{t}'\}$, and we need to pass to order three. We have $h_2 = e_2 = f_2 = 1$. Take $\phi_3(x) = x^2 - 2x - 2$ as a representative of $\mathbf{t}'$. The $\phi_3$-adic development of $f(x)$ is

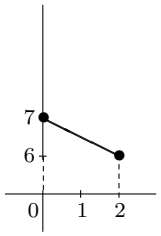
$$f(x) = \phi_3(x)^2 + (4x + a + 8)\phi_3(x) + (b + 2a + 16)x + c + 2a + 12.$$

By Proposition 2.8 and (13), we have

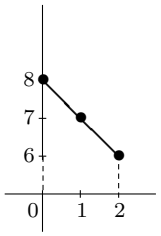
$$v_3(x) = 1,\ v_3(\phi_3) = 3,\ v_3(4x) = 5,\ v_3(c+2a+12) \geq 8,$$

$$v_3(4x+a+8) = \begin{cases} 4, & \text{if } v(a) = 2, \\ 5, & \text{if } v(a) \geq 3, \end{cases} \qquad v_3((b+2a+16)x) = \begin{cases} \geq 9, & \text{if } v(a) = 2, \\ 7, & \text{if } v(a) \geq 3. \end{cases}$$

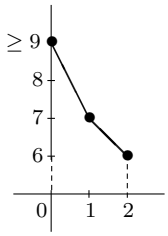
We have now three possibilities for the Newton polygon of third order




$v(a) \geq 3$




$v(a) = 2,$
$v(c+2a+12) = 4$




$v(a) = 2,$
$v(c+2a+12) \geq 5$

If $v(a) \geq 3$, $N_3(f)$ is one-sided with slope $\lambda_3 = -1/2$, and $R_3(f)(y) = y+1$. The type $\mathbf{t}'' := (x; -1/2, \phi_2(x); -1, \phi_3(x); -1/2, y+1)$ is $f$-complete and $\mathbf{t}_3(f) = \{\mathbf{t}''\}$. We have $e_3 = 2, f_3 = 1$. Thus, $f(x)$ is irreducible over $\mathbb{Z}_2[x]$, and it generates an extension $L/\mathbb{Q}_2$ with $e(L/\mathbb{Q}_2) = e_1e_2e_3 = 4$, $f(L/\mathbb{Q}_2) = f_0f_1f_2f_3 = 1$. Also, $\mathrm{ind}_3(f) = 0$, so that $\mathrm{ind}(f) = \mathrm{ind}_1(f) + \mathrm{ind}_2(f) = 3$.

If $v(a)=2$ and $v(c+2a+12)=4$, $N_3(f)$ is one-sided with slope $\lambda_3=-1$, and $R_3(f)(y)=y^2+y+1$. The type $\mathbf{t}''=(x;-1/2,\phi_2(x);-1,\phi_3(x);-1,y^2+y+1)$ is $f$-complete and $\mathbf{t}_3(f)=\{\mathbf{t}''\}$. We have $e_3=1, f_3=2$. Thus, $f(x)$ is irreducible over $\mathbb{Z}_2[x]$, and it generates an extension $L/\mathbb{Q}_2$ with $e(L/\mathbb{Q}_2)=e_1e_2e_3=2$, $f(L/\mathbb{Q}_2)=f_0f_1f_2f_3=2$. Also, $\operatorname{ind}_3(f)=1$, so that $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\operatorname{ind}_2(f)+\operatorname{ind}_3(f)=4$.

If $v(a)=2$ and $v(c+2a+12)\geq 5$, $N_3(f)$ has two sides with slopes $\lambda_3\leq -2$, $\lambda_3'=-1$, and $R_{\lambda_3}(f)(y)=R_{\lambda_3'}(f)(y)=y+1$. There are two types extending $\mathbf{t}'$:

$$\mathbf{t}_1''=(x;-1/2,\phi_2(x);-1,\phi_3(x);\lambda_3,y+1),$$
$$\mathbf{t}_2''=(x;-1/2,\phi_2(x);-1,\phi_3(x);-1,y+1).$$

Both types have $e_3=f_3=1$, they are both $f$-complete and $\mathbf{t}_3(f)=\{\mathbf{t}_1'',\mathbf{t}_2''\}$. Thus, $f(x)$ has two irreducible factors of degree two over $\mathbb{Z}_2[x]$, and both generate extensions $L/\mathbb{Q}_2$ with $e(L/\mathbb{Q}_2)=2$, $f(L/\mathbb{Q}_2)=1$. Finally, $\operatorname{ind}_3(f)=1$, so that $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\operatorname{ind}_2(f)+\operatorname{ind}_3(f)=4$.

In the final design of Montes' algorithm, this polynomial $f(x)$ is factorized already in order two. In the case $v(c+2a+4)=3$ the algorithm considers $\phi_3(x)=x^2-2x-2$ as a different representative of the type $\mathbf{t}$, in order to avoid the increase of recursivity caused by the work in a higher order. See [GMN08a] for more details on this optimization.

4.4. **Proof of the Theorem of the index.** Our first aim is to prove Theorem 4.18 for $f(x)\in\mathcal{O}[x]$ a monic irreducible polynomial of degree $n$, such that $\mathbf{t}_{r-1}(f)$ is non-empty, say $\mathbf{t}_{r-1}(f)=\{\mathbf{t}\}$, and $f(x)$ is not equal to the representative $\phi_r(x)$ of $\mathbf{t}$. Let $\mathbf{t}=(\phi_1(x);\cdots,\phi_{r-1}(x);\lambda_{r-1},\psi_{r-1}(y))$.

