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## On a kind of Noether symmetries and conservation laws in $k$ -cosymplectic field theory

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This paper is devoted to studying symmetries of certain kinds of  $k$ -cosymplectic Hamiltonian systems in first-order classical field theories. Thus, we introduce a particular class of symmetries and study the problem of associating conservation laws to them by means of a suitable generalization of Noether's theorem. © 2011 American Institute of Physics. [doi:10.1063/1.3545969]

### I. INTRODUCTION

The  $k$ -cosymplectic formalisms is one of the simplest geometric frameworks for describing first-order classical field theories. It is the generalization to field theories of the standard cosymplectic formalism for nonautonomous mechanics,<sup>9,10</sup> and it describes field theories involving the coordinates in the basis on the Lagrangian and on the Hamiltonian. The foundations of the  $k$ -cosymplectic formalism are the  $k$ -cosymplectic manifolds.<sup>9,10</sup>

Historically, it is based on the so-called *polysymplectic formalism* developed by Günther,<sup>16</sup> who introduced *polysymplectic manifolds*. A refinement of this concept led to define  *$k$ -symplectic manifolds*,<sup>2-4</sup> which are polysymplectic manifolds admitting *Darboux-type coordinates*.<sup>8</sup> (Other different polysymplectic formalisms for describing field theories have been also proposed.<sup>13,17,22,25,26,29</sup>)

The natural extension of the  $k$ -symplectic manifolds is the  $k$ -cosymplectic manifolds. All of this is discussed in Sec. II, which is devoted to make a review on the main features and characteristics of  $k$ -cosymplectic manifolds and of  $k$ -cosymplectic Hamiltonian systems. We also introduce the notions of *almost standard  $k$ -cosymplectic manifold*, which are those that we are interested in this paper.

The main objective of this paper is to study symmetries and conservation laws on the first-order classical field theories, from the Hamiltonian viewpoint, using the  $k$ -cosymplectic description, and considering only the regular case. These problems have been treated for  $k$ -symplectic field theories in Refs. 23 and 28, generalizing the results obtained for nonautonomous mechanical systems (see, in particular, Ref. 6, and references quoted therein). We further remark that the problem of symmetries in field theory has also been analyzed using other geometric frameworks, such as the *multisymplectic models* (see, for instance, Refs. 5, 7, 12, 14, 15, 18, and 19).

In this way, in Sec. III we recover the idea of *conservation law* or *conserved quantity*. Then, we introduce a particular kind of symmetries for (almost-standard)  $k$ -cosymplectic Hamiltonian

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systems, essentially those transformations preserving the  $k$ -cosymplectic structure, which allows us to state a generalization of Noether's theorem. The definition of these so-called  $k$ -cosymplectic Noether symmetries is inspired in the ideas introduced by Albert in his study of symmetries for the cosymplectic formalism of autonomous mechanical systems.<sup>1</sup>

Finally, as an example, in Sec. IV we describe briefly the  $k$ -cosymplectic quadratic Hamiltonian systems and we analyze some Noether symmetries for these kinds of systems (in particular, for the wave equation).

In this paper, manifolds are real, paracompact, connected and  $C^\infty$ , maps are  $C^\infty$ , and sum over crossed repeated indices is understood.

## II. GEOMETRIC ELEMENTS: HAMILTONIAN $k$ -COSYMPLECTIC FORMALIS

(The contents of this section can be seen in more detail in Ref. 10.)

### A. $k$ -vector fields and integral sections

Let  $M$  be an arbitrary manifold,  $T_k^1 M$  be the Whitney sum  $TM \oplus \dots \oplus TM$  of  $k$  copies of  $TM$ , and  $\tau : T_k^1 M \rightarrow M$  be its canonical projection.  $T_k^1 M$  is usually called the tangent bundle of  $k^1$ -velocities of  $M$ .

*Definition 1:* A  $k$ -vector field on  $M$  is a section  $\mathbf{X} : M \rightarrow T_k^1 M$  of the projection  $\tau$ .

Giving a  $k$ -vector field  $\mathbf{X}$  is equivalent to giving a family of  $k$  vector fields  $X_1, \dots, X_k$  on  $M$  obtained by projecting  $\mathbf{X}$  onto every factor; that is,  $X_A = \tau_A \circ \mathbf{X}$ , where  $\tau_A : T_k^1 M \rightarrow TM$  is the canonical projection onto the  $A$ th-copy  $TM$  of  $T_k^1 M$  ( $1 \leq A \leq k$ ). For this reason we will denote a  $k$ -vector field by  $\mathbf{X} = (X_1, \dots, X_k)$ .

*Definition 2:* An integral section of the  $k$ -vector field  $(X_1, \dots, X_k)$  passing through a point  $x \in M$  is a map  $\psi : U_0 \subset \mathbb{R}^k \rightarrow M$ , defined on some neighborhood  $U_0$  of  $0 \in \mathbb{R}^k$ , such that

$$\psi(0) = x, \quad \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = X_A(\psi(t)) \text{ (for every } t \in U_0),$$

A  $k$ -vector field  $\mathbf{X}$  is integrable if every point of  $M$  belongs to the image of an integral section of  $\mathbf{X}$ .

In coordinates, if  $X_A = X_A^i \frac{\partial}{\partial q^i}$ , then  $\psi$  is an integral section of  $\mathbf{X}$  if, and only if, the following system of partial differential equations holds

$$\frac{\partial \psi^i}{\partial t^A} = X_A^i \circ \psi.$$

We remark that a  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  is integrable if, and only if, the vector fields  $X_1, \dots, X_k$  generate a completely integrable distribution of rank  $k$ . This is the geometric expression of the integrability condition of the preceding differential equation (see, for instance, Refs. 11 and 21).

Observe that, in case  $k = 1$ , this definition coincides with the definition of integral curve of a vector field.

### B. $k$ -symplectic manifolds

The polysymplectic structures were introduced in Ref. 16 and the  $k$ -symplectic structures in Refs. 2 and 14.

*Definition 3:* Let  $M$  be a differentiable manifold of dimension  $N = n + kn$ .

1. A polysymplectic structure on  $M$  is a family  $(\omega_0^A)$  ( $1 \leq A \leq k$ ), where each  $\omega_0^A \in \Omega^2(M)$  is a closed form, such that,

$$\bigcap_{A=1}^k \ker \omega_0^A = \{0\}.$$

Then  $(M, \omega_0^A)$  is called a polysymplectic manifold.

2. A  $k$ -symplectic structure on  $M$  is a family  $(\omega_0^A, V)$  ( $1 \leq A \leq k$ ), such that  $(M, \omega_0^A)$  is a polysymplectic manifold and  $V$  is an integrable  $nk$ -dimensional tangent distribution on  $M$  satisfying that

$$\omega_0^A|_{V \times V} = 0, \text{ for every } A.$$

Then  $(M, \omega_0^A, V)$  is called a  $k$ -symplectic manifold.

The  $k$ -symplectic (respectively, polysymplectic) structure is exact if  $\omega_0^A = d\theta_0^A$ , for all  $A$ .

**Theorem 1:** (Ref. 8). Let  $(\omega_0^A, V)$  be a  $k$ -symplectic structure on  $M$ . For every point of  $M$ , there exists a neighborhood  $U$  and local coordinates  $(q^i, p_i^A)$  ( $1 \leq i \leq n, 1 \leq A \leq k$ ) such that, on  $U$ ,

$$\omega_0^A = dq^i \wedge dp_i^A, \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1, \dots, n}.$$

These are called Darboux or canonical coordinates of the  $k$ -symplectic manifold.

The canonical model of a  $k$ -symplectic manifold is  $((T_k^1)^*Q, \omega_0^A, V)$ , where  $Q$  is an  $n$ -dimensional differentiable manifold and  $(T_k^1)^*Q = T^*Q \oplus \dots \oplus T^*Q$  is the Whitney sum of  $k$  copies of the cotangent bundle  $T^*Q$ , which is usually called the bundle of  $k^1$ -covelocities of  $Q$  (see Ref. 20). We have the natural projections

$$\begin{aligned} \pi^A : (T_k^1)^*Q &\rightarrow T^*Q \\ (q; \alpha_q^1, \dots, \alpha_q^k) &\mapsto (q; \alpha_q^A), \end{aligned}$$

$$\begin{aligned} (\pi_Q)_1 : (T_k^1)^*Q &\rightarrow Q \\ (q; \alpha_q^1, \dots, \alpha_q^k) &\mapsto q. \end{aligned}$$

The manifold  $(T_k^1)^*Q$  can be identified with the manifold  $J^1(Q, \mathbb{R}^k)_0$  of 1-jets of mappings from  $Q$  to  $\mathbb{R}^k$  with target at  $0 \in \mathbb{R}^k$ , that is,

$$\begin{aligned} J^1(Q, \mathbb{R}^k)_0 &\equiv T^*Q \oplus \dots \oplus T^*Q \\ j_{q,0}^1 \sigma_Q &\equiv (d\sigma_Q^1(q), \dots, d\sigma_Q^k(q)), \end{aligned}$$

where  $\sigma_Q^A = \pi_0^A \circ \sigma_Q : Q \rightarrow \mathbb{R}$  is the  $A$ th component of  $\sigma_Q$ , and  $\pi_0^A : \mathbb{R}^k \rightarrow \mathbb{R}$  is the canonical projection onto the  $A$  component.

Here,  $(T_k^1)^*Q$  is endowed with the canonical forms

$$\theta^A = (\pi^A)^* \theta_0, \quad \omega_0^A = (\pi^A)^* \omega_0 = -(\pi^A)^* d\theta_0,$$

where  $\theta_0$  and  $\omega_0 = -d\theta_0$  are the Liouville 1-form and the canonical symplectic form on  $T^*Q$ . Obviously  $\omega_0^A = -d\theta_0^A$ .

