THERMAL NOISE in a Finite Bandwidth

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The thermal noise of an arbitrary resistor in an arbitrary linear circuit diminishes when the resistor value increases [7]. Giannetti derived the value of the resistor for which a maximum in the spectral density of the noise voltage occurs at the output [2]. In seeming contradiction, the mean square thermal noise voltage generated by a resistor depends only on the temperature and the total capacitance shunting the resistor [3]. Its value is

$$V_{\mathrm{t}} = \sqrt{\frac{kT}{C}} \tag{1}$$

where $k \approx 1.38 \times 10^{-23}$ J/K is Boltzmann's constant. Hence the term "kT/C noise" [4]. This means that the thermal noise does not depend on the resistor value. How can a short circuit and an open circuit yield the same thermal noise if capacitance, C, is the same? Equation (1) applies only in an ideal case involving infinite bandwidth [5]. Abbot derived the noise voltage only for the special case of finite bandwidth that begins at zero frequency [1]. Furthermore, Pyati derived an exact expression for the noise voltage across a resistance-capacitance (RC) network by accounting for quantum effects that cause the spectral density of thermal noise to drop off around frequencies of 100 GHz; therefore, it is not white, as assumed in deriving (1) [6].

In this article, we propose an approach that yields appropriate results in the frequency band where the thermal noise is white. In particular, we show that the thermal noise voltage of a resistor in a finite frequency bandwidth is maximum for a given resistor value. The thermal noise depends on the frequency limits and decreases for both smaller and larger resistor values. Finally, we show that the kT/C approach overestimates the thermal noise contributed by a resistor.

Development

Consider a resistor, R, shunted by a capacitance, C, as shown in Fig. 1. It can be either a parasitic or an actual capacitor. A true rms voltmeter having an infinite bandwidth and input impedance would measure the mean square thermal output noise voltage given by (1), where T is the absolute temperature of the resistor. Equation (1) is obtained by integrating the output noise voltage spectral density, v_{on}, over an infinite bandwidth. However, the frequency can never approach zero or infinite. The mean square thermal noise voltage in the frequency band from f_{L} to f_{H} is

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$$V_t = \sqrt{\int_{f_L}^{f_H} v_{on}^2(f)\,df} \quad (2)$$

where $v_{on}(f)$ is the noise voltage spectral density at the output of the circuit in Fig. 1 whose expression is

In a practical circuit, the output noise due to the thermal noise of a resistor will depend on the actual transfer function of the noise source.

$$v_{on}(f) = \sqrt{\frac{4kTR}{1+(2\pi fRC)^2}} \quad (3)$$

for frequencies below 100 GHz where thermal noise can be considered white [6]. Replacing (3) in (2) and integrating leads to

$$V_t = \sqrt{\frac{kT}{C}\cdot\frac{2}{\pi}\cdot\arctan\left(\frac{2\pi(f_H - f_L)RC}{1+\left(2\pi\sqrt{f_H f_L}RC\right)^2}\right)} \quad (4)$$

whose maximum is at $R_{max} = 1/(2\pi C\sqrt{f_L f_H})$ and it is

$$V_{t(max)} = \sqrt{\frac{kT}{C}\cdot\frac{2}{\pi}\cdot\arctan\left(\frac{f_H - f_L}{2\sqrt{f_L f_H}}\right)}. \quad (5)$$

Fig. 2 shows $V_t^2/(kT/C)$ as a function of R/R_{max} for different f_H/f_L values. If $R > R_{max}$, the mean square output noise diminishes as R increases. If $R < R_{max}$, the converse is true. If $R << 1/(2\pi f_H C)$, then (4) reduces to

$$V_t \approx \sqrt{4kTR(f_H - f_L)} \quad (6)$$

which is the familiar equation for the thermal noise of a resistor in the bandwidth from f_L to f_H. In this case, C has a negligible influence.