For $1\leq s\leq r$, let $E_s,H_s,d_s$ be the length, height and degree of the unique side of $N_s(f)$. Note that $E_s>0$ (because $f(x)$ is of type $\mathbf{t}$), and $0<H_s<\infty$ (because $f(x)=\phi_s(x)$ implies $\mathbf{t}_s(f)=\emptyset$, and $f(x)=\phi_r(x)$ is excluded by hypothesis). Let $\lambda_r=-h_r/e_r$ be the slope of $N_r(f)$, where $h_r,e_r$ are positive coprime integers. By the Theorem of the residual polynomial, $R_{\lambda_r}(f)\approx\psi_r(y)^{a_r}$, for some monic irreducible polynomial $\psi_r(y)$. Let $f_r:=\deg\psi_r$.

Let $\theta\in\overline{\mathbb{Q}}_p$ a root of $P(x)$, $L=K(\theta)$, and let us fix an embedding (34). Denote $z_r=\overline{\gamma_r(\theta)}$. We introduce now some other notations.

$$\nu_s:=v(\phi_s(\theta))=\textstyle\sum_{i=1}^{s}e_if_i\cdots e_{s-1}f_{s-1}\dfrac{h_i}{e_1\dots e_i},\ \text{for all } 1\leq s\leq r,$$

$$\nu_{\mathbf{j}}:=j_1\nu_1+\cdots+j_r\nu_r\in\mathbb{Q},\ \text{for all } \mathbf{j}=(j_0,\dots,j_r)\in\mathbb{N}^{r+1},$$

$$\Phi(\mathbf{j}):=\frac{\theta^{j_0}\phi_1(\theta)^{j_1}\dots\phi_r(\theta)^{j_r}}{\pi^{\lfloor\nu_{\mathbf{j}}\rfloor}}\in\mathcal{O}_L,\ \text{for all } \mathbf{j}=(j_0,\dots,j_r)\in\mathbb{N}^{r+1},$$

$$b_0:=f_0;\quad b_s:=e_sf_s,\ \text{ for } 1\leq s<r;\quad b_r:=e_rf_ra_r,$$

$$J=\{\mathbf{j}\in\mathbb{N}^{r+1}\mid 0\leq j_s<b_s, 0\leq s\leq r\}.$$

**Lemma 4.20.** *Let $\mathcal{O}_L'$ be the sub-$\mathcal{O}$-module of $\mathcal{O}_L$ generated by $\{\Phi(\mathbf{j})\mid\mathbf{j}\in J\}$. Then,*

1. *$\mathcal{O}_L'$ is a free $\mathcal{O}$-module of rank $n$, with basis $\{\Phi(\mathbf{j})\mid\mathbf{j}\in J\}$,*
2. *$\mathcal{O}[\theta]\subseteq\mathcal{O}_L'$, and $(\mathcal{O}_L':\mathcal{O}[\theta])=q^{\sum_{\mathbf{j}\in J}\lfloor\nu_{\mathbf{j}}\rfloor}$.*

*Proof.* Clearly, $|J|=n$, and the numerators of $\Phi(\mathbf{j})$, for $\mathbf{j}\in J$, are monic polynomials of degree $0,1,\dots,n-1$. Thus, the family $\{\Phi(\mathbf{j})\mid\mathbf{j}\in J\}$ is $\mathcal{O}$-linearly

independent. This proves item 1 and $\mathcal{O}[\theta] \subseteq Z'_L$. Finally,

$$\mathcal{O}'_L/\mathcal{O}[\theta] \simeq \prod_{\mathbf{j}\in J} \pi^{-\lfloor \nu_{\mathbf{j}} \rfloor}\mathcal{O}/\mathcal{O} \simeq \prod_{\mathbf{j}\in J} \mathcal{O}/\pi^{\lfloor \nu_{\mathbf{j}} \rfloor}\mathcal{O},$$

and since $|\mathcal{O}/\pi^a\mathcal{O}| = q^a$, we get $(\mathcal{O}'_L\colon \mathcal{O}[\theta]) = q^{\sum_{\mathbf{j}\in J}\lfloor \nu_{\mathbf{j}} \rfloor}$. □

Our next step is to prove that $\mathcal{O}'_L$ is actually an order of $\mathcal{O}_L$. To this end we need a couple of auxiliary results.

**Lemma 4.21.** *Let $Q(x) = \sum_{\mathbf{j}=(j_0,\dots,j_{r-1},0)\in J} a_{\mathbf{j}} x^{j_0}\phi_1(x)^{j_1}\dots\phi_{r-1}(x)^{j_{r-1}}$, for some $a_{\mathbf{j}} \in \mathcal{O}$. Then, for all $\mathbf{j} = (j_0,\dots,j_{r-1},0) \in J$,*

$$v(a_{\mathbf{j}}) + \nu_{\mathbf{j}} \geq v(Q(\theta)).$$

*Proof.* Since $\deg Q < m_r$, we have $v(Q(\theta)) = v_r(Q)/e_1\cdots e_{r-1}$ by Lemma 2.5 and Proposition 2.10. Let us prove $v(a_{\mathbf{j}})+\nu_{\mathbf{j}} \geq v_r(Q)/e_1\cdots e_{r-1}$ by induction on $r \geq 1$. If $r = 1$ this is obvious because $v_1(Q) = \min\{v(a_{\mathbf{j}})\}$. Let $r \geq 2$ and suppose the result is true for $r-1$. For each $0 \leq j_{r-1} < b_{r-1}$, consider the polynomial

$$Q_{j_{r-1}}(x) = \sum_{(j_0,\dots,j_{r-2},0,0)\in J} a_{\mathbf{j}} x^{j_0}\phi_1(x)^{j_1}\dots\phi_{r-2}(x)^{j_{r-2}},$$

