Homoclinic Connections in the Restricted Three Body Problem and the Scattering Map

*CELMEC, September 2005*

Elisabet Canalias, Amadeu Delshams, Josep Masdemont, Pablo Roldán
elisabet.canalias@upc.edu, amadeu.delshams@upc.edu,
josep@barquins.upc.edu, pablo.roldan@upc.edu

Universitat Politècnica de Catalunya
Contents

- I: The restricted three body problem:
  1. Equations of motion.
  2. Hyperbolic manifolds.
  3. Homoclinic connections.

- II: The scattering map:
  1. Definition.
  2. Computation for planar RTBP.
  3. Applications.

- References
Restricted three body problem

- (Planar) equations of motion:

\[ \ddot{X} - 2\dot{Y} = \frac{\partial \Omega}{\partial X}, \quad \ddot{Y} + 2\dot{X} = \frac{\partial \Omega}{\partial Y} \]  \hspace{1cm} (1)

where,

\[ \Omega(X, Y) = \frac{1}{2}(X^2 + Y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu(1 - \mu). \]

- First integral: Jacobi constant

\[ \mathcal{C}(X, Y, \dot{X}, \dot{Y}) = -(\dot{X}^2 + \dot{Y}^2) + 2\Omega(X, Y) \]

- 5 equilibrium points. We will focus on \( L_1 \) and \( L_2 \).
The vicinity of $L_1$ and $L_2$

- Linear behaviour around $L_1$ and $L_2$ is of the type centre $\times$ saddle.
- $\forall \mathcal{C} \in [\mathcal{C}_{\text{min}}, \mathcal{C}_{\text{max}}]$ $\exists$ one periodic motion around each $L_i \implies$ Lyapunov (planar) orbit $\gamma = \{\gamma(t)\}$. 

The vicinity of $L_1$ and $L_2$

- Linear behaviour around $L_1$ and $L_2$ is of the type centre-$\times$ saddle.
- $\forall C \in [C_{min}, C_{max}] \ni \exists$ one periodic motion around each $L_i \implies$ Lyapunov (planar) orbit $\gamma = \{\gamma(t)\}$.
- Hyperbolic manifolds of $\gamma$: unstable and stable:

$$W^u_\gamma = \{x \in \mathbb{R}^4 \mid \lim_{t \to -\infty} ||\Phi_t(x) - \gamma||=0 \},$$

$$W^s_\gamma = \{x \in \mathbb{R}^4 \mid \lim_{t \to +\infty} ||\Phi_t(x) - \gamma||=0 \}$$

where $\Phi_t(x) \equiv$ orbit of $x$ in the flow of equations (1).
The vicinity of $L_1$ and $L_2$
Homoclinic connections

- If $x \in W^{u}_{\gamma_1} \cap W^{s}_{\gamma_2}$, then $\Phi_t(x)$ asymptotically tends to a Lyapunov orbit both in forward and backward time.
Homoclinic connections

- If $x \in W^{u}_{\gamma_1} \cap W^{s}_{\gamma_2}$, then $\Phi_t(x)$ assymptotically tends to a Lyapunov orbit both in forward and backward time.
- If $\gamma_1 = \gamma_2$, $\Phi_t(x) \equiv$ homoclinic connection.
- If $\gamma_1 \neq \gamma_2$, $\Phi_t(x) \equiv$ heteroclinic connection.
Homoclinic connections (II)

- Space of solutions of equations (1) for each value of $C$ is 3-dimensional.
Homoclinic connections (II)

- Space of solutions of equations (1) for each value of $C$ is 3-dimensional.
- $W^u_\gamma, W^s_\gamma$ are 2-dimensional tubes.
Homoclinic connections (II)

- Space of solutions of equations (1) for each value of $C$ is 3-dimensional.
- $W^u_\gamma, W^s_\gamma$ are 2-dimensional tubes.
- A Poincaré section can be used, $\Sigma = \{X = \text{constant}\}$. 
Homoclinic connections (III)

- $\forall C$, each manifold intersects the Poincaré section in a curve.
- Homoclinic connections can be found on the section by intersecting two of these curves:
Families of connections

- 2 intersecting points \( \implies \) 2 connections \( \Gamma_1(C), \Gamma_2(C) \).

\[ C = 3.0008044 \]
Families of connections

- 2 intersecting points $\implies$ 2 connections $\Gamma_1(C), \Gamma_2(C)$. 

$C = 3.00083$
Families of connections

- 2 intersecting points $\iff$ 2 connections $\Gamma_1(C), \Gamma_2(C)$.

$c = 3.000862$
Families of connections

- 2 intersecting points $\Longleftrightarrow$ 2 connections $\Gamma_1(C), \Gamma_2(C)$.

