Classical Solutions for a nonlinear Fokker-Planck equation arising in Computational Neuroscience

José A. Carrillo[1], María d. M. González[2], Maria P. Gualdani[3] and Maria E. Schonbek[4]

[1] Institució Catalana de Recerca i Estudis Avançats and Departament de Matemàtiques
Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain

[2] ETSEIB - Departament de Matematica Aplicada I
Universitat Politècnica de Catalunya, E-08028 Barcelona, Spain

[3] Department of Mathematics
UT Austin, Austin, TX 78712, USA

[4] Department of Mathematics
UC Santa Cruz, Santa Cruz, CA 95064, USA

Abstract

In this paper we analyze the global existence of classical solutions to the initial boundary-value problem for a nonlinear parabolic equation describing the collective behavior of an ensemble of neurons. These equations were obtained as a diffusive approximation of the mean-field limit of a stochastic differential equation system. The resulting Fokker-Planck equation presents a nonlinearity in the coefficients depending on the probability flux through the boundary. We show by an appropriate change of variables that this parabolic equation with nonlinear boundary conditions can be transformed into a non standard Stefan-like free boundary problem with a source term given by a delta function. We prove that there are global classical solutions for inhibitory neural networks, while for excitatory networks we give local well-posedness of classical solutions together with a blow up criterium. Finally, we will also study the spectrum for the linear problem corresponding to uncoupled networks and its relation to Poincaré inequalities for studying their asymptotic behavior.

1 Introduction

The basic models for the collective behavior of large ensemble of interacting neurons are based on systems of stochastic differential equations. Each subsystem describes an individual neuron in the network as an electric circuit model with a choice of parameters such as the membrane potential v, the conductances, the proportion of open ion channels and their type. The individual description of each neuron includes an stochastic current due to the spike events produced by other neurons at the network received through the presynaptic connections. We refer to the classical references [14, 10, 23] and the nice brief introduction [13] for a wider overview of this area and further references. As a result of the coupling network, the collective behavior of the stochastic differential system can lead to complicated dynamics: several stationary states with different stability properties and bifurcations, synchronization, and so on, see [1, 17, 18] for instance.

To understand this behavior the evolution in time of the potential through the cell membrane $v(t)$ has been modeled by several authors [1, 2, 20, 6, 22, 19]. The neurons relax towards their resting potential v_L (leak potential) in the absence of any interaction. All the interactions of the neuron within the network are modeled by an incoming presynaptic current $I(t)$ given by an stochastic process to be specified below. Therefore, the evolution of the membrane potential is assumed to follow the equation

$$C_m \frac{dv}{dt} = -g_L(v - v_L) + I(t)\,, \tag{1.1}$$

where C_m is the capacitance of the membrane and, g_L the leak conductance. If the voltage achieves the so-called threshold voltage (maximum voltage), the neuron voltage is instantaneously reset to a fixed voltage v_R. At each reset time the process produces a spike, which builds up the incoming presynaptic current $I(t)$ and is added to the mean firing rate produced by the network $N(t)$ defined as the average number of spikes per unit time produced in the network.

Most of the microscopic models for neuron dynamics assume that the spike appearance times in the network follow an independent discrete Poisson process with constant probability of emitting a spike per unit time ν. We will assume that there are two types of neurons: inhibitory and excitatory, and each produce a spike of strength J_E and J_I respectively at their spike times. The total presynaptic current $I(t)$ in (1.1), coming from the spikes within the network, is computed as the difference of the total spike strengths received through the synapsis by a neuron at the network composed by C_E excitatory and C_I inhibitory neurons. This stochastic process $I(t)$ has mean given by $\mu_C = B\nu$ with $B = C_E J_E - C_I J_I$ and, and variance $\sigma_C^2 = (C_E J_E^2 + C_I J_I^2)\nu$. We will say that the network is excitatory if $B > 0$ (inhibitory respectively if $B < 0$). Dealing with these discrete Poisson processes can be difficult and thus, an approximation was proposed in the literature. This approximation consists in substituting the stochastic process $I(t)$ by a standard drift-diffusion process with the same mean and variance

$$I(t)\,dt \approx \mu_C\,dt + \sigma_C\,d\mathcal{W}_t$$

where $\mathcal{W}_t$ is the standard Brownian motion. We refer for more details of this approximation to [1, 2, 20, 6, 22, 19, 15]. The approximation to the original Leaky Integrate&Fire neuron model (1.1) is then given by

$$dv = (-v + v_L + \mu_C)\,dt + \sigma_C\,d\mathcal{W}_t \tag{1.2}$$

where we choose the units such that $C_m = g_L = 1$, for $v \le v_{th}$ with the jump process: $v(t_o^+) = v_R$ whenever at t_o the voltage achieves the threshold value $v(t_o^-) = v_{th}$; with $v_L < v_R < v_{th}$. The last ingredient of the model is given by the probability of firing per unit time of the Poissonian spike train ν, i.e., the so-called total firing rate. The firing rate depends on the activity of the network and some external stimuli, it is given by $\nu = \nu_{ext} + N(t)$ where $N(t)$ is the mean firing rate produced by the network and $\nu_{ext} \ge 0$ is the external firing rate. The value of $N(t)$ is then computed as the flux of neurons across the threshold or firing voltage v_{th}.

Studying the stochastic problem (1.2) with the jump process specified above can be written in terms of a partial differential equation for the evolution of the probability density $p(v,t) \ge 0$ of finding neurons at a voltage $v \in (-\infty, v_{th}]$ at a time $t \ge 0$. This PDE has the structure of a backward Kolmogorov or Fokker-Planck equation with sources and given by

$$\frac{\partial p}{\partial t}(v,t) = \frac{\partial}{\partial v}\left[\big(v - v_L - \mu_C\big)p(v,t)\right] + \frac{\sigma_C^2}{2}\frac{\partial^2 p}{\partial v^2}(v,t) + N(t)\,\delta_{v=v_R}, \qquad v \le v_{th}\,. \tag{1.3}$$

A Delta Dirac source term in the right-hand side appears due to the firing at time $t \ge 0$ for neurons whose voltage is immediately reset to v_R. Imposing the condition that no neuron should have the firing voltage due to their instantaneous discharge, we complement (1.3) with Dirichlet and initial boundary conditions

$$p(v_{th}, t) = 0, \qquad p(-\infty, t) = 0, \qquad p(v,0) = p_I(v) \ge 0\,. \tag{1.4}$$

The mean firing rate $N(t)$ is implicitly given by

$$N(t) := -\frac{\sigma_C^2}{2}\frac{\partial p}{\partial v}(v_{th}, t) \ge 0\,, \tag{1.5}$$

that is the flux of probability of neuron's voltage, is the least at v_{th}. It is easy to check that this definition implies, at least formally, that the evolution of (1.3) is a probability density for all times, that is

$$\int_{-\infty}^{v_{th}} p(v,t)\,dv = \int_{-\infty}^{v_{th}} p_I(v)\,dv = 1$$

for all $t \geq 0$. Let us note that in most of the computational neuroscience literature [1, 15], equation (1.3) is specified on the intervals $(-\infty, v_R)$ or (v_R, v_{th}) with no source term but rather a boundary condition relating the values of the fluxes from the right and the left at $v = v_R$. The formulation presented here is equivalent and more suitable for mathematical treatment. Other more complicated microscopic models including the conductance and leading to kinetic-like Fokker-Planck equations have been studied recently, see [4] and the references therein.

Finally, the nonlinear Fokker-Planck equation can be rewritten as

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial v^2} + \frac{\partial}{\partial v}[(v-\bar{\mu})P] + N(t)\,\delta_{v=v_R},\ v \leq v_{th}$$

where $\sigma^2 = 2a_0^2 + a_1 N(t)$, with $a_0 > 0$, $a_1 \geq 0$ and $\bar{\mu} = B\nu_{ext} + BN(t)$. We will focus only on the simplest case in which the nonlinearity in the diffusion coefficient is neglected by assuming $a_1 = 0$. Without loss of generality, we can choose a new voltage variable $\tilde{v} \leq 0$ and an scaled density $\tilde{p}$ defined by

$$\tilde{p}(t,\tilde{v}) = \beta p(t, \beta\tilde{v} + v_{th})$$

where $\beta = a_0$. Then our main equation, after dropping the tildes, reads

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial v^2} + \frac{\partial}{\partial v}\left[(v-\mu)p\right] + N(t)\,\delta_{v=v_R}, \qquad v \leq 0\,, \tag{1.6}$$

where the drift term, source of the nonlinearity, is given by

$$\mu = b_0 + bN(t) \qquad \text{with } N(t) = -\frac{\partial p}{\partial v}(0,t) \geq 0 \tag{1.7}$$

with $b_0 = (B\nu_{ext} - v_{th})/a_0$ and $b = B/a_0^3$. Let us remark that the sign of b_0 determines if the neurons due only to external stimuli may produce a spike or not, therefore it controls the strength of the external stimuli.

In a recent work [3], it was shown that the problem (1.6)-(1.4)-(1.7) can lead to finite-time blow up of solutions for excitatory networks $b > 0$ and for initial data concentrated close enough to the threshold voltage. Here, we give a characterization of the maximal time of existence of the classical solution, if finite, and thus, of the blow up time. We show that if the maxima existence time is finite, it coincides with the time in which the firing rate $N(t)$ diverges. This divergence in finite time of the firing rate has no clear biological significance. It could mean that some sort of synchronization of the whole network happens, see [3] for a deeper discussion. This is an scenario that does not show up in the typical reported applications [1, 2]. In the rest of this work, we concentrate in studying the existence of classical solutions to the initial boundary value problem (1.6)-(1.4)-(1.7). We show that solutions exists globally in time for inhibitory networks $b < 0$ and, we give a characterization of the blow up time for the case when $b > 0$. Although the precise notion of classical solution will be discussed in the next section, the main theorem of this work can be summarized as follows.

