Universitat Politècnica de Catalunya

Escola Tècnica Superior d'Enginyeria Industrial de Barcelona

Stability of (A, B)-invariant subspaces

Marta Peña, Ferran Puerta, Xavier Puerta

Departament de Matemàtica Aplicada I

Diagonal 647, Barcelona, Spain

ETSEIB, UPC

Regina

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Context

- Characterization of stable invariant subspaces
  - of an endomorphism: done
  - of a pair of matrices: open problem

Notation

$$\mathcal{M} = \{(A, B, W, F) : A \in M_n(\mathbb{F}),\ B \in M_{n,m}(\mathbb{F}),\ W \in \mathrm{Gr}_d(\mathbb{F}^n),$$
$$F \in M_{m,n}(\mathbb{F}),\ (A + BF)W \subset W\}$$

$\cap$

$$\mathcal{N} = M_n(\mathbb{F}) \times M_{n,m}(\mathbb{F}) \times \mathrm{Gr}_d(\mathbb{F}^n) \times M_{m,n}(\mathbb{F})$$

$\mathrm{Gr}_d(\mathbb{F}^n)$ set of d-dimensional subspaces of $\mathbb{F}^n$

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathcal{P}_d$ set of orthogonal projector operators of rank d,

$$\mathcal{P}_d = \{P \in M_n(\mathbb{F}) : P^* = P\ ,\ \ P^2 = P\ ,\ \ \mathrm{rank}\, P = d\}$$

Aim

To obtain computable conditions of stability of (A,B)-invariant subspaces

- Local coordinate charts of $\mathcal{M}$ and $\mathcal{N}$
- $\mathcal{M}$ is a manifold
- Sufficient condition of stability

Endomorphisms

- A-invariant subspace

Definition $A \in M_n(\mathbb{F})$, $W \in \mathrm{Gr}_d(\mathbb{F}^n)$

$$A(W) \subset W$$

- A-stable invariant subspace

Definition $A \in M_n(\mathbb{F})$, $W \in \mathrm{Gr}_d(\mathbb{F}^n)$ invariant is stable if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|A' - A\| < \delta$ for a linear map $A' \in M_n(\mathbb{F})$ implies that A' has an invariant subspace W' with $\Theta(W, W') < \varepsilon$

Pairs of matrices

- (A,B)-invariant subspace

Definition $A \in M_n(\mathbb{F})$, $B \in M_{n,m}(\mathbb{F})$, $W \in \mathrm{Gr}_d(\mathbb{F}^n)$

$$A(W) \subset W + \operatorname{Im} B$$

- (A,B)-stable invariant subspace

Definition $A \in M_n(\mathbb{F})$, $B \in M_{n,m}(\mathbb{F})$, $W \in \mathrm{Gr}_d(\mathbb{F}^n)$ (A,B)-invariant is (A,B)-stable if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|(A', B') - (A, B)\| < \delta$ for a pair $A' \in M_n(\mathbb{F})$, $B' \in M_{n,m}(\mathbb{F})$ implies that (A', B') has an (A', B')-invariant subspace W' with $\Theta(W, W') < \varepsilon$

Differentiable structure of $\mathcal{M}$

- Suggestion of U. Helmke $(\mathcal{P}_d)$
- Preliminary results

Proposition

(i) $\mathcal{P}_d$ submanifold of $M_n(\mathbb{F})$

$$\dim \mathcal{P}_d = d(n-d)$$

(ii) $T_P\mathcal{P}_d = \{[P,\Omega] : \Omega = -\Omega^*, \Omega \in M_n(\mathbb{F})\}$

$$[P,\Omega] = P\Omega - \Omega P$$

(iii) $\mathcal{P}_d \approx \mathrm{Gr}_d(\mathbb{F}^n)$

Proposition

$$\mathcal{N} = M_n(\mathbb{F}) \times M_{n,m}(\mathbb{F}) \times \mathcal{P}_d \times M_{m,n}(\mathbb{F})$$

$$\boxed{\mathcal{M} = \{(A,B,P,F) \in \mathcal{N} : (A+BF)P = P(A+BF)P\}}$$

Differentiable structure of $\mathcal{M}$

- Local coordinate charts of $\mathcal{M}$ and $\mathcal{N}$

We consider $(A_0, B_0, W_0, F_0) \in \mathcal{N}$ such that $W_0 = \operatorname{Im}\begin{bmatrix} I_d \\ 0 \end{bmatrix}$

Lemma

$$\mathcal{N}_0 = \Big\{(A, B, W, F) \;:\; A \in M_n(\mathbb{F}),\, B \in M_{n,m}(\mathbb{F}),\\ W = \operatorname{Im}\begin{bmatrix} I_d \\ Q \end{bmatrix}, Q \in M_{n-d,d}(\mathbb{F}),\, F \in M_{m,n}(\mathbb{F})\Big\}$$

is an open set of $\mathcal{N}$ that contains (A_0, B_0, W_0, F_0)

