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Eigenvalue interlacing and weight parameters of graphs [1]

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Abstract

Eigenvalue interlacing is a versatile technique for deriving results in algebraic combinatorics. In particular, it has been successfully used for proving a number of results about the relation between the (adjacency matrix or Laplacian) spectrum of a graph and some of its properties. For instance, some characterizations of regular partitions, and bounds for some parameters, such as the independence and chromatic numbers, the diameter, the bandwidth, etc., have been obtained. For each parameter of a graph involving the cardinality of some vertex sets, we can define its corresponding weight parameter by giving some "weights" (that is, the entries of the positive eigenvector) to the vertices and replacing cardinalities by square norms. The key point is that such weights "regularize" the graph, and hence allow us to define a kind of regular partition, called "pseudo-regular," intended for general graphs. Here we show how to use interlacing for proving results about some weight parameters and pseudo-regular partitions of a graph. For instance, generalizing a well-known result of Lovász, it is shown that the weight Shannon capacity Θ^* of a connected graph Γ, with n vertices and (adjacency matrix) eigenvalues $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$, satisfies

$$\Theta \leqslant \Theta^* \leqslant \frac{\|\boldsymbol{\nu}\|^2}{1-\lambda_1/\lambda_n},$$

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where Θ is the (standard) Shannon capacity and $\mathbf{v}$ is the positive eigenvector normalized to have smallest entry 1. In the special case of regular graphs, the results obtained have some interesting corollaries, such as an upper bound for some of the multiplicities of the eigenvalues of a distance-regular graph. Finally, some results involving the Laplacian spectrum are derived.

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1. Introduction

As has been shown by Haemers [25,26] and other authors, eigenvalue interlacing is a powerful technique for deriving results about combinatorial structures from the spectra of their associated matrices. A good and quite complete survey on this topic is Haemers' paper [26]. In particular, this technique allows us to infer a number of properties of a graph, such as bounds for its diameter, independence and chromatic numbers, bandwidth, etc, from (part of) its spectrum. Before explaining the contents of this work, we will introduce some basic terminology.

Let $\Gamma = (V, E)$ be a (finite and simple) graph on $n := |V|$ vertices. Throughout the paper, Γ will be supposed to be non-trivial, that is $E \neq \emptyset$. The distance between two vertices $u, v \in V$ will be denoted by $\partial(u, v)$. Then, the distance between two subsets $U_1, U_2 \subset V$ is $\partial(U_1, U_2) := \min_{u \in U_1,\, v \in U_2}\{\partial(u, v)\}$. For a given vertex subset $C \subset V$ and some integer $k \geqslant 0$, we denote by $N_k(C)$ the set of vertices at distance k from (some vertex of) C. Similarly, Γ_k stands for the graph with vertex set V where two vertices are adjacent whenever they are at distance k in Γ. Thus, $N_0(C) = C$, $N_1(\{u\}) = \Gamma(u)$, the set of vertices adjacent to u, and $\Gamma_1 = \Gamma$. The *eccentricity* of C, denoted ε_C, can be defined as the maximum distance of any vertex of Γ from C. (In Coding Theory this correspond to the "covering radius" of C.) The notation $\overline{C}$ will be used to denote the complement of the set C in V.

The eigenvalues of (the adjacency matrix $A(\Gamma)$ of) Γ will be denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ (including multiplicities). If Γ is connected, the theorem of Perron–Frobenius assures that λ_1 is simple, positive (in fact, it coincides with the *spectral radius* of $A(\Gamma)$), and with positive eigenvector. If Γ is not connected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. Throughout the paper, it is supposed that the eigenvalue λ_1 has indeed a positive eigenvector, denoted by $\mathbf{v}$, which is normalized in such a way that its minimum entry (in each connected component of Γ) is 1. For instance, if Γ is regular, we just have $\mathbf{v} = \mathbf{j}$, the all-1 vector. Usually, vectors and matrices are indexed by the vertices of V, so that the above condition reads $\min_{u \in U}\{v_u\} = 1$ for every vertex set

$U \subseteq V$ of a connected component. When we are interested in the set of distinct eigenvalues, the notation $\mathrm{ev}\,\Gamma \equiv \mathrm{ev}\,A(\Gamma) = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$ will be used (note that $\theta_0 = \lambda_1$ and $\theta_d = \lambda_n$).

Consider the map $\boldsymbol{\rho} : \mathscr{P}(V) \to \mathbb{R}^n$ defined by $\boldsymbol{\rho} U := \sum_{u \in U} v_u\, \mathbf{e}_u$ for any $U \neq \emptyset$, where $\mathbf{e}_u$ represents the uth canonical (column) vector, and $\boldsymbol{\rho}\emptyset := \mathbf{0}$. Note that, with $\boldsymbol{\rho} u := \boldsymbol{\rho}\{u\}$, we have $\|\boldsymbol{\rho} u\| = v_u$, so that we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of Γ. In doing so we "regularize" the graph, in the sense that the *average weight degree* of each vertex $u \in V$ becomes a constant:

$$\delta^*(u) := \frac{1}{v_u} \sum_{v \in \Gamma(u)} v_v = \lambda_1. \tag{1}$$

Using these weights, we can also consider the so-called "pseudo-regular partitions" of a graph, defined in the next section, which generalize the standard notion of regular (or equitable) partitions.

In this context, the author [14] introduced the notion of a "weight parameter" of a graph, defined as follows. For each parameter of a graph Γ, say ξ, defined as the maximum cardinality of a set $U \subset V$ satisfying a given property P, we define the corresponding *weight parameter*, denoted by ξ^*, as the maximum value of $\|\boldsymbol{\rho} U\|^2$ of a vertex set U satisfying P. Note that, when the graph is regular, we have $\boldsymbol{v} = \boldsymbol{j}$ and then $\xi^* \equiv \xi$. Otherwise, when we are dealing with non-regular graphs, the weight parameters are sometimes more convenient to work with, as we will see later. As an instance of weight parameter, let us consider the *weight independence number* of Γ, defined as

$$\alpha^* := \max_{U \subset V} \{\|\boldsymbol{\rho} U\|^2 : U \text{ is an independent set}\}.$$

The main concern of this paper is the use of eigenvalue interlacing for obtaining results on some weight parameters and pseudo-regular partitions. It is shown that this approach leads sometimes to simple proofs for some results concerning standard parameters, such as the chromatic index and the Shannon capacity of a (not necessarily regular) graph. The basic tools for our study are explained in the following section. The remaining sections are devoted to applying the technique in different situations where either the adjacency matrix or the Laplacian matrix is considered.

2. Interlacing and pseudo-regular partitions

Our starting point is the following theorem, proved by Haemers in Refs. [25,26]. That author alludes to the first part of the theorem as a classical result, referring the reader to the book written by Courant and Hilbert [9].

Theorem 2.1. *Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. For some integer $m < n$, let S be a real $n \times m$ matrix with orthonormal columns, that is $S^T S = I$, and consider the matrix $B := S^T A S$, with eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$. Then the following statements hold.*

(a) *The eigenvalues of B interlace the eigenvalues of A. That is,*

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i} \quad (1 \leqslant i \leqslant m).$$

(b) *If the interlacing is tight, that is, for some $0 \leqslant k \leqslant m$, $\lambda_i = \mu_i$ $(1 \leqslant i \leqslant k)$ and $\mu_i = \lambda_{n-m-i}$ $(k+1 \leqslant i \leqslant m)$, then $SB = AS$.*

Let $\Gamma = (V, E)$ be a graph with adjacency matrix $A := A(\Gamma)$ and positive eigenvector v with elements indexed by the vertices of Γ. A partition $\mathscr{P}$ of the vertex set $V = V_1 \cup \cdots \cup V_m$ is called *pseudo-regular* (or *pseudo-equitable*) whenever the *(pseudo-)intersection numbers*

$$b^*_{ij}(u) := \frac{1}{v_u} \sum_{v \in \Gamma(u) \cap V_j} v_v \quad (1 \leqslant i, j \leqslant m) \tag{2}$$

do not depend on the chosen vertex $u \in V_i$, but only on the subsets V_i and V_j. In this case, such numbers are simply written as b^*_{ij}, and the $m \times m$ matrix $B^* = (b^*_{ij})$ is referred to as the *pseudo-quotient matrix* of A with respect to the (pseudo-regular) partition $\mathscr{P}$. Pseudo-regular partitions were introduced by Garriga and the author [17], as a generalization of the so-called regular partitions, where the above numbers are just defined by $b^*_{ij}(u) := |\Gamma(u) \cap V_j|$ $(u \in V_i)$. A detailed study of regular partitions can be found in Refs. [22,23]. (See also Refs. [6,32].) A vertex subset $C \subset V$ is said to be a *completely pseudo-regular code* if the distance partition around C, that is $V = C \cup N_1(C) \cup \cdots \cup N_{\varepsilon_C}(C)$, is pseudo-regular. A spectral characterization of such codes can be found in Ref. [17].

Of course we can also define, in the same way, a pseudo-regular partition of (the rows and columns of) any matrix A with a positive eigenvector. For instance, by the Perron–Frobenius theorem, this is the case when A is an $n \times n$ non-negative irreducible matrix. (In this case the corresponding eigenvalue is simple, non-zero ($n > 1$), and coincides with the spectral radius.) Another example is when A is the *Laplacian matrix* of a graph Γ, denoted by $L \equiv L(\Gamma)$, and defined as $L(\Gamma) := D - A(\Gamma)$, where $D = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_n)$ and δ_i stands for the degree of the ith vertex. (See Ref. [33] for a comprehensive survey on the properties and applications of such a matrix, and Ref. [34] for some recent results involving it.) Indeed, L has eigenvalues $\lambda_1 = 0 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ (they are usually enumerated in non-decreasing order) and the eigenvalue 0 has the eigenvector j. Then, although in this paper we limit ourselves to the adjacency and Laplacian matrices, most of the results obtained remain valid for appropriate matrices which satisfy the hypotheses of Theorem 2.1 and have a positive eigenvector. Here appropriate means that the considered matrices give some information about the structure of the graph, which is relevant to the parameter(s) under consideration. (In other words, matrices with an appropriate underlying graph.) For instance, in the study of the weight independence number, undertaken in the next section, we need to use a matrix $\boldsymbol{A}$ such that if u, v are non-adjacent vertices then $(\boldsymbol{A})_{uv} = 0$.