Theorem 1.1. *Let $p_I(x)$ be a non-negative $\mathcal{C}^1((-\infty, v_{th}])$ function such that $p_I(v_{th}) = 0$. Suppose that $p_I, (p_I)_x$ decay at $-\infty$, then there exists a unique classical solution to the problem (1.6)-(1.4)-(1.7) on the time interval $[0, T^*)$ with $T^* = \infty$ for $b \leq 0$ and, for $b > 0$, $T^* > 0$ can be characterized by*

$$T^* = \sup\{t > 0\,:\, N(t) < \infty\}\,.$$

Furthermore, for $b > 0$ there exist classical solutions blowing up in finite time, and thus, with diverging mean firing rate in finite time.

Let us remark that the last statement is merely obtained by combining the result in [3] with our classical solutions existence result and the characterization of the maximal time of existence.

The main strategy of the proof as is shown in section 2, is given by an equivalence. This equivalence, through an explicit time-space change of variables, transforms our problem into a Stefan-like free boundary problem with Delta Dirac source terms, resembling price-formation models studied in [11]. In section 3, we will use ideas and arguments in Stefan problems [7, 8] to show local existence of a solution. Next, in section 4 we will prove global existence of classical solutions for inhibitory networks ($b < 0$) and give a characterization of the blow up time for excitatory networks ($b > 0$). The difference between the cases $b < 0$ and $b > 0$ corresponds to the well studied Stefan-problem in the normal and in the undercooled cases, see [16] for classical references in the Stefan problem. The final section is devoted to study the spectrum of the linear version of (1.6) ($b = 0$) that has some interesting features and properties connected to classical Fokker-Planck equations.

2 Relation to the Stefan problem

The main aim of this section is to rewrite equation (1.6) as a free boundary Stefan problem with a nonstandard right hand side. For this we recall a well known change of variables, [5], that transforms Fokker-Planck type equations into a non-homogeneous heat equation. This change of variables is given by

$$y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1),$$

that yields

$$p(v,t) = e^t w\left(e^t v, \frac{1}{2}(e^{2t} - 1)\right),$$

or equivalently

$$w(y,\tau) = (2\tau+1)^{-1/2} p\left(\frac{y}{\sqrt{2\tau+1}}, \frac{1}{2}\log(2\tau+1)\right).$$

In the sequel, to simplify the notation, we use $\alpha(\tau) = (2\tau+1)^{-1/2} = e^{-t}$. A straightforward computation gives for w:

$$w_\tau = w_{yy} - \mu(\tau)\alpha(\tau)w_y + M(\tau)\delta_{y=\frac{v_R}{\alpha(\tau)}} \tag{2.1}$$

where $M(\tau) = \alpha^2(\tau)N(t) = -\left.\frac{\partial w}{\partial y}\right|_{y=0}$. The additional change of variables:

$$u(x,\tau) = w(y,\tau) \;\; \text{where}\, x = y - \int_0^\tau \mu(s)\alpha(s)\,ds = y - b_0\left(\sqrt{1+2t} - 1\right) - b\int_0^\tau M(s)\alpha^{-1}(s)\,ds,$$

removes the term with w_y in (2.1). For boundary conditions at initial time, denote $s_I = v_{th}(=0)$. We have the following equivalent equation

Lemma 2.1. *System* (1.6)-(1.4)-(1.5) *is equivalent to the following problem*

$$\begin{cases} u_t = u_{xx} + M(t)\delta_{x=s_1(t)}, & x < s(t), t > 0, \\ s(t) = s_I - b_0\left(\sqrt{1+2t} - 1\right) - b\displaystyle\int_0^t M(s)\alpha^{-1}(s)\,ds, & t > 0, \\ s_1(t) = s(t) + \dfrac{v_R}{\alpha(t)}, & t > 0, \\ M(t) = -\left.\dfrac{\partial u}{\partial x}\right|_{x=s(t)}, & t > 0, \\ u(-\infty,t) = 0, \;\; u(s(t),t) = 0, & t > 0, \\ u(x,0) = u_I(x), & x < s_I. \end{cases} \tag{2.2}$$

Proof. The proof is straightforward by the changes of variables specified above and, as such is omitted. □

We now give a definition the concept of classical solution. In what follows we work with the Stefan-like free boundary problem (2.2). It is immediate to translate this to a concept of classical solution to the original problem (1.6)-(1.4)-(1.5) by substituting u by p, x by v, $M(t)$ by $N(t)$, $s_1(t)$ by v_R, and $s(t)$ by v_{th}.

Definition 2.2. *Let $u_I(x)$ be a non-negative $\mathcal{C}^1((-\infty, s_I])$ function such that $u_I(s_I) = 0$. Suppose that $u_I, (u_I)_x$ decay at $-\infty$. We say that $(u(x,t), s(t))$ is a solution of (2.2) with initial data $u_I(x)$ on the time interval $J = [0,T)$ or $J = [0,T]$, for a given $0 < T \le \infty$, if:*

1. *$M(t)$ is a continuous function for all $t \in J$,*
2. *u is continuous in the region $\{(x,t) : -\infty < x \le s(t), t \in J\}$,*
3. *u_{xx} and u_t are continuous in the region $\{(x,t) : -\infty < x < s_1(t), t \in J\backslash\{0\}\} \cup \{(x,t) : s_1(t) < x < s(t), t \in J\backslash\{0\}\}$,*
4. *$u_x(s_1(t)^-, t)$, $u_x(s_1(t)^+, t)$, $u_x(s(t)^-, t)$ are well defined,*
5. *u_x decays at $-\infty$,*
6. *Equations (2.2) are satisfied.*

The next lemma presents some of the a priori properties of the solution to (2.2).

Lemma 2.3. *Let $u(x,t)$ be a solution to (2.2) in the sense of Definition 2.2.Then*

i) The mass is conserved,

$$\int_{-\infty}^{s(t)} u(x,t)\, dx = \int_{-\infty}^{s_I} u_I(x) dx,$$

for all $t > 0$.

ii) The flux across the free boundary s_1 is exactly the strength of the source term:

$$M(t) := -u_x(s(t),t) = u_x(s_1(t)^-, t) - u_x(s_1(t)^+, t).$$

iii) If $b_0 < 0$ and $b < 0$ (resp. $b_0 > 0$ and $b > 0$), the free boundary $s(t)$ is a monotone increasing (resp. decreasing) function of time.

Proof. *i)* Mass conservation, follows by integration of the equation and straightforward integration by parts.
ii) To establish the jump across the free boundary, i.e. part *ii)*, integrate the first equation in (2.2) over the interval $(-\infty, s_1(t))$, yielding

$$\int_{-\infty}^{s_1(t)} u_t dx - \int_{-\infty}^{s_1(t)} u_{xx} dx = 0.$$

Hence,

$$\frac{\partial}{\partial t} \int_{-\infty}^{s_1(t)} u(x,t) dx = u_x(s_1(t)^-, t) + \dot{s}_1(t) u(s_1(t), t). \tag{2.3}$$

Similarly, an integration of the first equation in (2.2) in the interval $(s_1(t), s(t))$ gives

$$\frac{\partial}{\partial t} \int_{s_1(t)}^{s(t)} u(x,t) dx + \dot{s}_1(t) u(s_1(t),t) - \dot{s}(t) u(s(t),t) = u_x(s(t),t) - u_x(s_1(t)^+, t).$$

If we substitute $u(s(t),t) = 0$ in the previous line it follows

$$\frac{\partial}{\partial t} \int_{s_1(t)}^{s(t)} u dx + \dot{s}_1(t) u(s_1(t),t) = u_x(s(t),t) - u_x(s_1(t)^+, t). \tag{2.4}$$

Adding (2.3) to (2.4) and recalling that the mass is preserved we get

$$0 = \frac{\partial}{\partial t}\int_{-\infty}^{s(t)} u(x,t)dx = u_x(s_1(t)^-,t) + u_x(s(t),t) - u_x(s_1(t)^+,t).$$

It follows that

$$u_x(s(t),t) = u_x(s_1(t)^+,t) - u_x(s_1(t)^-,t),$$

as desired.

Let us prove the last part *iii)*. The free boundary is a monotone increasing function of time since $b_0 < 0$, $b < 0$, $\alpha > 0$, and

$$s(t) = s_I - b_0\left(\sqrt{1+2t} - 1\right) - b\int_0^t M(s)\alpha^{-1}(s)\,ds, \quad t > 0,$$

while the fact that $M(t)$ is strictly positive follows by the classical Hopf's lemma. □

3 Local existence and uniqueness

In this section we prove local existence of solution. Our method is inspired by the theory developed by Friedman in [7, 8] for the Stefan problem. We first derive an integral formulation for the problem. A derivative with respect to x yields an integral equation for the flux M, where a fixed point argument can be used to obtain short time existence. Once M is known the equation for u decouples and it is solved as a linear equation.

Theorem 3.1. *Let $u_I(x)$ be a non-negative $\mathcal{C}^1((-\infty, s_I])$ function such that $u_I(s_I) = 0$, and $u_I, (u_I)_x$ decay at $-\infty$. Then there exists a time $T > 0$, and a unique solution $(u(x,t), s(t))$ of the equation* (2.2) *in the sense of Definition* 2.2 *for $t \in [0,T]$ with initial data u_I. Moreover, the existence time T is an inversely proportional function of*

$$\sup_{-\infty < x \le s_I} |u_I'(x)|\,.$$

The proof of Theorem 3.1 will be divided in several steps. The first step deals with an integral formulation of the solution, which is used to show the existence of M.