Lemma

$$\mathcal{M}_0 = \left\{\left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \operatorname{Im}\begin{bmatrix} I_d \\ Q \end{bmatrix}, [F_1 \quad F_2]\right) : \\ A_3 = QA_1 - A_4Q + QA_2Q + QB_1F_1 + QB_1F_2Q - B_2F_1 - B_2F_2Q\right\}$$

Differentiable structure of $\mathcal{M}$

- Local coordinate charts of $\mathcal{M}$ and $\mathcal{N}$

Proposition $\gamma: \mathbb{F}^{n^2+d(n-d)+2nm} \to \mathcal{N}_0$

$$\gamma(A_1, A_2, A_3, A_4, B_1, B_2, Q, F_1, F_2) =$$
$$= \left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \operatorname{Im} \begin{bmatrix} I_d \\ Q \end{bmatrix}, \begin{bmatrix} F_1 & F_2 \end{bmatrix} \right)$$

$(\mathcal{N}_0, \gamma)$ coordinate system of the manifold $\mathcal{N}$

Theorem $\psi : \mathbb{F}^{n^2+2nm} \to \mathbb{F}^{n^2+d(n-d)+2nm}$

$$\psi(A_1, A_2, A_4, B_1, B_2, Q, F_1, F_2) = (A_1, A_2, QA_1 - A_4 Q + QA_2 Q + + QB_1 F_1 + QB_1 F_2 Q - B_2 F_1 - B_2 F_2 Q, A_4, B_1, B_2, Q, F_1, F_2)$$

$\theta := \gamma \circ \psi$

$(\mathcal{M}_0, \theta)$ coordinate system of $\mathcal{M}$

$\Rightarrow$ $\mathcal{M}$ is a manifold, $\boxed{\dim \mathcal{M} = n^2 + 2nm}$

Differentiable structure of $\mathcal{M}$

$$\mathbb{F}^{n^2+d(n-d)+2nm} \xrightarrow{\gamma} \mathcal{N}_0$$

$$(A_1, A_2, A_3, A_4, B_1, B_2, Q, F_1, F_2) \qquad \left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \mathrm{Im} \begin{bmatrix} I_d \\ Q \end{bmatrix}, [F_1 \quad F_2] \right)$$

$$(A_1, A_2, QA_1 - A_4 Q + QA_2 Q + QB_1 F_1 + QB_1 F_2 Q - B_2 F_1 - B_2 F_2 Q, A_4, B_1, B_2, Q, F_1, F_2)$$

$$\cup$$

$$\mathcal{M}_0$$

$$\psi \uparrow \qquad \nearrow \theta = \gamma \circ \psi$$

$$\mathbb{F}^{n^2+2nm}$$

$$(A_1, A_2, A_4, B_1, B_2, Q, F_1, F_2)$$

Theorem $\mathcal{M}$ submanifold of $\mathcal{N}$

Proposition $\chi = (A, B, P, F) \in \mathcal{M}$,

$$T_\chi \mathcal{M} = \{ (\dot{A}, \dot{B}, \dot{P}, \dot{F}) : \dot{A} \in M_n(\mathbb{F}), \dot{B} \in M_{n,m}(\mathbb{F}), \dot{P} \in T_P \mathcal{P}_d, \dot{F} \in M_{m,n}(\mathbb{F}), (I-P)(A\dot{P} + \dot{A}P + BF\dot{P} + B\dot{F}P + \dot{B}FP) - \dot{P}(A + BF)P = 0 \}$$

Differentiable structure of $\mathcal{M}$

Proof (Tangent space of $\mathcal{M}$)

Smooth map φ

$$\varphi: \ \mathcal{N} \to M_n(\mathbb{F})$$
$$\chi = (A, B, P, F) \ \mapsto \ (A + BF)P - P(A + BF)P$$

$$\mathcal{M} = \varphi^{-1}(0)$$

$$\mathrm{d}\varphi_\chi(\dot{A}, \dot{B}, \dot{P}, \dot{F}) = (\dot{A} + \dot{B}F + B\dot{F})P + (A + BF)\dot{P}$$
$$- \dot{P}(A + BF)P - P(\dot{A} + \dot{B}F + B\dot{F})P - P(A + BF)\dot{P}$$

$$\langle X, L^* \rangle = \mathrm{tr}\big(XL\big) \ ; \ \ L^* \in \big(\mathrm{Im}\ \mathrm{d}\varphi_\chi\big)^\perp \Leftrightarrow$$

$$\mathrm{tr}\Big((\dot{A} + \dot{B}F + B\dot{F})P + (A + BF)\dot{P} - \dot{P}(A + BF)P +$$
$$+[P, \Omega](L(A + BF) - LP(A + BF) - (A + BF)PL)\Big) = 0$$
$$\forall \dot{A} \in M_n(\mathbb{F}), \forall \dot{B} \in M_{n,m}(\mathbb{F}), \forall \dot{F} \in M_{m,n}(\mathbb{F}),$$
$$\forall \Omega \in M_n(\mathbb{F}), \Omega = -\Omega^*$$