A matrix characterization of pseudo-regular partitions can be done via the following matrix associated with (any) partition $\mathscr{P}$: $V_1 \cup \cdots \cup V_m$. The *weight-characteristic* matrix of $\mathscr{P}$ is the $n \times m$ matrix $\boldsymbol{S}^* = (s^*_{uj})$ with entries

$$s^*_{uj} = \begin{cases} v_u & \text{if } u \in V_j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let $\Gamma = (V, E)$ be a graph with adjacency matrix $\boldsymbol{A}$ and positive eigenvector $\mathbf{v}$, and consider a vertex partition $\mathscr{P}$ with weight-characteristic matrix $\boldsymbol{S}^*$. Then $\mathscr{P}$ is pseudo-regular if and only if there exists an $(m \times m)$ matrix $\boldsymbol{C}$ such that $\boldsymbol{S}^*\boldsymbol{C} = \boldsymbol{A}\boldsymbol{S}^*$. Moreover, in this case $\boldsymbol{C} = \boldsymbol{B}^*$, the pseudo-quotient matrix of $\boldsymbol{A}$ with respect to $\mathscr{P}$.*

Proof. Let $\boldsymbol{C} = (c_{ij})$ be an $m \times m$ matrix. Let $u \in V_i$ and $1 \leqslant j \leqslant m$. Then, the result follows from the equalities:

$$(\boldsymbol{S}^*\boldsymbol{C})_{uj} = \sum_{k=1}^{m} s^*_{uk} c_{kj} = v_u c_{ij};$$

$$(\boldsymbol{A}\boldsymbol{S}^*)_{uj} = \sum_{v \in V} a_{uv} s^*_{vj} = \sum_{v \in \Gamma(u) \cap V_j} v_v = v_u b^*_{ij}(u),$$

where we have used the definition of $b^*_{ij}(u)$. □

Most of the results about regular partitions can be generalized for pseudo-regular partitions. For instance, using the above lemma it can be proved that all the eigenvalues of the pseudo-quotient matrix $\boldsymbol{B}^*$ are also eigenvalues of $\boldsymbol{A}$ (see Ref. [21]).

Let us now consider a new $n \times m$ matrix, $\boldsymbol{S} = (s_{uj})$, obtained by just normalizing the columns of $\boldsymbol{S}^*$. Namely,

$$s_{uj} = \begin{cases} v_u / \|\boldsymbol{\rho} V_j\| & \text{if } u \in V_j, \\ 0 & \text{otherwise} \end{cases}$$

and, hence, satisfying $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S} = \boldsymbol{I}$. From such a matrix we define the *weight-quotient* matrix of $\boldsymbol{A}$, with respect to $\mathscr{P}$, as $\boldsymbol{B} := \boldsymbol{S}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{S}$. Notice that this matrix has entries

$$b_{ij} = \sum_{u,v\in V} s_{ui}a_{uv}s_{vj} = \sum_{u\in V_i, v\in V_j} a_{uv}\frac{v_u}{\|\boldsymbol{\rho} V_i\|}\frac{v_v}{\|\boldsymbol{\rho} V_j\|} = \frac{1}{\|\boldsymbol{\rho} V_i\|\|\boldsymbol{\rho} V_j\|}\sum_{uv\in E(V_i,V_j)} v_u v_v = b_{ji},$$

where $E(V_i, V_j)$ stands for the set of edges with endpoints in V_i and V_j (when $V_i = V_j$ each edge counts twice). In particular, note that when Γ is regular $b_{ij} = |E(V_i, V_j)|/\sqrt{|V_i||V_j|}$, so that if $|V_i| = |V_j|$ for any $1 \leqslant i, j \leqslant m$, then $\boldsymbol{B}$ coincides with the "quotient matrix" used by Haemers [26] (with entries being the average row sums of the submatrices induced by the partition). In the case $N_1(V_i) \subset V_j$ we get, by Eq. (1),

$$b_{ij} = \frac{1}{\|\boldsymbol{\rho} V_i\|\|\boldsymbol{\rho} V_j\|}\sum_{u\in V_i} v_u \sum_{v\in\Gamma(u)} v_v = \frac{\lambda_1}{\|\boldsymbol{\rho} V_i\|\|\boldsymbol{\rho} V_j\|}\sum_{u\in V_i} v_u^2 = \frac{\lambda_1\|\boldsymbol{\rho} V_i\|}{\|\boldsymbol{\rho} V_j\|}.$$

In addition, we will also use the fact that $\boldsymbol{B}$ has eigenvalue λ_1, with corresponding eigenvector $\boldsymbol{\mu} := \boldsymbol{S}^{\mathrm{T}}\mathbf{v} = (\|\boldsymbol{\rho} V_1\|, \ldots, \|\boldsymbol{\rho} V_m\|)^{\mathrm{T}}$. Indeed,

$$\begin{aligned}(\boldsymbol{B\mu})_i &= \sum_{j=1}^{m}\sum_{u\in V_i, v\in V_j}\frac{a_{uv}v_u v_v}{\|\boldsymbol{\rho} V_i\|\|\boldsymbol{\rho} V_j\|}\|\boldsymbol{\rho} V_j\| = \frac{1}{\|\boldsymbol{\rho} V_i\|}\sum_{u\in V_i} v_u \sum_{v\in\Gamma(u)} v_v \\ &= \frac{\lambda_1}{\|\boldsymbol{\rho} V_i\|}\sum_{u\in V_i} v_u^2 = \lambda_1\boldsymbol{\mu}_i \quad (1 \leqslant i \leqslant m).\end{aligned}$$

The following result, which is basic to our study, is a direct consequence of Theorem 2.1, and can be thought of as a generalization of Corollary 2.3 in Ref. [26].

Lemma 2.3. *Let $\Gamma = (V, E)$ be graph with adjacency matrix $\boldsymbol{A}$ and positive eigenvector $\mathbf{v}$, and consider a partition $\mathscr{P}$ of V inducing the weight-quotient matrix $\boldsymbol{B}$. Then the following hold:*

(a) *The eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$;*

(b) *If the interlacing is tight, then the partition $\mathscr{P}$ is pseudo-regular.*

Proof. We only need to prove (b). If the interlacing is tight we already know, by Theorem 2.1(b), that $\boldsymbol{SB} = \boldsymbol{AS}$. Moreover, $\boldsymbol{S} = \boldsymbol{S}^*\boldsymbol{D}$, with $\boldsymbol{D} := \mathrm{diag}(\|\boldsymbol{\rho} V_1\|^{-1}, \ldots, \|\boldsymbol{\rho} V_m\|^{-1})$. Hence,

$$\boldsymbol{SS}^{\mathrm{T}}\boldsymbol{AS} = \boldsymbol{S}^*(\boldsymbol{DS}^{\mathrm{T}}\boldsymbol{AS}^*)\boldsymbol{D} = \boldsymbol{AS}^*\boldsymbol{D} \quad\Rightarrow\quad \boldsymbol{S}^*\boldsymbol{B}^* = \boldsymbol{AS}^*$$

with $\boldsymbol{B}^* := \boldsymbol{DS}^{\mathrm{T}}\boldsymbol{AS}^* = \boldsymbol{DBD}^{-1}$ being the pseudo-quotient matrix of $\boldsymbol{A}$ with respect to $\mathscr{P}$. $\square$

3. The weight independence number

Using the results derived above, mainly Lemma 2.3, most of the results obtained for regular graphs can be extended to general graphs (with a positive eigenvector). The only difference is that we must now consider weight parameters and pseudo-equitable partitions. Inspired by Haemers' paper [26], we first derive an upper bound for both the weight independence number and the Shannon capacity of a graph. As a straightforward consequence of the former, we then obtain the well-known Hoffman's upper bound for the chromatic number.

Theorem 3.1. *Let Γ be a graph with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and positive eigenvector $\boldsymbol{v}$. Then, its weight independence number satisfies*

$$\alpha^* \leqslant \frac{\|\boldsymbol{v}\|^2}{1 - \lambda_1/\lambda_n}. \tag{3}$$

If the bound is attained for some independent set C, then C is a completely pseudo-regular code with eccentricity $\varepsilon_C = 2$.

Proof. Let $C \subset V$ such that $\alpha^* = \|\rho C\|^2$, and let $\mathscr{P}$ be the partition $V_1 \cup V_2 = C \cup \overline{C}$, where $\overline{C} := V \setminus C$. Then, the weight-quotient matrix of $\boldsymbol{A} := \boldsymbol{A}(\Gamma)$ with respect to $\mathscr{P}$ turns out to be

$$\boldsymbol{B} = \lambda_1 \begin{pmatrix} 0 & \frac{\|\rho C\|^2}{\|\rho C\|\|\rho\overline{C}\|} \\ \frac{\|\rho C\|^2}{\|\rho C\|\|\rho\overline{C}\|} & \frac{\|\rho\overline{C}\|^2 - \|\rho C\|^2}{\|\rho\overline{C}\|^2} \end{pmatrix} \tag{4}$$

with eigenvalues $\mu_1 = \lambda_1$ and

$$\mu_2 = \operatorname{tr}\boldsymbol{B} - \lambda_1 = \frac{-\lambda_1\|\rho C\|^2}{\|\boldsymbol{v}\|^2 - \|\rho C\|^2} = \frac{-\lambda_1\alpha^*}{\|\boldsymbol{v}\|^2 - \alpha^*}.$$

Hence, since $\mu_2 \geqslant \lambda_n$ by Lemma 2.3, the result follows. In addition, if equality holds, then the interlacing is tight (since $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_n$) and therefore the partition is pseudo-regular. In particular, from the corresponding pseudo-quotient matrix $\boldsymbol{B}^* = \boldsymbol{D}\boldsymbol{B}\boldsymbol{D}^{-1}$, we get that, for every vertex $u \in \overline{C}$,

$$b^*_{21}(u) = \frac{1}{\nu_u} \sum_{v \in \Gamma(u) \cap C} \nu_v = \frac{\lambda_1\|\rho C\|^2}{\|\rho\overline{C}\|^2} = -\lambda_n \neq 0.$$

Consequently, $\varepsilon_C = 2$ and $\mathscr{P}$ is the distance partition around C. $\square$

Let $v_{\max} := \max_{u\in V}\{v_u\}$. Then, since clearly $\alpha^* \geqslant v_{\max}^2$, the above theorem gives

$$1 - \frac{\lambda_1}{\lambda_n} \leqslant \frac{\|\boldsymbol{v}\|^2}{v_{\max}^2} \leqslant n$$

for any such graph Γ, with equality holding in both iff Γ is the complete graph K_n.

