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GEOMETRIC STRUCTURE OF SINGLE/COMBINED EQUIVALENCE CLASSES OF A CONTROLLABLE PAIR*

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Abstract. Given a pair of matrices representing a controllable linear system, its equivalence classes by the single or combined action of feedbacks, change of state and input variables, as well as their intersection are studied. In particular, it is proved that they are differentiable manifolds and their dimensions are computed. Some remarks concerning the effect of different kinds of feedbacks are derived.

Key words. Controllable pairs, Linear systems, Orbits by feedback, Orbits by variables change, System perturbations.

AMS subject classifications. 37A20, 93C05, 93C73.

1. Introduction.

Several equivalence relations between pairs of matrices representing linear systems are considered in the literature. For example, the one corresponding to a change of basis in the state variables, or the so-called block similarity, which also involves changes in the input space and feedbacks. It seems natural to consider the equivalence classes related to each one of these transformations and to several of them. We study the geometric relations between the different equivalence classes, such as the relative codimension and especially their nontrivial intersections.

Partial approaches to this subject appear in the literature (see, for example, [1], [3], [7], [14], [15]). Here we tackle a unified treatment in order to simplify the proofs and to present a full panorama of the geometric hierarchy of these equivalence classes. Some surprising remarks concerning the effect of the feedbacks will be derived.

The starting point is the differentiable structure of each equivalence class, which follows from the Closed Orbit Lemma (see for example [10]). The computation of its dimension is based on Arnold's technique of the so-called versal deformations, that is to say, transversal manifolds to the considered classes (or orbits) in some other coarser one. In fact, we use the results in [3] and [15] to obtain "adapted" deformations having similar patterns, in such a way that different families of parameters are responsible for

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the corresponding deformation. Moreover, it gives a local adapted parameterization of the different equivalence classes.

Regarding orbit intersections, they may not be orbits, not even differentiable manifolds. In our case, these intersections are differentiable manifolds because the transversality conditions hold and it is possible to give a particular description as orbits with regard to suitable subgroups. Even more, in some cases this subgroup is just the intersection of those generating the intersecting orbits.

The study of differentiable equivalence classes is tackled in Section 3 (simple and multiple actions), whereas Section 4 is devoted to their intersections. In the previous Section 2, we introduce some definitions and notations.

2. Preliminaries. Let $\mathcal{M} = \mathbb{C}^{n^2} \times \mathbb{C}^{n\times m}$ be the differentiable manifold of pairs of matrices $\mathcal{M} = \{(A,B) : A \in \mathbb{C}^{n^2}, B \in \mathbb{C}^{n\times m}\}$ and $\mathcal{M}^*$ be the open dense subset of $\mathcal{M}$ formed by the controllable pairs with $\operatorname{rank} B = m$, that is to say, the *full rank controllable pairs*. When no confusion is possible, we identify the pair (A,B) with the matrix $[A \quad B] \in \mathbb{C}^{n\times(n+m)}$.

The usual *block similarity* (or *BK-equivalence*) is induced by the group action:

$$\mathcal{G} = \{g = \begin{bmatrix} S & 0 \\ R & T \end{bmatrix} : S \in Gl_n, T \in Gl_m, R \in \mathbb{C}^{m\times n}\}.$$

$$g * (A,B) = S^{-1}(A,B) \begin{bmatrix} S & 0 \\ R & T \end{bmatrix} = (S^{-1}AS + S^{-1}BR, S^{-1}BT),$$

so that the *BK-equivalence class* of a pair (A,B) is the orbit

$$O_{BK}(A,B) = \{g * (A,B) : g \in \mathcal{G}\}.$$

The actions of S, T, R are called a change of state variables, a change of input variables and a feedback, respectively. In a natural way, we can also consider the subgroups relative to only some of these actions and their corresponding orbits.

DEFINITION 2.1. Let $(A,B) \in \mathcal{M}$. We consider the following suborbits of $O_{BK}(A,B)$ defined by:

1. $O_{ST}(A,B)$, $O_{SR}(A,B)$, $O_{TR}(A,B)$ when $R = 0$, $T = I_m$, $S = I_n$, respectively.
2. $O_S(A,B)$, $O_T(A,B)$, $O_R(A,B)$ when $R = 0$ and $T = I_m$, $R = 0$ and $S = I_n$, $S = I_n$ and $T = I_m$, respectively.

We will refer to them as the *ST-orbit, …, R-orbit* of (A,B), respectively. Or, equivalently, the *orbit of* (A,B) *with regard to the ST-action, …, R-action.*

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Our aim is to study these orbits and their intersections. It follows directly from the Closed Orbit Lemma (see for example [10]) that all the above orbits are differentiable manifolds. Their dimensions will be computed in Section 3. Concerning their intersections, we notice that the intersection of two differentiable manifolds may not be a differentiable manifold even if they are group orbits. However, in Section 4 we will see that in our case this fact follows from transversality conditions.

As we have pointed out, we restrict ourselves to the generic case of full rank controllable pairs. Several canonical forms and complete invariants are well-known with regard to block similarity (for a survey, see [12] and [13]). We will use the following *BK-form*, defined by means of the controllability indices. We write $E_q = (\,0 \cdots 0\;1\;0 \cdots 0\,)^t$, where the 1-valued entry is in the q-position and the size corresponds to the context, and $N_p = (0, E_1, \ldots, E_{p-1})$ is the upper nilpotent p-block.

Definition 2.2. Given a full rank controllable pair of matrices $(A, B) \in \mathcal{M}^*$ and $k_1 \geq k_2 \geq \cdots \geq k_m > 0$ its controllability indices, it is known that there is another pair (A_c, B_c) in its BK-orbit such that

$$(A_c, B_c) = (\mathrm{diag}(N_{k_1}, N_{k_2}, \ldots, N_{k_m}), (E_{l_1}, E_{l_2}, \ldots, E_{l_m})),$$

where $l_i = \sum_{j=1}^{i} k_j$.