3.1 The integral formulation

Let G be the the Green's function for the heat equation on the real line:

$$G(x,t,\xi,\tau) = \frac{1}{[4\pi(t-\tau)]^{1/2}}\exp\left\{-\frac{|x-\xi|^2}{4(t-\tau)}\right\}.$$

To obtain an integral formulation of the solution u of (2.2), recall the following Green's identity

$$\frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi}\right) - \frac{\partial}{\partial \tau}(Gu) = 0. \tag{3.1}$$

To recover u we first integrate the identity (3.1) in the two regions

$$-\infty < \xi < s_1(\tau),\ 0 < \tau < t, \quad \text{and} \quad s_1(\tau) < \xi < s(\tau),\ 0 < \tau < t,$$

and then add up the results from the integration. We split the resulting expression into the following four terms; the only problematic one is the one containing $u_{\xi\xi}$:

$$I = \int_0^t\int_{-\infty}^{s_1(\tau)} \frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right)d\xi d\tau, \qquad II = \int_0^t\int_{s_1(\tau)}^{s(\tau)} \frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right)d\xi d\tau,$$

$$III = \int_0^t\int_{-\infty}^{s(\tau)} \frac{\partial}{\partial \xi}\left(u\frac{\partial G}{\partial \xi}\right)d\xi d\tau, \qquad IV = \int_0^t\int_{-\infty}^{s(\tau)} \frac{\partial}{\partial \tau}(Gu)\,d\xi d\tau.$$

Each term will be analyzed separately. Note that u and G have enough decay as $|\xi| \to \infty$ to justify the following computations due to Definition 2.2. Since $G(x,t,-\infty,\tau) = 0$ it holds

$$I = \int_0^t G \frac{\partial u}{\partial \xi}\Big|_{\xi=-\infty}^{\xi=s_1(\tau)} d\tau = \int_0^t G(x,t,s_1(\tau),\tau)\, \frac{\partial u}{\partial \xi}\Big|_{s_1(\tau)^-} d\tau. \tag{3.2}$$

Next, we obtain

$$II = \int_0^t \left\{ G \frac{\partial u}{\partial \xi}\Big|_{\xi=s(\tau)} - G\, \frac{\partial u}{\partial \xi}\Big|_{\xi=s_1(\tau)^+} \right\} d\tau = -\int_0^t \left\{ G|_{\xi=s(\tau)}\, M(\tau) + G\, \frac{\partial u}{\partial \xi}\Big|_{\xi=s_1(\tau)^+} \right\} d\tau. \tag{3.3}$$

Here we have used that $\frac{\partial u}{\partial \xi}\Big|_{\xi=s(\tau)} = -M(\tau)$. For the third integral we have

$$\begin{aligned} III &= -\int_0^t \left\{ \left(u \frac{\partial G}{\partial \xi} \right)\Big|_{\xi=s(\tau)} - \left(u \frac{\partial G}{\partial \xi} \right)\Big|_{\xi=-\infty} \right\} d\tau \\ &= -\int_0^t \left\{ (u(s(\tau),\tau) \frac{\partial G}{\partial \xi}\Big|_{\xi=s(\tau)} - u(-\infty,\tau) \frac{\partial G}{\partial \xi}\Big|_{\xi=-\infty} \right\} d\tau = 0, \end{aligned} \tag{3.4}$$

taking into account that $u(s(\tau),\tau) = u(-\infty,\tau) = 0$. Finally, using that $u(s(\tau),\tau) = 0$, we have

$$IV = \int_0^t \frac{\partial}{\partial \tau} \int_{-\infty}^{s(\tau)} G u d\xi d\tau = \int_{-\infty}^{s(t)} G u|_{\tau=t}\, d\xi - \int_{-\infty}^{s(0)} G u|_{\tau=0}\, d\xi.$$

Recall that $G(x,t,\xi,t) = \delta_{x=\xi}$, thus the last identity yields

$$IV = \int_{-\infty}^{s(t)} \delta_{\xi=x} u(\xi,t) d\xi - \int_{-\infty}^{s(0)} G(x,t,\xi,0) u_I(\xi) d\xi. \tag{3.5}$$

Combining (3.2), (3.3), (3.4), (3.5), and part *ii)* of Lemma 2.3, yields that the solution u reads as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{s(0)} G(x,t,\xi,0) u_I(\xi) d\xi + \int_0^t G(x,t,s_1(\tau))\, \frac{\partial u}{\partial \xi}\Big|_{\xi=s_1(\tau)^-} \\ &\quad - \int_0^t M(\tau) G(x,t,s(\tau),\tau) d\tau - \int_0^t G(x,t,s_1(\tau)) \frac{\partial u}{\partial \xi}\Big|_{\xi=s_1(\tau)^+} \\ &= \int_{-\infty}^{s(0)} G(x,t,\xi,0) u_I(\xi) d\xi - \int_0^t M(\tau) G(x,t,s(\tau),\tau) d\tau + \int_0^t M(\tau) G(x,t,s_1(\tau),\tau) d\tau \\ &=: \; I_1 - I_2 + I_3. \end{aligned} \tag{3.6}$$

The term I_1 represents the solution to the heat equation with data u_I. Indeed,

$$\int_{-\infty}^{s(0)} G(x,t,\xi,0) u_I(\xi) d\xi$$

is the solution to the homogeneous heat equation with initial data

$$u_0(\xi) = \begin{cases} u_I(\xi) & \xi \le s(0), \\ 0 & \xi > s(0). \end{cases}$$

All the calculations up to here are formal assuming that u is a solution of the equation (2.2) as in Definition 2.2. We now derive an equation for M which will be solved for short time using a fixed point argument. The first step is to obtain the space derivatives of the terms $I_i,\ i=1,2,3$ and evaluate them at $x=s(t)^-$:

$$\frac{\partial I_1}{\partial x}\bigg|_{x=s(t)^-} = \int_{-\infty}^{s(0)} G_x(x,t,\xi,0)u_I(\xi)d\xi = -\int_{-\infty}^{s(0)} G(x,t,\xi,0)u_I'(\xi)d\xi.$$

To get the derivative of I_2, we use [7, Lemma 1, pag 217]: this lemma states that for any continuous function ρ,

$$\lim_{x\to s(t)^-}\frac{\partial}{\partial x}\int_0^t \rho(\tau)G(x,t,s(\tau),\tau)\,d\tau = \frac{1}{2}\rho(t) + \int_0^t \rho(\tau)\frac{\partial G}{\partial x}(s(t),t,s(\tau),\tau)\,d\tau. \tag{3.7}$$

As a consequence,

$$\frac{\partial I_2}{\partial x}\bigg|_{x=s(t)^-} = \frac{1}{2}M(t) + \int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau.$$

For the derivative of I_3 note that problems can only occur if $t=\tau$ and $s(t)=s_1(\tau)$, but this is not possible by the definition of s_1 . Thus,

$$\frac{\partial I_3}{\partial x}\bigg|_{x=s(t)^-} = \int_0^t G_x(s(t),t;s_1(\tau),\tau)M(\tau)d\tau.$$

Substituting the estimates on I_1, I_2 and I_3 into (3.6) we get

$$\begin{aligned}-M(t) = &\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)u_I'(\xi)d\xi - \frac{1}{2}M(t)\\ &- \int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau + \int_0^t M(\tau)G_x(s(t),t;s_1(\tau),\tau)d\tau.\end{aligned}$$

After reordering yields,

$$\begin{aligned}M(t) = &-2\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)u_I'(\xi)\,d\xi\\ &+2\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)\,d\tau - 2\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau)\,d\tau.\end{aligned} \tag{3.8}$$

3.2 Local existence and uniqueness for M

Theorem 3.2. *Let $u_I(x)$ be a non-negative $\mathcal{C}^1((-\infty,s_I])$ function such that $u_I(s_I)=0$. Suppose $u_I,(u_I)_x$ decay to zero as $x\to-\infty$. Then there exists a time $T>0$ such that $M(t)$ defined by the integral formulation* (3.8) *exists for $t\in[0,T]$ and is unique in $\mathcal{C}([0,T])$. The existence time T satisfies*

$$T \le \left(\sup_{-\infty<x\le s_I}|u_I'(x)|\right)^{-1}.$$

Proof. The local in time existence of $M(t)$ is obtained via a fixed point argument. For this, we modify the classical argument for the Stefan problem to account for the additional source term given by $M(t)\,\delta_{x=s_1(t)}$. For $\sigma,m>0$, consider the norm

$$\|M\| := \sup_{0\le t\le\sigma}|M(t)|$$

in the space

$$C_{\sigma,m} := \{M\in\mathcal{C}([0,\sigma]) : \|M\|\le m\}.$$

Set

$$
\begin{aligned}
T(M)(t) :=& -2\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)u_I'(\xi)d\xi \\
&+2\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau - 2\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau \\
:=& J_1 + J_2 + J_3.
\end{aligned}
\tag{3.9}
$$

In order to apply fixed point arguments, it is necessary to show that for sufficiently small σ we have: $T : C_{\sigma,m} \to C_{\sigma,m}$ and that T is a contraction. Define

$$
m := 1 + 2 \sup_{-\infty < x \le s(0)} |u_I'(x)|. \tag{3.10}
$$

Step 1.- We show that for σ sufficiently small $T : C_{\sigma,m} \to C_{\sigma,m}$. For simplicity, we focus on the proof in the case $b < 0$. At the end we make the necessary corrections for $b > 0$. Choose σ sufficiently small so that

i. $\alpha^{-1}(t) \le 2,\ \forall\, t \le \sigma$,

ii. $\frac{m(|b_0|+2m|b|)}{\sqrt{\pi}}\sigma^{1/2} \le 1/2$,

iii. $|v_R| - |b_0|\sigma > 0$,

iv. $\dfrac{2m}{\sqrt{\pi}} \displaystyle\int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^{\infty} z^{-1}\exp\{-z^2\}dz \le 1/2.$

We obtain first an auxiliary estimate. Since σ has been chosen so small that condition *i.* holds and $\alpha^{-1}\sqrt{1+2t}$ is a 1-Lipschitz function for $t \ge 0$, if $M \in C_{\sigma,m}$ then

$$
|s(t) - s(\tau)| \le |b_0||t-\tau| + |b|\int_\tau^t M(s)\alpha^{-1}(s)\, ds \le (|b_0| + 2|b|m)\, |t-\tau|, \tag{3.11}
$$

i.e., $s(t)$ is a Lipschitz continuous function of time.