As another simple corollary of Theorem 3.1 we can get the known result of Hoffman [27], which provides a lower bound on the chromatic number χ of any graph Γ. (Recall that χ is the minimum number of independent sets – color classes – into which V can be partitioned.)

Corollary 3.2 [27]. *Let Γ be a graph with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Then, its chromatic number satisfies*

$$\chi \geqslant 1 - \frac{\lambda_1}{\lambda_n}. \tag{5}$$

Proof. Suppose first that Γ is connected, with positive eigenvector $\boldsymbol{v}$. Since, for any minimum coloring of Γ, each color class U_i, $1 \leqslant i \leqslant \chi$, is an independent set, we have $\|\rho U_i\|^2 \leqslant \alpha^*$. Hence, $\chi \geqslant \|\boldsymbol{v}\|^2/\alpha^*$ and Eq. (3) yields the result. Otherwise, if Γ is disconnected, we only need to apply Eq. (5) to any connected component with maximum eigenvalue λ_1. □

A direct proof of Eq. (5) was given by Haemers [24,26]. His proof also uses eigenvalue interlacing, and so it is different from Hoffman's original one. However, excepting for the regular case, Haemers' proof is not related to any independence-like number. As cited by that author in Ref. [26], his proof has become a common example of application of the interlacing technique (see, for instance, Ref. [22], p. 48 or Ref. [31], Problem 11.21).

When Γ is regular, Theorem 3.1 reduces to the following bound for the (standard) independence number:

$$\alpha \leqslant \frac{n}{1 - \lambda_1/\lambda_n}, \tag{6}$$

which, according to Haemers [25,26], is an unpublished result of Hoffman. The first published proof is due to Lovász [30] who derived the same upper bound for the so-called *Shannon capacity* of Γ [35], defined as

$$\Theta := \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(\Gamma^k)}.$$

Here $\alpha(\Gamma^k)$ denotes the independence number of Γ^k, the *product* of k copies of Γ, with vertex set $V \times \stackrel{k}{\cdots} \times V$ and adjacencies between distinct vertices

$(u_1,\dots,u_k)\sim(v_1,\dots,v_k)$ iff, for any $1\leqslant i\leqslant k$, either $u_i=v_i$ or $u_i\sim v_i$. Note that, since $\alpha(\Gamma^k)\geqslant\alpha^k$, the Shannon capacity always satisfies the bound $\Theta\geqslant\alpha$. For more details about this parameter, see also Knuth's paper [29]. The weight version of the *Shannon capacity* can be defined by just writing

$$\Theta^* := \sup_k \sqrt[k]{\alpha^*(\Gamma^k)}$$

and, as expected, it can be shown to be bounded above by the weight analogue of Lovász bound, as the next theorem shows. (To prove it, recall that the *Kronecker product* of two matrices $\boldsymbol{A}\otimes\boldsymbol{B}$ is obtained by replacing each entry $(\boldsymbol{A})_{uv}$ with the matrix $(\boldsymbol{A})_{uv}\boldsymbol{B}$. Then if $\boldsymbol{v}$ and $\boldsymbol{\eta}$ are eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, with corresponding eigenvalues λ and μ, respectively, then $\boldsymbol{v}\otimes\boldsymbol{\eta}$ – viewing $\boldsymbol{v}$ and $\boldsymbol{\eta}$ as 1-column matrices – is an eigenvector of $\boldsymbol{A}\otimes\boldsymbol{B}$, with eigenvalue $\lambda\mu$.)

Theorem 3.3. *Let Γ be a graph with eigenvalues $\lambda_1\geqslant\cdots\geqslant\lambda_n$ and positive eigenvector $\boldsymbol{v}$. Then, its weight Shannon capacity satisfies*

$$\Theta^* \leqslant \frac{\|\boldsymbol{v}\|^2}{1-\lambda_1/\lambda_n}. \tag{7}$$

Proof. The proof goes along the same lines as that given by Haemers [26] in the regular case. As commented in Section 2, the above results remain valid for any symmetric matrix $\boldsymbol{A}^*$ with $(\boldsymbol{A}^*)_{uv}=0$ if $u\not\sim v$, which has maximum eigenvalue with a positive eigenvector. Then the application of Theorem 3.1 to the matrix

$$\boldsymbol{A}^*(\Gamma^k) := (\boldsymbol{A}-\lambda_n\boldsymbol{I})\otimes\overset{k}{\cdots}\otimes(\boldsymbol{A}-\lambda_n\boldsymbol{I})-(-\lambda_n)^k,$$

with maximum eigenvalue $(\lambda_1-\lambda_n)^k-(-\lambda_n)^k$, positive eigenvector $\boldsymbol{v}\otimes\overset{k}{\cdots}\otimes\boldsymbol{v}$, and minimum eigenvalue $-(-\lambda_n)^k$ gives

$$\alpha^*(\Gamma^k)\leqslant\left(\frac{\|\boldsymbol{v}\|^2}{1-\lambda_1/\lambda_n}\right)^k,$$

whence the result follows. □

Notice that, since $\alpha^*\leqslant\Theta^*$ and $\Theta\leqslant\Theta^*$, the above result yields also bounds for both α^* (that is Theorem 3.1) and Θ, the (standard) Shannon capacity of a (not necessarily regular) graph.

4. The weight odd-independence numbers

The concepts of odd and even distance were introduced by Bond and Delorme in Ref. [5], and they are based on looking at the parity of the lengths of the walks considered. Thus, the *odd distance* between two (not necessarily different) vertices u, v of a graph Γ, denoted by $\partial_o(u, v)$, is the length of a shortest walk of odd length between them. By using odd distances, we can now consider other related metric parameters, such as the *odd diameter* D_o and the *odd girth* g_o, defined as expected (in D_o we must also consider $\partial_o(u, u)$ when looking at maximum odd distance between pairs of vertices). Thus, since Γ has no loops, $1 < g_o \leqslant \partial_o(u, u) \leqslant D_o$ for any vertex $u \in V$ and, if Γ bipartite, $g_o = D_o = \infty$. Otherwise, the above-mentioned authors proved that $D_o \leqslant 2D + 1$, with D being the standard diameter of Γ. In fact it can be shown that, if Γ is a non-bipartite connected graph, then $D_o \leqslant d^\star \leqslant 2d + 1$, where $d^\star$ is the number of points of the "symmetrized mesh" $\mathcal{M}^\star := \mathcal{M} \cup \{0\} \cup (-\mathcal{M})$, with $\mathcal{M} = \operatorname{ev} \Gamma \setminus \{\lambda_1\}$, and $d = |\mathcal{M}|$ (see Ref. [14]).

By using odd distances, the author [14] introduced a new measure of independence as follows. Let $k \geqslant 1$ be an odd integer. Then, the *odd-k-independence number* α_{ok} is defined as the maximum number of vertices which are at odd distance greater than k from each other (including the odd distance from one vertex to itself). Note that, in particular, α_{o1} coincides with the independence number α. Any set of vertices satisfying such a condition is called an *odd-k-independent set*, so that the corresponding weight parameter is

$$\alpha^*_{ok} := \max_{U \subset V}\{\|\boldsymbol{\rho} U\|^2 \colon U \text{ is an odd-}k\text{-independent set}\}.$$

Basically the same proof used in Theorem 3.1 yields the following result, whose first part was also proved in Ref. [14] by using another technique.