It is said that (A_c, B_c) is the *Brunovsky canonical form of* (A, B) or that (A_c, B_c) is a *BK-pair*.

We express the pairs $(C, D) \in \mathcal{M}$ linked to a full rank controllable pair (A, B) with controllability indices $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ divided into blocks: $C = (C_{i,j})_{1 \leq i,j \leq m}$, $\quad C_{i,j} \in \mathbb{C}^{k_i \times k_j}$, $\quad D = (D_1, D_2, \ldots, D_m)$, $\quad D_i \in \mathbb{C}^n$.

When 0 appears in a block matrix, it will be a null block of the suitable size (it could be empty).

The following families of parameterized matrices divided into the above blocks will be widely used in the following sections:

Definition 2.3.

1. We consider $A_\alpha, A_\beta \in \mathbb{C}^{n\times n}$ and $B_\gamma, B_\delta \in \mathbb{C}^{n\times m}$ with $\alpha_{i,j,p}, \beta_{i,j,p}, \gamma_{i,j}, \delta_{i,j,l} \in \mathbb{C}$ such that
$A_{\alpha_{i,j}} = E_{k_i} \cdot (\alpha_{i,j,1}, \alpha_{i,j,2}, \ldots, \alpha_{i,j,\min\{k_i,k_j\}}, 0)$,
$A_{\beta,i,j} = E_{k_i} \cdot (0, \beta_{i,j,k_j-k_i+1}, \ldots, \beta_{i,j,k_j})$ if $k_i < k_j$, and $A_{\beta,i,j} = 0$ otherwise,
$B_{\gamma,j} = \sum_{k_i > k_j} \gamma_{i,j} E_{k_i}$,
$B_{\delta,j} = \sum_{1\leq i<j\leq m} \sum_{1\leq l\leq k_{i,j}} \delta_{i,j,l} E_{k_i - l}$, where $k_{i,j} = \max\{0, k_i - k_j - 1\}$.
2. Notice that A_α and A_β can be seen as feedbacks $B_c R$. We will refer to them as *α-feedbacks* and *β-feedbacks*, respectively.

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3. We denote the number of parameters of $A_\alpha, A_\beta, B_\gamma, B_\delta$ by
   $n_\alpha = \sum_{i,j=1}^{m} \min\{k_i, k_j\} = \sum_{i=1}^{m}(2i-1)k_i$.
   $n_\beta = \sum_{i,j=1}^{m} \max\{0, k_i - k_j\}$.
   $n_\gamma = \sum_{i,j=1}^{m} \Gamma_{i,j}$, where $\Gamma_{i,j} = 1$ if $k_i > k_j$, and 0 otherwise.
   $n_\delta = \sum_{i,j=1}^{m} \max\{0, k_i - k_j - 1\}$.

Notice that $n_\alpha + n_\beta = nm$, $\quad n_\gamma + n_\delta = n_\beta$, $\quad n_\alpha + n_\gamma + n_\delta = nm$.

EXAMPLE 2.4. If $k_1 = 6$, $k_2 = 3$, $k_3 = k_4 = 2$, and the pair (A_c, B_c) is the corresponding Brunovsky canonical form, then $A_c + A_\alpha + A_\beta$ and $B_c + B_\gamma + B_\delta$ are, respectively,

$$\left[\begin{array}{cccccc|ccc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{1,1,3} & \alpha_{1,1,4} & \alpha_{1,1,5} & \alpha_{1,1,6} & \alpha_{1,2,1} & \alpha_{1,2,2} & \alpha_{1,2,3} & \alpha_{1,3,1} & \alpha_{1,3,2} & \alpha_{1,4,1} & \alpha_{1,4,2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\alpha_{2,1,1} & \alpha_{2,1,2} & \alpha_{2,1,3} & \beta_{2,1,4} & \beta_{2,1,5} & \beta_{2,1,6} & \alpha_{2,2,1} & \alpha_{2,2,2} & \alpha_{2,2,3} & \alpha_{2,3,1} & \alpha_{2,3,2} & \alpha_{2,4,1} & \alpha_{2,4,2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\alpha_{3,1,1} & \alpha_{3,1,2} & \beta_{3,1,3} & \beta_{3,1,4} & \beta_{3,1,5} & \beta_{3,1,6} & \alpha_{3,2,1} & \alpha_{3,2,2} & \beta_{3,2,3} & \alpha_{3,3,1} & \alpha_{3,3,2} & \alpha_{3,4,1} & \alpha_{3,4,2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\alpha_{4,1,1} & \alpha_{4,1,2} & \beta_{4,1,3} & \beta_{4,1,4} & \beta_{4,1,5} & \beta_{4,1,6} & \alpha_{4,2,1} & \alpha_{4,2,2} & \beta_{4,2,3} & \alpha_{4,3,1} & \alpha_{4,3,2} & \alpha_{4,4,1} & \alpha_{4,4,2}
\end{array}\right]$$

$$\left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \delta_{1,3,3} & \delta_{1,4,3} \\
0 & \delta_{1,2,2} & \delta_{1,3,2} & \delta_{1,4,2} \\
0 & \delta_{1,2,1} & \delta_{1,3,1} & \delta_{1,4,1} \\
1 & \gamma_{1,2} & \gamma_{1,3} & \gamma_{1,4} \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & \gamma_{2,3} & \gamma_{2,4} \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]$$

In what follows, the linear space of pairs $\{(A_\alpha, 0), \alpha \in \mathbb{C}^{n_\alpha}\}$ will be simply denoted by $\{(A_\alpha, 0)\}_\alpha$; and similarly for $\{(A_\beta, 0)\}_\beta$, $\{(0, B_\delta)\}_\delta$ and $\{(0, B_\gamma)\}_\gamma$.

