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Master's thesis

Coordinate systems used to study the Elliptic Restricted Three Body Problem

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Abstract

There have been many studies and papers devoted to the three body problem over the years with a lot of interesting results found within the many variations of the problem. This paper will look at the Elliptic Restricted Three Body Problem (ER3BP) and discuss the differences found when using different co-ordinate systems to evaluate the problem. I will focus on the rotating pulsating coordinates and sidereal coordinates. I will be looking to find whether there is a preferential coordinate system to use when you are searching for certain orbits or properties for the ER3BP.

Keywords

ER3BP, pulsating coordinates, sidereal coordinates

1. Introduction

For the ER3BP, we consider two primaries, the smallest primary having a mass of μ and the largest primary having a mass of 1-μ, $\mu \in [0, 0.5]$, moving on a plane in Keplerian ellipses with eccentricity e, $0 < e < 1$, around their centre of mass. The ER3BP then describes the motion of a third infinitesimal particle which is governed by the gravitational field created by the two primaries yet obviously, the third infinitesimal particle does not affect the binary system of the two primaries.

For $e = 0$ the system corresponds to the Circular Restricted Three Body Problem (CR3BP). In this case the Hamiltonian is a first integral (the Jacobian Integral) as it does not depend on the independent variable (time). This is not the case for the ER3BP, which goes some way to explain the greater difficulties faced when studying the ER3BP.

Equations of motion

We can formulate the problem in either the inertial frame or in the pulsating frame. Szebehely 1967 [1] thoroughly shows the switch from the equations of motion in sidereal coordinates to the pulsating coordinates by going through an intermediate rotating frame before finally arriving at the pulsating coordinates which present the equations of motion for the ER3BP in the simplest form. We can also see a similar approach to the derivation of the pulsating coordinates in Gawlik 2007 [2], where he solely uses elementary properties of elliptical orbits obtained from the general solution to the two-body problem. We will bypass this derivation of the pulsating co-ordinates and just show the equations of motion and the corresponding Hamiltonian for each coordinate system.

We start by looking at the inertial frame where we fix the centre of mass of the system at the origin. In this frame the equations of motion for the third particle are:

$$\frac{d^2X}{dt^2} = \frac{-\mu(X - X_1)}{R_1^3} - \frac{(1-\mu)(X - X_2)}{R_2^3}$$

$$\frac{d^2Y}{dt^2} = \frac{-\mu(Y - Y_1)}{R_1^3} - \frac{(1-\mu)(Y - Y_2)}{R_2^3}$$

$$\frac{d^2Z}{dt^2} = \frac{-\mu(Z - Z_1)}{R_1^3} - \frac{(1-\mu)(Z - Z_2)}{R_2^3}$$

where $R_i^2 = (X - X_i)^2 + (Y - Y_i)^2 + (Z - Z_i)^2$ and obviously X,Y,Z are the axis in a Cartesian coordinate system. Then, the Hamiltonian in this frame with the origin as such at the centre of mass is given with q,p being the position and momenta respectively as;

$$H(q, p, f_p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1-\mu}{\sqrt{[q_1 + \mu\rho\cos(f_p)]^2 + [q_2 + \mu\rho\sin(f_p)]^2 + q_3^2}}$$
$$- \frac{\mu}{\sqrt{[q_1 - (1-\mu)\rho\cos(f_p)]^2 + [q_2 - (1-\mu)\rho\sin(f_p)]^2 + q_3^2}}$$

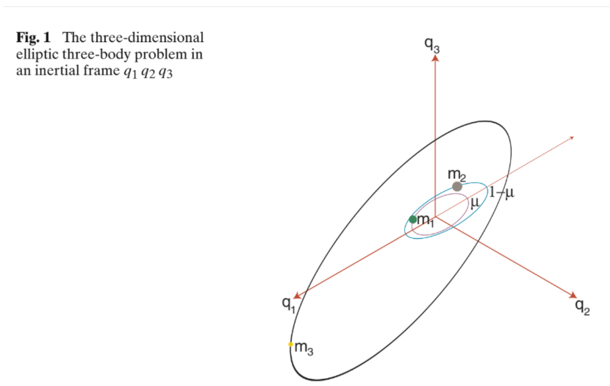

Figure 1: (Lifted from Palacian and Yanguas 2006) [3]