Theorem 4.1. *Let Γ be a graph with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and positive eigenvector $\boldsymbol{v}$. Let q be a polynomial with only odd powers, degree k, $q(\lambda_1) > 0$, and $q_{\min} := \min_{2 \leqslant i \leqslant n}\{q(\lambda_i)\}$. Then, provided that $\alpha^*_{ok} \neq 0$, we have*

$$\alpha^*_{ok} \leqslant \frac{\|\boldsymbol{v}\|^2}{1 - q(\lambda_1)/q_{\min}}. \tag{8}$$

If the bound is attained for some odd-k-independent set C, then

$$q(\boldsymbol{A})\boldsymbol{\rho} C = -q_{\min}\boldsymbol{\rho}\overline{C}. \tag{9}$$

Proof. Let $C \subset V$ be an odd-k-independent set with $\alpha^*_{ok} = \|\boldsymbol{\rho} C\|^2$. By the hypotheses on the polynomial q, the matrix $q(\boldsymbol{A})$ has minimum eigenvalue $\zeta_n := \min\{q(\lambda_1), q_{\min}\}$ and $(q(\boldsymbol{A}))_{uv} = 0$ for any $u, v \in C$. Hence, the weight-quotient matrix of $q(\boldsymbol{A})$, denoted by $\boldsymbol{B}_q$, with respect to the partition $V = C \cup \overline{C}$ is

$$\boldsymbol{B}_q = \begin{pmatrix} 0 & \frac{\|\boldsymbol{\rho} C\|}{\|\boldsymbol{\rho}\overline{C}\|} \\ \frac{\|\boldsymbol{\rho} C\|}{\|\boldsymbol{\rho}\overline{C}\|} & \frac{\|\boldsymbol{\rho}\overline{C}\|^2 - \|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\rho}\overline{C}\|^2} \end{pmatrix} q(\lambda_1) \tag{10}$$

(compare with the matrix $\boldsymbol{B}$ in Eq. (4)), with eigenvalues $\mu_1 = q(\lambda_1)$ and $\mu_2 = -q(\lambda_1)\|\boldsymbol{\rho} C\|^2/(\|\boldsymbol{v}\|^2 - \|\boldsymbol{\rho} C\|^2)$ satisfying

$$0 > \frac{-q(\lambda_1)\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{v}\|^2 - \|\boldsymbol{\rho} C\|^2} \geqslant \zeta_n = q_{\min}, \tag{11}$$

where we have used again Lemma 2.3, and the hypotheses $q(\lambda_1) > 0$, $\|\boldsymbol{v}\| > \|\boldsymbol{\rho} C\| > 0$. Hence the first statement follows. Furthermore, if we get equality for some set C, we have $q(\lambda_1) = -q_{\min}\|\boldsymbol{\rho}\overline{C}\|^2/\|\boldsymbol{\rho} C\|^2$ and the interlacing is tight: $\boldsymbol{S}\boldsymbol{B}_q = q(\boldsymbol{A})\boldsymbol{S}$. But $\boldsymbol{S}$ consists of the two (column) vectors $(1/\|\boldsymbol{\rho} C\|)\boldsymbol{\rho} C$ and $(1/\|\boldsymbol{\rho}\overline{C}\|)\boldsymbol{\rho}\overline{C}$, so that the above matrix equation reads:

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\|\boldsymbol{\rho} C\|}\boldsymbol{\rho} C & \frac{1}{\|\boldsymbol{\rho}\overline{C}\|}\boldsymbol{\rho}\overline{C} \end{pmatrix} \begin{pmatrix} 0 & \frac{\|\boldsymbol{\rho}\overline{C}\|}{\|\boldsymbol{\rho} C\|} \\ \frac{\|\boldsymbol{\rho}\overline{C}\|}{\|\boldsymbol{\rho} C\|} & \frac{\|\boldsymbol{\rho}\overline{C}\|^2 - \|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\rho} C\|^2} \end{pmatrix} (-q_{\min}) \\ &\quad = q(\boldsymbol{A}) \begin{pmatrix} \frac{1}{\|\boldsymbol{\rho} C\|}\boldsymbol{\rho} C & \frac{1}{\|\boldsymbol{\rho}\overline{C}\|}\boldsymbol{\rho}\overline{C} \end{pmatrix} \end{aligned}$$

giving $\boldsymbol{\rho}\overline{C}(-q_{\min}) = q(\boldsymbol{A})\boldsymbol{\rho} C$, as claimed. □

From the above proof, note that, if $q_{\min} \geqslant 0$, then Eq. (11) gives a contradiction and hence it must be $\alpha^*_{\mathrm{o}k} = 0$. This implies the existence of an odd closed walk of length at most k through any vertex and, therefore, $g_{\mathrm{o}} \leqslant k$. From these facts, we easily deduce that, if Γ is not bipartite ($\lambda_n \neq -\lambda_1$), then $g_{\mathrm{o}} \leqslant d^\star$, where $d^\star$ is the number of points of the symmetrized mesh $\mathscr{M}^\star$ defined above. (Just let q be any polynomial having such points as its roots and taking positive value at λ_1.)

As another consequence of Theorem 4.1, and reasoning as in the previous section, we can now derive an upper bound for a chromatic-like number, which we could call the "odd-k-chromatic number." Let Γ be a graph with n vertices and odd girth g_{o}. For each odd integer k, $1 \leqslant k < g_{\mathrm{o}}$, the *odd-k-chromatic number* of Γ, denoted by $\chi_{\mathrm{o}k} = \chi_{\mathrm{o}k}(\Gamma)$, is the minimum number of colors that can be assigned to the vertices of Γ in such a way that any two (not necessarily different) vertices having the same color are at odd distance greater than k from each other. Notice that, with this definition,

$$\chi_{\mathrm{o}1}(\equiv \chi) \leqslant \chi_{\mathrm{o}3} \leqslant \cdots \leqslant \chi_{\mathrm{o}g_{\mathrm{o}}-2} \leqslant n$$

(if Γ is bipartite, $\chi_{\mathrm{o}k} = 2$ for any odd $k \geqslant 1$). In other words, we can say that $\chi_{\mathrm{o}k}$ is the minimum number of odd-k-independent sets into which V can be partitioned. Within this framework, the following result could be seen as a generalization of Hoffman's bound (5).

Corollary 4.2. *Let Γ be a graph with odd girth g_o and eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Let q be a polynomial of degree k as above. Then, for any odd integer k, $1 \leqslant k < g_o$, the odd-k-chromatic number satisfies*

$$\chi_{ok} \geqslant 1 - \frac{q(\lambda_1)}{q_{\min}}. \tag{12}$$

Proof. Since $k < g_o$, we have $\alpha^*_{ok} \geqslant 1$. Then Theorem 4.1 applies and the result follows from $\chi_{ok} \geqslant \|\boldsymbol{v}\|^2/\alpha^*_{ok}$. □

In particular, if we take $q(x) = (x/-\lambda_n)^k$, with $q_{\min} = -1$, we get

$$\chi_{ok} \geqslant 1 - \left(\frac{\lambda_1}{\lambda_n}\right)^k, \tag{13}$$

a result to be compared with Eq. (5).

Of course, we can do better if we look for the (odd) polynomials, with degree at most k, that maximize the quotient $q(\lambda_1)/(-q_{\min})$ or, alternatively, we can try to maximize $q(\lambda_1)$ among the polynomials that satisfy $q_{\min} \geqslant -1$. These polynomials were studied with some detail in Ref. [14]. Also, a method to compute them, based on solving a linear programming problem, was proposed. They will be referred to as the odd polynomials and denoted by Q_k. To discuss some of their properties, it is better to consider only the distinct eigenvalues of the graph: $\operatorname{ev}\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$. As before, set $\mathcal{M} = \operatorname{ev}\Gamma \setminus \{\theta_0\}$ and consider the symmetrized mesh $\mathcal{M}^\star = \mathcal{M} \cup \{0\} \cup (-\mathcal{M})$, with $d^\star := |\mathcal{M}^\star|$ points. Then, for any odd integer k, $1 \leqslant k \leqslant d^\star - 2$, the *odd polynomial* Q_k satisfies

$$Q_k(\theta_0) = \max_{q \in \mathbb{R}_k^-[\theta]} \{q(\theta_0) : q_{\min} \geqslant -1\}, \tag{14}$$

where $\mathbb{R}_k^-[\theta]$ stands for the set of real polynomials with only odd powers and degree at most k, and $q_{\min} = \min_{\theta \in \mathcal{M}}\{q(\theta)\}$. If Γ is not bipartite ($\pm\theta_0 \notin \mathcal{M}^\star$) it was shown that there is a unique odd polynomial of degree k, satisfying $(Q_k)_{\min} = -1$, and

$$Q_1(\theta_0) < Q_3(\theta_0) < \cdots < Q_{d^\star-2}(\theta_0). \tag{15}$$

In particular, the extremal cases $k = 1$ and $k = d^\star - 2$ admit closed expressions. Namely, $Q_1(x) = x/-\theta_d$, and $Q_{d^\star-2}$ being the polynomial which takes alternating values ± 1 at $\mathcal{M}^\star \setminus \{0\} = \{\vartheta_1 > \vartheta_2 > \cdots > \vartheta_{d^\star}\}$. This gives (using Lagrange interpolation in the second case)

$$Q_1(\theta_0) = \frac{\theta_0}{-\theta_d}, \qquad Q_{d^\star-2}(\theta_0) = \sum_{i=1}^{d^\star} \frac{\pi_0}{\pi_i}, \tag{16}$$

where $\pi_i := \prod_{j=0, j\neq i} |\vartheta_i - \vartheta_j|$ $(\vartheta_0 = \theta_0)$.

Then, in terms of these polynomials and using the new notation for the eigenvalues, Theorem 4.1 reads

$$\alpha^*_{ok} \leqslant \frac{\|\boldsymbol{v}\|^2}{1+Q_k(\theta_0)} \tag{17}$$

and, in the case of equality for some vertex subset C,

$$Q_k(A)\rho C = \rho\overline{C}; \tag{18}$$

whereas Corollary 4.2 yields

$$\chi_{ok} \geqslant 1 + Q_k(\theta_0). \tag{19}$$

Example 4.3. *Let $\Gamma = O_4$, the (regular) "odd graph" with degree* 4, 35 *vertices, and eigenvalues* $\operatorname{ev} O_4 = \{4 > 2 > -1 > -3\}$. *(The odd graph O_k has the $(k-1)$-subsets of a $(2k-1)$-subset as vertices, and two vertices are adjacent iff their corresponding subsets are disjoint; see Refs.* [3,4].*) Then the corresponding symmetrized mesh is $\mathcal{M}^\star = \{0, \pm 1, \pm 2, \pm 3\}$ and hence $g_o \leqslant D_o \leqslant 7$ (in fact $g_o = 7$). The corresponding odd polynomials and their values at $\theta_0 = 4$ are:*

- $Q_5(x) = \frac{1}{12}(x^5 - 11x^3 + 22x)$, 34;
- $Q_3(x) = \frac{1}{6}(x^3 - 7x)$, 6;
- $Q_1(x) = \frac{1}{3}x$, 4/3.

Hence, the respective bounds for the odd-k-independence numbers, given by (17), turn out to be $\alpha_{o1} = \alpha \leqslant 15$, $\alpha_{o3} \leqslant 5$, and $\alpha_{o5} \leqslant 1$. In fact, all these bounds are tight, as can be easily shown by using the known formulas for the distances between vertices in the odd graphs (see Ref. [3]).