where $\mathbf{j} = (j_0,\dots,j_{r-2},j_{r-1},0)$ in each summand. Clearly,

$$Q(x) = \sum_{0\leq j_{r-1}<b_{r-1}} Q_{j_{r-1}}(x)\phi_{r-1}(x)^{j_{r-1}},$$

is the $\phi_{r-1}$-adic development of $Q(x)$. By item 4 of Proposition 2.8, the Theorem of the polygon and the induction hypothesis we get

$$\begin{aligned} v_r(Q)/e_{r-1} &= \min_{0\leq j_{r-1}<b_{r-1}} \{v_{r-1}(Q_{j_{r-1}}) + j_{r-1}(v_{r-1}(\phi_{r-1}) + |\lambda_{r-1}|)\} \\ &= \min_{0\leq j_{r-1}<b_{r-1}} \{v_{r-1}(Q_{j_{r-1}}) + j_{r-1}e_1\cdots e_{r-2}\nu_{r-1}\} \\ &\leq e_1\cdots e_{r-2}\left(v(a_{\mathbf{j}}) + j_1\nu_1 + \cdots + j_{r-2}\nu_{r-2} + j_{r-1}\nu_{r-1}\right). \end{aligned}$$

□

**Lemma 4.22.** *Let $\mathbf{j} = (j_0,\dots,j_r) \in \mathbb{N}^{r+1}$.*

(1) *For all $0 \leq s < r$,*

$$\begin{aligned} \Phi(j_0,\dots,j_{s-1},j_s+b_s,j_{s+1},\dots,j_r) &= \pi^{\delta_{\mathbf{j},s}}\Phi(j_0,\dots,j_s,j_{s+1}+1,j_{s+2},\dots,j_r) \\ &\quad + \sum_{\mathbf{j}'=(j'_0,\dots,j'_s,0,\dots,0)\in J} c_{\mathbf{j},\mathbf{j}'}\Phi(\mathbf{j}+\mathbf{j}'), \end{aligned}$$

*for some nonnegative integer $\delta_{\mathbf{j},s}$ and some $c_{\mathbf{j},\mathbf{j}'} \in \mathcal{O}$.*

(2) *$\Phi(j_0,\dots,j_{r-1},j_r+b_r) = \sum_{\mathbf{j}'\in J} c_{\mathbf{j},\mathbf{j}'}\Phi(\mathbf{j}+\mathbf{j}')$, for some $c_{\mathbf{j},\mathbf{j}'} \in \mathcal{O}$.*

*Proof.* Let $0 \leq s < r$. The polynomial $Q(x) = \phi_s(x)^{b_s} - \phi_{s+1}(x)$ has degree less than $m_{s+1} = b_s m_s$; hence, it admits a development

$$Q(x) = \sum_{\mathbf{j}'=(j'_0,\dots,j'_s,0,\dots,0)\in J} a_{\mathbf{j}'}\, x^{j'_0}\phi_1(x)^{j'_1}\dots\phi_s(x)^{j'_s},$$

for some $a_{\mathbf{j}'} \in \mathcal{O}$. If we substitute $\phi_s(x)^{b_s} = \phi_{s+1}(x) + Q(x)$ in $\Phi(j_0,\dots,j_{s-1},j_s+b_s,j_{s+1},\dots,j_r)$ we get the identity of item 1, with

$$\delta_{\mathbf{j},s} = \lfloor \nu_{\mathbf{j}} + \nu_{s+1} \rfloor - \lfloor \nu_{\mathbf{j}} + b_s\nu_s \rfloor, \qquad c_{\mathbf{j},\mathbf{j}'} = a_{\mathbf{j}'}\,\pi^{\lfloor \nu_{\mathbf{j}}+\nu_{\mathbf{j}'} \rfloor - \lfloor \nu_{\mathbf{j}}+b_s\nu_s \rfloor}.$$

The Theorem of the polygon and the usual relationships $v_{s+1}(\phi_{s+1}) = b_s v_{s+1}(\phi_s) = b_s(e_s v_s(\phi_s) + h_s)$ (cf. Proposition 2.8 and (13)), show that $\nu_{s+1} > b_s\nu_s$. Therefore, $v(Q(\theta)) = b_s\nu_s$, and by the above lemma we have $v(a_{\mathbf{j}'}) + \nu_{\mathbf{j}'} \geq b_s\nu_s$. This shows that $\delta_{\mathbf{j},s} \geq 0$ and $v(c_{\mathbf{j},\mathbf{j}'}) \geq 0$.

Item 2 follows by identical arguments, starting with $Q(x) = \phi_r(x)^{b_r} - f(x)$. □

**Proposition 4.23.** *The $\mathcal{O}$-module $\mathcal{O}'_L$ is a subring of $\mathcal{O}_L$.*

*Proof.* For all $\mathbf{j}, \mathbf{j}' \in J$ we have $\Phi(\mathbf{j})\Phi(\mathbf{j}') = \pi^{\delta}\Phi(\mathbf{j}+\mathbf{j}')$, with $\delta = \lfloor \nu_{\mathbf{j}} + \nu_{\mathbf{j}'} \rfloor - \lfloor \nu_{\mathbf{j}} \rfloor - \lfloor \nu_{\mathbf{j}'} \rfloor \in \{0, 1\}$. Thus, it is sufficient to check that $\Phi(\mathbf{j}) \in \mathcal{O}'_L$, for all $\mathbf{j} \in \mathbb{N}^{r+1}$.