To estimate the image of the operator $T(M)$ as defined in (3.9) for $M \in C_{\sigma,m}$ we find separately a bound for each term J_1, J_2, J_3. First, for J_1, note that

$$
\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)\, d\xi \le 1.
$$

Then, it is straightforward to check

$$
|J_1| \le 2\left\{\sup_{-\infty<x\le s(0)} |u_I'(x)|\right\}\int_{-\infty}^{s(0)} G(x,t,\xi,0)d\xi \le 2\sup_{-\infty<x\le s(0)} |u_I(x)|.
$$

We bound J_2 as

$$
|J_2| \le 2m\int_0^t |G_x(s(t),t,s(\tau),\tau)|d\tau.
$$

Substituting

$$
G_x(x,t,\xi,\tau) = -\frac{1}{2\sqrt{4\pi}}\frac{(x-\xi)}{(t-\tau)^{3/2}}\exp\left\{-\frac{|x-\xi|^2}{4(t-\tau)}\right\}
$$

in (3.11) and taking into account the choice of σ given by *ii.*, it follows that:

$$
\begin{aligned}
|J_2| &\le \frac{m}{\sqrt{4\pi}}\int_0^t \frac{|s(t)-s(\tau)|}{(t-\tau)^{3/2}}\exp\left\{-\frac{|s(t)-s(\tau)|^2}{4(t-\tau)}\right\}d\tau \\
&\le \frac{m(|b_0|+2m|b|)}{\sqrt{4\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}d\tau = \frac{2m(|b_0|+2m|b|)}{\sqrt{4\pi}}t^{1/2} \le \frac{2m(|b_0|+2m|b|)}{\sqrt{4\pi}}\sigma^{1/2} \le \frac{1}{2}.
\end{aligned}
$$

Before we consider J_3, we need the following auxiliary estimates. The inequality $ye^{-y^2} \le e^{-\frac{y^2}{2}}$ implies that

$$|G_x(x,t,\xi,\tau)| \le \frac{1}{\sqrt{4\pi}(t-\tau)} \exp\left\{-\frac{|x-\xi|^2}{8(t-\tau)}\right\}. \tag{3.12}$$

The definitions of $s(t)$ and $s_1(\tau) = s(\tau) + v_R\alpha^{-1}(\tau)$, using that $b<0$, and by the condition *iii*, yield

$$|s(t) - s_1(\tau)| \ge |v_R| - |b_0|\sigma > 0. \tag{3.13}$$

If we integrate (3.12) we get

$$\begin{aligned}
\int_0^t |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau &\le \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{t-\tau} \exp\left\{-\frac{|s(t)-s_1(\tau)|^2}{8(t-\tau)}\right\} d\tau \\
&\le \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{t-\tau} \exp\left\{-\frac{(|v_R|-|b_0|\sigma)^2}{8(t-\tau)}\right\} d\tau \\
&= \frac{1}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8t}}}^{\infty} \frac{1}{z} e^{-z^2}\, dz \le \frac{1}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^{\infty} \frac{1}{z} e^{-z^2}\, dz,
\end{aligned} \tag{3.14}$$

where we used the change of variables $z = \frac{|v_R|-|b_0|\sigma}{\sqrt{8(t-\tau)}}$. By the last estimate and by condition *iv*

$$|J_3| \le 2m \int_0^t |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau \le \frac{2m}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^{\infty} \frac{1}{z} e^{-z^2} dz \le \frac{1}{2}. \tag{3.15}$$

The estimates for $J_i, i=1,2,3$ establish that $T(M) \in C_{\sigma,m}$ since

$$\|T(M)\| \le J_1 + J_2 + J_3 \le m, \quad \forall M \in C_{\sigma,m},$$

by the choice of m in (3.10).

It remains to consider the case $b>0$. It is clear that the only modification needed is the estimte (3.13). For this use

$$|s(t) - s_1(\tau)| = |s(t) - s(\tau) - v_R\alpha^{-1}(\tau)| \ge \big||v_R|\alpha^{-1}(\tau) - |s(t)-s(\tau)|\big| \ge |v_R| - (|b_0|+m)\sigma, \tag{3.16}$$

which may be estimated from below by a positive constant for some σ small enough. Then, we have the same result as in the case $b<0$ assuming analogous conditions to *i., ii., iii., iv.* above. The main difference between the cases $bleq0$ and $b>0$ is that in the case $b>0$ in (3.13) the difference between the free boundary $s(t)$ and the source $s_1(\tau)$ for $0 \le \tau \le t$ now depends on the bound of the initial data (3.10).

Step 2.- The mapping $T : C_{\sigma,m} \to C_{\sigma,m}$ defined in (3.9) is a contraction for σ small enough. In the sequel the constant C is arbitrary and may change from line to line. Let $M, \tilde{M} \in C_{\sigma,m}$, and

$$\begin{aligned}
s(t) &= s_I - b_0\left(\sqrt{1+2t} - 1\right) - b\int_0^t M(\tau)\alpha^{-1}(\tau)\, d\tau, \,. \\
\tilde{s}(t) &= s_I - b_0\left(\sqrt{1+2t} - 1\right) - b\int_0^t \tilde{M}(\tau)\alpha^{-1}(\tau)\, d\tau\,.
\end{aligned} \tag{3.17}$$

The following auxiliary estimate holds:

$$\begin{aligned}
|s(t) - \tilde{s}(t)| &\le |b| \int_0^t |M(\tau) - \tilde{M}(\tau)|\alpha^{-1}(\tau)\, d\tau \le |b|\, \|M - \tilde{M}\| \int_0^t \sqrt{2\tau+1} d\tau \\
&= \frac{|b|}{3} \|M - \tilde{M}\| \left[(2t+1)^{3/2} - 1\right].
\end{aligned} \tag{3.18}$$

It is straightforward from (3.17) that

$$|\dot{s}(t) - \dot{\tilde{s}}(t)| \leq 2\,|b|\,\|M - \tilde{M}\|, \quad 0 < t \leq \sigma < 1. \tag{3.19}$$

By condition $i.$ on σ and (3.11) it follows that

$$\max\{|s(t) - s(\tau)|, |\tilde{s}(t) - \tilde{s}(\tau)|\} \leq (|b_0| + 2m\,|b|)|t - \tau| \leq (|b_0| + 2\,|b|)m|t - \tau|\,. \tag{3.20}$$

To show that T is a contraction we proceed as follows.

$$\begin{aligned}
|T(M) - T(\tilde{M})| \leq & \, 2\left[\int_{-\infty}^{s(0)} |u_I'(\xi)||G(s(t), t, \xi, 0) - G(\tilde{s}(t), t, \xi, 0)|\, d\xi\right] \\
& + 2\left|\int_0^t M(\tau)G_x(s(t), t, s(\tau), \tau) - \tilde{M}(\tau)G_x(\tilde{s}(t), t, \tilde{s}(\tau), \tau)\, d\tau\right| \\
& + 2\left|\int_0^t M(\tau)G_x(s(t), t, s_1(\tau), \tau) - \tilde{M}(\tau)G_x(\tilde{s}(t), t, \tilde{s}_1(\tau), \tau)\, d\tau\right| \\
=: & \, \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.
\end{aligned}$$

Without loss of generality assume that $\tilde{s}(t) > s(t)$. The mean value theorem applied to the kernel $G(x, t, \xi, 0)$ gives for some $\bar{s} \in [s(t), \tilde{s}(t)]$

$$|G(s(t), t, \xi, 0) - G(\tilde{s}(t), t, \xi, 0)| \leq |G_x(\bar{s}, t, \xi, 0)| \cdot |s(t) - \tilde{s}(t)|. \tag{3.21}$$

Recall that

$$|G_x(\bar{s}, t, \xi, 0)| = \frac{|\bar{s} - \xi|}{2t}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{|\bar{s} - \xi|^2}{4t}\right\} \leq \frac{1}{\sqrt{t}}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{|\bar{s} - \xi|^2}{8t}\right\},$$

where we have used the relation $ye^{-y^2} \leq e^{-y^2/2}$. Hence (3.21) simply reduces to

$$|G(s(t), t, \xi, 0) - G(\tilde{s}(t), t, \xi, 0)| \leq \frac{C}{\sqrt{t}}G(\bar{s}(t), 2t, \xi, 0))|s(t) - \tilde{s}(t)|.$$

Integrating in ξ, together with (3.18) yields

$$\mathcal{A}_1 \leq C|b|\|u_I'\|\|M - \tilde{M}\|\left\{\frac{(1 + 2t)^{3/2} - 1}{t^{1/2}}\right\}.$$

Since $\lim_{t\to 0} t^{-1/2}((1 + 2t)^{3/2} - 1) = 0$, for σ sufficiently small we have $\mathcal{A}_1 \leq \frac{1}{6}\|M - \tilde{M}\|$. To estimate $\mathcal{A}_2$ we proceed as follows

$$\begin{aligned}
|\mathcal{A}_2| \leq & \, 2\left|\int_0^t M(\tau)G_x(s(t), t, s(\tau), \tau) - \tilde{M}(\tau)G_x(s(t), t, s(\tau), \tau)d\tau\right| \\
& + 2\left|\int_0^t \tilde{M}(\tau)G_x(s(t), t, s(\tau), \tau) - \tilde{M}(\tau)G_x(\tilde{s}(t), t, \tilde{s}(\tau), \tau)d\tau\right| \\
=: & \, \mathcal{A}_{21} + \mathcal{A}_{22}.
\end{aligned}$$