3. The orbits O_S, O_T, O_R, O_{BK}, O_{ST}, and O_{SR}.

It is easy to prove that, for full rank controllable pairs, the single actions S, T, and R generate orbits of constant

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dimension.

PROPOSITION 3.1. *Let* $(A, B) \in \mathcal{M}$.

1. *If* (A, B) *is controllable,* $\dim O_S(A, B) = n^2$.
2. *If* B *has full rank,* $\dim O_R(A, B) = nm$.
3. *If* B *has full rank,* $\dim O_T(A, B) = m^2$.

Proof. In general (see, for example, [3, Section I.1]) if a group $\mathcal{G}$ acts on a manifold $\mathcal{M}$, the dimension of the orbit $O_{\mathcal{G}}(x)$ of a point $x \in \mathcal{M}$ is the codimension of the stabilizer subgroup $\{g \in \mathcal{G} : g * x = x\}$. In [8] and [15] (Part IV) one proves that the pair is controllable if and only if the stabilizer with regard to the S-action is trivial. Next, we conclude it from the fact that the differential (at the identity element) of the action $g \longrightarrow g * x$ is injective. This argument will also work for assertions 2 and 3.

For the (S, T, R)-action we have

$$\varphi : \mathbb{C}^{n^2} \times \mathbb{C}^{m^2} \times \mathbb{C}^{m\times n} \longrightarrow \mathcal{M}, \quad (S, T, R) \longmapsto (S^{-1}AS + S^{-1}BR, S^{-1}BT);$$

$$D\varphi_{(Id_n,Id_m,0)}(S', T', R') = (AS' - S'A + BR', BT' - S'B).$$

In particular, if only S varies,

$$\frac{\partial\varphi}{\partial S}_{(Id_n,Id_m,0)}(S') = (AS' - S'A, -S'B),$$

which is injective if the pair (A, B) is controllable: $AS' = S'A$ and $S'B = 0$ implies $S'(B, AB, A_2B, \ldots, A_{n-1}B) = 0$ and, because of the controllability of (A, B), $S' = 0$.

Analogously,

$$\frac{\partial\varphi}{\partial R}_{(Id_n,Id_m,0)}(R') = (BR', 0), \quad \frac{\partial\varphi}{\partial T}_{(Id_n,Id_m,0)}(T') = (0, BT'),$$

are injective if B has full column rank. ◻

As we have pointed out, we are interested in the intersections of the above orbits and the mixed actions. In fact, we focus on the cases when the S-action is involved because of the trivial fact that:

PROPOSITION 3.2. *Let* $(A, B) \in \mathcal{M}^*$ *be a full rank controllable pair of matrices. Then*

1. $O_T(A, B) \cap O_R(A, B) = \{(A, B)\}$.
2. $\dim O_{TR}(A, B) = \dim O_T(A, B) + \dim O_R(A, B) = m^2 + nm$.

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We recall that the triple action of S, T and R corresponds to the usual block similarity. The geometric structure of the BK-orbits has been studied in [3] and [4] by means of Arnold's techniques introduced in [2]. See, for example, [3, Sections (I.2) and (II.1)] for the basic definitions and results concerning versal/miniversal deformations. We will refer to BK-miniversal, S-miniversal, ... the miniversal deformations with regard to the BK-action, S-action, ... In particular, for a full rank controllable pair, we have:

THEOREM 3.3. [3] *Given a full rank controllable pair* $(A, B) \in \mathcal{M}^*$ *with controllability indices* $k_1 \geq k_2 \geq \cdots \geq k_m > 0$*, then*

$$\dim O_{BK}(A, B) = n^2 + nm - n_\delta = n^2 + n_\alpha + n_\gamma.$$

If (A_c, B_c) *is its Brunovsky canonical form, a BK-miniversal deformation of* (A_c, B_c) *in* $\mathcal{M}^*$ *is the* n_δ*-dimensional linear manifold* $(A_c, B_c) + \{(0, B_\delta)\}_\delta$.

Canonical forms with regard to the change of states have been obtained for controllable pairs by several authors ([1],[7],[9],[11],[14],[15]). In fact, we will base the study of the orbits O_{ST} and O_{SR} on the following result, which is a direct consequence of Theorems (2.2) and (2.3) in [15], jointly with the above Theorem 3.3 and (1) of Proposition 3.1:

THEOREM 3.4. *Let* $(A, B) \in \mathcal{M}^*$ *be a full rank controllable pair with controllability indices* $k_1 \geq k_2 \geq \cdots \geq k_m > 0$*, and* (A_c, B_c) *its Brunovsky canonical form. Then:*

An S-miniversal deformation of (A_c, B_c) *in its BK-orbit is given by the* $(n_\alpha + n_\gamma)$*-dimensional linear manifold* $(A_c, B_c) + \{(A_\alpha, B_\gamma)\}_{\alpha,\gamma}$.