For any $0 \leq s \leq r+1$, let $J_s := \{\mathbf{j} = (j_0, \ldots, j_r) \in \mathbb{N}^{r+1} \mid 0 \leq j_t < b_t, \text{ for } t \geq s\}$. Note that $J_0 = J$, $J_{r+1} = \mathbb{N}^{r+1}$. Consider the condition

$$(i_s) \qquad \Phi(\mathbf{j}) \in \mathcal{O}'_L, \text{ for all } \mathbf{j} \in J_s.$$

By the definition of $\mathcal{O}'_L$, the condition $(i_0)$ holds, and our aim is to show that $(i_{r+1})$ holds. Thus, it is sufficient to show that $(i_s)$ implies $(i_{s+1})$, for all $0 \leq s \leq r$. Let us prove this implication by induction on $j_s$. Take $\mathbf{j}_0 = (j_0, \ldots, j_r) \in J_{s+1}$. If $0 \leq j_s < b_s$, condition $(i_{s+1})$ holds for $\mathbf{j}_0$. Let $j_s \geq b_s$ and suppose that $\Phi(j'_0, \ldots, j'_{s-1}, j, j'_{s+1}, \ldots, j'_r) \in \mathcal{O}'_L$, for all $j'_0, \ldots, j'_{s-1} \in \mathbb{N}$, all $0 \leq j < j_s$, and all $0 \leq j'_t < b_t$, for $t > s$.

By item 2 of the last lemma, applied to $\mathbf{j} = (j_0, \ldots, j_{s-1}, j_s - b_s, 0, \ldots, 0)$:

$$\Phi(j_0, \ldots, j_{s-1}, j_s - b_s, 0, \ldots, 0, b_r) = \sum_{\mathbf{j}' \in J} c_{\mathbf{j},\mathbf{j}'}\Phi(\mathbf{j}+\mathbf{j}'), \tag{39}$$

if $s < r$, and $\Phi(j_0, \ldots, j_r) = \sum_{\mathbf{j}' \in J} c_{\mathbf{j},\mathbf{j}'}\Phi(\mathbf{j}+\mathbf{j}')$, if $s = r$. In both cases, the terms $\Phi(\mathbf{j}+\mathbf{j}')$ belong to $\mathcal{O}'_L$, because the $s$-th coordinate of $\mathbf{j}+\mathbf{j}'$ is $j_s - b_s + j'_s < j_s$. In particular, if $s = r$ we are done. If $s < r$ we apply item 1 of the last lemma to $\mathbf{j} = (j_0, \ldots, j_{s-1}, j_s - b_s, j_{s+1}, \ldots, j_r)$ and we get

$$\Phi(\mathbf{j}_0) = \pi^{\delta_{\mathbf{j},s}}\Phi(j_0, \ldots, j_s - b_s, j_{s+1}+1, j_{s+2}, \ldots, j_r) + \sum_{\mathbf{j}' = (j'_0, \ldots, j'_s, 0, \ldots, 0) \in J} c_{\mathbf{j},\mathbf{j}'}\Phi(\mathbf{j}+\mathbf{j}').$$

The last sum belongs to $\mathcal{O}'_L$ by the same argument as above. Thus, we need only to show that the term $\Phi(j_0, \ldots, j_s - b_s, j_{s+1}+1, j_{s+2}, \ldots, j_r)$ belongs to $\mathcal{O}'_L$ too. If $s = r-1$ this is clear by (39). If $s < r-1$, it is also clear if $j_{s+1} + 1 < b_{s+1}$. Finally, if $s < r-1$ and $j_{s+1} + 1 = b_{s+1}$, we can apply item 1 of the last lemma again to see that it is sufficient to check that $\Phi(j_0, \ldots, j_s - b_s, 0, j_{s+2}+1, \ldots, j_r)$ belongs to $\mathcal{O}'_L$. In this iterative process we conclude either by (39), or because we find some $j_t + 1 < b_t$. □

We need still some auxiliary lemmas. The first one is an easy remark about integral parts.

**Lemma 4.24.** *For all $x \in \mathbb{R}$ and $e \in \mathbb{Z}_{>0}$, we have $\sum_{0 \leq k < e} \left\lfloor \dfrac{x+k}{e} \right\rfloor = \lfloor x \rfloor$.*

*Proof.* The identity is obvious when $x$ is an integer, $0 \leq x < e$, because $\left\lfloor \dfrac{x+k}{e} \right\rfloor = 1$ for the $x$ values of $k$ such that $e - x \leq k < e$, and it is zero otherwise.

Write $x = n + \epsilon$, with $n = \lfloor x \rfloor$ and $0 \leq \epsilon < 1$; clearly, $\lfloor (x+k)/e \rfloor = \lfloor (n+k)/e \rfloor$, because $\epsilon/e < 1/e$. Consider the division with remainder, $n = Qe + r$, with

$0 \leq r < e$. Then,

$$\sum_{0\leq k<e} \Big\lfloor \frac{n+k}{e} \Big\rfloor = \sum_{0\leq k<e} \left( Q + \Big\lfloor \frac{r+k}{e} \Big\rfloor \right) = eQ + r = n.$$

□

**Lemma 4.25.** *Take $e_0 = 1$, $h_0 = 0$ by convention. Every $\mathbf{j} \in \mathbb{N}^{r+1}$ can be written in a unique way: $\mathbf{j} = \mathbf{j}' + \mathbf{j}''$, with $\mathbf{j}'$, $\mathbf{j}''$ belonging respectively to the two sets:*

$$J' := \{\mathbf{j}' = (j'_0, \dots, j'_r) \in \mathbb{N}^{r+1} \mid 0 \leq j'_s < e_s, \text{ for all } 0 \leq s \leq r\} \subseteq J,$$
$$J'' := \{\mathbf{j}'' = (j''_0, \dots, j''_r) \in \mathbb{N}^{r+1} \mid j''_s \equiv 0 \pmod{e_s}, \text{ for all } 0 \leq s \leq r\}.$$