When Γ is bipartite we have $Q_k(\theta_0) = -Q_k(\theta_d) = 1$ for any k, and Eq. (17) yields $\alpha^*_{ok} \leqslant \|\boldsymbol{v}\|^2/2$, as expected. In the case of regular non-bipartite connected graphs, and since $\alpha_{og_o-2} \geqslant 1$, Eq. (17) gives the following result.

Corollary 4.4. *The order n of a non-bipartite regular connected graph Γ, with eigenvalues $\operatorname{ev}\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$ and odd girth g_o, satisfies the bound*

$$n \geqslant Q_{g_o-2}(\theta_0) + 1, \tag{20}$$

where Q_{g_o-2} is the odd $(g_o - 2)$-polynomial.

From the example above, note that the bound (20) is tight for O_4. In fact, using the value of $Q_{d^\star-2}(\theta_0)$ given in Eq. (16), it can be shown that this property is shared by all odd graphs O_k. Notice also that Eq. (20) still holds if we replace g_o by the standard girth g (since $g_o \geqslant g$ and the odd polynomials satisfy Eq. (15)).

The next straightforward consequence of Theorem 4.1 is also given in terms of the odd polynomials. Let Γ be a graph on n vertices. Given any integer $1 \leqslant t \leqslant n$, let us define the *odd t-diameter* of a graph Γ as

$$D_{ot} := \max_{U \subset V}\{\min_{u,v \in U} \partial_o(u, v) \colon |U| = t\}$$

so that the following inequalities hold:

$$D_o \geqslant D_{o1} \geqslant D_{o2} \geqslant \cdots \geqslant D_{on} (= 1),$$

where D_o is the above-mentioned odd diameter.

Corollary 4.5. *Let Γ be a graph as above, with odd polynomials Q_k. Then*

$$Q_k(\theta_0) > \frac{\|\boldsymbol{v}\|^2}{t} - 1 \quad \Rightarrow \quad D_{ot} \leqslant k. \tag{21}$$

Proof. Under the hypothesis, Eq. (17) gives $\alpha^*_{ok} < t$. Consequently, between any t vertices, there must be some walk of odd length $\leqslant k$ (perhaps between a vertex and itself). □

5. The weight set independence numbers

In this section we study another generalization of the concept of independence, which concerns the elements considered (sets instead of single vertices) rather than the type of distance involved. Indeed, we can extend the notion of k-independence to vertex subsets if we require that they must be at distance greater than k from each other. We will first suppose that all such subsets have the same weight. Afterwards, we shall pay attention to the simplest case of (two) subsets with different weights.

5.1. Subsets with equal weights

Assume that the graph Γ, on n vertices, has some vertex subset U with weight $w := \|\rho U\|^2$. (Note that, if $U = \{u\}$, the "weight" of vertex u is now v_u^2.) Then, given some integer $k \geqslant 0$, we define the (w, k)-*independence number*, denoted by α_k^w, as the maximum number of k-independent subsets with common weight w. As in the standard notion of independence, we assume that the set U is k-independent from itself, so that $\alpha_k^w \geqslant 1$. Notice that, when Γ is regular, α_k^1 is the maximum number of vertices which are mutually at distance greater than k. This parameter, denoted just by α_k, has been recently considered in the literature by Delorme and Tillich [13], and Garriga, Yebra and the author [15,16,21], and it is called the *k-independence number*. Thus, $\alpha_0 = n$, $\alpha_1 \equiv \alpha$, and α_k is, in fact, the independence number of the kth power of Γ (that is the graph

$\Gamma_{\leqslant k}$ with vertex set V and where two vertices are adjacent whenever their distance in Γ is at most k). In order to give bounds for α_k, it is useful to consider the so-called alternating polynomials, introduced in Ref. [18], which can be thought of as the discrete version of the Chebychev polynomials. As above, let Γ be a graph with $\mathrm{ev}\,\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$. For any integer $0 \leqslant k \leqslant d-1$, the *k-alternating polynomial* P_k, is the (unique) polynomial of degree k satisfying

$$P_k(\theta_0) = \max_{p\in\mathbb{R}_k[x]}\{p(\theta_0)\colon \|p\|_\infty \leqslant 1\}, \tag{22}$$

where $\|p\|_\infty := \max_{1\leqslant i\leqslant d}|p(\theta_i)|$. Thus, we obviously have $P_0 = 1$. Otherwise, for $k \geqslant 1$, it was proved in Ref. [18] that the k-alternating polynomial is characterized by taking $k+1$ alternating values ± 1 at $\mathrm{ev}\,\Gamma\setminus\{\theta_0\}$, with $P_k(\theta_1) = 1$ and $P_k(\theta_d) = (-1)^k$. Moreover,

$$1 < P_1(\theta_0) < P_2(\theta_0) < \cdots < P_{d-1}(\theta_0). \tag{23}$$

In particular, for the values $k = 1$ and $k = d-1$, the above characterization gives $P_1(x) = 2(x - \theta_1/\theta_1 - \theta_d) + 1$, and the $(d-1)$-alternating polynomial is defined by $P_{d-1}(\theta_i) = (-1)^{i+1}$, $1 \leqslant i \leqslant d$. Thus, (using again Lagrange interpolation) we get

$$P_1(\theta_0) = 2\frac{\theta_0 - \theta_1}{\theta_1 - \theta_d} + 1, \quad P_{d-1}(\theta_0) = \sum_{i=1}^{d}\frac{\pi_0}{\pi_i}, \tag{24}$$

where $\pi_i := \prod_{j=0,j\neq i}^{d}|\theta_i - \theta_j|$, $0 \leqslant i \leqslant d$. Some particular cases of these polynomials were also considered by Van Dam and Haemers in Ref. [12]. In fact, as noted by Van Dam [10], they had already been considered in the theory of uniform approximations of continuous functions.

In terms of the alternating polynomials, the author [15] showed that, for a regular connected graph Γ on n vertices, the k-independence number is bounded above by

$$\alpha_k \leqslant \frac{2n}{P_k(\theta_0) + 1}. \tag{25}$$

In the next theorem the above result is generalized by giving a similar bound for α_k^w of any (connected) graph. The case $w > \|\boldsymbol{v}\|^2/2$ can be excluded since then $\alpha_k^w = 1$ for any k.

Theorem 5.1. *Let Γ be a connected graph with eigenvalues* $\mathrm{ev}\,\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$, *positive eigenvector* $\boldsymbol{v}$, *and k-alternating polynomials* P_k, $0 \leqslant k \leqslant d-1$. *Assume that, for some weight* $w \geqslant 1$, the *(w,k)-independence number satisfies* $\alpha_k^w \geqslant 2$. *Then*

$$\alpha_k^w \leqslant \frac{2\|\boldsymbol{v}\|^2}{w(P_k(\theta_0) + 1)}. \tag{26}$$

Proof. We can suppose that $k \geqslant 1$ since, otherwise, $P_0 = 1$ and the result trivially holds. Then, let $r := \alpha_k^w < \|\boldsymbol{v}\|^2/w$, and assume that U_i, $1 \leqslant i \leqslant r$, are some k-independent sets with common weight $w = \|\boldsymbol{\rho} U_i\|^2$. Take the polynomial $q := (r/2)P_k + (r-2)/2$, which satisfies $-1 \leqslant q(\theta_i) \leqslant r-1$ for any $1 \leqslant i \leqslant d$. Then, as $P_k(\theta_0) > 1$ and Γ is connected, the matrix $q(\boldsymbol{A}(\Gamma))$ has eigenvalues $\{-1 < \cdots < r-1 < q(\theta_0)\}$ and $q(\theta_0)$ has multiplicity 1. Moreover, the complete graph K_r with vertex set $\{1, 2, \ldots, r\}$ has eigenvalues $\operatorname{ev} K_r = \{-1 < r-1\}$. Consequently, the matrix $\boldsymbol{K}$ obtained as the Kronecker product $\boldsymbol{A}(K_r) \otimes q(\boldsymbol{A}(\Gamma))$ has eigenvalues

$$\{-q(\theta_0) < -(r-1) < \cdots < (r-1)^2 < (r-1)q(\theta_0)\}.$$

Let us now consider the partition $\bigcup_{i=1}^{r}[(i, U_i) \cup (i, \overline{U_i})]$ of (the rows and columns of) $\boldsymbol{K}$. Since, for $i \neq j$, $\partial(U_i, U_j) > k$ in Γ, the weight-quotient matrix with respect to such a partition turns out to be again a Kronecker product, namely $\boldsymbol{B} := \boldsymbol{A}(K_r) \otimes \boldsymbol{B}_q$, with $\boldsymbol{B}_q$ as in Eq. (10) – that is, the weight-quotient matrix of $q(\boldsymbol{A}(\Gamma))$ with respect to any partition $U_i \cup \overline{U_i}$ – with eigenvalues $\{-q(\theta_0)w/(\|\boldsymbol{v}\|^2 - w) < q(\theta_0)\}$. Therefore, $\boldsymbol{B}$ has eigenvalues

$$\left\{-q(\theta_0) < -\frac{(r-1)w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0) < \frac{w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0) < (r-1)q(\theta_0)\right\}.$$

Thus, since for both matrices $\boldsymbol{K}$ and $\boldsymbol{B}$ the minimum eigenvalue $-q(\theta_0)$ has multiplicity $r-1$ (that is the multiplicity of -1 as eigenvalue of $\boldsymbol{A}(K_r)$), we have, by Lemma 2.3, that their rth smallest eigenvalues satisfy

$$-(r-1) \leqslant -\frac{(r-1)w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0).$$

Hence, using the expression for q,

$$\frac{r}{2}(P_k(\theta_0) + 1) - 1 = q(\theta_0) \leqslant \frac{\|\boldsymbol{v}\|^2}{w} - 1 \tag{27}$$

whence we get Eq. (26). □

As in the previous section, let us now give some straightforward consequences of the above theorem. First, from its proof we get the following simple corollary.