This S-miniversal deformation of (A_c, B_c) has the following property: the γ-parameters, and only they, can be eliminated by the T-action; and analogously, the α-parameters, and only they, by the R-action. Therefore:

PROPOSITION 3.5. *In the conditions of* Theorem 3.4*:*

1. *The* n_α*-dimensional linear manifold* $(A_c, B_c) + \{(A_\alpha, 0)\}_\alpha \subset O_{SR}(A_c, B_c)$ *is an ST-miniversal deformation of* (A_c, B_c) *in its BK-orbit.*
2. *The* n_γ*-dimensional linear manifold* $(A_c, B_c) + \{(0, B_\gamma)\}_\gamma \subset O_{ST}(A_c, B_c)$ *is an SR-miniversal deformation of* (A_c, B_c) *in its BK-orbit.*

As a first direct consequence of this result, we have:

COROLLARY 3.6. *Given a full rank controllable BK-pair* $(A_c, B_c) \in \mathcal{M}^*$*, we have*

$$\dim O_{ST}(A_c, B_c) = n^2 + n_\gamma, \qquad \dim O_{SR}(A_c, B_c) = n^2 + n_\alpha.$$

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EXAMPLE 3.7. Let us consider $n = 5$ and $m = 2$. Then, there are only two kinds of orbits, according to the controllability indices being $(4, 1)$ or $(3, 2)$ (see Table 3.1). Let us obtain the bifurcation diagram in Figure 3.1, that is to say, the partition in the $(\delta_1, \delta_2, \gamma)$-space induced by the partition of the space of pairs of matrices into the considered orbits (in our case, the BK-orbits, and the ST and SR-suborbits).

k	n_α	n_γ	n_δ	$\operatorname{codim} O_{BK}$	$\operatorname{codim} O_{ST}$	$\operatorname{codim} O_{SR}$	$\operatorname{codim} O_S$
4,1	7	1	2	2	9	3	10
3,2	9	1	0	0	9	1	10

TABLE 3.1
The orbits O_{BK}, O_{ST}, and O_{SR}.

An SR-miniversal deformation in $\mathcal{M}^*$ of the pair

$$(A_c, B_c) = \left(\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array} \right] \right) \in O_{BK}(4,1)$$

is the 3-dimensional linear manifold formed by the pairs

$$(A_c, B_c(\delta_1, \delta_2, \gamma)) = \left(\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & \delta_1 \\ 0 & \delta_2 \\ 1 & \gamma \\ \hline 0 & 1 \end{array} \right] \right).$$

Here, the orbit $O_{BK}(4, 1)$ appears as the γ-axis, formed by pairs belonging to different SR-orbits of type $k = (4, 1)$, parameterized by γ.

The remaining points correspond to the orbit $O_{BK}(3, 2)$. In particular, the points in the plane (δ_1, δ_2), or equivalently $\gamma = 0$, with $(\delta_1, \delta_2) \neq (0, 0)$ can be S-transformed according to Theorem 3.4. A quite laborious computation shows that

- If $\delta_2 \neq 0$, then $(A_c, B_c(\delta_1, \delta_2, 0))$ is S-equivalent to

$$\left(\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\delta_2 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ -\delta_1/\delta_2 2 & 1 \\ \hline 0 & 0 \\ 1 & 0 \end{array} \right] \right) \in O_{BK}(3,2).$$

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- If $\delta_2 = 0$ and $\delta_1 \neq 0$, then $(A_c, B_c(\delta_1, \delta_2, 0))$ is S-equivalent to

$$\left(\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1/\delta_1 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array} \right] \right) \in O_{BK}(3,2).$$

Each one lies in a different ST-orbit, because they correspond to different values of the α-parameters.

Since the SR-orbits correspond to the γ-parameters, they are the axes $\delta_2 \neq 0$, $\delta_1 = 0$, and $\delta_1 \neq 0$, $\delta_2 = 0$, and the parabolas $\frac{\delta_1}{\delta_2^2}$ = constant, $\delta_2 \neq 0$.

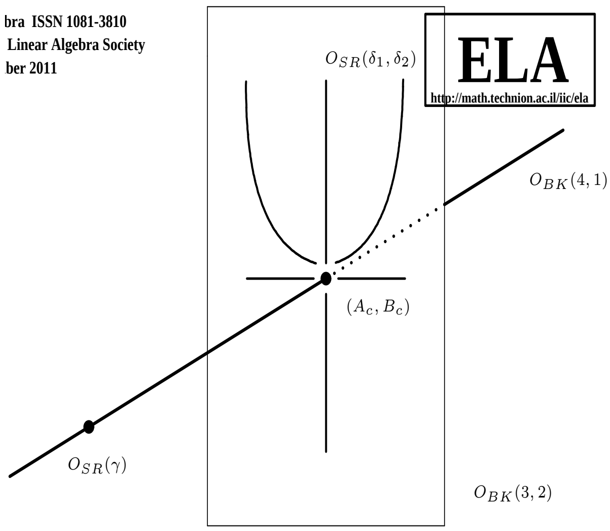

FIG. 3.1. *The orbits O_{BK}, O_{ST}, and O_{SR}.*

4. The intersections $O_S \cap O_T$, $O_R \cap O_S$, and $O_{SR} \cap O_{ST}$. Next we study the intersection of the orbits in Section 3. As we have pointed out, an orbit intersection may not be an orbit, not even a differentiable manifold. However, in our case we have:

PROPOSITION 4.1. *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair with controllability indices $k_1 \geq \cdots \geq k_m > 0$. Then the orbit intersections $O_S \cap O_T$ and $O_S \cap O_R$ are differentiable submanifolds of $O_{BK}(A, B)$ and:*

$$\dim(O_S(A, B) \cap O_T(A, B)) = n^2 + m^2 - \dim O_{ST}(A, B),$$

$$\dim(O_S(A, B) \cap O_R(A, B)) = n^2 + nm - \dim O_{SR}(A, B).$$

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If (A,B) is a BK-pair, the dimensions of the above intersections are $m^2 - n_\gamma$ and $nm - n_\alpha$, respectively.

Proof. It is well known (see, for example, [6, page 30]) that if two submanifolds (of a manifold) are transversal, then their intersection is also a submanifold and its codimension is the sum of the codimensions of the given ones.