where ρ stands for the radial distance between m_2 and m_1 and it depends on the time through the true anomaly by;

$$\rho = \frac{1 - e_p^2}{1 + e_p \cos(f_p)}$$

and where f is the true anomaly and is satisfied by;

$$\frac{df}{dt} = \frac{(1 + e\cos f)^2}{(1 - e^2)^{3/2}}$$

For the pulsating coordinate system we introduce the true anomaly f and semi-major axis $a = 1$ and let χ, η, ζ be the pulsating coordinates. As presented in Szebehely 1967, with f being an independent variable, the set of governing equations describing the ER3BP in pulsating coordinates is written in the form;

$$\frac{d^2\chi}{df^2} - 2\frac{d\eta}{df} = \omega_\chi$$

,

$$\frac{d^2\eta}{df^2} + 2\frac{d\chi}{df} = \omega_\eta$$

$$and$$

$$\frac{d^2\zeta}{df} + \frac{d\zeta}{df} = \omega_\zeta$$

where $\omega = \frac{\Omega}{1+ecosf}$,$\Omega = \frac{(\chi^2+\eta^2+\zeta^2)}{2} + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$
Also, $r_1^2 = (\chi - \mu)^2 + \eta^2 + \zeta^2$ and $r_2^2 = (\chi - \mu + 1) + \eta^2 + \zeta^2$. If we only want to consider the planar ER3BP then we simply set $\zeta = 0$, removing the third equation from the system.
In order to obtain the Hamiltonian of the system in pulsating coordinates, as done by Llibre and Pinol 1990 [4], we introduce $q_1 = -\chi + \mu$, $q_2 = -\eta$, $q_3 = \zeta$ and $p_1 = -\chi' + \eta$, $p_2 = -\eta' - \chi + \mu$, $p_3 = \zeta'$ where prime indicates d/df. With these new equations, we can transform the equations of motion into the Hamiltonian system;

$$\frac{dq_i}{df} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{df} = -\frac{\partial H}{\partial q_i}$$

where i = 1,2 and 3 and H is the time independent Hamiltonian $H = H_0 + \mu H_1$
Where we have that

$$H_0 = \frac{1}{2}[(p_1 + q_2)^2 + (p_2 - q_1)^2 + p_3^2 + q_3^2] - \frac{1}{1 + ecosf}[\frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + \frac{1}{r_1}]$$

$$and$$

$$H_1 = \frac{1}{1 + ecosf}(q_1 + \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{2})$$

2. Pulsating co-ordinates

By looking at the equations of motion for the pulsating coordinate system, Szebehely 1967 was able to easily find the stationary solutions for this system. From the pulsating equations of motion he showed that for when,

$$\frac{d\zeta}{df} = \frac{d\eta}{df} = \frac{d^2\zeta}{df^2} = \frac{d^2\eta}{df^2} = 0$$
$$\frac{\partial\omega}{\partial\zeta} = \frac{\partial\omega}{\partial\eta} = 0$$

or

$$\frac{\partial\Omega}{\partial\zeta} = \frac{\partial\Omega}{\partial\eta} = 0$$

We have that, in fact, the five libration points for the ER3BP are the same as for the CR3BP. Of the five libration points, we have z_1, z_2, z_3 colinear to the primaries and then the triangular points;

$$z_{4,5} = \mu - \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

These equilibria if expressed in the inertial frame would be solutions of the Hamiltonian H which describe ellipses in the phase space and as such are relatively harder to find. So even at this early stage it proves useful to work in pulsating coordinates to find the equilibria of the ER3BP.