Corollary 5.2. *Assume that the connected graph Γ has a vertex subset U with weight $w_U \geqslant w$, for some $1 \leqslant w < \|\boldsymbol{v}\|^2$. Then, if $P_k(\theta_0) > (\|\boldsymbol{v}\|^2/w) - 1$, all the other subsets with weight w_U, if any, are at distance at most k from U.*

Proof. From the hypotheses, we have $P_k(\theta_0) > \|\boldsymbol{v}\|^2/w_U - 1$. Consequently, Eq. (27) gives $r < 2$, a contradiction. Hence, it must be $\alpha_k^{w_U} = 1$ and the result follows. $\square$

Consider now the specialization of the above results to regular graphs. In this case, for any integer w, $1 \leqslant w < n$, we can consider subsets of any weight (cardinality) w. Then, with the notation $\alpha_k \equiv \alpha_k^1$, we clearly have $\alpha_k^w \geqslant \lfloor \alpha_k / w \rfloor$. Moreover, Theorem 5.1 gives:

Corollary 5.3. *Let Γ be a δ-regular connected graph on n vertices,* $\operatorname{ev}\Gamma = \{\theta_0(=\delta) > \theta_1 > \cdots > \theta_d\}$, *and k-alternating polynomials P_k, $0 \leqslant k \leqslant d-1$. Then, for any integer w, $1 \leqslant w < n$, the (w,k)-independence number satisfies*

$$\alpha_k^w \leqslant \max\left\{1, \frac{2n}{w(P_k(\theta_0)+1)}\right\}. \tag{28}$$

In particular, if Γ contains at least two (1-)independent w-sets, then α_1^w satisfies the bound

$$\alpha_1^w \leqslant \frac{n(\theta_1 - \theta_d)}{w(\theta_0 - \theta_d)}, \tag{29}$$

where we have used the value of $P_1(\theta_0)$ in Eq. (24). Notice that, for all non-complete connected graphs with $\theta_1 > 0$ (that is, those different from the complete multipartite graphs), the bound for the independence number α ($\geqslant$ $\geqslant 2$) obtained by taking $w = 1$ in Eq. (29) is worse than $\alpha \leqslant n(-\theta_d)/(\theta_0 - \theta_d)$, given in Eq. (6). Another particular case of Eq. (28) worth mentioning is the following upper bound for the maximum number of w-sets of a regular connected graph which are pairwise at (spectrally maximum) distance d.

$$\alpha_{d-1}^w \leqslant \max\left\{1, \frac{2n}{w\sum_{i=0}^{d} \pi_0/\pi_i}\right\}$$

where we have used the value of $P_{d-1}(\theta_0)$ in Eq. (24).

Assume now that the (not necessarily regular) graph Γ has at least $t \geqslant 2$ vertex subsets $U_1, \ldots, U_t$ with the same weight w, say. Then, we can define the *(w,t)-diameter by*

$$D_t^w := \max_{U_1,\ldots,U_t \subset V}\left\{\min_{1 \leqslant i < j \leqslant t} \partial(U_i, U_j) : \|\boldsymbol{\rho} U_i\|^2 = w, 1 \leqslant i \leqslant t\right\}. \tag{30}$$

Thus, if Γ is regular, the (w,t)-diameter coincides with the parameter $D_{t\times w}$, studied by Garriga and the author in Ref. [16] (there we consider the minimum distance between families of t subsets on w vertices). Similarly, the t-diameter D_t considered by Chung et al. [7] corresponds to D_t^1 (that is, the largest pairwise minimum distance of a set of t vertices), so that the (standard) diameter is just D_2^1.

Now, Theorem 5.1 gives the following result which can be seen as an extension of Corollary 5.2 (the case $t = 2$).

Corollary 5.4. *Let Γ be a graph as above containing at least $t \geqslant 2$ vertex subsets U with weight $w = \|\boldsymbol{\rho} U\|^2$. Then*

$$P_k(\theta_0) > \frac{2\|\boldsymbol{v}\|^2}{wt} - 1 \quad \Rightarrow \quad D_t^w \leqslant k. \tag{31}$$

When Γ is a regular graph on n vertices (with $\binom{n}{w} \geqslant t$) we have the following result concerning subsets of w vertices:

$$P_k(\theta_0) > \frac{2n}{wt} - 1 \quad \Rightarrow \quad D_t^w \leqslant k. \tag{32}$$

The particular case $w = 1$ was proved in Ref. [15] by using a different technique. Notice that the above results still hold if we replace P_k by the Chebychev polynomial T_k "shifted" from $[-1, 1]$ to $[\theta_d, \theta_1]$, that is $T_k^*(x) := T_k((2x - \theta_1 - \theta_d)/(\theta_1 - \theta_d))$ (since $\|T_k^*\|_\infty = \|P_k\|_\infty = 1$ and $P_k(\theta_0) \geqslant T_k^*(\theta_0)$). Then, using that $T_k(x) = \cosh(k \cosh^{-1} x)$, Eq. (32) yields

$$D_t^w \leqslant \left\lfloor \frac{\cosh^{-1}(2n/wt - 1)}{\cosh^{-1}((2\theta_0 - \theta_1 - \theta_d)/(\theta_1 - \theta_d))} \right\rfloor + 1. \tag{33}$$

A result to be compared with that given by Kahale [28], who proved that if Γ is a regular connected graph on n vertices, and $\vartheta_1 (= \theta_0), \vartheta_2, \ldots, \vartheta_n$ represent its eigenvalues with absolute value in non-increasing order, $|\vartheta_1| > |\vartheta_2| \geqslant \cdots \geqslant |\vartheta_n|$, then

$$D_t^w \leqslant \left\lceil \frac{\cosh^{-1}(n/w - 1)}{\cosh^{-1}(\vartheta_1/|\vartheta_t|)} \right\rceil + 1. \tag{34}$$

5.2. Subgraphs with different weights

When we consider vertex subsets with different weights, we can still apply the same techniques as above. However the complexity of the analysis steadily (dramatically) increases with the number of sets considered. By way of example, we analyze below the simplest case of two subsets. In this context, the following theorem was also proved in Ref. [17] without using eigenvalue interlacing. The corresponding results for either regular graphs or Laplacian spectrum were also proved by Van Dam and Haemers [12] and Van Dam [11], respectively.

Theorem 5.5. *Let Γ be a connected graph with eigenvalues* $\operatorname{ev}\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$ *and k-alternating polynomials* P_k, $0 \leqslant k \leqslant d-1$. *Let* X, Y *be two subsets of vertices such that* $\partial(X,Y) > k$. *Then,*

$$P_k(\theta_0) \leqslant \frac{\|\rho\overline{X}\|\|\rho\overline{Y}\|}{\|\rho X\|\|\rho Y\|}.$$

Proof. Since $\partial(X,Y) \geqslant 1$, we have $\|\rho X\| \leqslant \|\rho\overline{Y}\|$ and $\|\rho Y\| \leqslant \|\rho\overline{X}\|$, and hence the result is trivial for $k = 0$. The proof for $k \geqslant 1$ is similar to that of Theorem 5.1, but taking $r = 2$. Then, the polynomial q is just P_k and hence we consider the matrix $\boldsymbol{K} := A(K_2) \otimes P_k(A(\Gamma))$ with eigenvalues $\pm P_k(\theta_0), \ldots, \pm P_k(\theta_d)$, satisfying $|P_k(\theta_i)| \leqslant \|P_k\|_\infty = 1 < P_k(\theta_0)$, for $1 \leqslant i \leqslant d$, and $P_k(\theta_0)$ having multiplicity 1. Moreover, the weight-quotient matrix of $\boldsymbol{K}$, with respect to the partition $(1,X) \cup (1,\overline{X}) \cup (2,Y) \cup (2,\overline{Y})$, is now

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 & \frac{\|\rho X\|}{\|\rho\overline{Y}\|} \\ 0 & 0 & \frac{\|\rho Y\|}{\|\rho\overline{X}\|} & \frac{\|\rho\overline{X}\|^2 - \|\rho Y\|^2}{\|\rho\overline{X}\|\|\rho\overline{Y}\|} \\ 0 & \frac{\|\rho Y\|}{\|\rho\overline{X}\|} & 0 & 0 \\ \frac{\|\rho X\|}{\|\rho\overline{Y}\|} & \frac{\|\rho\overline{Y}\|^2 - \|\rho X\|^2}{\|\rho\overline{Y}\|\|\rho\overline{X}\|} & 0 & 0 \end{pmatrix} P_k(\theta_0)$$

with eigenvalues $\pm P_k(\theta_0)$ and $\pm P_k(\theta_0)\|\rho X\|\|\rho Y\|/\|\rho\overline{X}\|\|\rho\overline{Y}\|$. Hence, the result follows from

$$P_k(\theta_0)\frac{\|\rho X\|\|\rho Y\|}{\|\rho\overline{X}\|\|\rho\overline{Y}\|} \leqslant \|P_k\|_\infty = 1. \quad \square$$

Some consequences of this theorem, together with the study of the case in which equality is attained, can be found in Refs. [17,20]. For instance, a straightforward reasoning gives an upper bound for the so-called *conditional* (s, t)*-diameter* of Γ, defined in Ref. [1] by

$$D_{(s,t)} = \max_{U_1,U_2 \subset V} \{\partial(U_1,U_2) : |U_1| = s, |U_2| = t\}$$

for some integers $1 \leqslant s,t \leqslant n$. Namely,

$$P_k(\theta_0) > \sqrt{\left(\frac{\|\boldsymbol{v}\|^2}{s} - 1\right)\left(\frac{\|\boldsymbol{v}\|^2}{t} - 1\right)} \quad \Rightarrow \quad D_{(s,t)} \leqslant k.$$

(See also Ref. [18] for the case $s = t = 1$ corresponding to the standard diameter.) Using weights instead of cardinalities we also get another generalization of Corollary 5.2, which, roughly speaking, tell us that vertex subsets with large weight tend to be close together. More precisely if $P_k(\theta_0) >$

$\|\boldsymbol{v}\|^2/w - 1$, then all subsets with weight at least w are at most k apart from each other.