Equation (1) overestimates the thermal noise of R in Fig. 1. Equation (4) yields the actual thermal noise in a given frequency band below 100 GHz. That noise depends not only on C and T but also on R. Fig. 2 shows that the mean square noise is maximal for $R = R_{max} = 1/(2\pi C\sqrt{f_L f_H})$ and never reaches kT/C. The mean square noise of small resistors does not depend on C. Because all resistors have a parasitic capacitance in parallel, the preceding analysis applies whenever considering resistor noise.

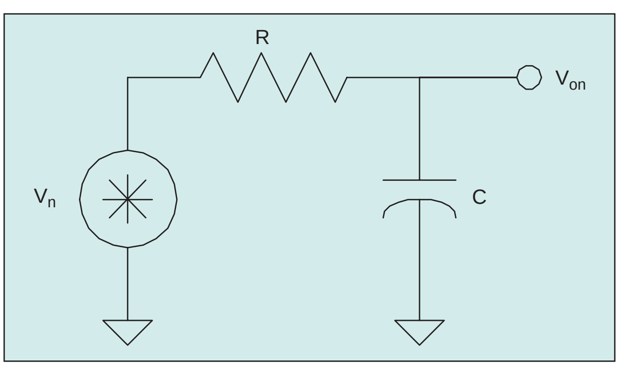

Fig. 1. Circuit showing the resistor, R, its thermal noise voltage density, v_n, a shunt capacitance, C, and the resulting output thermal noise voltage spectral density, v_{on}.

If the noise bandwidth is limited by a first-order bandpass filter with –3 dB cutoff frequencies, f_L and f_H, the mean square thermal noise voltage is given by

$$V_t = \sqrt{\frac{kT}{C}\cdot\frac{f_H^2\, 2\pi RC}{(f_L + f_H)\cdot(1+2\pi f_H RC)\cdot(1+2\pi f_L RC)}} \quad (7)$$

whose maximum is also at $R_{max} = 1/(2\pi C\sqrt{f_L f_H})$, and it is

$$V_{t(max)} = \sqrt{\frac{kT}{C}\cdot\frac{f_H^2\cdot\sqrt{f_L f_H}}{(f_H + f_L)\cdot(f_L + \sqrt{f_L f_H})\cdot(f_H + \sqrt{f_L f_H})}}. \quad (8)$$

Expressions (4) and (7) may look quite different, but they differ by less than 6% relative to their maximum value $(kT/C)^{1/2}$ if $f_L < 1/(2\pi RC) << f_H$.

Conclusion

In a practical circuit, the output noise due to the thermal noise of a resistor will depend on the actual transfer function of the noise source. The bandwidth will never be infinite because the transfer function of either the noise source or the device measuring the output noise will cause limitations. Consequently, the thermal noise voltage of a resistor in a finite bandwidth is maximum for a given resistor value. That maximum thermal noise voltage depends on the stop frequencies and decreases for both smaller and larger resistor values.

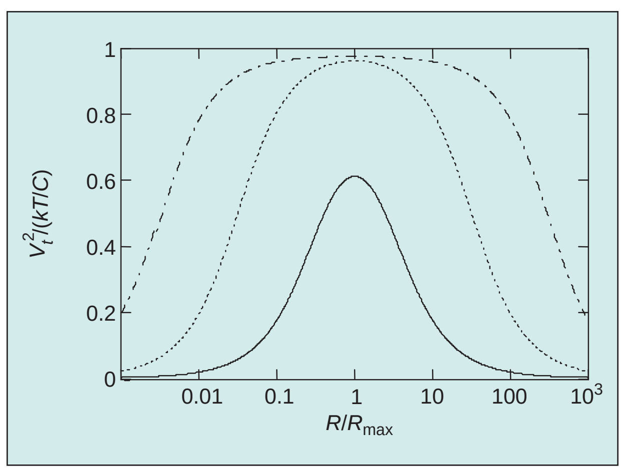

Fig. 2. Normalized mean square noise voltage of a resistor shunted by a capacitance, C, as a function of R/R_{max} for different values of f_H/f_L. R_{max} is the value of the resistor, R, that gives the maximal noise. Legend: ______ $f_H/f_L = 10$; .... $f_H/f_L = 10^3$; – · – · $f_H/f_L = 10^5$.

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