If  $(q^i)$  are local coordinates on  $U \subset Q$ , the induced coordinates  $(q^i, p_i^A)$  on  $(\pi_Q^1)^{-1}(U)$  are given by

$$\begin{aligned} q^i(q; \alpha_q^1, \dots, \alpha_q^k) &= q^i(q) \\ p_i^A(q; \alpha_q^1, \dots, \alpha_q^k) &= \alpha_q^A \left( \frac{\partial}{\partial q^i} \Big|_q \right). \end{aligned}$$

Then we have

$$\theta_0^A = p_i^A dq^i, \quad \omega_0^A = dq^i \wedge dp_i^A.$$

Thus, the triple  $((T_k^1)^*Q, \omega_0^A, V)$ , where  $V = \ker T(\pi_Q)_1$ , is a  $k$ -symplectic manifold, and the natural coordinates in  $(T_k^1)^*Q$  are Darboux coordinates.

**C. k-cosymplectic manifolds**

*Definition 4:* Let  $\mathcal{M}$  be a differentiable manifold of dimension  $N = k + n + kn$ .

1. A polycosymplectic structure in  $\mathcal{M}$  is a family  $(\eta^A, \omega^A)$ , where  $\eta^A \in \Omega^1(\mathcal{M})$  and  $\omega^A \in \Omega^2(\mathcal{M})$  are closed forms satisfying that

- (a)  $\eta^1 \wedge \dots \wedge \eta^k \neq 0$ .
- (b)  $(\cap_{A=1}^k \ker \omega^A \cap_{A=1}^k \ker \eta^A) = \{0\}$ .

Then,  $(\mathcal{M}, \eta^A, \omega^A)$  is said to be a polycosymplectic manifold.

2. A  $k$ -cosymplectic structure in  $\mathcal{M}$  is a family  $(\eta^A, \omega^A, \mathcal{V})$  such that  $(\mathcal{M}, \eta^A, \omega^A)$  is a polycosymplectic manifold, and  $\mathcal{V}$  is an  $nk$ -dimensional integrable distribution on  $\mathcal{M}$ , satisfying that

- (a)  $\eta^A|_{\mathcal{V}} = 0$ .
- (b)  $\omega^A|_{\mathcal{V} \times \mathcal{V}} = 0$ .

Then,  $(\mathcal{M}, \eta^A, \omega^A, \mathcal{V})$  is said to be a  $k$ -cosymplectic manifold.

The  $k$ -cosymplectic (respectively, polycosymplectic) structure is exact if  $\omega^A = d\theta^A$ , for all  $A$ .

For every  $k$ -cosymplectic structure  $(\eta^A, \omega^A, \mathcal{V})$  on  $\mathcal{M}$ , there exists a family of  $k$  vector fields  $\{R_A\}_{1 \leq A \leq k}$ , which are called *Reeb vector fields*, characterized by the following conditions:<sup>9</sup>

$$i(R_A)\eta^B = \delta_A^B, \quad i(R_A)\omega^B = 0; \quad 1 \leq A, B \leq k.$$

**Theorem 2:** (Darboux Theorem).<sup>9</sup> If  $\mathcal{M}$  is a  $k$ -cosymplectic manifold, then for every point of  $\mathcal{M}$  there exists a local chart of coordinates  $(t^A, q^i, p_i^A)$ ,  $1 \leq A \leq k$ ,  $1 \leq i \leq n$ , such that

$$\eta^A = dt^A, \quad \omega^A = dq^i \wedge dp_i^A, \quad R_A = \frac{\partial}{\partial t^A}$$

$$\mathcal{V} = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1, \dots, n}.$$

These are called *Darboux* or *canonical coordinates* of the  $k$ -cosymplectic manifold.

The canonical model for  $k$ -cosymplectic manifolds is  $(\mathbb{R}^k \times (T_k^1)^*Q, \eta^A, \omega^A, \mathcal{V})$ . The manifold  $J^1\pi_Q$  of 1-jets of sections of the trivial bundle  $\pi_Q : \mathbb{R}^k \times Q \rightarrow Q$  is diffeomorphic to  $\mathbb{R}^k \times (T_k^1)^*Q$ . We use also the following notation for the canonical projections:

$$(\pi_Q)_1 : \mathbb{R}^k \times (T_k^1)^*Q \xrightarrow{(\pi_Q)_{1,0}} \mathbb{R}^k \times Q \xrightarrow{\pi_Q} Q$$

given by

$$\pi_Q(t, q) = q, \quad (\pi_Q)_{1,0}(t, \alpha_q^1, \dots, \alpha_q^k) = (t, q),$$

$$(\pi_Q)_1(t, \alpha_q^1, \dots, \alpha_q^k) = q,$$

with  $t \in \mathbb{R}^k$ ,  $q \in Q$ , and  $(\alpha_q^1, \dots, \alpha_q^k) \in (T_k^1)^*Q$ .

If  $(q^i)$  are local coordinates on  $U \subseteq Q$ , then the induced local coordinates  $(t^A, q^i, p_i^A)$  on  $[(\pi_Q)_1]^{-1}(U) = \mathbb{R}^k \times (T_k^1)^*U$  are given by

$$t^A(t, \alpha_q^1, \dots, \alpha_q^k) = t^A; \quad q^i(t, \alpha_q^1, \dots, \alpha_q^k) = q^i(q);$$

$$p_i^A(t, \alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A \left( \frac{\partial}{\partial q^i} \Big|_q \right).$$

On  $\mathbb{R}^k \times (T_k^1)^*Q$ , we define the differential forms

$$\eta^A = (\pi_1^A)^* dt^A, \quad \theta^A = (\pi_2^A)^* \theta_0, \quad \omega^A = (\pi_2^A)^* \omega_0,$$

where  $\pi_1^A : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow \mathbb{R}$  and  $\pi_2^A : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow T^*Q$  are the projections defined by

$$\pi_1^A(t, (\alpha_q^1, \dots, \alpha_q^k)) = t^A, \quad \pi_2^A(t, (\alpha_q^1, \dots, \alpha_q^k)) = \alpha_q^A,$$

In local coordinates, we have

$$\eta^A = dt^A, \quad \theta^A = \sum_{i=1}^n p_i^A dq^i, \quad \omega^A = \sum_{i=1}^n dq^i \wedge dp_i^A.$$

Moreover, let  $\mathcal{V} = \ker T(\pi_Q)_{1,0}$ . Then  $\mathcal{V} = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1, \dots, n}$ .

Hence  $(\mathbb{R}^k \times (T_k^1)^*Q, \eta^A, \omega^A, \mathcal{V})$  is a  $k$ -cosymplectic manifold, and the natural coordinates of  $\mathbb{R}^k \times (T_k^1)^*Q$  are Darboux coordinates for this canonical  $k$ -cosymplectic structure. Furthermore,  $\left\{ \frac{\partial}{\partial t^A} \right\}$  are the Reeb vector fields of this structure.

Now, let  $\varphi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$  be a diffeomorphism of  $\pi_Q$ -fiber bundles, and let  $\varphi_Q : Q \rightarrow Q$  be the diffeomorphism induced on the base. We can lift  $\varphi$  to a diffeomorphism  $j^{1*}\varphi : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow \mathbb{R}^k \times (T_k^1)^*Q$  such that the following diagram commutes:

$$\begin{array}{ccc} J^1\pi_Q \equiv \mathbb{R}^k \times (T_k^1)^*Q & \xrightarrow{j^{1*}\varphi} & J^1\pi_Q \equiv \mathbb{R}^k \times (T_k^1)^*Q \\ \downarrow (\pi_Q)_{1,0} & & \downarrow (\pi_Q)_{1,0} \\ \mathbb{R}^k \times Q & \xrightarrow{\varphi} & \mathbb{R}^k \times Q \\ \downarrow \pi_Q & & \downarrow \pi_Q \\ Q & \xrightarrow{\varphi_Q} & Q \end{array}$$

*Definition 5:* Let  $\varphi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$  be a  $\pi_Q$ -bundles morphism in the above conditions. The canonical prolongation of the diffeomorphism  $\varphi$  is the map  $j^{1*}\varphi : J^1\pi_Q \rightarrow J^1\pi_Q$  given by

$$(j^{1*}\varphi)(j_q^1\sigma) := j_{\varphi_Q(q)}^1(\varphi \circ \sigma \circ \varphi_Q^{-1}),$$

where  $\sigma = (\sigma_{\mathbb{R}^k}, Id_Q)$  and  $\sigma_{\mathbb{R}^k} : Q \xrightarrow{\sigma} \mathbb{R}^k \times Q \xrightarrow{\pi_{\mathbb{R}^k}} \mathbb{R}^k$ .

It is clear that this definition is valid because choosing other representative  $\sigma'$  with the same 1-jet at  $q$  gives the same result, that is,  $j^{1*}\varphi(j_q^1\sigma)$  is well defined.

In local coordinates, if  $\varphi(t^B, q^j) = (\varphi^A(t^B, q^j), \varphi_Q^i(q^j))$ , then

$$j^{1*}\varphi(t^B, q^j, p_j^B) = \left( \varphi^A(t^B, q^j), \varphi_Q^j(q^j), \left( \frac{\partial \varphi^A}{\partial q^k} + p_k^B \frac{\partial \varphi^A}{\partial t^B} \right) \frac{\partial (\varphi_Q^{-1})^k}{\partial q^i} \Big|_{\varphi_Q(q^j)} \right).$$

*Definition 6:* Let  $Z \in \mathfrak{X}(\mathbb{R}^k \times Q)$  be a  $\pi_Q$ -projectable vector field, with local 1-parameter group of transformations  $\varphi_s : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$ . Then the local 1-parameter group of transformations  $j^{1*}\varphi_s : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow \mathbb{R}^k \times (T_k^1)^*Q$  generates a vector field  $Z^{1*} \in \mathfrak{X}(\mathbb{R}^k \times (T_k^1)^*Q)$ , which is called the complete lift of  $Z$  to  $\mathbb{R}^k \times (T_k^1)^*Q$ .