Differentiable structure of $\mathcal{M}$

$$\Leftrightarrow \quad PL(I-P)=0$$

$$\left(\mathrm{Im}\ \mathrm{d}\varphi_{\chi}\right)^{\perp}=\left\{L^{*}\in M_{n}(\mathbb{F}):\ PL(I-P)=0\right\}=$$
$$=\left\{L\in M_{n}(\mathbb{F}):\ (I-P)LP=0\right\}$$

$$P=\begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix},\ L=\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$$

$$\Leftrightarrow \quad L_3=0$$

$$\mathrm{rank}\ \mathrm{d}\varphi_{\chi}=d(n-d)$$

$$\dim\mathcal{M}=\dim\mathcal{N}-\mathrm{rank}\ \mathrm{d}\varphi_{\chi}=$$
$$=(n^2+d(n-d)+2nm)-d(n-d)=n^2+2nm$$

$$\dim\left(\mathrm{Ker}\ \mathrm{d}\varphi_{\chi}\right)=\dim\left(T_{\chi}\mathcal{M}\right)$$

$$T_{\chi}\mathcal{M}=\mathrm{Ker}\ \mathrm{d}\varphi_{\chi}$$

Stability of (*A*,*B*)-invariant subspaces

$$\begin{array}{ccc} & M_n(\mathbb{F}) \times M_{n,m}(\mathbb{F}) \times M_{m,n}(\mathbb{F}) & \\ \dot{\pi}_1 \uparrow & & \nwarrow \pi_1 \\ \mathcal{M} & \subset & \mathcal{N} \\ \dot{\pi}_2 \downarrow & & \swarrow \pi_2 \\ & \mathrm{Gr}_d(\mathbb{F}^n) & \end{array}$$

$$\pi_1(A, B, P, F) = (A, B, F)$$

$$\pi_2(A, B, P, F) = \mathrm{Im}\, P$$

- Properties of $\dot{\pi}_2$

Proposition

(i) $\chi = (A, B, P, F) \in \mathcal{M}$, $\boxed{\mathrm{rank}\ d\dot{\pi}_{2,\chi} = d(n-d)}$

(ii) $\dot{\pi}_2 : \mathcal{M} \to \mathrm{Gr}_d(\mathbb{F}^n)$ is a submersion

Stability of (*A*,*B*)-invariant subspaces

- Sufficient condition of stability

Theorem $\chi = (A, B, P, F) \in \mathcal{M}$,

$$\boxed{\mathrm{d}\dot{\pi}_{1,\chi} \ \text{ bijective } \ \Rightarrow \ W \ \ (A,B)\text{-stable}}$$

$$\Updownarrow$$

$$\left\{ \dot{P} \in T_P \mathcal{P}_d : \ (I-P)(A+BF)\dot{P} - \dot{P}(A+BF)P = 0 \right\} = \{0\}$$

Comparison with stability of *A*-invariant subspaces

Given an endomorphism,

$$\bar{\mathcal{M}} = \left\{ (A,W) : A \in M_n(\mathbb{F}),\, W \in \mathrm{Gr}_d(\mathbb{F}^n),\; A(W) \subset W \right\}$$

$$\bar{\mathcal{N}} = M_n(\mathbb{F}) \times \mathrm{Gr}_d(\mathbb{F}^n)$$

$$\boxed{\begin{array}{ccc} & M_n(\mathbb{F}) & \\ \dot{\bar{\pi}}_1 \uparrow & & \nwarrow \bar{\pi}_1 \\ \bar{\mathcal{M}} & \subset & \bar{\mathcal{N}} \\ \dot{\bar{\pi}}_2 \downarrow & & \swarrow \bar{\pi}_2 \\ & \mathrm{Gr}_d(\mathbb{F}^n) & \end{array}}$$

$$\bar{\pi}_1(A,W) = A$$

$$\bar{\pi}_2(A,W) = W$$

- Sufficient condition of stability

Theorem $(A,W) \in \mathcal{M}$,

$$\boxed{d\dot{\bar{\pi}}_{1,(A,W)} \ \text{bijective} \ \Rightarrow \ W \ \ A-\text{stable}}$$

$$\Updownarrow$$

$$\left. \begin{cases} -P\Omega AP + \Omega AP - A\Omega P + PA\Omega P = 0 \\ \Omega \in M_n(\mathbb{F}),\; \Omega = -\Omega^* \end{cases} \right\} \Rightarrow P\Omega = \Omega P$$

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