*Then, for any $\mathbf{j}'' = (k_0, e_1k_1, \dots, e_rk_r) \in J''$, there is a unique $\mathbf{j}' = (j'_0, \dots, j'_r) \in J'$ such that $v(\Phi(\mathbf{j}' + \mathbf{j}'')) = 0$. Moreover, $j'_r = 0$, and $j'_s$ depends only on $k_{s+1}, \dots, k_r$, for $0 \leq s < r$.*

*Proof.* For any $\mathbf{j} \in \mathbb{N}^{r+1}$ denote by $\lambda_{\mathbf{j}}$ the positive integer

$$\lambda_{\mathbf{j}} := e_1 \cdots e_r \, \nu_{\mathbf{j}} = \sum_{i=1}^{r} \left( \sum_{t=i}^{r} j_t e_i f_i \cdots e_{t-1} f_{t-1} \right) e_{i+1} \cdots e_r h_i.$$

Clearly,

$$v(\Phi(\mathbf{j})) = \nu_{\mathbf{j}} - \lfloor \nu_{\mathbf{j}} \rfloor = \frac{\lambda_{\mathbf{j}}}{e_1 \cdots e_r} - \Big\lfloor \frac{\lambda_{\mathbf{j}}}{e_1 \cdots e_r} \Big\rfloor. \tag{40}$$

Thus, we are interested in the elements $\mathbf{j} \in \mathbb{N}^{r+1}$ such that $\lambda_{\mathbf{j}} \equiv 0 \pmod{e_1 \cdots e_r}$. Define now, for each $0 \leq s \leq r$,

$$\lambda_{\mathbf{j},s} := j_s h_s e_{s+1} \cdots e_r + \sum_{i=s+1}^{r} \left( \sum_{t=i}^{r} j_t e_i f_i \cdots e_{t-1} f_{t-1} \right) e_{i+1} \cdots e_r h_i.$$

Note that $\lambda_{\mathbf{j},s}$ depends only on $j_s, \dots, j_r$, and $\lambda_{\mathbf{j},0} = \lambda_{\mathbf{j}}$, $\lambda_{\mathbf{j},r} = j_r h_r$. Clearly,

$$\lambda_{\mathbf{j},s} - \lambda_{\mathbf{j},s+1} = j_s h_s e_{s+1} \cdots e_r + \left( \sum_{t=s+2}^{r} j_t e_{s+1} f_{s+1} \cdots e_{t-1} f_{t-1} \right) e_{s+2} \cdots e_r h_{s+1},$$

for all $0 \leq s \leq r$. In particular, $\lambda_{\mathbf{j},s} \equiv \lambda_{\mathbf{j},s+1} \pmod{e_{s+1} \cdots e_r}$, and

$$\lambda_{\mathbf{j}} \equiv 0 \pmod{e_1 \cdots e_r} \iff \lambda_{\mathbf{j},s} \equiv 0 \pmod{e_s \cdots e_r}, \text{ for all } 1 \leq s \leq r.$$

The condition $\lambda_{\mathbf{j},r} \equiv 0 \pmod{e_r}$ is equivalent to $j_r \equiv 0 \pmod{e_r}$. On the other hand, for $1 \leq s < r$, the condition $\lambda_{\mathbf{j},s} \equiv 0 \pmod{e_s \cdots e_r}$ is equivalent to

$$\lambda_{\mathbf{j},s+1} \equiv 0 \pmod{e_{s+1} \cdots e_r}, \text{ and}$$
$$j_s h_s + \left( \textstyle\sum_{t=s+2}^{r} j_t (f_{s+1} \cdots f_{t-1})(e_{s+2} \cdots e_{t-1}) \right) h_{s+1} + \frac{\lambda_{\mathbf{j},s+1}}{e_{s+1} \cdots e_r} \equiv 0 \pmod{e_s}.$$

Thus, the class of $j_s$ modulo $e_s$ is uniquely determined, and it depends only on $j_{s+1}, \dots, j_r$. □

**Corollary 4.26.** *Let $\kappa = (k_0, \dots, k_r) \in \mathbb{N}^{r+1}$, and let $\mathbf{j} = \mathbf{j}' + (k_0, e_1k_1, \dots, e_rk_r)$, where $\mathbf{j}'$ is the unique element in $J'$ such that $v(\Phi(\mathbf{j})) = 0$. Then,*

$$\Phi(\mathbf{j}) = \gamma_0(\theta)^{k_0} \cdots \gamma_r(\theta)^{k_r} \gamma_1(\theta)^{i_1} \cdots \gamma_{r-1}(\theta)^{i_{r-1}},$$

*for some integers $i_1, \dots, i_{r-1}$. Moreover, each $i_s$ depends only on $k_{s+1}, \dots, k_r$.*