If the local expression of  $Z \in \mathfrak{X}(\mathbb{R}^k \times Q)$  is  $Z = Z^A \frac{\partial}{\partial t^A} + Z^i \frac{\partial}{\partial q^i}$ , then

$$Z^{1*} = Z^A \frac{\partial}{\partial t^A} + Z^i \frac{\partial}{\partial q^i} + \left( \frac{dZ^A}{dq^i} - p_j^A \frac{dZ^j}{dq^i} \right) \frac{\partial}{\partial p_i^A},$$

where  $\frac{d}{dq^i}$  denotes the total derivative, that is,  $\frac{d}{dq^i} = \frac{\partial}{\partial q^i} + p_i^B \frac{\partial}{\partial t^B}$ .

#### D. *k*-cosymplectic Hamiltonian systems

Along this paper we are interested only in a kind of *k*-cosymplectic manifolds: those which are of the form  $\mathcal{M} = \mathbb{R}^k \times M$ , where  $(M, \omega_0^A, V)$  is a generic *k*-symplectic manifold. Then, denoting by

$$\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k, \quad \pi_M : \mathbb{R}^k \times M \rightarrow M$$

the canonical projections, we have the differential forms

$$\eta^A = \pi_{\mathbb{R}^k}^* dt^A, \quad \omega^A = \pi_M^* \omega_0^A,$$

and the distribution  $V$  in  $M$  defines a distribution  $\mathcal{V}$  in  $\mathcal{M} = \mathbb{R}^k \times M$  in a natural way. All the conditions given in Definition 4 are verified, and hence  $\mathbb{R}^k \times M$  is a *k*-cosymplectic manifold. From the Darboux Theorem 1, we have local coordinates  $(t^A, q^i, p_i^A)$  in  $\mathbb{R}^k \times M$ .

Observe that the standard model is a particular class of these kinds of *k*-cosymplectic manifolds, where  $M = (T_k^1)^*Q$ .

*Definition 7:* These kinds of *k*-cosymplectic manifolds will be called almost-standard *k*-cosymplectic manifolds.

Consider an almost-standard *k*-cosymplectic manifold  $(\mathbb{R}^k \times M, \eta^A, \omega^A, \mathcal{V})$ , and let  $H \in C^\infty(\mathbb{R}^k \times M)$  be a Hamiltonian function. The couple  $(\mathbb{R}^k \times M, H)$  is called a *k*-cosymplectic Hamiltonian system.

We denote by  $\mathfrak{X}_H^k(\mathbb{R}^k \times M)$  the set of (local) *k*-vector fields  $\mathbf{X} = (X_1, \dots, X_k)$  on  $\mathbb{R}^k \times M$  which are solutions to the equations

$$\eta^A(X_B) = \delta_B^A, \quad \sum_{A=1}^k i(X_A)\omega^A = dH - \sum_{A=1}^k R_A(H)\eta^A. \tag{1}$$

Since  $R_A = \partial/\partial t^A$  and  $\eta^A = dt^A$ , then we can write locally the above equations as follows:

$$dt^A(X_B) = \delta_B^A, \quad \sum_{A=1}^k i(X_A)\omega^A = dH - \sum_{A=1}^k \frac{\partial H}{\partial t^A} dt^A.$$

Furthermore, for a section  $\psi : I \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  of the projection  $\pi_{\mathbb{R}^k}$ , the *Hamilton–de Donder–Weyl equations* for this system are

$$\sum_{A=1}^k i \left( \psi_{*}(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) \right) (\omega^A \circ \psi) = \left[ dH - \sum_{A=1}^k R_A(H)\eta^A \right] \circ \psi, \tag{2}$$

In Darboux coordinates, if  $\psi(t) = (\psi^A(t), \psi^i(t), \psi_i^A(t))$ , as  $\psi$  is a section of the projection  $\pi_{\mathbb{R}^k}$ , it implies that  $\psi^A(t) = t^A$  the above equations lead to the equations

$$\frac{\partial H}{\partial q^i} = - \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A}, \quad \frac{\partial H}{\partial p_i^A} = \frac{\partial \psi^i}{\partial t^A}. \tag{3}$$

The relation between Equations (1) and (2) is given by the following:

**Theorem 3:** Let  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  [i.e., it is a  $k$ -vector field on  $\mathbb{R}^k \times M$  which is a solution to the geometric Hamiltonian equations (1)]. If a section  $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  of  $\pi_{\mathbb{R}^k}$  is an integral section of  $\mathbf{X}$ , then  $\psi$  is a solution to the Hamilton–de Donder–Weyl field equations (2).

*Proof:* Let  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  be locally given by

$$X_A = (X_A)^B \frac{\partial}{\partial t^B} + (X_A)^i \frac{\partial}{\partial q^i} + (X_A)_i^B \frac{\partial}{\partial p_i^B},$$

then, from (1) we obtain

$$(X_A)^B = \delta_A^B, \quad \frac{\partial H}{\partial p_i^A} = (X_A)^i, \quad \frac{\partial H}{\partial q^i} = - \sum_{A=1}^k (X_A)_i^A, \quad (4)$$

and if  $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$ , locally given by  $\psi(t) = (t^A, \psi^i(t), \psi_i^A(t))$ , is an integral section of  $\mathbf{X}$ , then

$$\frac{\partial \psi^i}{\partial t^B} = (X_B)^i, \quad \frac{\partial \psi_i^A}{\partial t^B} = (X_B)_i^A.$$

Therefore, from (4) we obtain that  $\psi(t)$  is a solution to the Hamiltonian field equations (3). ■

And, conversely, we have the following:

*Lemma 1:* If a section  $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  of  $\pi_{\mathbb{R}^k}$  is a solution to the Hamilton–de Donder–Weyl equation (2) and  $\psi$  is an integral section of  $\mathbf{X} = (X_1, \dots, X_k)$ , then  $\mathbf{X} = (X_1, \dots, X_k)$  is solution to the equations (1) at the points of the image of  $\psi$ .

*Proof:* We must prove that

$$\begin{aligned} \frac{\partial H}{\partial p_i^A}(\psi(t)) &= (X_A)^i(\psi(t)), \\ \frac{\partial H}{\partial q^i}(\psi(t)) &= - \sum_{A=1}^k (X_A)_i^A(\psi(t)), \end{aligned} \quad (5)$$

now as  $\psi(t) = (t^A, \psi^i(t), \psi_i^A(t))$  is integral section of  $\mathbf{X}$  we have that

$$\frac{\partial \psi^i}{\partial t^B}(t) = (X_B)^i(\psi(t)), \quad \frac{\partial \psi_i^A}{\partial t^B}(t) = (X_B)_i^A(\psi(t)). \quad (6)$$

As  $\psi$  is a solution to the Hamilton–de Donder–Weyl equation (3) then, from (6), we deduce (5). ■

We cannot claim that  $\mathbf{X} \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  because we cannot assure that  $\mathbf{X}$  is a solution to the equations (1) everywhere in  $\mathbb{R}^k \times M$ .

*Proposition 1:* If  $\psi : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  is a solution to the Hamilton–de Donder–Weyl equation (2), then for each  $t \in U_0$  there exist a neighborhood  $U_t$  of  $t$  and a  $k$ -vector field  $\mathbf{X}^t = (X_1^t, \dots, X_k^t)$  on  $\psi(U_t)$  which is solution to the equations (1) in  $\psi(U_t)$ .

*Proof:* If  $\psi : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  is a solution to the Hamilton–de Donder–Weyl equation (2) then for every  $t \in U_0$  there exists a neighborhood  $U_t \subset U_0$  of  $t$ , and a neighborhood coordinate system  $(W_t, s^A, q^i, p_i^A)$  of  $\psi(t)$ , such that  $\psi(U_t) = W_t \subset \psi(U_0)$ , and  $\psi(s) = (s, \psi^i(s), \psi_i^A(s))$  for every  $s \in U_t$ .

As  $\psi|_{U_t} : U_t \rightarrow W_t$  is an injective immersion ( $\psi$  is a section and hence its image is an embedded submanifold), we can define a  $k$ -vector field  $\mathbf{X}^t = (X_1^t, \dots, X_k^t)$  in  $\psi(U_t)$  as follows:

$$X_A^t(\psi(s)) = \psi_*(s) \left( \frac{\partial}{\partial s^A} \Big|_s \right), \quad s \in U_t,$$

and so  $\psi|_{U_t}$  is an integral section of  $\mathbf{X}^t$ . Then, from Lemma 1, one obtains that  $\mathbf{X}^t$  is solution to the equations (1) in  $\psi(U_t)$ . ■



*Remark:* It should be noticed that, in general, equations (1) do not have a single solution. In fact, if  $(\mathcal{M}, \eta^A, \omega^A, \mathcal{V})$  is a  $k$ -cosymplectic manifold we can define the vector bundle morphism,

$$\begin{aligned} \omega^\sharp : T_k^1 \mathcal{M} &\longrightarrow T^* \mathcal{M} \\ (X_1, \dots, X_k) &\mapsto \sum_{A=1}^k i(X_A) \omega^A \end{aligned}$$

and, denoting by  $\mathbb{M}_k(\mathbb{R})$  the space of matrices of order  $k$  whose entries are real numbers, the vector bundle morphism

$$\begin{aligned} \eta^\sharp : T_k^1 \mathcal{M} &\longrightarrow \mathcal{M} \times \mathbb{M}_k(\mathbb{R}) \\ (X_1, \dots, X_k) &\mapsto (\tau(X_1, \dots, X_k), \eta^A(X_B)). \end{aligned}$$

We denote by the same symbols  $\omega^\sharp, \eta^\sharp$  their natural extensions to vector fields and forms.