In our case, $O_S(A,B)$ and $O_T(A,B)$ are transversal in $O_{ST}(A,B)$ at (A,B) because the tangent space

$$T_{(A,B)}O_{ST}(A,B) = \{(AS' - S'A, BT' - S'B); S' \in \mathbb{C}^{n^2}, T' \in \mathbb{C}^{m^2}\}$$

(see the proof of Proposition 3.1) is clearly spanned by the analogously described $T_{(A,B)}O_S(A,B)$ and $T_{(A,B)}O_T(A,B)$.

Therefore, their intersection is a submanifold of $O_{ST}(A,B)$ (and hence also of $O_{BK}(A,B)$) with dimension:

$$\dim(O_S(A,B) \cap O_T(A,B)) = \dim O_S(A,B) + \dim O_T(A,B) - \dim O_{ST}(A,B)$$

and the results follow from Proposition 3.1 and Corollary 3.6.

Analogously for the second assertion. ☐

Given two subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ of a group $\mathcal{G}$, the intersection of their orbits may not be the orbit corresponding to $\mathcal{G}_1 \cap \mathcal{G}_2$. For example, the intersection of the S-subgroup and the T-subgroup is simply the identity matrix. It may not even be an orbit. However, let us see that the intersections in Proposition 4.1 are indeed orbits with regard to the action of suitable subgroups which will depend on the pair (A,B). We will describe it explicitly when (A,B) is a BK-pair.

For the first intersection this description is obtained by means of the following class of Toeplitz matrices:

Definition 4.2.

(1) $X \in \mathbb{C}^{p\times q}$ is called an *upper triangular Toeplitz matrix* if
 (i) It is constant along the diagonals $x_{i,j} = x_{i+1,j+1}$,
 (ii) $x_{i,1} = 0$ if $i > 1$,
 (iii) $x_{1,i} = 0$ if $i \leq q - p$.

(2) If $k_1, \ldots, k_m$ is a partition of n (that is, $k_1 + \cdots + k_m = n$), $X \in \mathbb{C}^{n\times n}$ is called a *block upper triangular Toeplitz matrix* if

$$X = (X_{i,j})_{1\leq i,j\leq m}, \qquad X_{i,j} \in \mathbb{C}^{k_i\times k_j}$$

and each $X_{i,j}$ is an upper triangular Toeplitz matrix. We will denote the set of these matrices by $UTT(k_1, \ldots, k_m)$.

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We recall also that if A is a block diagonal nilpotent matrix

$$A = \mathrm{diag}(N_{k_1}, \dots, N_{k_m})$$

then a nonsingular matrix $S \in Gl_n$ belongs to its *centralizer* (that is: $S^{-1}AS = A$) if and only if $S \in UTT(k_1, \dots, k_m)$.

With this notation, we have:

PROPOSITION 4.3. *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.*

(1) The submanifold $O_S(A, B) \cap O_T(A, B)$ is the orbit of (A, B) with regard to the action of $\mathcal{G}_{ST}(A, B)$ formed by the $S \in Gl_n$ such that $AS = SA$, and there is $T \in Gl_m$ such that: $SB = BT$.
(2) In particular, if $(A_c, B_c) \in \mathcal{M}^*$ is a BK-pair with controllability indices $k_1 \geq \cdots \geq k_m > 0$, then:

$$\mathcal{G}_{ST}(A_c, B_c) = \{S \in Gl_n \cap UTT(k_1, \dots, k_m) : S_{i,j} = 0 \quad \textit{if} \quad k_i > k_j,$$

$$\textit{and} \quad S_{i,j} = (0, s_{i,j} I_{k_i}) \ \textit{if} \quad k_i \leq k_j\}.$$

Proof.

(1) It is obvious that $O_T(A, B) \cap O_S(A, B) = \{(A, SB) : S \in \mathcal{G}_{ST}\}$. Hence, it is sufficient to check that $\mathcal{G}_{ST}$ is a subgroup of the centralizer of A: if for $i = 1, 2$, $S_i \in \mathcal{G}_{ST}$ and $T_i \in Gl_m$ are such that $S_iB = BT_i$, then left multiplying by S_i^{-1} and right multiplying by T_i^{-1} we have $S_i^{-1}B = BT_i^{-1}$ and $S_1S_2^{-1}B = S_1BT_2^{-1} = BT_1T_2^{-1}$.
(2) We have noticed that the set of nonsingular matrices S such that $AS = SA$ is $UTT(k_1, \dots, k_m)$.
On the other hand, if $B_c = (E_{l_1}, E_{l_2}, \dots, E_{l_m})$, the columns of SB_c will be the columns $l_1, l_2, \dots, l_m$ of S. Moreover, because $SB_c = B_cT$, these columns must be linear combinations of the columns of B_c. Hence, $(S_{i,j})_{t,k_j} = 0$ if $1 \leq t < k_i$, and the proposition is proved. ∎

Concerning the second intersection in Proposition 4.1, we have again a general description as an orbit with regard to a group depending on the pair (A, B), and an explicit description for BK-pairs (A_c, B_c):

THEOREM 4.4. *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.*

(1) The submanifold $O_S(A, B) \cap O_R(A, B)$ is the orbit of (A, B) with regard to the action of $\mathcal{G}_{SR}(A, B)$ formed by the $S \in Gl_n$ such that $S^{-1}B = B$, and there is $R \in \mathbb{C}^{m\times n}$ such that: $S^{-1}AS = A + BR$.

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(2) In particular, if $(A_c, B_c) \in \mathcal{M}^*$ *is a BK-pair with controllability indices* $k_1 \geq \cdots \geq k_m > 0$*, it is a* n_β*-dimensional linear manifold:*

$$O_S(A_c, B_c) \cap O_R(A_c, B_c) = (A_c, B_c) + \{(A_\beta, 0)\}_\beta.$$

Proof.