Szebehely shows that the triangular libration points pulsate together with their own coordinate system. He shows that this pulsation occurs when the system ζ is projected to the system z by the equation

$$z = \frac{1 - e^2}{1 + ecosf}\zeta$$

Now the coordinates of the triangular libration points in the ER3BP in the system (ζ, η) are then;

$$\zeta_{4,5} = \mu - \frac{1}{2}$$

$$\eta_{4,5} = \pm\frac{\sqrt{3}}{2}$$

The linearized equations of motions of the elliptic problem around the libration points in the system are then established and Szebehely reveals the linearized stability analysis regarding the effect of the eccentricity on the linear stability of the triangular points, which he took from Danby 1964 [5].

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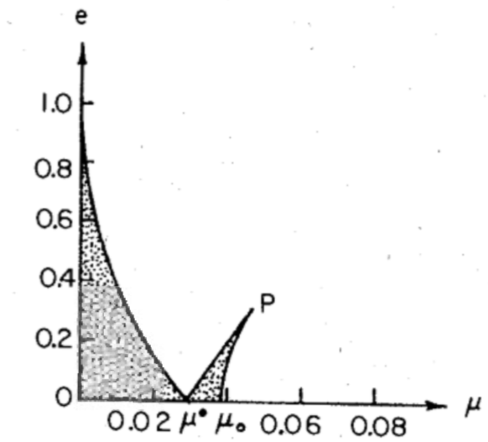

Figure 2: Linear stability of the triangular points in the ER3BP. (Lifted from Danby 1964)

The shaded areas denote linear stability. We have the value $\mu_0 = 0.0385209$ on the μ axis which represents the point at which instability occurs thereon for e=0. Then, $\mu^* = 0.02859548$ which is the first point for which there is linear instability for any nonzero eccentricity and finally, the coordinates of P are μ=0.04698 and e=0.3143

It is clear why Szebehely uses a pulsating coordinate system as he is easily able to extract the triangular libration points and then from there can linearize the equations of motion and evaluate the stability of those libration points for varying eccentricity and mass parameter. Additionally, he comments that the colinear libration points, akin to in the CR3BP, are unstable for all values of μ and e.

We now take a look at Delva [6], who uses the method of lie series to construct a solution for the ER3BP. In a pulsating coordinate system, the Lie operator for the motion of the third infinitesimal body is derived as a function of coordinates, velocities and the true anomaly of the primaries.

Delva starts by taking the equations of motion of the third infinitesimal particle as presented by Szebehely and then writes these equations as a system of four first order differential equations in the coordinates (ζ, η) and velocities (ζ', η') with f as independent variables.

$$\frac{d\zeta}{df} = \zeta'$$

$$\frac{d\eta}{df} = \eta'$$

$$\frac{d\zeta'}{df} = 2\eta' + \frac{1}{1+ecosf}\{\mu_1(\zeta-\zeta_1) + \mu_2(\zeta-\zeta_2) - \frac{\mu_1(\zeta-\zeta_1)}{\rho_1^3} - \frac{\mu_2(\zeta-\zeta_2)}{\rho_2^3}\}$$

$$\frac{d\eta'}{df} = -2\eta' + \frac{1}{1+ecosf}\{\mu_1\eta + \mu_2\eta - \frac{\mu_1\eta}{\rho_1^3} - \frac{\mu_2\eta}{\rho_2^3}\}$$

The Linear Lie operator D has the general form

$$D = \frac{d}{df} = \frac{\partial}{\partial\zeta}\frac{d\zeta}{df} + \frac{\partial}{\partial\eta}\frac{d\eta}{df} + \frac{\partial}{\partial\zeta'}\frac{d\zeta'}{df} + \frac{\partial}{\partial\eta'}\frac{d\eta'}{df} + \frac{\partial}{\partial f}\frac{df}{df}$$

Now Delva can plug the four first order differential equations into the equation for the linear lie operator D and from there the solution $\mathbf{X}(\zeta, \eta, \zeta', \eta', f)$ is given by the Lie series:

$$\mathbf{X}(\zeta, \eta, \zeta', \eta', f) = [\{exp[(f - f_0)D]\}\mathbf{X}]_{X=X_0} = \sum_{j=1}^{\infty}[D^j X]_{X_0}\frac{(f - f_0)^j}{j!}$$

where $D^j X$ is to be evaluated for the initial condition $X_0(\zeta_0, \eta_0, \zeta_0', \eta_0', f_0)$.