6. Subgraphs

For a given integer $k \geqslant 0$, let us consider the graph $\Gamma_{>k}$ with the same vertex set as Γ, and where two vertices are adjacent iff their distance apart in Γ is greater than k. In other words, if $k \geqslant 1$, $\Gamma_{>k}$ is the complement of $\Gamma_{\leqslant k}$, the kth power of Γ, and $\Gamma_{>0} = K_n$. The next result can be seen as a generalization of Theorem 5.1.

Theorem 6.1. *Let Γ be a connected graph with eigenvalues* $\operatorname{ev}\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$, *positive eigenvector* $\boldsymbol{v}$, *and k-alternating polynomials P_k, $0 \leqslant k \leqslant d-1$. Let H be a (non-trivial) subgraph of the complement of the k-th power of Γ, $H \subseteq \Gamma_{>k}$, with all its vertices u having equal weight $w = v_u^2 \leqslant \|\boldsymbol{v}\|^2/2$ (in Γ), and eigenvalues* $\operatorname{ev} H = \{\eta_0 > \eta_1 > \cdots > \eta_e\}$. *Then*

$$1 - \frac{\eta_0}{\eta_e} \leqslant \frac{2\|\boldsymbol{v}\|^2}{w(P_k(\theta_0) + 1)}. \tag{35}$$

Proof. Reason as in the proof of Theorem 5.1, but using now the polynomial $q := ((\eta_0 - \eta_e)/(2)P_k + (\eta_0 + \eta_e)/2$, with $\eta_e \leqslant q(\theta_i) \leqslant \eta_0 \leqslant q(\theta_0)$, $1 \leqslant i \leqslant d$. Moreover, suppose that $k \geqslant 1$ (the extreme case $k = 0$ is proved similarly). Then the matrix $q(\boldsymbol{A}(\Gamma))$ has maximum eigenvalue $q(\theta_0) > \eta_0$, with multiplicity 1. Hence the matrix $\boldsymbol{K} := \boldsymbol{A}(H) \otimes q(\boldsymbol{A}(\Gamma))$ has eigenvalues

$$\{q(\theta_0)\eta_e < q(\theta_0)\eta_{e-1} < \cdots < q(\theta_0)\eta_{e-i} \leqslant \eta_e\eta_0 < \cdots < q(\theta_0)\eta_0\} \tag{36}$$

for some $0 \leqslant i < e$; whereas its weight-quotient matrix $\boldsymbol{B} := \boldsymbol{A}(H) \otimes \boldsymbol{B}_q$ – with $\boldsymbol{B}_q$ being the weight-quotient matrix of $q(\boldsymbol{A}(\Gamma))$ with respect to any partition $\{u\} \cup (V \setminus u)$ – has eigenvalues

$$\begin{Bmatrix} q(\theta_0)\eta_e < q(\theta_0)\eta_{e-1} < \cdots q(\theta_0)\eta_{e-j} \\ \leqslant -\dfrac{\eta_0 w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0) < \cdots < q(\theta_0)\eta_0 \end{Bmatrix} \tag{37}$$

for some $0 \leqslant j < e$. Furthermore, note that the multiplicity of $q(\theta_0)\eta_h$, $0 \leqslant h \leqslant e$, in Eqs. (36) and (37) is the same, and coincides with the multiplicity of η_h in $\boldsymbol{A}(H)$. Then, assuming that $j \geqslant i$, the eigenvalue inequality (coming again from Lemma 2.3)

$$\eta_e\eta_0 \leqslant -\frac{\eta_0 w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0)$$

gives the result. Now, it only remains to show that the other case is impossible. Indeed, if $j < i$, the same lemma would give

$$q(\theta_0)\eta_{e-(j+1)} \leqslant -\frac{\eta_0 w}{\|\boldsymbol{v}\|^2 - w} q(\theta_0)$$

contradicting Eq. (37). □

From this result we can derive a number of consequences. For instance, notice that if we take $H = K_r$, with $r = \alpha_k^w > 1$, then $1 - \eta_0/\eta_e = r$ and Eq. (35) becomes the bound (26) for the (w,k)-independence number. As other examples, we next give some results, for regular graphs and distance-regular graphs, considering the extreme cases $k = 0, 1$ and $k = d - 1$, respectively. First, if Γ is regular and $k = 0$, we can take $H = \Gamma$ and Theorem 6.1 – with $w = 1$ and $P_0(\theta_0) = 1$ – gives again $1 - \theta_0/\theta_d \leqslant n$. Still in the regular case, but taking now $k = 1$, we also have the following corollary.

Corollary 6.2. *Let Γ be a regular graph with n vertices and* $\operatorname{ev}\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$, $d \geqslant 2$, *such that both Γ and its complement $\overline{\Gamma}$ are connected. Then*

$$\frac{\theta_0 - \theta_d}{\theta_1 - \theta_d} \leqslant \frac{n}{n - (\theta_0 - \theta_1)} \min\{-\theta_d, \theta_1 + 1\}.$$

Proof. Let $H = \Gamma_{>1} \equiv \overline{\Gamma}$. Then $\operatorname{ev} H = \{\eta_0 > \eta_1 > \cdots > \eta_d\}$ with $\eta_0 = n - \theta_0 - 1$ and $\eta_i = -\theta_{d-i+1} - 1$, $1 \leqslant i \leqslant d$. Thus, using Eq. (35) with $w = 1$ and the value of $P_1(\theta_0)$ given in Eq. (24), we get

$$1 + \frac{\theta_0 - \theta_1}{\theta_1 - \theta_d} = \frac{\theta_0 - \theta_d}{\theta_1 - \theta_d} \leqslant \frac{n(\theta_1 + 1)}{n - (\theta_0 - \theta_1)}.$$

Similarly, interchanging the roles of Γ and $\overline{\Gamma}$, we obtain

$$\frac{\theta_0 - \theta_d}{\theta_1 - \theta_d} \leqslant \frac{n(-\theta_d)}{n - (\theta_0 - \theta_1)},$$

and the result follows. □

Let us assume now that Γ is a distance-regular graph (see Refs. [4] or [6]). In this case, Theorem 6.1 can be used to derive the following upper bound for the multiplicity of some of its eigenvalues.

Corollary 6.3. *Let Γ be a distance-regular graph on n vertices and with eigenvalues* $\mathrm{ev}\,\Gamma = \{\theta_0 > \theta_1 > \cdots > \theta_d\}$. *Then the multiplicity of some eigenvalue θ_i with odd index i satisfies the bound*

$$m(\theta_i) \leqslant \frac{\pi_0}{\pi_i}\left(\frac{2n}{\sum_{j=0}^{d}\frac{\pi_0}{\pi_j}} - 1\right), \tag{38}$$

where $\pi_j := \prod_{k=0,k\neq j} |\theta_j - \theta_k|$.

Proof. Apply Theorem 6.1 with $H = \Gamma_d$ $(k = d-1)$ and $w = 1$. Then, if p_d denotes the distance-d polynomial, satisfying $p_d(A) = A(\Gamma_d)$, we have $\eta_0 = p_d(\theta_0)$ and $\eta_e = p_{d_{\min}} := \min_{1\leqslant i\leqslant d} p_d(\theta_i)$. Moreover, by Eq. (24), $P_{d-1}(\theta_0) + 1 = \sum_{j=0}^{d}(\pi_0/\pi_j)$. Consequently, Eq. (35) becomes

$$\frac{p_d(\theta_0)}{-p_{d_{\min}}} \leqslant \frac{2n}{\sum_{j=0}^{d}\pi_0/\pi_j} - 1. \tag{39}$$

Then the result follows from the known formula for the multiplicities of a distance-regular graph in terms of p_d (see, for instance, Ref. [2]), namely

$$m(\theta_i) = (-1)^i \frac{\pi_0 p_d(\theta_0)}{\pi_i p_d(\theta_i)} \quad (1 \leqslant i \leqslant d), \tag{40}$$

and the fact that, for some odd i, $1 \leqslant i \leqslant d$, we must have $p_{d_{\min}} = p_d(\theta_i)$. $\square$

This result is best possible in the sense that, for some distance-regular graphs, some of the "odd multiplicities" equal the upper bound in Eq. (38). In fact, we have examples where all of such multiplicities equal the bound. Indeed, if Γ is an r-antipodal distance-regular graph (see Ref. [4]), it was shown in Refs. [19,21] that $P(\theta_0) + 1 = \sum_{j=0}^{d}\pi_0/\pi_j = 2n/r$. Then, Corollary 6.3 assures that, for some odd index i, $1 \leqslant i \leqslant d$, we have $m(\theta_i) \leqslant (r-1)\pi_0/\pi_i$ but, using that $\mathrm{ev}\,\Gamma_d = \{r-1 > -1\}$, it is easy to prove that, in fact, $m(\theta_i) = (r-1)\pi_0/\pi_i$ for every $i = 1, 3, \ldots$ (see Refs. [15,21] for more details). In fact, in these references it was proved that a distance-regular graph is r-antipodal, for some $r \geqslant 2$, if and only if the multiplicities of its eigenvalues are:

$$m(\theta_i) = \frac{\pi_0}{\pi_i} \quad (i \text{ even}), \qquad m(\theta_i) = (r-1)\frac{\pi_0}{\pi_i} \quad (i \text{ odd}). \tag{41}$$

Notice that, since the distance-d matrix $A(\Gamma_d)$ has positive eigenvector $\boldsymbol{j}$, with eigenvalue $p_d(\theta_0)$, then $p_d(\theta_0) \geqslant |p_d(\theta_i)|$ for any $1 \leqslant i \leqslant d$. Consequently, Eq. (40) gives the following general lower bound for the "even multiplicities"

$$m(\theta_i) \geqslant \frac{\pi_0}{\pi_i} \quad (i \text{ even}) \tag{42}$$

and the above example shows that this is also best possible. Also, a direct proof of Corollary 6.3 can be obtained from Eq. (42).