Now, let  $H : \mathcal{M} \rightarrow \mathbb{R}$  be a real  $C^\infty$ -function on  $\mathcal{M}$ . Then, as in the case of an almost-standard  $k$ -cosymplectic manifold, we can consider the set  $\mathfrak{X}_H^k(\mathcal{M})$  of the (local)  $k$ -vector fields  $\mathbf{X} = (X_1, \dots, X_k)$  on  $\mathcal{M}$  which are solutions to the equations

$$\eta^A(X_B) = \delta_B^A, \quad \sum_{A=1}^k i(X_A) \omega^A = dH - \sum_{A=1}^k R_A(H) \eta^A. \tag{7}$$

Moreover, we may prove the following result:

*Proposition 2:* The solutions to Eqs. (7) are the sections of an affine bundle of rank  $(k - 1)(kn + n)$  which is modeled on the vector sub-bundle  $\ker \omega^\sharp \cap \ker \eta^\sharp$  of  $T_k^1 \mathcal{M}$ .

*Proof:* We consider the vector sub-bundle  $\ker \eta^\sharp$  of  $T_k^1 \mathcal{M}$  and the vector bundle morphism

$$\omega^\sharp|_{\ker \eta^\sharp} : \ker \eta^\sharp \rightarrow T^* \mathcal{M}.$$

It is clear that this morphism takes values in the vector sub-bundle  $\cap_{A=1}^k \langle R_A \rangle^0$  of  $T^* \mathcal{M}$ , where  $\langle R_A \rangle^0$  is the vector sub-bundle of  $T^* \mathcal{M}$  whose fiber at the point  $x \in \mathcal{M}$  is  $\{\alpha \in T_x^* \mathcal{M} / \alpha(R_A(x)) = 0\}$ . Furthermore, we have that

$$\ker(\omega^\sharp|_{\ker \eta^\sharp}) = \ker \omega^\sharp \cap \ker \eta^\sharp.$$

We will prove that

$$\omega^\sharp|_{\ker \eta^\sharp} : \ker \eta^\sharp \rightarrow \cap_{A=1}^k \langle R_A \rangle^0$$

is an epimorphism of vector bundles. For this purpose, we will see that the dual morphism

$$(\omega^\sharp|_{\ker \eta^\sharp})^* : (\cap_{A=1}^k \langle R_A \rangle^0)^* \rightarrow (\ker \eta^\sharp)^*$$

is a monomorphism of vector bundles.

First, it is clear that the dual bundle to  $\cap_{A=1}^k \langle R_A \rangle^0$  (respectively,  $\ker \eta^\sharp$ ) may be identified with the vector bundle whose fiber at the point  $x \in \mathcal{M}$  is  $\cap_{A=1}^k \langle \eta^A(x) \rangle^0$  [respectively,  $\{(\alpha_1, \dots, \alpha_k) \in ((T_k^1)^* \mathcal{M})_x / \alpha_A(R_B(x)) = 0, \text{ for all } A, B\}$ ]. Under these identifications, the morphism  $\omega^\sharp|_{\ker \eta^\sharp}$  is given by

$$(\omega^\sharp|_{\ker \eta^\sharp})^*(v) = (i(v)\omega^1(x), \dots, i(v)\omega^k(x)),$$

for  $v \in \cap_{A=1}^k \langle \eta^A(x) \rangle^0$ . Thus,  $(\omega^\sharp|_{\ker \eta^\sharp})^*$  is a monomorphism of vector bundles. Then  $\omega^\sharp|_{\ker \eta^\sharp} : \ker \eta^\sharp \rightarrow \cap_{A=1}^k \langle R_A \rangle^0$  is an epimorphism of vector bundles.

So, as the rank of the vector bundle  $\ker \eta^\sharp$  (respectively,  $\cap_{A=1}^k \langle R_A \rangle^0$ ) is  $k(kn + n)$  (respectively,  $kn + n$ ), we deduce that the rank of the vector bundle  $\ker \omega^\sharp \cap \ker \eta^\sharp$  is  $(k - 1)(kn + n)$ .

Furthermore, if  $(X_1, \dots, X_k)$  is a particular solution of Eqs. (1) and  $\mathbf{Z}$  is a section of the vector bundle  $\ker \omega^\sharp \cap \ker \eta^\sharp \rightarrow \mathcal{M}$  then  $(X_1, \dots, X_k) + \mathbf{Z}$  also is a solution of these equations. In addition, if  $\mathbf{X}'$  and  $\mathbf{X}$  are solutions of Eqs. (1) then  $\mathbf{Z} = \mathbf{X}' - \mathbf{X}$  is a section of the vector bundle  $\ker \omega^\sharp \cap \ker \eta^\sharp \rightarrow \mathcal{M}$ .

Finally, if  $(t^A, q^i, p_i^A)$  are Darboux coordinates in a neighborhood  $U_x$  of each point  $x \in \mathcal{M}$ , then we may define a local  $k$ -vector field on  $U_x$  that satisfies (7). For instance, we can put

$$(X_1)_i^1 = \frac{\partial H}{\partial q^i}, \quad (X_A)_i^B = 0 \text{ (for } A \neq 1 \neq B), \quad (X_A)^i = \frac{\partial H}{\partial p_i^A}.$$

Now one can construct a global  $k$ -vector field, which is a solution of (1), by using a partition of unity in the manifold  $\mathcal{M}$  (see Ref. 9). ■

### III. SYMMETRIES FOR $k$ -COSYMPLECTIC HAMILTONIAN SYSTEMS

#### A. Symmetries and conservation laws

Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system. First, following Ref. 27, we introduce the next definition:

*Definition 8:* A conservation law for the Hamilton–de Donder–Weyl equations (2) is a map  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k) : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$  such that the divergence of

$$\mathcal{F} \circ \psi = (\mathcal{F}^1 \circ \psi, \dots, \mathcal{F}^k \circ \psi) : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is zero for every solution  $\psi$  to the Hamilton–de Donder–Weyl equations (2); that is for all  $t \in U_0 \subset \mathbb{R}^k$ ,

$$\begin{aligned} 0 &= [Div(\mathcal{F} \circ \psi)](t) = \sum_{A=1}^k \frac{\partial(\mathcal{F}^A \circ \psi)}{\partial t^A} \Big|_t \\ &= \sum_{A=1}^k \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) (\mathcal{F}^A). \end{aligned} \quad (8)$$

*Proposition 3:* The map  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k) : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$  defines a conservation law if, and only if, for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  which is a solution to equations (1), we have that

$$\sum_{A=1}^k L(X_A) \mathcal{F}^A = 0. \quad (9)$$

*Proof:* (8)  $\Rightarrow$  (9) Let  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)$  be a conservation law and  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  an integrable  $k$ -vector field. If  $\psi : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  is an integral section of  $\mathbf{X}$ , by Lemma 1, we have that  $\psi$  is a solution to the Hamilton–de Donder–Weyl equation (2), and by definition of integral section we have that  $X_A(\psi(t)) = \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right)$ . Therefore, from (8) we obtain (9).

Conversely, (9)  $\Rightarrow$  Eq. (8). In fact, we must prove that for every solution  $\psi : U_0 \rightarrow \mathbb{R}^k \times M$  to the Hamilton–de Donder–Weyl equations (2) the identity (8) holds. From Proposition 1 there exist a  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  on  $\psi(U_0)$  which is solution to the equations (1) and  $\psi$  is an integral section of  $\mathbf{X}$ . We know that

$$X_A(\psi(t)) = \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right), \quad t \in U_0.$$

Then for all  $\psi(t) \in \psi(U_0)$

$$0 = \sum_{A=1}^k L(X_A) \mathcal{F}^A(\psi(t)) = \sum_{A=1}^k \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) (\mathcal{F}^A).$$

■

*Definition 9:*

1. A symmetry of the  $k$ -cosymplectic Hamiltonian system  $(\mathbb{R}^k \times M, H)$  is a diffeomorphism  $\Phi : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k \times M$  verifying the following conditions:
  - (a) It is a fiber preserving map for the trivial bundle  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ ; that is,  $\Phi$  induces a diffeomorphism  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that  $\pi_{\mathbb{R}^k} \circ \Phi = \phi \circ \pi_{\mathbb{R}^k}$ .
  - (b) For every section  $\psi$  solution to the Hamilton–de Donder–Weyl equations (2), we have that the section  $\Phi \circ \psi \circ \phi^{-1}$  is also a solution to these equations.
2. An infinitesimal symmetry of the  $k$ -cosymplectic Hamiltonian system  $(\mathbb{R}^k \times M, H)$  is a vector field  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  whose local flows are local symmetries.

As a consequence of the definition, all the results that we state for symmetries also hold for infinitesimal symmetries.

Symmetries can be used to generate new conservation laws from a given conservation law, In fact, a first straightforward consequence of Definitions 8 and 9 is:

*Proposition 4:* If  $\Phi : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k \times M$  is a symmetry of a  $k$ -cosymplectic Hamiltonian system and  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k) : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$  is a conservation law, then so is  $\Phi^*\mathcal{F} = (\Phi^*\mathcal{F}^1, \dots, \Phi^*\mathcal{F}^k)$ .