*Proof.* By Lemma 4.25, $\mathbf{j} = (k_0, j_1' + e_1k_1, \dots, j_{r-1}' + e_{r-1}k_{r-1}, e_rk_r)$. By (14),

$$\gamma_s(\theta)^{k_s} = \pi^{n_{s,0}}\phi_1(\theta)^{n_{s,1}} \cdots \phi_s(\theta)^{e_sk_s},$$

for all $1 \le s \le r$, with integers $n_{s,i}$ that depend only on $k_s$. Hence,

$$\Phi(\mathbf{j})\gamma_0(\theta)^{-k_0} \cdots \gamma_r(\theta)^{-k_r} = \pi^{n_0}\phi_1(\theta)^{n_1} \cdots \phi_{r-1}(\theta)^{n_{r-1}},$$

for integers $n_s$ that depend only on $j_s'$ and $k_{s+1}, \dots, k_r$; hence they depend only on $k_{s+1}, \dots, k_r$. By Lemma 3.4, $v(\pi^{n_0}\phi_1(\theta)^{n_1} \cdots \phi_{r-1}(\theta)^{n_{r-1}}) = 0$, and by Propositions 2.10 and 2.16 we have $v_r(\pi^{n_0}\phi_1(x)^{n_1} \cdots \phi_{r-1}(x)^{n_{r-1}}) = 0$. By Lemma 2.17, this rational function can be expressed as a product $\gamma_1(x)^{i_1} \cdots \gamma_{r-1}(x)^{i_{r-1}}$, with integers $i_1, \dots, i_{r-1}$ such that each $i_s$ depends only on $n_s, \dots, n_{r-1}$, that is, on $k_{s+1}, \dots, k_r$. □

**Corollary 4.27.** *Let $\mathbf{j}_1 = \mathbf{j}_1' + \mathbf{j}''$, $\mathbf{j}_2 = \mathbf{j}_2' + \mathbf{j}''$, for some $\mathbf{j}_1', \mathbf{j}_2' \in J'$, $\mathbf{j}'' \in J''$. Then, $v(\Phi(\mathbf{j}_1)) = v(\Phi(\mathbf{j}_2))$ if and only if $\mathbf{j}_1 = \mathbf{j}_2$.*

*In particular, $\{v(\Phi(\mathbf{j})) \mid \mathbf{j} \in J'\} = \{k/e_1 \cdots e_r \mid 0 \le k < e_1 \cdots e_r\}$.*

*Proof.* Let $\mathbf{j}_1 = (j_{1,0}, \dots j_{1,r})$, $\mathbf{j}_2 = (j_{2,0}, \dots j_{2,r})$. With the above notations,

$$\begin{aligned} v(\Phi(\mathbf{j}_1)) = v(\Phi(\mathbf{j}_2)) &\iff \lambda_{\mathbf{j}_1} \equiv \lambda_{\mathbf{j}_2} \pmod{e_1 \cdots e_r} \\ &\iff \lambda_{\mathbf{j}_1,s} \equiv \lambda_{\mathbf{j}_2,s} \pmod{e_s \cdots e_r}, \text{ for all } 1 \le s \le r. \end{aligned}$$

For $s = r$ this is equivalent to $j_{1,r} = j_{2,r}$. By a recursive argument analogous to the one used in the proof of the lemma, once we know that $j_{1,t} = j_{2,t}$ for all $t > s$, then $\lambda_{\mathbf{j}_1,s} \equiv \lambda_{\mathbf{j}_2,s} \pmod{e_s \cdots e_r}$ is equivalent to $j_{1,s} = j_{2,s}$.

Finally, it is clear that $|J'| = e_1 \cdots e_r$, and we have just shown that the elements $v(\Phi(\mathbf{j}))$, $\mathbf{j} \in J'$, take $e_1 \cdots e_r$ different values, all of them contained in the set $\{k/e_1 \cdots e_r \mid 0 \le k < e_1 \cdots e_r\}$ by (40). □

**Proposition 4.28.** *Let $f(x) \in \mathcal{O}[x]$ be a monic irreducible polynomial such that $\mathbf{t}_{r-1}(f) = \{\mathbf{t}\}$. Let $\lambda_r$ be the slope of $N_r(f)$, and suppose that $R_{\lambda_r}(y)$ is a separable polynomial. Then, $\mathcal{O}_L' = \mathcal{O}_L$. Moreover, the family of all $\Phi(j)\Phi(\mathbf{j}')$, for $\mathbf{j} \in J_0 := \{\mathbf{j} \in J \mid v(\Phi(\mathbf{j})) = 0\}$ and $\mathbf{j}' \in J'$ is an $\mathcal{O}$-basis of $\mathcal{O}_L$.*

*Proof.* We have $a_r = 1$ by hypothesis, and Corollary 3.8 shows that $e(L/K) = e_1 \cdots e_r$, $f(L/K) = f_0f_1 \cdots f_r$. By Corollary 4.27, we have $\{v_L(\Phi(\mathbf{j})) \mid \mathbf{j} \in J'\} = \{0, 1, \dots, e(L/K)-1\}$. By Lemma 4.25, $|J_0| = f_0f_1 \cdots f_r = \dim_{\mathbb{F}_K} \mathbb{F}_L$, and each $\mathbf{j} \in J_0$ is parameterized by a sequence $(k_0, \dots, k_r)$, with $0 \le k_s < f_s$ for all $0 \le s \le r$. By item 4 of Proposition 3.5, $\mathbb{F}_L = \mathbb{F}_K(\overline{\gamma_0(\theta)}, \dots, \overline{\gamma_r(\theta)})$, where $\gamma_0(x) := x$. Recall that $z_i = \overline{\gamma_i(\theta)}$ for all $0 \le i \le r$, under our identification of $\mathbb{F}_{r+1} := \mathbb{F}_r[y]/\psi_r(y)$ with $\mathbb{F}_L$.