(1) It is obvious that:

$$O_S(A, B) \cap O_R(A, B) = \{(S^{-1}AS, B) : S \in \mathcal{G}_{SR}(A, B)\}.$$

Hence, it is sufficient to check that $\mathcal{G}_{SR}$ is a subgroup of Gl_n: if for $i = 1, 2$ one has $S_i^{-1}AS_i = A + BR_i$ and $S_i^{-1}B = B$, then

$$(S_1S_2^{-1})^{-1}B = S_2S_1^{-1}B = S_2B = B$$

$$(S_1S_2^{-1})^{-1}A(S_1S_2^{-1}) = S_2(A + BR_1)S_2^{-1} =$$
$$A + BR_2 + S_2BR_1S_2^{-1} = A + B(R_2 + R_1S_2^{-1})$$

(2) For simplicity we will refer to the given pair as (A, B), to its orbits as O_R and O_S and to the n_β-linear manifold as L_β.

Firstly, we prove that $L_\beta \subset O_R$: taking the rows $R^j = A_\beta^{l_j}$ and bearing in mind that $B^{l_j} = E_j^T$ and $B^i = 0$ otherwise, it is easy to see that $A_\beta = BR$.

Secondly, to prove that $L_\beta \subset O_S$ we must see that there is $X \in Gl_n$ such that $XAX^{-1} = A + A_\beta$ and $XB = B$ or, equivalently, $XA = AX + A_\beta X$ and $XB = B$.

Expressing these conditions in blocks, we have
(a) $X_{i,j}N_{k_j} = N_{k_i}X_{i,j} + \sum_{p=1}^{i-1} A_{\beta,i,p}X_{p,j}$,
(b) $(X_{i,j})_{k_j} = \delta_{i,j}E_{k_i}$.

We define
(a) $S \in Gl_n$ such that $S_{i,i} = I_{k_i}$, $S_{i,j} = 0$ if $k_i \geq k_j$ and $(S_{i,j})^p = \sum_{q=1}^{k_j-k_i} \beta_{i,j,q}E_{p+q-1}^T$ if $k_i < k_j$.
(b) $X \in Gl_n$ such that $X_{i,i} = I_{k_i}$, $X_{i,j} = 0$ if $i < j$, $X_{i,i-1} = S_{i,i-1}$ and $X_{i,j} = \sum_{p=i-1}^{j} S_{i,p}X_{p,j}$ if $i > j$.

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In Example 2.4, the matrix S is

$$\left[\begin{array}{cccccc|ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
\beta_{2,1,4} & \beta_{2,1,5} & \beta_{2,1,6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \beta_{2,1,4} & \beta_{2,1,5} & \beta_{2,1,6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \beta_{2,1,4} & \beta_{2,1,5} & \beta_{2,1,6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
\hline
\beta_{3,1,3} & \beta_{3,1,4} & \beta_{3,1,5} & \beta_{3,1,6} & 0 & 0 & \beta_{3,2,3} & 0 & 0 & 1 & 0 & 0 & 0\\
0 & \beta_{3,1,3} & \beta_{3,1,4} & \beta_{3,1,5} & \beta_{3,1,6} & 0 & 0 & \beta_{3,2,3} & 0 & 0 & 1 & 0 & 0\\
\hline
\beta_{4,1,3} & \beta_{4,1,4} & \beta_{4,1,5} & \beta_{4,1,6} & 0 & 0 & \beta_{4,2,3} & 0 & 0 & 0 & 0 & 1 & 0\\
0 & \beta_{4,1,3} & \beta_{4,1,4} & \beta_{4,1,5} & \beta_{4,1,6} & 0 & 0 & \beta_{4,2,3} & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

Notice that obviously $SB = B$. In addition, if $i > j$, we have

- $S_{i,j}A_{\beta,j,p} = 0$.
- $S_{i,j}N_{k_j} = N_{k_i}S_{i,j} + A_{\beta,i,j}$.
- $X_{i,j}N_{k_j} = \sum_{p=i-1}^{j} S_{i,p}X_{p,j}N_{k_j} = \sum_{p=i-1}^{j} S_{i,p}N_{k_p}X_{p,j} = \sum_{p=i-1}^{j}(N_{k_i}S_{i,p} + A_{\beta,i,p})X_{p,j} = N_{k_i}X_{i,j} + \sum_{p=i-1}^{j} A_{\beta,i,p}X_{p,j} = N_{k_i}X_{i,j} + \sum_{p=1}^{i-1} A_{\beta,i,p}X_{p,j}$.
- $(X_{i,i})_{k_i} = E_{k_i}$, $(X_{i,i-1})_{k_{i-1}} = (S_{i,i-1})_{k_{i-1}} = 0$ and by recurrence $(X_{i,j})_{k_j} = \delta_{i,j}E_{k_i}$.

The last equalities prove that $L_\beta \subset O_S$. Then, $L_\beta \subset O_S \cap O_R$. The two manifolds have the same dimension n_β. Let us see that the above inclusion is an equality. It is a straightforward computation that if a pair $(A_0, B_0) = (A, B) + (A_\alpha, 0) + (A_\beta, 0) \in O_R(A, B)$ belongs to $O_S(A, B)$, then $\alpha = 0$ and hence it lies in L_β; the condition $\operatorname{rank} A_0 = \operatorname{rank} A$, implies that the α-parameters in the first column of each block (i.e., in the columns 1, $q_1 + 1$, $q_2 + 1, \dots$) must be zero; then, the condition $\operatorname{rank} A_0^2 = \operatorname{rank} A_c^2$ implies that the α-parameters in the second column of each block (i.e., in the columns 2, $q_1 + 2, \dots$) must be zero; and so on. □

Remark 4.5.