*Proof:* For every section  $\psi$  solution to the Hamilton–de Donder–Weyl equations and for every  $t \in \mathbb{R}^k$ , we have that

$$\begin{aligned} (\Phi^*\mathcal{F} \circ \psi)(t) &= (\mathcal{F} \circ \Phi \circ \psi)(t) = (\mathcal{F} \circ \Phi \circ \psi \circ \phi^{-1} \circ \phi)(t) \\ &= (\mathcal{F} \circ \Phi \circ \psi \circ \phi^{-1})(\phi(t)), \end{aligned}$$

and, therefore,

$$Div(\Phi^*\mathcal{F} \circ \psi) = 0 \Leftrightarrow Div(\mathcal{F} \circ \Phi \circ \psi \circ \phi^{-1}) = 0$$

on the corresponding domains. But the last equality holds since  $\mathcal{F}$  is a conservation law and  $\Phi \circ \psi \circ \phi^{-1}$  is also a solution to the Hamilton–de Donder–Weyl equations. ■

The following proposition gives a characterization of symmetries in terms of  $k$ -vector fields.

*Proposition 5:* Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system and  $\Phi : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k \times M$  a fiber preserving diffeomorphism for the trivial bundle  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ .

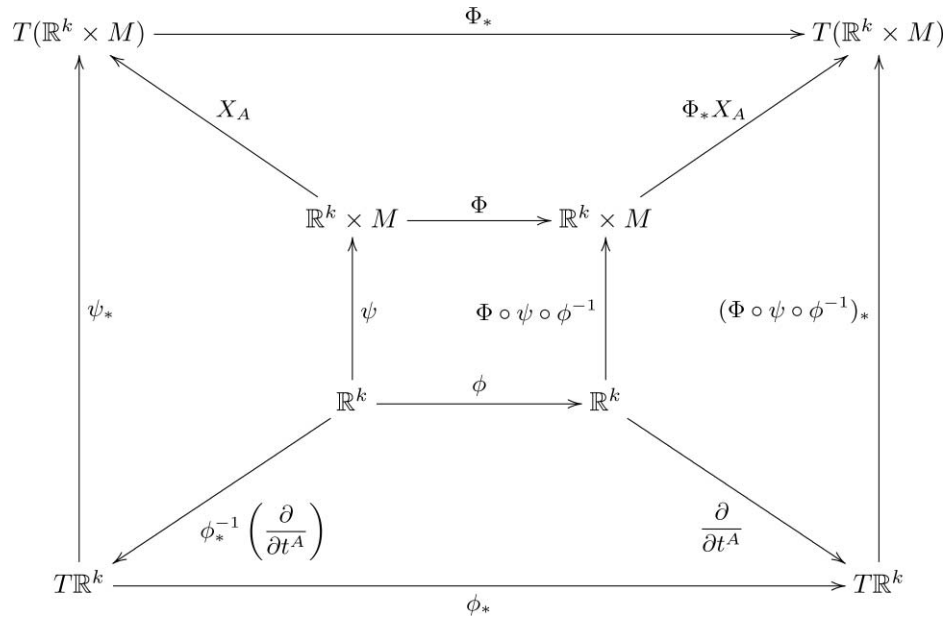
1. For every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  and for every integral section  $\psi$  of  $\mathbf{X}$ , the section  $\Phi \circ \psi \circ \phi^{-1}$  is an integral section of the  $k$ -vector field  $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k)$ , and hence  $\Phi_*\mathbf{X}$  is integrable.
2.  $\Phi$  is a symmetry if, and only if, for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$ , then  $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$ .

*Proof:*

1. Given  $x \in \mathbb{R}^k \times M$ , let  $\psi : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  an integral section of  $\mathbf{X}$  passing through  $x$ ; that is  $\psi(0) = x$ , then  $\Phi \circ \psi \circ \phi^{-1} : \phi(U_0) \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  is a section passing through  $\Phi(x)$ ; that is, if  $t_0 = \phi(0)$ , then  $(\Phi \circ \psi \circ \phi^{-1})(t_0) = \Phi(x)$ .  
Next we have to prove that  $\Phi \circ \psi \circ \phi^{-1}$  is an integral section of  $\Phi_*\mathbf{X}$ ; that is, for every  $t \in \phi(U_0)$ , and for every  $A = 1, \dots, k$ ,

$$(\Phi \circ \psi \circ \phi^{-1})_* (t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = (\Phi_* X_A)((\Phi \circ \psi \circ \phi^{-1})(t)),$$

or, what is equivalent, that the following diagram is commutative



First, we must take into account that the diffeomorphism  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$  makes a change of global coordinates in  $\mathbb{R}^k$ ; that is,  $\phi(\tilde{t}^A) = (t^A)$ , and then, if  $\psi$  is an integral section of  $\mathbf{X}$ , we have that

$$\psi_*(\phi^{-1}(t)) \left( \frac{\partial}{\partial \tilde{t}^A} \Big|_{\phi^{-1}(t)} \right) = \psi_*(\phi^{-1}(t)) \left( \phi_*^{-1}(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) \right) = X_A(\psi \circ \phi^{-1})(t).$$

Then, we obtain

$$\begin{aligned} & (\Phi \circ \psi \circ \phi^{-1})_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = \\ & \Phi_*(\psi(\phi^{-1}(t)) \left( \psi_*(\phi^{-1}(t)) \left( \phi_*^{-1}(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) \right) \right)) = \\ & \Phi_*(\psi(\phi^{-1}(t)) (X_A(\psi(\phi^{-1}(t)))) = \\ & (\Phi_* X_A)((\Phi \circ \psi \circ \phi^{-1})(t)). \end{aligned}$$

- ( $\Rightarrow$ ) Now, let  $x$  be an arbitrary point of  $\mathbb{R}^k \times M$  and  $\psi$  be an integral section of  $\mathbf{X}$  passing through the point  $\Phi^{-1}(x)$ , that is  $\psi(0) = \Phi^{-1}(x)$ . We know that  $\psi$  is a solution to the Hamilton–de Donder–Weyl equations (2). Since  $\Phi$  is a symmetry,  $\Phi \circ \psi \circ \phi^{-1}$  is a solution to the Hamilton–de Donder–Weyl equations (2) and, by the item 1, it is an integral section of  $\Phi_* \mathbf{X}$  passing through the point  $\Phi(\psi(0)) = \Phi(\Phi^{-1}(x)) = x$  (this means that  $(\Phi \circ \psi \circ \phi^{-1})(\phi(0)) = (\Phi \circ \psi)(0) = x$ ). Hence, from Lemma 1, we deduce that  $\Phi_* \mathbf{X} \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  at the points  $(\Phi \circ \psi)(t)$ , in particular at the arbitrary point  $(\Phi \circ \psi)(0) = x$ .

( $\Leftarrow$ ) Conversely, let  $\psi : U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times M$  be a solution to the Hamilton–de Donder–Weyl equations (2), then (see Proposition 1) there exists a  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  on  $\psi(U_0)$  which is solution to the equations (1) and  $\psi$  is an integral section of  $\mathbf{X}$  in  $\psi(U_0)$ . Since  $\mathbf{X}$  is solution to (1), then  $\Phi_* \mathbf{X} = (\Phi_* X_1, \dots, \Phi_* X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  by hypothesis, and then, as a consequence of the item 1 and Theorem 3,  $\Phi \circ \psi \circ \phi^{-1}$  is a solution to the Hamilton–de Donder–Weyl equations (2). ■

As a consequence of this, if  $\Phi$  is a symmetry and  $\mathbf{X}$  is an integrable  $k$ -vector field in  $\mathfrak{X}_H^k(\mathbb{R}^k \times M)$ , we have that  $\Phi_*\mathbf{X} - \mathbf{X} \in \ker \omega^\sharp \cap \ker \eta^\sharp$ .

*Proposition 6:* Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system. If  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  is an infinitesimal symmetry, then for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  we have that  $[Y, \mathbf{X}] = ([Y, X_1], \dots, [Y, X_k]) \in \ker \omega^\sharp \cap \ker \eta^\sharp$ .

*Proof:* Denote by  $F_t$  the local 1-parameter groups of diffeomorphisms generated by  $Y$ . As  $Y$  is an infinitesimal symmetry, as a consequence of Proposition 5, we have  $F_{t*}\mathbf{X} - \mathbf{X} = \mathbf{Z} \in \ker \omega^\sharp \cap \ker \eta^\sharp$ . Then, taking a local basis of sections  $\{\mathbf{Z}^1, \dots, \mathbf{Z}^r\} = \{(Z_1^1, \dots, Z_k^1), \dots, (Z_1^r, \dots, Z_k^r)\}$  of the vector bundle  $\ker \omega^\sharp \cap \ker \eta^\sharp \rightarrow \mathbb{R}^k \times M$ , we have that  $F_{t*}\mathbf{X} - \mathbf{X} = g_\alpha \mathbf{Z}^\alpha$ ,  $\alpha = 1, \dots, r$ , with  $g_\alpha : \mathbb{R} \times (\mathbb{R}^k \times M) \rightarrow \mathbb{R}$  (they are functions that depend on  $t$ , some of them different from 0); that is,

$$\begin{aligned} F_{t*}\mathbf{X} - \mathbf{X} &= (F_{t*}X_1 - X_1, \dots, F_{t*}X_k - X_k) \\ &= (g_\alpha Z_1^\alpha, \dots, g_\alpha Z_k^\alpha) = g_\alpha \mathbf{Z}^\alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} [Y, \mathbf{X}] &= L(Y)\mathbf{X} = (L(Y)X_1, \dots, L(Y)X_k) \\ &= \left( \lim_{t \rightarrow 0} \frac{F_{t*}X_1 - X_1}{t}, \dots, \lim_{t \rightarrow 0} \frac{F_{t*}X_k - X_k}{t} \right) \\ &= \left( \lim_{t \rightarrow 0} \frac{g_\alpha}{t} Z_1^\alpha, \dots, \lim_{t \rightarrow 0} \frac{g_\alpha}{t} Z_k^\alpha \right) \\ &= (f_\alpha Z_1^\alpha, \dots, f_\alpha Z_k^\alpha) = f_\alpha \mathbf{Z}^\alpha \in \ker \omega^\sharp \cap \ker \eta^\sharp, \end{aligned}$$

where  $f_\alpha : \mathbb{R}^k \times M \rightarrow \mathbb{R}$ . ■

### B. $k$ -cosymplectic Noether symmetries. Noether's theorem

As it is well known, the existence of symmetries is associated with the existence of conservation laws. How to obtain these conservation laws depends on the symmetries that we are considering. In particular, for Hamiltonian and Lagrangian systems, Noether's theorem gives a rule for doing it, for certain kinds of symmetries: those that preserve both the physical information (given by the Hamiltonian or the Lagrangian function), and some geometric structures of the system. For  $k$ -cosymplectic Hamiltonian field theories a reasonable choice consists in taking those symmetries preserving the  $k$ -cosymplectic structure as well as the Hamiltonian function. Bearing this in mind, first we prove the following:

*Proposition 7:* Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system.