By Corollary 4.26,

$$\overline{\Phi(\mathbf{j})} = z_0^{k_0} z_1^{k_1+i_1} \cdots (z_{r-1})^{k_{r-1}+i_{r-1}} z_r^{k_r} = z_0^{k_0} z_1^{k_1} \Gamma_2(k_2, \dots, k_r) \cdots \Gamma_r(k_r),$$

where $\Gamma_s(k_s, \dots, k_r) = z_s^{k_s}(z_{s-1})^{i_{s-1}}$, for $s \ge 2$. Now, the family of all $\overline{\Phi(\mathbf{j})}$ for $\mathbf{j} \in J_0$ is an $\mathbb{F}_K$-basis of $\mathbb{F}_L$. In fact, the set of all $\Gamma_r(k_r)$ for $0 \le k_r < f_r$, is a $\mathbb{F}_r$-basis of $\mathbb{F}_L = \mathbb{F}_{r+1}$, because they are obtained from the basis $z_r^{k_r}$, just by multyplying every element by the nonzero scalar $z_{r-1}^{i_{r-1}} \in \mathbb{F}_r$, which depends only on $k_r$. Then, the set of all $\Gamma_{r-1}(k_{r-1}, k_r)\Gamma_r(k_r)$ for $0 \le k_{r-1} < f_{r-1}$, $0 \le k_r < f_r$, is a $\mathbb{F}_{r-1}$-basis of $\mathbb{F}_r$, because they are obtained from the basis $(z_{r-1})^{k_{r-1}}\Gamma_r(k_r)$, just by multyplying every element by the nonzero scalar $z_{r-2}^{i_{r-2}} \in \mathbb{F}_{r-1}$, which depends only on $k_{r-1}, k_r$, etc.

Therefore, the $e(L/K)f(L/K)$ elements $\Phi(\mathbf{j})\Phi(\mathbf{j}')$, $\mathbf{j} \in J_0$, $\mathbf{j}' \in J'$, are a $\mathcal{O}$-basis of $\mathcal{O}_L$. By Proposition 4.23, all these elements are contained in $\mathcal{O}'_L$, and we have necessarily $\mathcal{O}'_L = \mathcal{O}_L$. □

*Proof of the Theorem of the index.* Suppose first that $f(x) \in \mathcal{O}[x]$ is a monic irreducible polynomial, such that $\mathbf{t}_{r-1}(f) = \{\mathbf{t}\}$, and $f(x) \neq \phi_r(x)$. In this case we have built an order $\mathcal{O}[\theta] \subseteq \mathcal{O}'_L \subseteq \mathcal{O}_L$, and we have

$$(\mathcal{O}_L : \mathcal{O}[\theta]) = q^{\mathrm{ind}(f)}, \quad (\mathcal{O}'_L : \mathcal{O}[\theta]) = q^{\sum_{\mathbf{j}\in J}\lfloor \nu_{\mathbf{j}} \rfloor}, \tag{41}$$

the last equality by Lemma 4.20. Therefore, in order to prove item 1 of Theorem 4.18 it is sufficient to show that

$$f_0 \sum_{\mathbf{j}=(0,j_1,\dots,j_r)\in J} \lfloor \nu_{\mathbf{j}} \rfloor = \mathrm{ind}_1(f) + \cdots + \mathrm{ind}_r(f). \tag{42}$$

Let us prove this identity by induction on $r \geq 1$. For $r = 1$ this was proved already in (38). From now on, let $r \geq 2$. Both sides of the identity depend only on $a_r$ and the vectors $\mathbf{e} = (e_1, \dots, e_r)$, $\mathbf{f} = (f_1, \dots, f_{r-1})$, $\mathbf{h} = (h_1, \dots, h_r)$. Recall that

$$\nu_s = \nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) := \sum_{i=1}^{s} e_i f_i \cdots e_{s-1} f_{s-1} \frac{h_i}{e_1 \dots e_i}.$$

If we denote $\mathbf{e}' = (e_2, \dots, e_r)$, $\mathbf{f}' = (f_2, \dots, f_{r-1})$, $\mathbf{h}' = (h_2, \dots, h_r)$, it is easy to check that, for every $2 \leq s \leq r$:

$$\nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) - \frac{m_s}{m_2} f_1 h_1 = \frac{1}{e_1} \nu_{s-1}(\mathbf{e}', \mathbf{f}', \mathbf{h}'). \tag{43}$$

Let us show that the identity

$$f_0 \sum_{\mathbf{j}=(0,j_1,\dots,j_r)\in J} \Big\lfloor \sum_{s=1}^{r} j_s \nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) \Big\rfloor = \mathrm{ind}_1(f) + \cdots + \mathrm{ind}_r(f),$$

holds for any choice of $a_r$ and $\mathbf{e}, \mathbf{f}, \mathbf{h}$, under the assumption that this is true for $r - 1$. Write $j_1 = je_1 + k$, with $0 \leq j < f_1$, $0 \leq k < e_1$, and let $0 \leq s_k < e_1$ be determined by $kh_1 \equiv s_k \pmod{e-1}$. Then, by (43),

$$\begin{aligned}
\Big\lfloor \sum_{s=1}^{r} j_s \nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) \Big\rfloor &= \Big\lfloor jh_1 + k\frac{h_1}{e_1} + \sum_{s=2}^{r} j_s \nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) \Big\rfloor = \\
&= \sum_{s=2}^{r} j_s \frac{m_s}{m_2} f_1 h_1 + jh_1 + \Big\lfloor k\frac{h_1}{e_1} + \sum_{s=2}^{r} j_s \left( \nu_s(\mathbf{e}, \mathbf{f}, \mathbf{h}) - \frac{m_s}{m_2} f_1 h_1 \right) \Big\rfloor = \\
&= \sum_{s=2}^{r} j_s \frac{m_s}{m_2} f_1 h_1 + jh_1 + \Big\lfloor k\frac{h_1}{e_1} + \frac{1}{e_1} \sum_{s=2}^{r} j_s \nu_{s-1}(\mathbf{e}', \mathbf{f}', \mathbf{h}') \Big\rfloor = \\
&= \sum_{s=2}^{r} j_s \frac{m_s}{m_2} f_1 h_1 + jh_1 + \Big\lfloor k\frac{h_1}{e_1} \Big\rfloor + \Big\lfloor \frac{s_k}{e_1} + \frac{1}{e_1} \sum_{s=1}^{r-1} j_{s+1} \nu_s(\mathbf{e}', \mathbf{f}', \mathbf{h}') \Big\rfloor.
\end{aligned}$$