1. It is well known that, if (A, B) is controllable, the eigenvalues of A can be shifted by means of suitable feedbacks. Even more, Rosenbrock's theorem details the effects of feedbacks on the Jordan form of A. The last assertion in Theorem 4.4 shows that in fact the β-feedbacks do not change the Jordan invariants of A_c. On the other hand, Arnold's theory shows that $A_c + \{A_\alpha\}_\alpha$ is a Jordan-miniversal deformation of A_c, so that any nonzero α-feedback of (A_c, B_c) modifies the Jordan invariants of A_c.

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2. More explicitly, in the above proof one has described, for each β-feedback, the change of basis $S_\beta \in Gl_n$ such that

$$S_\beta^{-1} A_c S_\beta = A_c + A_\beta = A_c + B_c R_\beta, \qquad S_\beta^{-1} B_c = B_c.$$

It gives an alternative explicit description of $O_S \cap O_R$:

$$O_S(A_c, B_c) \cap O_R(A_c, B_c) = (A_c, B_c) + \{(A_\beta, 0)\}_\beta = \{(S_\beta^{-1} A_c S_\beta, B_c)\}_\beta.$$

As above, let us see that $O_{SR}(A,B) \cap O_{ST}(A,B)$ is a differentiable submanifold of $O_{BK}(A,B)$. Notice that obviously $O_S(A,B) \subset O_{SR}(A,B) \cap O_{ST}(A,B)$. We will see that in fact the converse is also true if (A,B) is a BK-pair. Previously we prove a similar result concerning $O_R(A_c,B_c) \cap O_{ST}(A_c,B_c)$:

Lemma 4.6. *Let $(A,B) \in \mathcal{M}^*$ be a full rank controllable pair.*

(1) The intersection $O_{ST}(A,B) \cap O_R(A,B)$ is a submanifold of $O_{BK}(A,B)$.
(2) In particular, if $(A_c,B_c) \in \mathcal{M}^*$ is a BK-pair,

$$(A_c, B_c) + \{(A_\beta, 0)\}_\beta = O_S(A_c, B_c) \cap O_R(A_c, B_c) = O_{ST}(A_c, B_c) \cap O_R(A_c, B_c).$$

Proof.

(1) As in Proposition 4.1.
(2) With the notation in the proof of Theorem 4.4 (2), we have shown that:

$$L_\beta = O_S \cap O_R(A_c, B_c) \subset O_{ST}(A_c, B_c) \cap O_R(A_c, B_c).$$

If $(A', B') \in O_{ST}(A_c, B_c) \cap O_R(A_c, B_c)$, then:

$$A' = S^{-1} A S = A + A_\alpha + A_\beta, \quad B' = S^{-1} B T = B,$$

for some S, T, α, β. We have also proved that the first relation implies $\alpha = 0$, and that then (see Remark 4.5):

$$A + A_\beta = S_\beta^{-1} A S_\beta, \quad B = S_\beta^{-1} B.$$

Hence, $(A', B') \in O_S \cap O_R$. □

Theorem 4.7. *Let$(A,B) \in \mathcal{M}^*$ be a full rank controllable pair.*

(1) The intersection $O_{SR}(A,B) \cap O_{ST}(A,B)$ is a differentiable submanifold of $O_{BK}(A,B)$ and

$$n^2 \le \dim(O_{SR}(A,B) \cap O_{ST}(A,B)) =$$
$$\dim O_{SR}(A,B) + \dim O_{ST}(A,B) - (n^2 + n_\alpha + n_\gamma).$$

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(2) In particular, if $(A_c, B_c) \in \mathcal{M}^*$ *is a BK-pair, then*

$$O_{SR}(A_c, B_c) \cap O_{ST}(A_c, B_c) = O_S(A_c, B_c).$$

Proof.

(1) As in Proposition 4.1.

(2) For BK-pairs, we have $\dim(O_{SR}(A,B) \cap O_{ST}(A,B)) = n^2 = \dim O_S(A,B)$. Again we refer to the given pair simply as (A,B). If (A',B') lies in the intersection, there will be S_1, S_2, R and T such that:

$$A' = S_1^{-1}AS_1 = S_2^{-1}(A+BR)S_2, B' = S_1^{-1}BT = S_2^{-1}B.$$

Let $A'' = S_2A'S_2^{-1} = S_3^{-1}AS_3 = A + BR$, where $S_3 = S_1S_2^{-1}$. Then, $B = S_2B' = S_3^{-1}BT$. Clearly

$$(A'', B) \in O_S(A', B'),$$

$$(A'', B) \in O_{ST}(A,B) \cap O_R(A,B) \supset O_S(A,B) \cap O_R(A,B).$$

But Lemma 4.6 ensures that the last inclusion is in fact an equality. Hence $(A'', B) \in O_S(A,B)$. From it and $(A'', B) \in O_S(A', B')$, one has $(A', B') \in O_S(A,B)$. □

Remark 4.8. The last assertion in Theorem 4.7 shows that, in this case, the intersection $O_{ST} \cap O_{SR}$ is just the orbit generated by the action of the intersection Lie subgroup of those generating O_{ST} and O_{SR}.

5. Some comments and remarks.

5.1. Summary diagram. We can summarize the above results in the way shown in Figure 5.1, where the squared quantities are the dimensions of the manifolds and the quantities on the arrows are the corresponding codimensions. We point out especially the nontrivial conclusions concerning $O_{SR} \cap O_{ST}$ and $O_R \cap O_S$.