1. If  $\Phi : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k \times M$  is a diffeomorphism satisfying that

- (a)  $\Phi^*\omega^A = \omega^A$ ,
- (b)  $\Phi^*\eta^A = \eta^A$ ,
- (c)  $\Phi^*H = H$ ,

then  $\Phi$  is a symmetry of the  $k$ -cosymplectic Hamiltonian system  $(\mathbb{R}^k \times M, H)$ .

(1) If  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  a vector field satisfying that

- (a)  $L(Y)\omega^A = 0$ ,
- (b)  $L(Y)\eta^A = 0$ ,
- (c)  $L(Y)H = 0$ ,

then  $Y$  is an infinitesimal symmetry of the  $k$ -cosymplectic Hamiltonian system  $(\mathbb{R}^k \times M, H)$ .

*Proof:*

1. First, from

$$dt^A = \eta^A = \Phi^* \eta^A = \Phi^* dt^A = d\Phi^* t^A$$

we conclude that  $\Phi^* t^A = t^A + k^A$  ( $k^A \in \mathbb{R}$ ). This result (together with the condition  $\Phi^* \omega^A = \omega^A$ ) means that the local expression of  $\Phi$  is  $\Phi(t^A, q^i, p_i) = (t^A + k^A, \Phi^i(q, p), \Phi_i^A(q, p))$ . Therefore it induces a diffeomorphism  $\phi: \mathbb{R}^k \rightarrow \mathbb{R}^k$  given by  $\phi(t^A) = t^A + k^A$ ; hence  $\pi_{\mathbb{R}^k} \circ \Phi = \phi \circ \pi_{\mathbb{R}^k}$  and  $\Phi$  is a fiber preserving map for the trivial bundle  $\pi_{\mathbb{R}^k}: \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ .

Now, as  $\eta^A(R_B) = dt^A \left( \frac{\partial}{\partial t^B} \right) = \delta_B^A$ , and  $\Phi^* \eta^A = \Phi^* dt^A = dt^A = \eta^A$ , we have

$$\delta_B^A = dt^A \left( \frac{\partial}{\partial t^B} \right) = (\Phi^* dt^A) \left( \frac{\partial}{\partial t^B} \right) = \Phi^* \left\{ dt^A \left( \Phi_* \left( \frac{\partial}{\partial t^B} \right) \right) \right\},$$

thus

$$\Phi_* \left( \frac{\partial}{\partial t^B} \right) = \frac{\partial}{\partial t^B} + \alpha^i \frac{\partial}{\partial q^i} + \beta_i^A \frac{\partial}{\partial p_i^A},$$

but, since  $\Phi^* \omega^A = \omega^A$ , for all  $A$ ,

$$0 = i \left( \frac{\partial}{\partial t^B} \right) \omega^A = i \left( \Phi_* \left( \frac{\partial}{\partial t^B} \right) \right) (\omega^A \circ \Phi)$$

and then

$$\frac{\partial}{\partial t^B} + \alpha^i \frac{\partial}{\partial q^i} + \beta_i^A \frac{\partial}{\partial p_i^A} = \Phi_* \left( \frac{\partial}{\partial t^B} \right) \in \bigcap_{A=1}^k \ker(\omega^A \circ \Phi) = \left\langle \frac{\partial}{\partial t^A} \circ \Phi \right\rangle_{A=1, \dots, k},$$

which implies that  $\Phi_* \left( \frac{\partial}{\partial t^B} \right) = \frac{\partial}{\partial t^B}$  that is,  $\Phi_*(R_B) = R_B$ .

Furthermore, for every  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$ , we obtain that

$$\Phi^*(\eta^A(\Phi_* X_B)) = (\Phi^* \eta^A)(X_B) = \eta^A(X_B) = \delta_B^A,$$

$$\begin{aligned} & \Phi^* \left[ \sum_{A=1}^k i(\Phi_* X_A) \omega^A - dH + (L(R_A)H) \eta^A \right] \\ &= \sum_{A=1}^k [i(X_A)(\Phi^* \omega^A) - \Phi^* dH + (\Phi^* L(R_A)H)(\Phi^* \eta^A)] \\ &= \sum_{A=1}^k [i(X_A) \omega^A - dH + (L(R_A)H) \eta^A] = 0. \end{aligned}$$

Hence, as  $\Phi$  is a diffeomorphism, these results are equivalent to demanding that

$$\eta^A(\Phi_* X_B) = \delta_B^A$$

$$\sum_{A=1}^k [i(\Phi_* X_A) \omega^A - dH + L(R_A)H \eta^A] = 0.$$

Thus  $\Phi_* \mathbf{X} = (\Phi_* X_1, \dots, \Phi_* X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$ . Finally, if  $\mathbf{X}$  is integrable, then  $\Phi_* \mathbf{X}$  is integrable too (as Proposition 5 claims), and thus  $\Phi$  is a symmetry.

2. It is a consequence of the above item, taking the local flows of  $Y$ . ■

Although the condition 2(b) of the hypothesis is sufficient to prove that these kinds of vector fields are infinitesimal symmetries, in order to achieve a good generalization of Noether's theorem, this condition must be hardened by demanding that  $i(Y)\eta^A = 0$  [observe that  $i(Y)dt^A = 0 \implies L(Y)dt^A = 0$ ]. This is equivalent to write  $L(Y)t^A = 0$  and hence, the equivalent global condition 1(b) for this case is  $\Phi^*t^A = t^A$ . This means that the induced diffeomorphism  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is the identity on  $\mathbb{R}^k$ .

Taking into account all of this, we introduce the following definitions:

*Definition 10: Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system.*

1. A  $k$ -cosymplectic Noether symmetry is a diffeomorphism  $\Phi : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k \times M$  satisfying the following conditions:
  - (a)  $\Phi^*\omega^A = \omega^A$ ,
  - (b)  $\Phi^*t^A = t^A$ ,
  - (c)  $\Phi^*H = H$ .

*If the  $k$ -symplectic structure is exact, a  $k$ -cosymplectic Noether symmetry is said to be exact if  $\Phi^*\theta^A = \theta^A$ .*

*In the particular case that  $M = (T_k^1)^*Q$  (the standard model), if  $\Phi = j^{1*}\varphi$  for some diffeomorphism  $\varphi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$ , then the  $k$ -cosymplectic Noether symmetry  $\Phi$  is said to be natural.*

2. Let  $(\mathbb{R}^k \times M, H)$  be a  $k$ -cosymplectic Hamiltonian system. An infinitesimal  $k$ -cosymplectic Noether symmetry is a vector field  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  whose local flows are local  $k$ -cosymplectic Noether symmetries; that is, it satisfies that:
  - (a)  $L(Y)\omega^A = 0$ ,
  - (b)  $i(Y)\eta^A = 0$ ,
  - (c)  $L(Y)H = 0$ .

*If the  $k$ -symplectic structure is exact, an infinitesimal  $k$ -cosymplectic Noether symmetry is said to be exact if  $L(Y)\theta^A = 0$ .*

*In the particular case that  $M = (T_k^1)^*Q$ , if  $Y = Z^{1*}$  for some  $Z \in \mathfrak{X}(\mathbb{R}^k \times Q)$ , then the infinitesimal  $k$ -cosymplectic Noether symmetry  $Y$  is said to be natural.*

*[Obviously natural (infinitesimal)  $k$ -cosymplectic Noether symmetries are exact].*

*Lemma 2: If  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  is an infinitesimal  $k$ -cosymplectic Noether symmetry, then  $[Y, R_A] = 0$ .*

*Proof:* In fact, for all  $A, B$ , we have that

$$i([Y, R_A])\omega^B = L(Y)i(R_A)\omega^B - i(R_A)L(Y)\omega^B = 0 \implies [Y, R_A] \in \ker \omega^B,$$

$$i([Y, R_A])\eta^B = L(Y)i(R_A)\eta^B - i(R_A)L(Y)\eta^B = L(Y)\delta_A^B = 0 \implies [Y, R_A] \in \ker \eta^B,$$

and then  $[Y, R_A] \in (\cap_B \ker \omega^B) \cap (\cap_B \ker \eta^B) = \{0\}$ . ■

*Remarks:*

- The condition  $\Phi^*t^A = t^A$  means that  $k$ -cosymplectic Noether symmetries generate transformations along the fibers of the projection  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ ; that is, they leave the fibers of the projection  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$  invariant or, what means the same thing,  $\pi_{\mathbb{R}^k} \circ \Phi = \pi_{\mathbb{R}^k}$ . As a consequence, in the particular case that  $M = (T_k^1)^*Q$ , if  $\Phi = j^{1*}\varphi$  (for some diffeomorphism  $\varphi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$ ) is a natural  $k$ -cosymplectic Noether symmetry, then the diffeomorphism  $\varphi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times Q$  must leave the fibers of the projection  $p_{\mathbb{R}^k} : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$  invariant necessarily; that is,  $p_{\mathbb{R}^k} \circ \varphi = p_{\mathbb{R}^k}$ .
- In the case of infinitesimal  $k$ -cosymplectic Noether symmetries the analogous condition is  $i(Y)dt^A = 0$ , which means that  $Y$  has the local expression  $Y = Y_i \frac{\partial}{\partial q^i} + Y_i^A \frac{\partial}{\partial p_i^A}$ . This means

that  $Y$  is tangent to the fibers of the projection  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ . Thus these infinitesimal symmetries only generate transformations along these fibers, or, what means the same thing, the local flows of the generators  $Y$  leave the fibers of the projection  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \times M \rightarrow \mathbb{R}^k$  invariant. Furthermore, as a consequence of the above Lemma and taking into account that  $R_A = \frac{\partial}{\partial t^A}$ , in this local expression for  $Y$  the component functions  $Y_i, Y_i^A$  do not depend on the coordinates  $(t^A)$ .