Therefore, it is sufficient to check the two identities:

$$f_0 \sum_{\substack{(0,\,0,\,j_2,\,\dots,\,j_r)\,\in\, J \\ 0 \leq j < f_1,\, 0 \leq k < e_1}} \left( \sum_{s=2}^{r} j_s \frac{m_s}{m_2} f_1 h_1 + jh_1 + \Big\lfloor k\frac{h_1}{e_1} \Big\rfloor \right) = \mathrm{ind}_1(f),$$

$$f_0 \sum_{\substack{(0,0,j_2,\dots,j_r)\in J\\ 0\le j<f_1, 0\le k<e_1}} \Big\lfloor \frac{s_k}{e_1} + \frac{1}{e_1}\sum_{s=1}^{r-1} j_{s+1}\nu_s(\mathbf{e}',\mathbf{f}',\mathbf{h}')\Big\rfloor = \operatorname{ind}_2(f)+\cdots+\operatorname{ind}_r(f).$$

The left-hand side of the first identity is equal to $f_0\sum_{0\le i<e_1f_1a_1}\lfloor i\frac{h_1}{e_1}\rfloor$. Hence, it has been proved in (38). The second identity follows from the induction hypothesis. In fact, the set $\{s_k \mid 0\le k<e_1\}$ coincides with $\{0,1,\dots,e_1-1\}$, and by Lemma 4.24 the left-hand side of the identity is equal to

$$f_0f_1 \sum_{(0,0,j_2,\dots,j_r)\in J} \Big\lfloor \sum_{s=1}^{r-1} j_{s+1}\nu_s(\mathbf{e}',\mathbf{f}',\mathbf{h}')\Big\rfloor.$$

Let us prove now the second part of the theorem. If $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_r(f)$, then $\operatorname{ind}_{r+1}(f)=0$ by item 1 of the theorem in order $r+1$. Conversely, if $\operatorname{ind}_{r+1}(f)=0$, then Lemma 4.16 shows that $\mathbf{t}_{r+2}(f)=\emptyset$, and by Lemma 3.11, for all $\mathbf{t}'\in\mathbf{t}_r(f)$ and all $\lambda_{r+1}$, the residual polynomial $R_{\mathbf{t}',\lambda_{r+1}}(f)(y)$ is separable.

If $\mathbf{t}$ is $f$-complete, we have $a_r=1$ by Lemma 4.5, and $\mathcal{O}'_L=\mathcal{O}_L$ by Proposition 4.28. By (41) and (42), we get $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_r(f)$. If $\mathbf{t}$ is not $f$-complete then $\mathbf{t}_r(f)=\{\mathbf{t}'\}$ for some type $\mathbf{t}'$ of order $r$. By Proposition 4.28 applied to $\mathbf{t}'$ in order $r$, we get $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_r(f)+\operatorname{ind}_{r+1}(f)$ by the same argument in order $r$. Since $\operatorname{ind}_{r+1}(f)=0$, we have $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_r(f)$, as desired. This ends the proof of the theorem in the particular case we were dealing with.

Let us prove now the theorem in the other instances where $f(x)$ is irreducible. If $f(x)=\phi_r(x)$ we have $\operatorname{ind}_r(f)=0$; in this case we can apply the theorem in order $r-1$, since $f(x)\ne\phi_{r-1}(x)$ and $\mathbf{t}_{r-2}(f)=\{\mathbf{t}_{r-2}\}$ is non-empty. Thus, $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_{r-1}(f)$. Finally, suppose that $\mathbf{t}_{r-1}(f)=\emptyset$ and let $s<r$ be maximal with the property $\mathbf{t}_{s-1}(f)\ne\emptyset$. We can apply the theorem in order $s$, and since $\mathbf{t}_s(f)=\emptyset$ we have $\operatorname{ind}_{s+1}(f)=0$ and $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_s(f)$. Since $\operatorname{ind}_t(f)=0$ for all $t>s$, this proves both statements of the theorem. This ends the proof of the theorem when $f(x)$ is irreducible.

In the general case, if $f(x)=F_1(x)\cdots F_k(x)$ is the factorization of $f(x)$ into a product of monic irreducible polynomials, we have by definition

$$\operatorname{ind}(f)=\sum_{i=1}^{k}\operatorname{ind}(F_i)+\sum_{1\le i<j\le k} v(\operatorname{Res}(F_i,F_j)).$$

By Lemma 4.17, an analogous relationship holds for every $\operatorname{ind}_s(f)$, $1\le s\le r$. Hence, item 1 of the theorem holds by the theorem applied to each $\operatorname{ind}(F_i)$, and by Theorem 4.10. Let us prove now item 2. By Lemma 4.17, $\operatorname{ind}_{r+1}(f)=0$ if and only if $\operatorname{ind}_{r+1}(F_i)=0$ and $\operatorname{Res}_{r+1}(F_i,F_j)=0$, for all $i$ and all $j\ne i$. By the theorem in the irreducible case and Theorem 4.10, this is equivalent to $\operatorname{ind}(f)=\operatorname{ind}_1(f)+\cdots+\operatorname{ind}_r(f)$. □

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