From here in order to find the series, it is necessary to calculate the multiple action of D, where the multiple action gives

$$D^j\zeta = D^{j-1}\zeta'$$
$$D^j\eta = D^{j-1}\eta'$$

for $j \geq 1$. After that Delva is able to find the terms of the lie series by evaluating the lie operator acting on the ζ-component and η-component respectively, to eventually come out with the solution of the ER3BP in terms of the Lie series and from here, orbits can be calculated using the initial conditions $\mathbf{X}(\zeta, \eta, \zeta', \eta', f)$ and since any number of terms can be found by iteration, the series can be continued until satisfactory convergence is reached. Delva then proceeded to use the long period librating Trojans in the Sun Jupiter system to demonstrate the effectiveness of his method.

From here, we can see that the reason Delva takes advantage of pulsating coordinates is due to the relatively simpler equations of motion that he uses to take advantage of the lie operator D. Another example of this type of method is demonstrated in Hamdy et al. 2004 [7] in which the authors are able to develop explicit analytical expressions for the orbits around the libration points. Alike Delva, Hamdy uses the pulsating coordinates to apply the Lie operator seemingly effortlessly and the fact that he is evaluating the libration points which are easily obtainable in pulsating coordinates as demonstrated by Szebehely makes this choice of coordinates even more poignant.

Within a different field, Gawlik demonstrates the existence of periodically pulsating Lagrangian coherent structures which are derived as time independent analogues of the stable and unstable manifolds of periodic orbits in the CR3BP. Rather interestingly these types of developments allow for the advancements in Interstellar transport networks and low fuel spacecraft missions.

The authors draw attention to the works done by Koon et al. [8] on the CR3BP. They linearized the equations of motion at the colinear libration points and were consequently able to show the existence of periodic orbits (called Lyapunov orbits) around each colinear libration point, whose stable and unstable manifolds form cylindrical tubes.

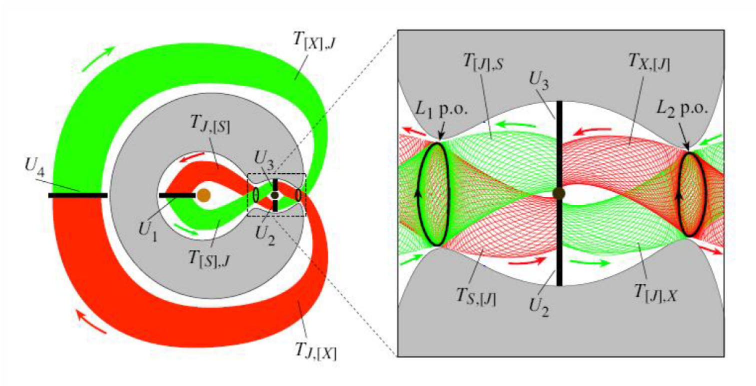

Figure 3: Projection of the stable (green) and unstable (red) manifold tubes in the CR3BP. (Lifted from Koon et al)

Gawlik explains that as the ER3BP is a non-autonomous system , the existence of exactly periodic motion disintegrates, as does the presence of static invariant manifolds, therefore, they use the theory of Lagrangian coherent Structures (LCS) to generalize the results found by Koon et al. LCS are essentially time varying 'ridges' that form barriers between the almost invariant sets of the domain.

Gawlik examines the time-independent analogues of $W^S_{z_1,P.O}$ and $W^U_{z_1,P.O}$ in the full four dimensional space, where y=0 is 3 dimensional. They choose a coordinate system (x,y,x',E) to parameterize phase space in the ER3BP and thereby allow for a natural extension of the qualitative results from the CR3BP, which they can then use to demonstrate the existence of periodically pulsating LCS.

It would seem to me, that Gawlik chooses to use pulsating coordinates as he is dealing with the Lyapunov orbits around the colinear libration points in the CR3BP, and to extend this notion to the ER3BP would possibly be easier if the libration points are relatively more attainable than if he was working in an inertial frame. Furthermore the fact that the results of his studies prove the the LCS pulsate periodically would sort of seem backwards intuitive with the fact that Szebehely showed that the libration points themselves pulsate together with their own coordinate system.