The Lipschitz bound (3.11) for s yields

$$|G_x(s(t), t, s(\tau), \tau)| \leq \frac{1}{2\sqrt{4\pi}}\frac{|s(t) - s(\tau)|}{(t - \tau)^{3/2}} \leq \frac{(|b_0| + 2m\,|b|)}{2\sqrt{4\pi}}\frac{1}{(t - \tau)^{1/2}},$$

and consequently

$$\begin{aligned}
|\mathcal{A}_{21}| \leq & \, 2\|M - \tilde{M}\|\int_0^t |G_x(s(t), t, s(\tau), \tau)|d\tau \\
\leq & \, C\|M - \tilde{M}\|\int_0^t \frac{1}{(t - \tau)^{1/2}}d\tau \leq Cm\|M - \tilde{M}\|\sigma^{1/2} \leq \frac{1}{12}\|M - \tilde{M}\|,
\end{aligned}$$

for σ small enough. To estimate $\mathcal{A}_{22}$ proceed as follows:

$$
\begin{aligned}
|G_x(s(t),&t,s(\tau),\tau) - G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\
&= C\left|\frac{s(t)-s(\tau)}{t-\tau}G(s(t),t,s(\tau),\tau) - \frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)\right| \\
&\le C\left|\frac{s(t)-s(\tau)}{t-\tau} - \frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}\right| G(s(t),t,s(\tau),\tau) \\
&\quad + C\frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}\,|G(s(t),t,s(\tau),\tau) - G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\
&=: \mathcal{B}_1 + \mathcal{B}_2.
\end{aligned}
$$

In order to estimate $\mathcal{B}_1$ we use the mean value theorem

$$
\frac{[s(t)-\tilde{s}(t)] - [s(\tau)-\tilde{s}(\tau)]}{t-\tau} = \dot{s}(\bar{\tau}) - \dot{\tilde{s}}(\bar{\tau}) \tag{3.22}
$$

for some $0 < \bar{\tau} < t$. By the previous equality and (3.19) we have

$$
\mathcal{B}_1 \le \frac{C}{(t-\tau)^{1/2}}|\dot{s}(\bar{\tau}) - \dot{\tilde{s}}(\bar{\tau})| \le \frac{C}{(t-\tau)^{1/2}}\|M-\tilde{M}\|.
$$

On the other hand, to handle the term $\mathcal{B}_2$, we first note that

$$
\begin{aligned}
&|G(s(t),t,s(\tau),\tau) - G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\
&\le G(s(t),t,s(\tau),\tau)\left|1-\exp\left\{\frac{-(\tilde{s}(t)-\tilde{s}(\tau))^2 + (s(t)-s(\tau))^2}{4(t-\tau)}\right\}\right|.
\end{aligned} \tag{3.23}
$$

Define now

$$
S := (s(t)-s(\tau))^2 - (\tilde{s}(t)-\tilde{s}(\tau))^2 = [s(t)-s(\tau)+\tilde{s}(t)-\tilde{s}(\tau)]\,[s(t)-\tilde{s}(t)-(s(\tau)-\tilde{s}(\tau))]. \tag{3.24}
$$

The mean value theorem (3.22) and the estimate (3.19) lead to

$$
|[s(t)-\tilde{s}(t)] - [s(\tau)-\tilde{s}(\tau)]| = \left|\dot{s}(\bar{\tau}) - \dot{\tilde{s}}(\bar{\tau})\right|(t-\tau) \le C\|M-\tilde{M}\|(t-\tau). \tag{3.25}
$$

On the other hand, we recall again the Lipschitz estimate (3.20), i.e.,

$$
\max\{|s(t)-s(\tau)|, |\tilde{s}(t)-\tilde{s}(\tau)|\} \le Cm(t-\tau), \tag{3.26}
$$

for a constant depending on $|b|$, $|b_0|$, which yields an estimate for (3.24),

$$
\frac{|S|}{t-\tau} \le Cm\sigma\|M-\tilde{M}\|.
$$

The combination of the above inequality with (3.23) together with the mean value theorem shows that

$$
|G(s(t),t,s(\tau),\tau) - G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \le G(s(t),t,s(\tau),\tau)Cm\sigma\|M-\tilde{M}\|,
$$

and thus the term $\mathcal{B}_2$ is estimated using (3.26)

$$
\mathcal{B}_2 \le Cm^2\|M-\tilde{M}\|\sigma\frac{1}{(t-\tau)^{1/2}}.
$$

Multiplying $\mathcal{B}_1 + \mathcal{B}_2$ by $\tilde{M}(\tau)$ and integrating over the interval $[0,t]$ yields

$$
\begin{aligned}
\mathcal{A}_{22} &\le Cm\int_0^t |G_x(s(t),t,s(\tau),\tau) - G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)|\,d\tau \\
&\le Cm\int_0^t (\mathcal{B}_1+\mathcal{B}_2)\,d\tau \le Cm^3\|M-\tilde{M}\|\sigma^{1/2} < \frac{1}{12}\|M-\tilde{M}\|,
\end{aligned}
$$

for σ small enough.

The next step is to estimate $\mathcal{A}_3$. Split the integral into two terms

$$\begin{aligned}|\mathcal{A}_3| &\leq 2\left|\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau) - \tilde{M}(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau\right| \\ &\quad + 2\left|\int_0^t \tilde{M}(\tau)G_x(s(t),t,s_1(\tau),\tau) - \tilde{M}(\tau)G_x(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)d\tau\right| \\ &=: \mathcal{A}_{31} + \mathcal{A}_{32}.\end{aligned}$$

The estimate for $\mathcal{A}_{31}$ is very similar to that of J_3 from (3.15). Indeed, we can analogously deduce

$$\begin{aligned}|\mathcal{A}_{31}| &\leq 2\|M - \tilde{M}\| \int_0^t |G_x(s(t),t,s_1(\tau),\tau)|d\tau \\ &\leq C\|M - \tilde{M}\| \int_{\frac{\Lambda}{\sqrt{8\sigma}}}^\infty \frac{1}{z}e^{-z^2}dz < \frac{1}{12}\|M - \tilde{M}\|,\end{aligned}$$

where we used that $\tilde{s}(t) - \tilde{s}_1(\tau) \geq \Lambda > 0$ for σ sufficiently small with

$$\Lambda := \begin{cases} |v_R| - |b_0|\sigma & \text{for } b < 0 \\ |v_R| - (|b_0| + m)\sigma & \text{for } b > 0 \end{cases}.$$

To bound $\mathcal{A}_{32}$ we split

$$\begin{aligned}|G_x(s(t),&t,s_1(\tau),\tau) - G_x(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)| \\ &= C\left|\frac{s(t)-s_1(\tau)}{t-\tau}G(s(t),t,s_1(\tau),\tau) - \frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\right| \\ &\leq C\left|\frac{s(t)-s_1(\tau)}{t-\tau} - \frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}\right|G(s(t),t,s_1(\tau),\tau) \\ &\quad + C\frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}\left|G(s(t),t,s_1(\tau),\tau) - G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\right| \\ &=: \mathcal{B}_1' + \mathcal{B}_2'.\end{aligned}$$

We observe that $\mathcal{B}_1'$ is estimated exactly the same way as $\mathcal{B}_1$. This is a consequence of

$$[s(t) - \tilde{s}(t)] - [s_1(\tau) - \tilde{s}_1(\tau)] = [s(t) - \tilde{s}(t)] - [s(\tau) - \tilde{s}(\tau)].$$

We can continue from (3.22) as before to obtain

$$|B_1'| \leq \frac{C}{(t-\tau)^{1/2}}\|M - \tilde{M}\|. \tag{3.27}$$

The estimate for $\mathcal{B}_2'$ is slightly more involved. We write

$$\mathcal{B}_2' = C\frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}|G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)|\left(1 - \exp\left\{\frac{S'}{4(t-\tau)}\right\}\right) \tag{3.28}$$

for

$$\begin{aligned}S' &:= -(s(t) - s_1(\tau))^2 + (\tilde{s}(t) - \tilde{s}_1(\tau))^2 \\ &= [\tilde{s}(t) - \tilde{s}_1(\tau) + s(t) - s_1(\tau)]\,[\tilde{s}(t) - \tilde{s}_1(\tau) - s(t) + s_1(\tau)].\end{aligned}$$

By the definitions of s_1 and $\tilde{s}_1$ (see (2.2)) we have that

$$|\tilde{s}(t) - \tilde{s}_1(\tau) - s(t) + s_1(\tau)| = |\tilde{s}(t) - \tilde{s}(\tau) - s(t) + s(\tau)| \leq C\|M - \tilde{M}\|(t-\tau), \tag{3.29}$$

where in the last inequality we have used estimate (3.25). On the other hand,

$$|[\tilde{s}(t) - \tilde{s}_1(\tau) + s(t) - s_1(\tau)]| \leq |s(t) - s(\tau)| + |\tilde{s}(t) - \tilde{s}(\tau)| + 2|v_R|\sqrt{2\tau+1} \leq Cm\sigma, \tag{3.30}$$

if we use again the Lipschitz estimate (3.26). Hence putting together (3.29) and (3.30) we get again that

$$\frac{|S'|}{t-\tau} \leq Cm\sigma\|M-\tilde{M}\|,$$

and consequently, (3.28) reduces to

$$|\mathcal{B}_2'| \leq Cm\frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\sigma\|M-\tilde{M}\|.$$

Integrating the previous expression, using the inequality $ye^{-y^2} \leq e^{-y^2/2}$, and noting that $\tilde{s}(t) - \tilde{s}_1(\tau) \geq \Lambda > 0$, we can give a very rough estimate that is enough to our purposes:

$$\int_0^t |\mathcal{B}_2'|\,d\tau \leq Cm\sigma\|M-\tilde{M}\|. \tag{3.31}$$

Thus, from the estimates for $\mathcal{B}_1'$ and $\mathcal{B}_2'$ from (3.27) and (3.31) respectively,

$$|\mathcal{A}_{32}| \leq Cm\int_0^t (\mathcal{B}_1'+\mathcal{B}_2')\,d\tau \leq C\|M-\tilde{M}\|\left(m\sigma^{1/2}+m^2\sigma\right) < \frac{1}{12}\|M-\tilde{M}\|,$$

for some suitable σ small enough. Then, adding the estimates obtained for $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ yields that T is a contraction satisfying for some σ small enough inversely proportional to m:

$$\|TM - T\tilde{M}\| \leq \frac{1}{2}\|M-\tilde{M}\|.$$

This concludes the proof of Theorem 3.2 as desired. □

3.3 Recovery of u

Theorem 3.2 shows that we have short time existence of a mild solution for problem (2.2) (i.e., a solution in the integral sense). However, one can easily show that:

Corollary 3.3. *There exists a unique solution of problem* (2.2) *in the sense of Definition* 2.2 *for* $t \in [0,T]$.