7. The Laplacian matrix

When we deal with a non-regular graph Γ, but still want to consider the cardinalities of the vertex subsets, rather than their weights, we can use the Laplacian matrix $\boldsymbol{L}$ of Γ. As commented in Section 2, this is because it always has the eigenvalue 0 with eigenvector $\boldsymbol{j}$ (the multiplicity of 0 being the number of connected components of Γ). Notice that $\boldsymbol{L}$ can be seen as the adjacency matrix of a weighted pseudograph, obtained from Γ by giving weight -1 to its edges and adding a loop with weight δ_i on each vertex v_i. Therefore, as when using the adjacency matrix $\boldsymbol{A}$ if the (distinct) vertices u, v are k-independent, then $(p(\boldsymbol{L}))_{uv} = 0$ for any polynomial p of degree k. This allows us to derive some results which are similar to those in the previous sections. For instance we next consider the analogues of the bounds given in Sections 3 and 5.

7.1. The independence number

Let Γ be a δ-regular graph on n vertices, with adjacency matrix eigenvalues $\lambda'_1 = \delta \geqslant \lambda'_2 \geqslant \cdots \geqslant \lambda'_n$. Since the Laplacian matrix of Γ is $\boldsymbol{L}(\Gamma) = \delta \boldsymbol{I} - \boldsymbol{A}(\Gamma)$, its Laplacian eigenvalues are $\lambda_i = \delta - \lambda'_i$, $1 \leqslant i \leqslant n$, and hence the Hoffman–Lovász' bound (6) becomes

$$\alpha \leqslant n\left(1 - \frac{\delta}{\lambda_n}\right). \tag{43}$$

In Ref. [33] Mohar extended this bound to the case of non-regular graphs by considering the degree average introduced below. In fact, as it is shown in the first part of the next theorem, Mohar's result can also be obtained reasoning as in the proof of Theorem 3.1. Let $\delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_n$ represent the degree sequence of Γ and set $\overline{\delta}_r := 1/r \sum_{i=1}^{r} \delta_i$, $1 \leqslant r \leqslant n$.

Theorem 7.1. *Let Γ be a graph on n vertices, with Laplacian eigenvalues $\lambda_1 = 0 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Then*

$$\alpha \leqslant n\left(1 - \frac{\overline{\delta}_\alpha}{\lambda_n}\right). \tag{44}$$

If the bound is attained for some independent set C, then Γ is δ-regular, with $\delta = \lambda_n(n - \alpha)/n$, and C is a completely regular code, with every vertex in $\overline{C}$ being adjacent to $\delta\alpha/(n - \alpha) = \lambda_n(\alpha/n)$ vertices of C.

Proof. Let $C \subset V$ be a maximum independent set, $\alpha = |C|$, with average degree $\overline{\delta}_C := (1/\alpha)\sigma_C = (1/\alpha)\sum_{u\in C}\delta_u$. Then, the weight-quotient matrix of $\boldsymbol{L}$ with respect to the partition $\mathscr{P}$: $V_1 \cup V_2 = C \cup \overline{C}$ is now

$$\boldsymbol{B} = \begin{pmatrix} \frac{1}{\alpha}\sigma_C & \frac{-\sigma_C}{\sqrt{\alpha(n-\alpha)}} \\ \frac{-\sigma_C}{\sqrt{\alpha(n-\alpha)}} & \frac{\sigma_C}{n-\alpha} \end{pmatrix} = \overline{\delta}_C \begin{pmatrix} 1 & \frac{-\sqrt{\alpha}}{\sqrt{n-\alpha}} \\ \frac{-\sqrt{\alpha}}{\sqrt{n-\alpha}} & \frac{\alpha}{n-\alpha} \end{pmatrix} \tag{45}$$

with eigenvalues $\mu_1 = 0$ and $\mu_2 = \overline{\delta}_C n/(n-\alpha) \leqslant \lambda_n$, by Lemma 2.3, whence Eq. (44) follows since $\overline{\delta}_\alpha \leqslant \overline{\delta}_C$.

When equality holds $\overline{\delta}_\alpha = \overline{\delta}_C = \lambda_n(n-\alpha)/n$, the interlacing is tight and, by Lemma 2.3, the partition is pseudo-regular with pseudo-quotient matrix (of $\boldsymbol{L}$ with respect to $\mathscr{P}$)

$$\boldsymbol{B}^* = \boldsymbol{DBD}^{-1} = \delta\begin{pmatrix} 1 & -1 \\ -\frac{\alpha}{n-\alpha} & \frac{\alpha}{n-\alpha} \end{pmatrix},$$

where $\delta := \overline{\delta}_\alpha$ and $\boldsymbol{D} = \operatorname{diag}(1/\sqrt{\alpha}, 1/\sqrt{n-\alpha})$. But now the pseudo-intersection numbers of Eq. (2) are simply

$$b^*_{ij}(u) = \begin{cases} \delta_u - \beta_{ij}(u) & \text{if } i = j \\ -\beta_{ij}(u) & \text{otherwise,} \end{cases} \quad (u \in V_i), \tag{46}$$

where $\beta_{ij}(u) := |\Gamma(u) \cap V_j|$ represent the (standard) intersections numbers of Γ with respect to $\mathscr{P}$. This gives: $\beta_{11}(u) = 0 = \delta_u - \delta$, whence $\delta_u = \delta$, $\beta_{12}(u) = \delta$ for any $u \in C$; and $\beta_{21}(v) = \delta\alpha/(n-\alpha)$, $\beta_{22}(v) = \delta - \delta\alpha/(n-\alpha)$ for any $v \in \overline{C}$; whence the result follows. □

In order to make inequality (44) more explicit, Mohar [33] presents his result by stating that, if r is the smallest positive integer for which $r > n(\lambda_n - \overline{\delta}_r)/\lambda_n$, then $\alpha \leqslant r - 1$.

The second part of the theorem, characterizing the case of equality, extends a result of Haemers [26] for regular graphs. (He uses the adjacency matrix and hence considers the – equivalent – case of equality in the Hoffman–Lovász' bound (6) – see the last comment in the proof of Theorem 3.1.)

7.2. The set independence number

Let us now see how the results of Section 5 look when the Laplacian spectrum is involved. We shall here omit the proofs, since they are very similar, and the use of the Laplacian matrix has already been illustrated above. Of course, since ev $\boldsymbol{L} = \{\theta_0 = 0 < \theta_1 < \cdots < \theta_d\}$, the k-alternating polynomial P_k, defined as in Eq. (22), must now attain maximum value at 0, that is on the left of the other eigenvalues.

Theorem 7.2. *Let Γ be a connected graph on n vertices, with Laplacian eigenvalues* ev $\boldsymbol{L} = \{0 < \theta_1 < \cdots < \theta_d\}$, *and corresponding k-alternating polynomials P_k, $0 \leqslant k \leqslant d-1$. Assume that, for some integer w, $1 \leqslant w \leqslant n/2$, the (w,k)-independence number satisfies $\alpha_k^w \geqslant 2$. Then*

$$\alpha_k^w \leqslant \frac{2n}{w(P_k(0)+1)}. \tag{47}$$

In particular, using that $P_1(0) = 2(-\theta_1)/(\theta_1 - \theta_d) + 1 = (\theta_d + \theta_1)/(\theta_d - \theta_1)$, we have that the independence and chromatic numbers of a connected graph Γ, in terms of its Laplacian eigenvalues, satisfy respectively

$$\alpha \leqslant n\left(1 - \frac{\theta_1}{\theta_d}\right), \qquad \chi \geqslant \frac{\theta_d}{\theta_d - \theta_1}.$$

Of course, the advantage of the first bound above, in comparison with Eq. (44), is its explicit form in terms of the Laplacian spectrum.

Corollary 7.3. *Let Γ be a connected graph on n vertices, $\binom{n}{w} \geqslant t$, with Laplacian eigenvalues* ev $\boldsymbol{L} = \{0 < \theta_1 < \cdots < \theta_d\}$. *Then*

$$P_k(0) > \frac{2n}{wt} - 1 \quad \Rightarrow \quad D_t^w \leqslant k. \tag{48}$$

As in Section 5, in the above results we can replace P_k by the Chebychev polynomial $T_k(-x)$ "shifted" from $[-1,1]$ to $[\theta_1, \theta_d]$, that is $T_k((\theta_1 + \theta_d - 2x)/(\theta_d - \theta_1))$, now giving:

$$D_t^w \leqslant \left\lfloor \frac{\cosh^{-1}(2n/wt - 1)}{\cosh^{-1}((\theta_d + \theta_1)/(\theta_d - \theta_1))} \right\rfloor + 1, \tag{49}$$

whereas the results of Chung et al. [7], proved by using the normalized Laplacian matrix (the so-called *Laplace operator*), correspond to

$$D_t^w \leqslant \left\lfloor \frac{\cosh^{-1}(n/w - 1)}{\cosh^{-1}((\theta_d + \theta_s)/(\theta_d - \theta_s))} \right\rfloor + 1, \tag{50}$$

where θ_s is the t-th smallest Laplacian eigenvalue λ_t, that is, s is the smallest integer satisfying $1 + m(\theta_1) + \cdots + m(\theta_s) \geqslant t$. See also Ref. [8]. In the way of comparing the above bounds, note that for $t = 2$ both results coincide. Otherwise, there is a general case in which Eq. (49) clearly supersedes Eq. (50), namely whenever the multiplicity of the second smallest Laplacian eigenvalue θ_1 – the so-called "algebraic connectivity" of Γ – satisfies $m(\theta_1) \geqslant t - 1$ (since then $s = 1$).

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