3. Sidereal coordinates

Palacian et al. 2006 [9] calculate new families of periodic orbits in the 3DER3BP. They overcome the non-integrability of the original Hamiltonian which is of 3 degrees of freedom by using a double averaging method over the time and the mean anomaly.

Their work focuses on the comet case, where the infinitesimal particle is very far from the primaries. They begin by formulating the Hamiltonian in the inertial frame and expanding it in terms of the small parameter ϵ , where they have set $a = \epsilon$. They arrive at;

$$H = H_0 + \epsilon H_1 + \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6} H_r$$

Which through simple arguments of calculus of series it can be shown that the Hamiltonian is convergent in all cases for $\frac{\epsilon}{\mathbf{q}} \leq \frac{1}{2}$ where $\mathbf{q} = (q_1^2 + q_2^2 + q_3^2)^{\frac{1}{2}}$

Most significantly about the reduced Hamiltonian is that

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

is the Hamiltonian of a kepler problem centred at the origin of the inertial frame. Therefore the infinitesimal particle can be treated as a perturbation of the two body problem when m_3 is far enough from the primaries.

They later remove the time variable by using the Lie transformations and then through the process of the Delaunay normalization and a further Lie transformation are able to remove the mean anomaly. At this point, it is expected that the resulting reduced system is of 2 degrees of freedom, however, the argument of the pericentre is eliminated too. This surprising feature allows for the authors to pass from 3 degrees of freedom to 1 degree of freedom in the second Lie transformation and thus allows them to analyze the existence of periodic orbits through computing the relative equilibria of the reduce system and then studying their linear and Lyapunov stability. From there they can reconstruct the corresponding invariant manifolds in the original Hamiltonian.

When looking at the comet case, the fact that the Hamiltonian in the inertial frame can be reduced to 1 degree of freedom and can be seen as a perturbation on the Kepler 2 body problem is what is most powerful and compelling a reason to use this coordinate system. Intuitively it makes sense as well as the further away an infinitesimal particle is from the two primaries, the two primaries gravitational affects would be combined and seem like one rather than a three body problem.

In later papers by Palacian and Yanguas, they similarly used the inertial frame to discuss the sub case of this study where $e_p = 0$, i.e the CR3BP, despite the generating function from Palacian et al, $W_3 \to \infty$ as $e \to 0$ and confirm that for the circular case it is in general more convenient to use rotating coordinates. However, they reveal that as they are working on the comet case and extending their previous study to the sub case, then it is more beneficial to work with sidereal coordinates.

With a similar method, Xingbo and Yanning [10] show the existence of a new class of symmetric periodic solutions of the 3DER3BP, where the infinitesimal particle is confined to the vicinity of only one of the primaries or the lunar case, as its known. Those periodic orbits move in near on perpendicular orbits to the plane of the primaries.

They express the Hamiltonian in the inertial frame as this paper is only interested in orbits very close to M_1. With this the Hamiltonian can once again be written as a perturbed Keplerian problem. However, in this case rather than the semi-major axis a being the small parameter ϵ, the authors choose to characterize the distance between m_3 and M_1 as the small parameter ϵ and do so by introducing a set of scaled variables

$$\mathbf{u} = \epsilon^{-2}\mu^{-\frac{1}{3}}\mathbf{x}$$

$$\mathbf{v} = \epsilon\mu^{-\frac{1}{3}}\mathbf{y}$$

In a similar fashion, the authors then average the new Hamiltonian first over the fast angular variable (the mean anomaly of m_3) and then over the slow angular variable (the mean anomaly M_2). The authors do also reveal that the desired Hamiltonian can also be achieved by making two von Ziepel transformations up to order ϵ^6. From the desired Hamiltonian, the authors eliminate terms containing M_2 thereby nullifying the gravitation of M_2 and getting the unperturbed system governing the motion of the infinitesimal body in the gravitational field of M_1 only. They then go on to prove the existence of such symmetric periodic solutions using a corollary of Arenstorfs fixed point theorem. Also, showing that thee orbits, were not conditionally upon any restrictions on the mass ratio or the eccentricity.