Proof. Once M is known, one can construct u from Duhamel's formula (3.6). The smoothness and decay of u follow immediately from here. One needs to check also that u has well defined side derivatives at s_1. But this follows from formula (3.7) applied to $s_1(t)$ and the estimate for $|G_x(s_1(t),t,s_1(\tau),\tau)|$ that follows similarly as the calculation in (3.14). □

This completes the proof of Theorem 3.1.

4 Proofs of the Main Results

From the previous arguments, see (3.10), it is clear that the obstacle for long time existence in this case is the possible blow up in time of $\|u_x(\cdot,t)\|_\infty$ particularly at the free boundary, i.e. the blow up of $M(t)$. We now formalize this idea by showing that we can extend the solution as long as the firing rate $M(t)$ is bounded.

Proposition 4.1. *Suppose that the hypotheses of Theorem* 3.1 *hold and that* $(u(t),s(t))$ *is a solution to* (2.2) *in the time interval* $[0,T]$. *Assume, in addition, that*

$$U_0 := \sup_{x\in(-\infty,s(t_0-\varepsilon)]} |u_x(x,t_0-\varepsilon)| < \infty \qquad \textit{and that} \qquad M^* = \sup_{t\in(t_0-\varepsilon,t_0)} M(t) < \infty\,,$$

for some $0<\varepsilon<t_0\leq T$. *Then*

$$\sup\left\{|u_x(x,t)| \textit{ with } x\in(-\infty,s(t)]\,,\, t\in[t_0-\varepsilon,t_0)\right\} < \infty\,,$$

with a bound depending only on the quantities M^* *and* U_0.

Proof. Differentiating (3.6) in x yields

$$\begin{aligned} u_x(x,t) =& \int_{-\infty}^{s(t_0-\varepsilon)} G(x,t,\xi,t_0-\varepsilon)u_x(\xi,t_0-\varepsilon)d\xi \\ &- \int_{t_0-\varepsilon}^{t} M(\tau)G_x(x,t,s(\tau),\tau)d\tau + \int_{t_0-\varepsilon}^{t} M(\tau)G_x(x,t,s_1(\tau),\tau)d\tau \\ =:& I_1 - I_2 + I_3. \end{aligned}$$

The estimate for I_1 is straightforward from heat kernel properties and it depends only on U_0. Let us deal with I_2. Since M is uniformly bounded in the whole interval $t_0-\varepsilon<t<t_0$, we get

$$|I_2| \le C\int_{t_0-\varepsilon}^{t} |G_x(x,t,s(\tau),\tau)|d\tau. \tag{4.1}$$

Next, it is shown in [7, Eq. (1.16), pag. 219] that for any Lipschitz continuous function s,

$$\int_{t-\varepsilon}^{t} \frac{|x-s(\tau)|}{(t-\tau)} G(x,t,s(\tau),\tau)d\tau \le C, \quad t\in(t_0-\varepsilon,t_0),$$

for some C depending on the Lipschitz constant of s, t_0 and ε. Then this formula allows to control the expression in (4.1) in order to bound the term I_2 for $t\in(t_0-\varepsilon,t_0)$. However, this bound may depend on t_0 and M^* since the Lipschitz constant of s does, see (3.17).

Finally, the same argument works for I_3, replacing s by s_1 in the previous calculations. □

With this result in hand, our solutions can be extended to a maximal time of existence and, we can characterize this maximal time. The following result holds no matter the sign of b.

Theorem 4.2. *Suppose that the hypotheses of Theorem* 3.1 *hold. Then the solution u can be extended up to a maximal time* $0<T^*\le\infty$ *given by*

$$T^* = \sup\{t>0 \,:\, M(t)<\infty\}\,.$$

Proof. Assume that the maximal time of existence of a classical solution $(u(t),s(t))$ to (2.2) in the sense of Definition 2.2 is $T^*<\infty$, if not there is nothing to show. By definition we have $T^*\le\sup\{t>0 \,:\, M(t)<\infty\}$. Let us show the equality by contradiction. Let us assume that $T^*<\sup\{t>0 \,:\, M(t)<\infty\}$ and then, there exists $0<\varepsilon<T^*$ such that

$$M^* = \sup_{t\in(T^*-\varepsilon,T^*)} M(t) < \infty\,.$$

Let U_0 be defined as in Proposition 4.1 with $t_0=T^*$. Applying Proposition 4.1, we deduce that $u_x(x,t)$ is also uniformly bounded for $x\in(-\infty,s(t)]$ and $t\in[T^*-\varepsilon,T^*)$ by a constant, denoted U^*. The same Proposition tells us that U^* only depends on M^* and also on U_0, i.e., the uniform bound of $u_x(x,T^*-\varepsilon)$ for $x\in(-\infty,s(T^*-\varepsilon)]$. Therefore, we can now restart by using the local in time existence Theorem 3.1 using as initial time any $t_0\in[T^*-\varepsilon,T^*)$ for a time interval whose length does only depend on U^*. Thus, we can extend the solution $(u(t),s(t))$ to (2.2) after T^* and find a continuous extension of $M(t)$ past T^*. We have reached a contradiction, hence the conclusion of the Theorem follows. □

We now show, following Friedman's ideas [7], that it is possible to extend the solution for a short (but uniform) time ε for $b<0$.

Proposition 4.3. *Suppose that the hypotheses of Theorem* 3.1 *hold and that* $(u(t),s(t))$ *is a solution to* (2.2) *in the time interval* $[0,t_0)$ *for* $b<0$. *There exists* $\varepsilon>0$ *small enough such that, if*

$$\sup_{x\in(-\infty,s(t_0-\varepsilon)]} |u_x(x,t_0-\varepsilon)| < \infty, \tag{4.2}$$

for $0 < \varepsilon < t_0$ *then*

$$\sup_{t_0-\varepsilon<t<t_0} M(t) < \infty.$$

Although the estimate depends on the bound (4.2), ε *does not depend on* t_0.

Proof. We use the integral formulation (3.8) for M, this time with initial condition at time $t_0-\varepsilon$ for some fixed ε chosen below, and $t \in (t_0-\varepsilon, t_0)$. Thus

$$\begin{aligned} M(t) =& -2\int_{-\infty}^{s(t_0-\varepsilon)} G(s(t),t,\xi,t_0-\varepsilon)u_x(\xi,t_0-\varepsilon)d\xi \\ &+2\int_{t_0-\varepsilon}^{t} M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau - 2\int_{t_0-\varepsilon}^{t} M(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau \\ =&: K_1+K_2+K_3. \end{aligned} \tag{4.3}$$

Since $s(t) \geq s(\tau)$, it follows that $G_x(s(t),t,s(\tau),\tau) \leq 0$. Moreover, $M \geq 0$, hence $K_2 \leq 0$ and this term can be discarded. To estimate K_3 let

$$\Phi(t) := \sup_{t_0-\varepsilon<\tau<t} M(\tau).$$

Note that

$$|K_3| \leq \Phi(t)\int_{t_0-\varepsilon}^{t} |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau. \tag{4.4}$$

To estimate the derivative $|G_x(s(t),t,s_1(\tau),\tau)|$ we use that the nonlinear part of s is an increasing function in the case $b<0$ as in (3.13). Thus, for ε small enough, we conclude that

$$s(t)-s_1(\tau) = s(t)-s(\tau)-v_R\alpha^{-1}(\tau) \geq |v_R| - |b_0|\varepsilon > 0\,. \tag{4.5}$$

Hence, we can recall the computations in (3.14) to estimate

$$\int_{t_0-\varepsilon}^{t} |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau \leq C\int_{\frac{|v_R|-|b_0|\varepsilon}{\sqrt{8(t-t_0+\varepsilon)}}}^{\infty} \frac{1}{z}e^{-z^2}\, dz \leq C\int_{\frac{|v_R|-|b_0|\varepsilon}{\sqrt{8\varepsilon}}}^{\infty} \frac{1}{z}e^{-z^2}\, dz.$$

It is clear that this last integral can be made less than $1/2$ for some small enough ε, which is independent of t_0. Substituting the above inequality into (4.4) we have the estimate $|K_3| \leq \frac{1}{2}\Phi(t)$. Finally, note that $|K_1| \leq C$ depending on $\sup|u_x(x,t_0-\varepsilon)|$ Combining the estimates for K_1, K_2, K_3 with (4.3) yields

$$M(t) \leq \frac{1}{2}\Phi(t) + C.$$

Taking the supremum on the left hand side, we get that $\Phi(t) \leq 2C$, for all $t \in (t_0-\varepsilon, t_0)$, as desired. □

Remark. Let us point out that the key estimate (4.5) to get the uniformity of the time interval with respect to t_0 comes from the fact that the nonlinear part of the free boundary $s(t)$ is monotone increasing in the case $b<0$. For the case $b>0$, instead of (4.5) we got (3.16), which makes impossible to get a uniform estimate with respect to t_0 since m will depend on it.

Combining Proposition 4.3, Theorem 4.2, and 4.1 gives global existence for $b<0$, as summarized in the following result:

Theorem 4.4. *Let* $u_I(x)$ *be a non-negative* $\mathcal{C}^1((-\infty, s_I])$ *function such that* $u_I(s_I)=0$, *and* $u_I, (u_I)_x$ *decay at* $-\infty$. *Then there exists a unique global classical solution* $(u(x,t), s(t))$ *of the equation* (2.2) *with* $b<0$ *in the sense of Definition* 2.2 *with initial data* u_I. *Furthermore, the function* $s(t)$ *is a monotone increasing function of* t *if both* b *and* b_0 *are negative.*

With this the proof of our main Theorem 1.1 is complete. We emphasize that our Theorem 4.2 characterizes the possible blow-up of classical solutions in finite time as the time of divergence of the firing rate $N(t)$.