Observe also that, in the particular case that  $M = (T_k^1)^*Q$ , if  $Y = Z^{1*}$  [for some  $Z \in \mathfrak{X}(\mathbb{R}^k \times Q)$ ] is a natural infinitesimal  $k$ -cosymplectic Noether symmetry, then  $i(Y)dt^A = 0$ , necessarily.

In addition, it is immediate to prove that, if  $Y_1, Y_2 \in \mathfrak{X}(\mathbb{R}^k \times M)$  are infinitesimal Noether symmetries, then so is  $[Y_1, Y_2]$ .

It is interesting to comment that, for infinitesimal  $k$ -cosymplectic Noether symmetries, the results in the item 2 of Proposition 5 and in Proposition 6 hold, not only for integrable  $k$ -vector fields in  $\mathfrak{X}_H^k(\mathbb{R}^k \times M)$ , but also for every  $k$ -vector field  $\mathbf{X} \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$ . In fact, for the first one, we have

$$\begin{aligned} \sum_{A=1}^k i([Y, X_A])\omega^A &= \sum_{A=1}^k \{L(Y)i(X_A)\omega^A - i(X_A)L(Y)\omega^A\} \\ &= \sum_{A=1}^k L(Y)(dH - (L(R_A)H)\eta^A) \\ &= \sum_{A=1}^k \{d(L(Y)(H)) - (L(Y)L(R_A)H)\eta^A \\ &\quad - (L(R_A)H)L(Y)\eta^A\} \\ &= - \sum_{A=1}^k (L(R_A)L(Y)H)\eta^A = 0 . \end{aligned}$$

Furthermore,

$$i([Y, X_A])\eta^B = L(Y)i(X_A)\eta^B - i(X_A)L(Y)\eta^B = 0 ,$$

and the proof for the second one is straightforward.

As infinitesimal  $k$ -cosymplectic Noether symmetries are vector fields in  $\mathbb{R}^k \times M$  whose local flows are local  $k$ -cosymplectic Noether symmetries, all the results that we state for  $k$ -cosymplectic Noether symmetries also hold for infinitesimal  $k$ -cosymplectic Noether symmetries. Hence, from now on we consider only the infinitesimal case.

A first relevant result is the following:

*Proposition 8: Let  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  be an infinitesimal  $k$ -cosymplectic Noether symmetry. Then, for every  $p \in \mathbb{R}^k \times M$ , there is an open neighborhood  $U_p \ni p$ , such that:*

1. *There exist  $\mathcal{F}^A \in C^\infty(U_p)$ , which are unique up to constant functions, such that*

$$i(Y)\omega^A = d\mathcal{F}^A, \quad (\text{on } U_p) . \tag{10}$$

2. *There exist  $\zeta^A \in C^\infty(U_p)$ , verifying that  $L(Y)\theta^A = d\zeta^A$ , on  $U_p$ ; and then*

$$\mathcal{F}^A = i(Y)\theta^A - \zeta^A , \quad (\text{up to a constant function, on } U_p) .$$

*Proof:*

1. It is a consequence of the Poincaré Lemma and the condition

$$0 = L(Y)\omega^A = i(Y)d\omega^A + di(Y)\omega^A = di(Y)\omega^A .$$



2. We have that

$$dL(Y)\theta^A = L(Y)d\theta^A = -L(Y)\omega^A = 0$$

and hence  $L(Y)\theta^A$  are closed forms. Therefore, by the Poincaré Lemma, there exist  $\zeta^A \in C^\infty(U_p)$ , verifying that  $L(Y)\theta^A = d\zeta^A$ , on  $U_p$ . Furthermore, as (10) holds on  $U_p$ , we obtain that

$$\begin{aligned} d\zeta^A &= L(Y)\theta^A = di(Y)\theta^A + i(Y)d\theta^A \\ &= di(Y)\theta^A - i(Y)\omega^A = d\{i(Y)\theta^A - \mathcal{F}^A\}, \end{aligned}$$

and thus 2 holds. ■

*Remark:* For exact infinitesimal  $k$ -cosymplectic Noether symmetries we have that  $\mathcal{F}^A = i(Y)\theta^A$  (up to a constant function).

Finally, the classical Noether's theorem can be stated for these kinds of symmetries as follows:

**Theorem 4:** (Noether's theorem). *If  $Y \in \mathfrak{X}(\mathbb{R}^k \times M)$  is an infinitesimal  $k$ -cosymplectic Noether symmetry then, for every  $p \in \mathbb{R}^k \times M$ , there is an open neighborhood  $U_p \ni p$  such that the functions  $\mathcal{F}^A = i(Y)\theta^A - \zeta^A$ , define a conservation law  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)$ .*

*Proof:* Let  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k(\mathbb{R}^k \times M)$  an integrable  $k$ -vector field. From (10), one obtains

$$\begin{aligned} \sum_{A=1}^k L(X_A)\mathcal{F}^A &= \sum_{A=1}^k i(X_A)d\mathcal{F}^A = \sum_{A=1}^k i(X_A)i(Y)\omega^A \\ &= -i(Y)\sum_{A=1}^k i(X_A)\omega^A \\ &= -i(Y)dH + \sum_{A=1}^k i(Y)((L(R_A)H)\eta^A) \\ &= -L(Y)H + \sum_{A=1}^k (L(R_A)H)i(Y)\eta^A = 0, \end{aligned}$$

that is,  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)$  is a conservation law for the Hamilton–de Donder–Weyl equations. ■

Observe that, using Darboux coordinates in  $\mathbb{R}^k \times M$ , the item 2 of Proposition 8 tells us that the conservation laws associated with infinitesimal  $k$ -cosymplectic Noether symmetries does not depend on the coordinates  $(t^A)$  (as it is obvious since the generators of these symmetries, the vector fields  $Y$ , neither depend on them).

## IV. EXAMPLE

### A. $k$ -cosymplectic quadratic Hamiltonian systems

Many Hamiltonian systems in field theories are of “quadratic” type and they can be modeled as follows.

Consider the  $k$ -cosymplectic manifold  $(\mathbb{R}^k \times (T_k^1)^*Q, \eta^A, \omega^A, \mathcal{V})$ . Let  $g_1, \dots, g_k$  be  $k$  semi-Riemannian metrics in  $Q$ . For every  $q \in Q$  we have the following isomorphisms:

$$\begin{aligned} g_A^b : T_q Q &\longrightarrow T_q^* Q \\ v &\longmapsto i(v)g_A, \end{aligned}$$

with  $A \in \{1, \dots, k\}$  and then we can introduce the dual metric of  $g_A$ , denoted by  $g_A^*$ , which is defined by

$$g_A^*(\alpha_q, \beta_q) := g_A((g_A^b)^{-1}(\alpha_q), (g_A^b)^{-1}(\beta_q)),$$

for every  $\alpha_q, \beta_q \in T_q^*Q$ , and  $A \in \{1, \dots, k\}$ . We can define a function  $K \in C^\infty(\mathbb{R}^k \times (T_k^1)^*Q)$  as follows: for every  $(t, q; \alpha_q^1, \dots, \alpha_q^k) \in \mathbb{R}^k \times (T_k^1)^*Q$ ,

$$K(t, q; \alpha_q^1, \dots, \alpha_q^k) := \frac{1}{2} \sum_{A=1}^k g_A^*(\alpha_q^A, \alpha_q^A).$$

Then, if  $V \in C^\infty(\mathbb{R}^k \times Q)$  we can introduce a Hamiltonian function  $H \in C^\infty(\mathbb{R}^k \times (T_k^1)^*Q)$  of quadratic type as follows

$$H = K + V \circ (\pi_Q)_{1,0}^*.$$

Using natural coordinates  $(t^A, q^i, p_i^A)$  on  $\mathbb{R}^k \times (T_k^1)^*Q$  the local expression of  $H$  is

$$H(t^A, q^i, p_i^A) = \frac{1}{2} \sum_{A=1}^k g_A^{ij}(q^k) p_i^A p_j^A + V(t^B, q^j),$$

where  $g_A^{ij}$  denote the coefficients of the matrix associated to  $g_A^*$ . Then

$$dH = \sum_{A=1}^k \left[ \frac{\partial V}{\partial t^A} dt^A + \left( \frac{1}{2} \frac{\partial g_A^{ij}}{\partial q^k} p_i^A p_j^A + \frac{\partial V}{\partial q^k} \right) dq^k + (g_A^{ij} p_i^A) dp_j^A \right]$$

Moreover, if  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H(\mathbb{R}^k \times (T_k^1)^*Q)$  with

$$X_A = \sum_{B=1}^k \left[ (X_A)^B \frac{\partial}{\partial t^B} + (X_A)^i \frac{\partial}{\partial q^i} + (X_A)_i^B \frac{\partial}{\partial p_i^B} \right]$$

the equations (1) lead to

$$\begin{aligned} (X_A)^B &= \delta_A^B, \quad (X_A)^i = g_A^{ij} p_j^A \quad (A \text{ fixed}), \\ - \sum_{A=1}^k (X_A)_i^A &= \frac{1}{2} \sum_{A=1}^k \frac{\partial g_A^{jk}}{\partial q^i} p_j^A p_k^A + \frac{\partial V}{\partial q^i}; \end{aligned} \quad (11)$$

that is, we have obtained

$$X_A = \left[ \frac{\partial}{\partial t^A} + g_A^{ij} p_j^A \frac{\partial}{\partial q^i} + (X_A)_i^B \frac{\partial}{\partial p_i^B} \right],$$

with  $(X_B)_i^B = -\frac{\partial V}{\partial q^i} - \frac{1}{2} \frac{\partial g_B^{jk}}{\partial q^i} p_j^B p_k^B$ .