Similarly to the previous study by Palacian et al, it shows that using the Hamiltonian in the inertial coordinates, it is possible to reduce the system and achieve a perturbation on the Kepler problem. Again this seems intuitive from a physics mindset as the infinitesimal particle is so close to one of the primaries that the affects of the other primary is negligible.

Dumas et al. [11] do a very similar study on the lunar case in the CR3BP but instead chose to use polar coordinates, which is as Palacian and Yanguas suggested in their later works, is the most conventional way to evaluate the CR3BP.

In both of these cases, we see the ER3BP being analysed in three dimensions. i.e the spatial ER3BP, however, I do not believe that this extra dimension brings any extra bearing on whether to start within an inertial or pulsating frame as the z-axis and the equations of motion in both instances are the same. Therefore I believe the position of the infinitesimal particle relative to the plane in which the two primaries lie is more significant than how far it is in the vertical component. As an example, the Sitnikov problem which can see a particle stretching off to infinity along the z-axis, however, there is not a need to use pulsating coordinates as the equations of motion for that particle are the same as for the inertial problem, yet as the particle is close to the primaries with regards to the plane they are lying on then, perhaps, its more poignant to keep to sidereal coordinates.

Using a slightly different method, Delshams et al. [12] study the comet case to show the existence of orbits whose angular momentum performs arbitrary excursions in a large region. The authors start in the inertial frame and then unlike the previous authors make a standard polar-canonical change of variables;

$$(q, p) \to (\rho, \alpha, P_\rho, P_\alpha)$$

$$q = (\rho\cos\alpha, \rho\sin\alpha), p = (P_\rho\cos\alpha - \frac{P_\alpha\sin\alpha}{\rho}, P_\rho\sin\alpha + \frac{P_\alpha cos\alpha}{\rho})$$

From there, the authors make the McGehee non-canonical change of variables

$$\rho = \frac{2}{x^2}, x > 0$$

and do so in order to bring the infinity $\rho = \infty$ to the origin $x = 0$ and as a result, arrive at a new Hamiltonian of the form;

$$H_\mu(x, \alpha, P_\rho, P_\alpha, t; e) = \frac{P_\rho^2}{2} + \frac{x^4 P_\alpha}{8} - U_\mu(x, \alpha, t; e)$$

where U_μ is the self potential. From this Hamiltonian, the authors are able to notice for $\mu = 0$ and $G > 0$ (which is the Kepler problem as the second mass is infinitesimal now) that the Hamiltonian becomes Duffing Hamiltonian and is a first integral, therefore they can analyse the Hamiltonian. Alike, the other papers they are therefore treating the ER3BP as a perturbation of the Kepler problem for when $\mu = 0$.

Using the well-known result of McGehee, that the union of future parabolic orbits is an analytic manifold P^{+}, then in a properly chosen coordinate system these manifolds are stable manifolds of a manifold at parabolic infinity. Using scattering maps they are then able to find the desired diffusive pseudo-orbits and show that there does exist orbits whose angular momentum performs arbitrary excursions in a large region.

As with the other examples, the fact that Delshams et al are studying the comet case precludes the need to use a pulsating coordinate system as at such a distance, where they are even looking at the infinity manifold and are taking the mass ratio to be zero and perturbing the system from there, it is rationale enough to use the inertial frame.

4. Conclusion

From reading through papers and trying to look for patterns as to why a mathematician would choose to use pulsating coordinates or sidereal coordinates for the ER3BP, the main factor is really upon distance from the primaries. It would seem that intuition of the physics of gravity is correct in governing the rational of your choice of coordinate systems. For the lunar and comet cases where the system can be seen as a perturbation of the Kepler 2 Body problem, then it is natural to use to sidereal coordinates. However, when looking for orbits near to the libration points or the Poincare problem as its known for orbits in the intermediate region, it is more beneficial to use pulsating coordinates.

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