In fact, in a neighborhood of a BK-pair (A_c, B_c), we have an adapted coordinate system $(S, \alpha, \gamma, \delta)$ of $\mathcal{M}^*$, where $S \in Gl_n$ and α, γ, δ are the parameters in Definition 2.3, in a such way that:

$$O_{BK} = \{\delta = 0\}, \quad O_{ST} = \{\delta = \alpha = 0\},$$
$$O_{SR} = \{\delta = \gamma = 0\}, \quad O_S = \{\delta = \alpha = \gamma = 0\}.$$

Clearly, (α, β) is a coordinate system of O_R, and one has proved that:

$$O_S \cap O_R = \{\alpha = 0\}.$$

Moreover, if $\delta = 0$, the parameters α, γ classify the ST-classes and the SR-classes, respectively, and the pair (α, γ), the S-classes.

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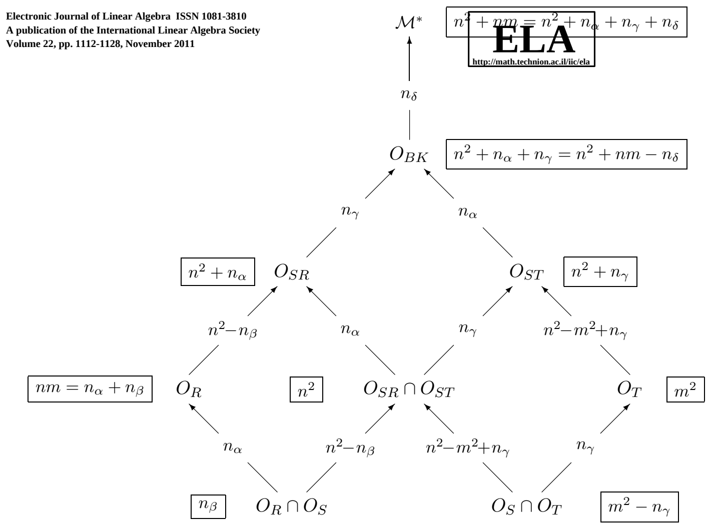

FIG. 5.1. *Summary diagram.*

5.2. Feedbacks. In control theory, feedbacks are used in different contexts, such as the study of conditioned invariant subspaces ([5]) or in the pole assignment problem, where any prescribed spectrum for $A+BR$ can be obtained provided that (A, B) is controllable. It seems natural (see, for example, [16]) to study which other characteristics of A can be changed by means of suitable feedbacks, and what kind of feedbacks are needed in each case. For instance, if one considers a BK-pair (A_c, B_c), it is clear that any pole assignment can be solved using only block-diagonal feedbacks. Let us see that, from our study in the above sections, one has the following surprising conclusions:

- the β-feedbacks do not change any Jordan invariant, and any α-feedback perturbs the Jordan type.
- any Jordan form near A_c can be obtained by means of a suitable α-feedback.

Indeed we have shown that *the β-feedbacks do not change any Jordan invariant* of A_c, because they are equivalent to changes of basis of the state space. On the other hand, we have proved that *any α-feedback perturbs the Jordan type* of A_c. We notice that, from a "local" approach, these feedbacks form a miniversal deformation of A_c in the sense of [2]. That is to say, that *any Jordan form near A_c can be obtained by means of a suitable α-feedback* or, even more, that they induce any local differentiable family of perturbations of A_c in the space $\mathbb{C}^{n^2}$.

We have seen in Section 5.1 that these parameters also play an important role from the point of view of classifying pairs of matrices; *the α-feedbacks parameter-*

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ize the ST-classes contained in $O_{BK}(A_c, B_c)$, and in particular they form an ST-miniversal deformation of (A_c, B_c) in its BK-orbit. This role of α-feedbacks as an ST-deformation, and their role as a Jordan-deformation in the above paragraph, is not evident. For instance, for $\alpha \neq \alpha'$ (small, nonzero), the pairs $(A_c + A_\alpha, B_c)$ and $(A_c + A_{\alpha'}, B_c)$ are not ST-equivalent, but the matrices $A_c + A_\alpha$ and $A_c + A_{\alpha'}$ may be similar.

Finally, we remark the quite surprising fact that the number of β-parameters, n_β, (more precisely, *the dimension of* $O_S \cap O_R$*) is just the codimension of* O_{SR} in $\mathcal{M}$. It is zero if and only if $k_1 = k_2 = \cdots = k_m$.

5.3. Dimension of orbits. Notice that

$$\dim O_{BK} = n^2 + m^2 + nm - \dim(O_S \cap O_T) - \dim(O_S \cap O_R)$$

(recall that $\dim(O_T \cap O_R) = 0$). This gives a nice geometric interpretation:

- the dimension of O_{BK} (as well as O_{ST} and O_{SR}) depends only on the "overlaps" $O_S \cap O_R$ and $O_S \cap O_T$.

We have already seen that $\dim O_S \cap O_R = 0$ if and only if $k_1 = k_2 = \cdots = k_m$. Then and only then, $O_S \cap O_T = O_T$ (or equivalently, $O_{ST} = O_T$) and $O_{SR} = O_{BK}$.

More generally, when $k_1 - k_m \leq 1$, and only then, the BK-orbit has null codimension, or equivalently, it is open (hence, dense, from the Closed Orbit Lemma (see for example [10])).

When changing the controllability indices and the indices are not minorized, the dimension of the BK-orbits may be equal. Nevertheless, in these cases there is not a clear pattern between this dimension and the dimension of the ST-orbits or SR-orbits. See, for instance, Table 5.1.

k	codim O_{BK}	codim O_{ST}	codim O_{SR}
6,3,3,1	10	47	15
5,5,2,1	10	47	15
5,4,2,2	6	47	11
4,4,4,1	6	49	9
5,3,3,2	4	47	9

TABLE 5.1
Dimension of Orbits

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