5 Study of the spectrum

In this section we study the spectrum of the linear version $\mu = 0$ of (1.6):

$$p_t - \partial_v(vp) - \partial_{vv}p = N(t)\delta_{v=v_R} \quad \text{on } (-\infty, 0),$$

where

$$N(t) = -p_v(0,t), \quad p(0,t) = 0.$$

The objective is to solve the eigenvalue problem

$$\begin{cases} \partial_{vv}p + \partial_v(vp) - p_v(0)\delta_{v=v_R} = \lambda\, p, & v \in (-\infty, 0), \\ p(0) = 0, \end{cases} \tag{5.1}$$

with eigenfunctions $p(v)$ in the space $L^2_{exp}(\mathbb{R})$ defined as

$$L^2_{exp}(\mathbb{R}) := \left\{ p \in L^2(\mathbb{R}) \,:\, \|p\|_{L^2_{exp}(\mathbb{R})} < \infty \right\},$$

with norm

$$\|p\|^2_{L^2_{exp}(\mathbb{R})} := \int_{\mathbb{R}} \left(e^{v^2/2}|p(v)| \right)^2 dv.$$

Note that although problem (5.1) is only defined in $(-\infty, 0)$, it can be easily extended to $\mathbb{R}$ by odd reflection. Following an idea developed in [11, 12], we consider the equivalent problem to (5.1) defined as

$$\partial_{vv}p_\lambda + \partial_v(vp_\lambda) = \lambda\, p_\lambda \quad \text{in } (-\infty, v_R) \cup (v_R, 0), \tag{5.2}$$

with p_λ satisfying the following properties:

(F1) $p_\lambda \in L^2_{exp}(\mathbb{R})$,

(F2) $p_\lambda(0) = 0$,

(F3) Matching condition: $p_\lambda(v_R^+) = p_\lambda(v_R^-)$,

(F4) Jump condition: $\partial_v p_\lambda(v_R^+) = \partial_v p_\lambda(v_R^-) + \partial_v p_\lambda(0)$.

We are going to define the solution for (5.1) of the form:

$$p_\lambda(v) = \chi_{(-\infty, v_R)} p^1(v) + \chi_{(v_R, 0)} p^2(v), \tag{5.3}$$

where each $p^i(v)$, $i = 1, 2$, is a linear combination of the two linearly independent solutions of (5.2) in $\mathbb{R}$, and such that the combination (5.3) satisfies **(F1)-(F4)**.

The functions $p^1(v)$ and $p^2(v)$ will be calculated by a standard classical method used to compute the spectrum for the classical Fokker-Planck equation given by $\mathcal{L}(p) = \lambda p$, $v \in \mathbb{R}$, with

$$\mathcal{L}(p) := \partial_{vv}p + \partial_v(vp). \tag{5.4}$$

Define first the eigenspace

$$L^2_m(\mathbb{R}) = \left\{ p \in L^2(\mathbb{R}) \,:\, \|p\|_{L^2_m(\mathbb{R})} < \infty \right\},$$

with norm

$$\|p\|^2_{L^2_m(\mathbb{R})} := \int_{\mathbb{R}} \left(1 + v^2\right)^m |p(v)|^2\, dv.$$

For completeness we recall a well known result on the spectrum for the classical operator $\mathcal{L}$, see for instance [9, 21]:

Lemma 5.1. *For any $m \geq 0$, the spectrum of the operator $\mathcal{L}$ defined in* (5.4) *on $L^2_m(\mathbb{R})$ is given by*

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbb{C} \,:\, \Re(\lambda) \leq \tfrac{1}{2} - m\right\} \cup \{-n \,:\, n \in \mathbb{N} \cup \{0\}\}\,.$$

Moreover, if $m > \frac{1}{2}$ and if $n \in \mathbb{N} \cup \{0\}$ satisfies $n + \frac{1}{2} < m$, then $\lambda_n = -n$ is an isolated eigenvalue of $\mathcal{L}$, with multiplicity one, and eigenfunction given by the n-th Hermite polynomial

$$H_n(v) = (-1)^n e^{v^2/2} \frac{d^n}{dv^n} e^{-v^2/2}.$$

In particular, the spectrum of the Fokker-Planck operator $\mathcal{L}$ in the space $L^2_{exp}(\mathbb{R})$ reduces to the eigenvalues

$$\lambda = -n, \quad n \in \mathbb{N} \cup \{0\}.$$

It will be very illustrative to give a sketch of the proof of the above Lemma in view of the computations for (5.1).

Proof of Lemma 5.1: Given $m \in \mathbb{N}$, we seek a solution $p \in L^2_m(\mathbb{R})$ for

$$p_{vv} + \partial_v(vp) = \lambda p, \quad \lambda \in \mathbb{C}. \tag{5.5}$$

The Fourier transform of a function p is defined as

$$\hat{p}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} p(v) e^{-iv\xi}\, dv.$$

Then, the Fourier transform of (5.5) yields the following first order differential equation

$$-\xi^2 \hat{p} - \xi\, \hat{p}_\xi = \lambda \hat{p}, \quad \xi \in \mathbb{R},$$

which has solutions given by

$$\hat{p}(\xi) = \begin{cases} \sqrt{2\pi}\,\alpha\, \xi^{-\lambda} e^{-\xi^2/2}, & \text{for } \xi > 0, \\ \sqrt{2\pi}\,\beta\, (-\xi)^{-\lambda} e^{-\xi^2/2}, & \text{for } \xi < 0, \end{cases} \tag{5.6}$$

for some constants $\alpha, \beta \in \mathbb{C}$. By inverse Fourier transform we get:

$$\begin{aligned} p(v) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iv\xi} \hat{p}(\xi)\, d\xi \\ &= \alpha \int_0^{+\infty} e^{iv\xi} \xi^{-\lambda} e^{-\xi^2/2}\, d\xi + \beta \int_{-\infty}^0 e^{iv\xi} (-\xi)^{-\lambda} e^{-\xi^2/2}\, d\xi. \end{aligned} \tag{5.7}$$

This is, for each $\lambda \in \mathbb{C}$, p is the linear combination of the two linearly independent solutions of (5.5). The constants α, β, λ will be determined from the boundary conditions.

Note that the Fourier transform is an isomorphism from $L^2_m(\mathbb{R})$ to $H^m(\mathbb{R})$. Thus, λ belongs to the spectrum of $\mathcal{L}$ if and only if the function $\hat{p}$ from (5.6) belongs to $H^m(\mathbb{R})$. Since $\hat{p}$ is sufficiently smooth and rapidly decaying for any m, provided we stay away from the origin, we just need to check if $\hat{p} \in H^m(0)$, i.e., all derivatives of $\hat{p}$ of order less or equal than m are square integrable near the origin.

Clearly, the values $\lambda = -n$, $n \in \mathbb{N} \cup \{0\}$ are special and will be considered later. In order to see if any other values of $\lambda \in \mathbb{C}$ are admissible, we compute a general n-derivative of the term $|\xi|^{-\lambda} e^{-\xi^2/2}$ around $\xi = 0$:

$$\frac{\partial^{(n)}}{\partial \xi^{(n)}} \left[|\xi|^{-\lambda} e^{-\xi^2/2} \right] \approx c^{\pm}_{\lambda,n} |\xi|^{-\lambda-n}.$$

It is easy to see that

$$\frac{\partial^{(n)}}{\partial \xi^{(n)}} \left[|\xi|^{-\lambda} e^{-\xi^2/2} \right] \in L^2(\mathbb{R})$$

if and only if $Re(\lambda) < \frac{1}{2} - m$. Since the spectrum of $\mathcal{L}$ is closed, this shows that

$$\sigma(\mathcal{L}) \supset \{\lambda \in \mathbb{C}\ \Re(\lambda) \leq \tfrac{1}{2} - m\},$$

as claimed.

Now we study the values $\lambda = -n$, $n = 1, \ldots, m-1$. Let p_n be the corresponding eigenfunction, with Fourier transform given by (5.6). Since $\hat{p}_n$ belongs to H^m near the origin, its m-derivative will be L^2 integrable, while the rest of the derivatives $\hat{p}_n, \partial_\xi \hat{p}_n, \ldots, \partial_\xi^{(m-1)}(\hat{p}_n)$ will be continuous at the origin. This forces to have a very precise values for α, β. In particular, for n even, $\alpha = \beta$, while for n odd we must have $\alpha = -\beta$. Then we have shown that $\lambda = -n$, $n = 1, \ldots, m-1$ are admissible eigenvalues and the corresponding eigenspaces are one dimensional, with eigenfunctions given by the well known Hermite polynomial H_n as

$$p_n(v) = e^{-v^2/2} H_n(v).$$

□

We consider now the original problem (5.1) and seek for solutions $p(v)$ of the form (5.3). Our first observation is that the values for λ are determined only by the decay of p as $v \to -\infty$. Consequently, if we impose that the function p^1 belongs to $L^2_{exp}(\mathbb{R})$, then this fixes the possible values of the eigenvalues λ as in Lemma 5.1. In particular, there is no continuous spectrum. Moreover, for each $\lambda_n = -n$, $n \in \mathbb{N}$, we must have

$$p^1(v) = \alpha H_n(v) e^{-v^2/2}, \tag{5.8}$$

for some $\alpha \in \mathbb{R}$.