Now, if  $\psi(t) = (t^A, \psi^i(t), \psi_i^A(t))$  is an integral section of  $\mathbf{X}$  then

$$X_A(\psi(t)) = \psi_{*}(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = \left[ \frac{\partial}{\partial t^A} + \frac{\partial \psi^i}{\partial t^A} \frac{\partial}{\partial q^i} + \frac{\partial \psi_i^B}{\partial t^A} \frac{\partial}{\partial p_i^B} \right]. \quad (12)$$

Thus, from (11) and (12), we obtain the Hamilton–de Donder–Weyl equations

$$\begin{aligned} - \frac{\partial V}{\partial q^i}(\psi(t)) - \frac{1}{2} \frac{\partial g_A^{ij}}{\partial q^i} \psi_j^A \psi_k^A &= \sum_{A=1}^k (X_A)_i^A(\psi(t)) = \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A} \\ g_A^{ij}(\psi(t)) \psi_j^A &= X_i^A(\psi(t)) = \frac{\partial \psi^i}{\partial t^A} \quad (A, i \text{ fixed}). \end{aligned} \quad (13)$$

Then, from these equations we conclude that

$$\psi_i^A = (g_A)_{ij} \frac{\partial \psi^j}{\partial t^A} \quad (A, i \text{ fixed}),$$

and hence the equations for the integral sections are

$$\sum_{A,j} (g_A)_{ij} \frac{\partial^2 \psi^j}{\partial (t^A)^2} = -\frac{\partial V}{\partial q^i} - \frac{1}{2} \sum_{A,j,k,l,m} \frac{\partial g_A^{jk}}{\partial q^i} (g_A)_{kl} (g_A)_{jm} \frac{\partial \psi^l}{\partial t^A} \frac{\partial \psi^m}{\partial t^A}. \tag{14}$$

We also may prove the following result:

*Proposition 9: Let  $X$  be a Killing vector field on  $Q$  for the semi-Riemannian metrics  $g_1, \dots, g_k$  (that is,  $L(X)g_A = 0$ , for all  $A \in \{1, \dots, k\}$ ) such that  $X(V) = 0$ . Then, the vector field  $X^{1*}$  on  $\mathbb{R}^k \times (T_k^1)^*Q$  is a natural infinitesimal symmetry for the  $k$ -cosymplectic Hamiltonian system  $(\mathbb{R}^k \times (T_k^1)^*Q, H)$ . Thus, if  $\mathcal{F} = (\hat{X}, \dots, \hat{X}) : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow \mathbb{R}^k$  is the map defined by*

$$\mathcal{F}(t, q; \alpha_q^1, \dots, \alpha_q^k) = (\alpha_q^1(X(q)), \dots, \alpha_q^k(X(q))),$$

for  $(t, q; \alpha_q^1, \dots, \alpha_q^k) \in \mathbb{R}^k \times (T_k^1)^*Q$ , we have that  $\mathcal{F}$  is a conservation law for the Hamiltonian system.

*Proof:* As we know

$$L(X^{1*})\theta^A = 0.$$

Moreover, it is clear that

$$i(X^{1*})\eta^A = 0.$$

So, it is sufficient to prove that

$$L(X^{1*})H = 0.$$

Now, using that  $X^{1*}$  is  $(\pi_Q)_{1,0}^*$ -projectable over  $X$  and the fact that  $L(X)V = 0$ , we deduce that

$$L(X^{1*})(V \circ (\pi_Q)_{1,0}^*) = 0.$$

Next, we will prove that

$$L(X^{1*})(K) = 0.$$

Assuming that the local expression of  $X$  is

$$X = X^i \frac{\partial}{\partial q^i},$$

then, as  $L(X)g_A = 0$ , we have that

$$X((g_A)_{jk}) = -\frac{\partial X^l}{\partial q^j} (g_A)_{kl} - \frac{\partial X^l}{\partial q^k} (g_A)_{jl},$$

which implies that

$$X(g_A^{ij}) = -\frac{\partial X^j}{\partial q^k} g_A^{ik} - \frac{\partial X^i}{\partial q^k} g_A^{jk}.$$

Therefore, using that the local expressions of  $X^{1*}$  and  $K$  are

$$X^{1*} = X^i \frac{\partial}{\partial q^i} - p_j^A \frac{\partial X^j}{\partial q^i} \frac{\partial}{\partial p_i^A}, \quad K = \frac{1}{2} \sum_{A,i,j} g_A^{ij} p_i^A p_j^A$$

we conclude that

$$L(X^{1*})K = 0.$$

Furthermore, if  $\hat{X} : T^*Q \rightarrow \mathbb{R}$  is the linear function on  $T^*Q$  associated with the vector field  $X$ , it follows that

$$(i(X^{1*})\theta^A)(t, q; \alpha_q^1, \dots, \alpha_q^k) = \hat{X}(\alpha_q^A).$$

Consequently,  $\mathcal{F} = (\hat{X}, \dots, \hat{X})$  is a conservation law (see Remark after Proposition 8). ■

### B. A particular case: The wave equation

As particular examples of these kinds of systems we can detach the following case (see Ref. 24 for a more detailed explanation):

Consider the three-dimensional wave equation,

$$\sigma \frac{\partial^2 \psi}{\partial t^2} - \tau \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = 0. \quad (15)$$

In this case  $M = \mathbb{R}^4 \times (T_2^1)^*Q$  (i.e.,  $k = 4$ ), with  $Q = \mathbb{R}$  ( $n = 1$ ), and  $g_i$ ,  $i = 1, \dots, 4$ , are the semi-Riemannian metrics on  $\mathbb{R}$ ,

$$g_1 = \sigma dq^2, \quad g_2 = g_3 = g_4 = -\tau dq^2,$$

$q$  being the standard coordinate on  $\mathbb{R}$ . We have done the identifications  $t^1 \equiv t$  and  $t^2 \equiv x$ ,  $t^3 \equiv y$ ,  $t^4 \equiv z$ , where  $t$  is time and  $x, y, z$  denote the position in space. Then,  $\psi(t, x, y, z)$  denotes the displacement of each point of the media where the wave is propagating, as function of the time and the position, and  $\sigma$  and  $\tau$  are physical constants.

Thus, the wave equation (15) is a particular case of the equation (14) for the quadratic Hamiltonian in  $\mathbb{R}^2 \times (T_2^1)^*\mathbb{R}$

$$H = \frac{1}{2} \left[ \frac{1}{\sigma} (p^1)^2 - \frac{1}{\tau} ((p^2)^2 + (p^3)^2 + (p^4)^2) \right].$$

We have that the canonical vector field,  $\frac{\partial}{\partial q}$  on  $\mathbb{R}$  is a Killing vector field for the semi-Riemannian metrics  $g_i$ ,  $i = 1, \dots, 4$ . Thus,

$$\mathcal{F} = (p^1, p^2, p^3, p^4) : \mathbb{R}^4 \times (T_4^1)^*\mathbb{R} \rightarrow \mathbb{R}^4$$

is a conservation law for the three-dimensional wave equation.

Note that if

$$\begin{aligned} \tilde{\psi} : (t, x, y, z) &\rightarrow (t, x, y, z, \psi(t, x, y, z)); \\ \psi^1(t, x, y, z), \psi^2(t, x, y, z), \psi^3(t, x, y, z), \psi^4(t, x, y, z) \end{aligned}$$

is a solution to the Hamilton–de Donder–Weyl equations then, from (14), it follows that

$$\psi^1 = \sigma \frac{\partial \psi}{\partial t}, \quad \psi^2 = -\tau \frac{\partial \psi}{\partial x}, \quad \psi^3 = -\tau \frac{\partial \psi}{\partial y}, \quad \psi^4 = -\tau \frac{\partial \psi}{\partial z}.$$

Thus, the conservation law leads to the starting field equations. In fact,

$$\text{Div}(\mathcal{F} \circ \tilde{\psi}) = \sigma \frac{\partial^2 \psi}{\partial t^2} - \tau \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = 0.$$

### V. CONCLUSIONS AND OUTLOOK

We have studied symmetries and reduction of  $k$ -cosymplectic Hamiltonian systems in classical field theories; in particular, those which are modeled on  $k$ -cosymplectic manifolds  $\mathcal{M} = \mathbb{R}^k \times M$ , with  $M$  being a generic  $k$ -symplectic manifold (which we have called almost-standard  $k$ -cosymplectic manifolds).

In particular, we have analyzed a kind of  $k$ -cosymplectic Noether symmetries for which there is a direct way to associate conservation laws by means of the application of the corresponding generalized version of the Noether theorem.

As discussed in Sec. III, for the almost-standard  $k$ -cosymplectic Hamiltonian systems, the symmetries that we have considered in this work have the following geometric characteristic: they generate transformations along the fibers of the projection  $\mathbb{R}^k \times M \rightarrow \mathbb{R}^k$ . As a consequence, in a local description, the associated conservation laws do not depend on the base coordinates ( $t^A$ ). This could seem to be a strong restriction but, really, many symmetries of field theories in physics are of

this type. In any case, a theory of symmetries, conservation laws, and reduction concerning to more general kinds of symmetries would have to be developed.

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