$C = 3.0008788$
Families of connections

- Tangency → bifurcation orbit.

\[ C = 3.0008839 \]
Families of connections (II)
Normally hyperbolic manifold

- Objects vary with energy:

\[ \gamma(C), \quad W_{\gamma}^{u/s}(C), \quad \Gamma_{1,2}(C). \]
Normally hyperbolic manifold

- Objects vary with energy:
  \[ \gamma(C), \quad W_{\gamma}^{u/s}(C), \quad \Gamma_{1,2}(C). \]

- Consider the sets
  \[ \Lambda = \bigcup_{C} \gamma(C) \]
  \[ W_{\Lambda}^{s} = \bigcup_{C} W_{\gamma}^{s}(C), \quad W_{\Lambda}^{u} = \bigcup_{C} W_{\gamma}^{u}(C) \]

  where \( C \in [C_{\text{min}}, C_{\text{max}}] \).
Normally hyperbolic manifold

- $\Lambda = \Lambda(\theta, C)$ is a normally hyperbolic invariant manifold (2D).
- $W^s_\Lambda, W^u_\Lambda$ are stable/unstable manifolds to $\Lambda$ (3D).
Normally hyperbolic manifold

Parametric representation of the manifolds
Homoclinic intersection

\[ \Gamma = \bigcup_{C \in [C_{min}', C_{max}']} \Gamma_1(C) \subset W^s_\Lambda \cap W^u_\Lambda \]

- The intersection of the stable/unstable manifolds is transversal along \( \Gamma \). Hence, \( dim(\Gamma) = dim(\Lambda) \).
Scattering map

- Given $\Gamma$, define $x_+ = S(x_-)$ if there exists a point $z \in \Gamma$ s.t.

$$\| \Phi_t(z) - \Phi_t(x_-) \| \to 0, \quad t \to -\infty$$

$$\| \Phi_t(z) - \Phi_t(x_+) \| \to 0, \quad t \to \infty.$$
Scattering map

- Scattering map $S : \mathcal{D} \subset \Lambda \rightarrow \Lambda$.
- $S$ is a dynamical system acting on $\Lambda$ associated to the homoclinic excursions along $\Gamma$.
- There exist different scattering maps associated to different $\Gamma$. 
Computation for planar RTBP

- Dynamics inside $\Lambda$ amounts to Lyapunov orbits $\gamma(C)$.
- Homoclinic orbits $\Phi_t(z) \to \gamma, \quad t \to \pm\infty$.
- $S$ can be computed independently for each $\gamma(C)$:

  $$x_+ = S(x_-)$$

  or

  $$\theta_+ = S(\theta_-)$$

  where $\theta_-, \theta_+ \in \gamma(C)$. 

Computation for planar RTBP
Computation for planar RTBP

- Lindstedt-Poincaré expansion of $W^u_\gamma$ (linear approximation):

$$\bar{X}(t) = \alpha_1 e^{\lambda t} + \alpha_3 \cos(\omega t + \theta)$$
$$\bar{Y}(t) = \kappa_2 \alpha_1 e^{\lambda t} + \kappa_1 \alpha_3 \sin(\omega t + \theta)$$

where $\kappa_1$, $\kappa_2$, $\omega$ and $\lambda$ are constant.
Computation for planar RTBP

- Lindstedt-Poincaré expansion of $W^u_\gamma$ (linear approximation):

\[
\begin{align*}
\tilde{X}(t) &= \alpha_1 e^{\lambda t} + \alpha_3 \cos(\omega t + \theta) \\
\tilde{Y}(t) &= \kappa_2 \alpha_1 e^{\lambda t} + \kappa_1 \alpha_3 \sin(\omega t + \theta)
\end{align*}
\]

where $\kappa_1$, $\kappa_2$, $\omega$, and $\lambda$ are constant.

- In the planar case, $S$ is simply a twist map

\[
(C, \Theta) \rightarrow (C, \Theta + \Delta(C))
\]

\[
\Delta'(C) \neq 0
\]
Spatial RTBP

- Linear behaviour around $L_1$ and $L_2$ is of the type centre $\times$ centre $\times$ saddle.
Spatial RTBP

- Linear behaviour around $L_1$ and $L_2$ is of the type centre $\times$ centre $\times$ saddle.
- Fixed $C$, there is a whole family $\{\gamma^n\}$ of libration orbits around $L_{1,2}$:
  - Lissajous orbits (quasi-periodic motions on $\mathbb{T}^2$);
  - Halo periodic orbits.
Spatial RTBP

- Linear behaviour around $L_1$ and $L_2$ is of the type centre × centre × saddle.
- Fixed $C$, there is a whole family $\{\gamma^n\}$ of libration orbits around $L_{1,2}$:
  - Lissajous orbits (quasi-periodic motions on $\mathbb{T}^2$);
  - Halo periodic orbits.
- Homoclinic/heteroclinic connections between libration orbits.
Spatial RTBP

- Linear behaviour around $L_1$ and $L_2$ is of the type centre $\times$ centre $\times$ saddle.
- Fixed $C$, there is a whole family $\{\gamma^n\}$ of libration orbits around $L_{1,2}$:
  - Lissajous orbits (quasi-periodic motions on $T^2$);
  - Halo periodic orbits.
- Homoclinic/heteroclinic connections between libration orbits.
- Each libration orbit $\gamma^i$ can be connected to many libration orbits $\gamma^{j_1}, \gamma^{j_2}, \ldots$. 

E.Canalias, A.Delshams, J.Masdemont, P.Roldan, CELMEC 2005 – p.20/22
Applications

- Scattering map is an efficient and convenient representation of homoclinic and heteroclinic connections.
- Combine scattering map(s) with dynamics inside $\Lambda$ to obtain interesting orbits:
  - Complex mission design.
  - Diffusion orbits.
References