The difference between our problem and the classical Fokker-Planck operator lies in the fact in the interval $(v_R, 0)$ all solutions to the ODE (5.5) for $\lambda = -n$ as given in (5.7) are admissible since the behavior at infinity does no play any role. Nevertheless, we can find a better expression for the two linearly independent solutions in (5.7). One of those is the well known (5.8). The other solution can be easily found by making the following ansatz:

$$p^2(v) = e^{-v^2/2} H_n(v) g(v).$$

By imposing that $p^2(v)$ satisfies (5.2) one can obtain an equation for $g(v)$ that reads as follows:

$$2H_n' g' - v g' H_n + H_n g'' = 0.$$

This equation has the following general solution:

$$g(v) = \beta_1 \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2,$$

for some constants $\beta_1, \beta_2 \in \mathbb{R}$, and where we have fixed any $v_0 \in (v_R, 0)$ such that $H_n(v_0) \neq 0$ for the integral to be well defined. Note that g is well defined for all v even where the denominator vanishes because the Hermite polynomials only have single roots. Consequently we define

$$p^2(v) := \beta_1 e^{-v^2/2} H_n(v) \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2 e^{-v^2/2} H_n(v),$$

and the eigenfunction corresponding to $\lambda = -n$ is simply

$$p_n(v) = \begin{cases} \alpha e^{-v^2/2} H_n(v), & v \in (-\infty, v_R), \\ \beta_1 e^{-v^2/2} H_n(v) \displaystyle\int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2 e^{-v^2/2} H_n(v), & v \in (v_R, 0]. \end{cases} \tag{5.9}$$

for some real constants α, β For simplicity, let

$$\theta_n(v) := H_n(v) \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds.$$

It is clear, by doing a careful Taylor expansion, that if v_1 is a root of H_n, then there exists a finite limit for

$$\Delta_{v_1,n} := \lim_{v \to v_1} \theta_n(v) \neq 0.$$

Now we are ready to check if (5.9) is an admissible eigenfunction. In the case n is odd integer, the Hermite polynomial H_{2n+1} vanishes at zero, but as we have mentioned,

$$\theta_{2n+1}(v) \to \Delta_{0,2n+1} \neq 0, \quad \text{as } v \to 0 \quad \text{for any } n \in \mathbb{N}.$$

Then in this case condition **(F2)** is satisfied only when $\beta_1 = 0$. Then, if we wish p_{2n+1} to be a continuous function as stated in condition **(F3)**, we must have $\alpha = \beta_2$ unless $H_{2n+1}(v_R) = 0$ that will be considered afterwards. The solution constructed this way does not satisfy condition **(F4)**, so we conclude that $2n+1$ is not an admissible eigenvalue.

On the other hand, let us check if p_{2n} is an admissible eigenvalue. For even integers it holds that $H_{2n}(0) \neq 0$. Thus we can simply take $v_0 = 0$. Consequently condition **(F2)** is satisfied if and only if

$$\beta_2 = 0. \tag{5.10}$$

The matching condition **(F3)** implies

$$\alpha H_{2n}(v_R) = \beta_1 H_{2n}(v_R) \int_0^{v_R} \frac{e^{s^2/2}}{H_n^2(s)}\, ds. \tag{5.11}$$

Here we do need to distinguish two cases: if v_R is not a root for any H_{2n}, then the above equality implies

$$\alpha = \beta_1 \int_0^{v_R} \frac{e^{s^2/2}}{H_n^2(s)}\, ds. \tag{5.12}$$

If instead $H_{2n}(v_R) = 0$ (note that Hermite polynomials only have single roots), one can repeat a Taylor expansion around v_R for $\theta_{2n}(v)$ and see that

$$\theta_{2n}(v) \to \Delta_{v_R,2n} \neq 0, \quad \text{as } v \to v_R.$$

Consequently (5.11) cannot be satisfied for these n such that $H_{2n}(v_R) = 0$.

Using conditions (5.10) and (5.12) for p_n we get

$$p_{2n}(v) = \beta_1 e^{-v^2/2} H_{2n}(v) \cdot \begin{cases} \displaystyle\int_0^{v_R} \frac{e^{s^2/2}}{H_{2n}^2(s)}\, ds, & v \in (-\infty, v_R), \\ \displaystyle\int_0^{v} \frac{e^{s^2/2}}{H_{2n}^2(s)}\, ds, & v \in (v_R, 0]. \end{cases} \tag{5.13}$$

One can easily check that the jump condition **(F4)** is satisfied if and only if

$$H_{2n}(0) = H_{2n}(v_R). \tag{5.14}$$

Summarizing, we have shown the following:

Theorem 5.2. *Consider the operator*

$$\begin{cases} \partial_{vv} p + \partial_v(vp) - p_v(0)\Delta_{v=v_R} = \lambda\, p, & v \in (-\infty, 0] \\ p(0) = 0, \end{cases}$$

subject to conditions **(F1)** - **(F4)**.

1. *There is no continuous spectrum.*
2. *The value $\lambda = 0$ is an eigenvalue with a one-dimensional eigenspace spanned by the function*

$$p_\infty(v) = \begin{cases} e^{-v^2/2} & v \in (-\infty, v_R), \\ \alpha_0 e^{-v^2/2} \displaystyle\int_v^0 e^{v^2/2}\, dv & v \in (v_R, 0], \end{cases}$$

for

$$\alpha_0 := \left(\int_{v_R}^0 e^{v^2/2}\, dv \right)^{-1}.$$

3. *There exists a countable set $S \subset R$ such that for all $v_R \notin S$, there are no other eigenvalues.*
4. *If n and v_R happen to satisfy the compatibility condition* (5.14)*, then $\lambda = -2n$ is an eigenvalue with eigenspace of finite dimension spanned by the eigenfunction $p_{2n}(v)$ defined in* (5.13)*.*

Remark. We remark that the steady state $p_\infty(v)$ was previously obtained in [11, 3]. In this last paper, it was also shown exponential decay towards equilibrium p_∞. However, the speed of convergence is unknown and the spectral analysis does not seem to give any insight.

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References

[1] N. Brunel and V. Hakim V: *Fast global oscillations in networks of integrate-and-fire neurons with long fiting rates. Neural Computation*, 11:1621–1671, 1999.

[2] N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking networks. J. Comp. Neurosci.*, 8:183–208, 2000.

[3] M. J. Cáceres, J. A. Carrillo, and B. Perthame, *Analysis of nonlinear noisy integrate&fire neuron models: blow-up and steady states*, to appear in J. Math. Neuroscience.

[4] M. J. Cáceres, J. A. Carrillo, and L. Tao, *A numerical solver for a nonlinear Fokker-Planck equation representation of neuronal network dynamics*, *J. Comp. Phys.*, 230:1084–1099, 2011.

[5] J. A. Carrillo and G. Toscani, *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, *Indiana Univ. Math. J.* 49:113–141, 2000.

[6] A. Compte, N. Brunel, P.S. Goldman-Rakic, and X.J. Wang, *Synaptic mechanisms and network dynamics underlying spatial working memory in a cortical network model*, *Cerebral Cortex*, 10:910–923, 2000.

[7] A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, inc, Englewood Cliffs, N.J. 1964.

[8] A. Friedman, *Free boundary problems for parabolic equations I. Melting of solids*, *J. of Math. and Mech.*, 8:499–518, 1959.

[9] T. Gallay and G. Raugel. Scaling variables and asymptotic expansions in damped wave equations. *J. Differential Equations*, 150(1):42–97, 1998.

[10] W. Gerstner and W. Kistler, *Spiking neuron models*, Cambridge: Cambridge Univ. Press 2002.

[11] M. d. M. González and M. P. Gualdani, *Asymptotics for a symmetric equation in price formation*, *Appl. Math. Optim.*, 59(2):233–246, 2009.

[12] M. d. M. González and M. P. Gualdani. *Asymptotics for a free-boundary problem in price formation*, Nonlinear Analysis, 74, pp. 3269-3294, 2011.

[13] T. Guillamon, *Una introducció a les matemàtiques de lactivitat neuronal*, *Butl. Soc. Catalana Mat.*, 19:25–45, 2004.

[14] L. Lapicque, *Recherches quantitatives sur l'excitation électrique des nerfs traitée comme une polarisation*, *J. Physiol. Pathol. Gen.*, 9:620–635, 1907.

[15] M. Mattia, P. Del Giudice, *Population dynamics of interacting spiking neurons*, *Phys. Rev. E*, 66:051917, 2002.

[16] A. M. Meirmanov, *The Stefan problem*, volume 3 of *de Gruyter Expositions in Mathematics*, Walter de Gruyter & Co., Berlin, 1992.

[17] K. Newhall, G. Kovačič, P. Kramer, A.V. Rangan, and D. Cai, *Cascade-Induced Synchrony in Stochastically-Driven Neuronal Networks*, *Phys. Rev. E.*, 82:041903, 2010.

[18] K. Newhall, G. Kovačič, P. Kramer, D. Zhou, A.V. Rangan, and D. Cai, *Dynamics of Current-Based, Poisson Driven, Integrate-and-Fire Neuronal Networks*, *Comm. in Math. Sci.*, 8:541–600, 2010.

[19] A. Omurtag, B.W. Knight, and L. Sirovich, *On the Simulation of Large Populations of Neurons*, *J. Comp. Neurosci.*, 8:51–63, 2000.

[20] A. Renart, N. Brunel, and X.J. Wang, *Mean-Field Theory of Irregularly Spiking Neuronal Populations and Working Memory in Recurrent Cortical Networks*, In *Computational Neuroscience: A comprehensive approach.* Edited by Feng J, Chapman & Hall/CRC Mathematical Biology and Medicine Series 2004.

[21] H. Risken, *The Fokker-Planck equation*, *Volume 18 of* Springer Series in Synergetics. Berlin: Springer-Verlag, second edition 1989. [Methods of solution and applications].

[22] L. Sirovich, A. Omurtag, and K. Lubliner, *Dynamics of neural populations: Stability and synchrony*, *Network: Computation in Neural Systems*, 17:3–29, 2006.

[23] H. Tuckwell, *Introduction to Theoretical Neurobiology.* Cambridge, Cambridge Univ. Press 1988.