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An Extension of Vickrey Commute Model by means of the Macroscopic Fundamental Diagram (MFD)

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Als pingüins, per aquests anys.

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1 Introduction

Congestion is a problem that arises in all major cities around the world. Both during the morning and evening peak-hours roads get extremely packed, the speed of vehicles quickly drops and a journey that takes only several minutes in free flow conditions may take more than double. To get an idea of the order of magnitude we are talking about, we can have a look at some figures provided by Tomtom, a world-wide GPS seller that computes a congestion index of the main cities across the world. According to their methodology, in the most congested cities in the world (like Mexico City, Bangkok or Jakarta), a regular commuter can have a 60% increase in overall travel time when compared to a free flow situation. In Spanish cities, even though they are not located among top positions in the ranking, the effect of congestion is still quite noticeable and, as an example, a regular commuter in Barcelona is expected to have approximately a 30% increase in overall travel times when compared to a free flow situation[1].

The objective of this work is to develop a mathematical model that is able to explain how congestion arises. The main characteristic of the model developed here is that it treats congestion as an endogenous problem, namely, as the result of the decisions made by different commuters who try to anticipate how others commuters behave and act consequently. In order to do so, we will rely on two main ingredients: the well-known Vickrey Morning Commute Model (Vickrey (1969)) and the recently discovered Macro Fundamental Diagrama (MFD) (Daganzo (2007)).

On the one hand, the Vickrey Morning Commute Model models congestion as a queue that builds up behing a bottleneck. Users want to cross the bottleneck to arrive at work at a desired time but due to the limited capacity they cannot cross it at the same time. Hence commuters anticipate that congestion will arise and they try to deviate to earlier or later times in order to encounter a less congested situation. On the other hand, a Macro Fundamental Diagram (MFD) is a function that tries to explain a simple yet difficult to prove idea: when the number of vehicles in a network increases, the rate at which journeys can be completed or *capacity* decreases because vehicles tend to disturb each other at crossings, traffic lights or intersections. These two ingredients combined together imply that we can imagine that congestion arises due to the strategic behaviour of users when trying to cross a bottleneck that has a time changing capacity. Around the desired crossing time, the network will have very little capacity because many users will try to cross it and this will create the incentive for users to shift to less desirable crossing time when the network is less crowded.

The structure of the works consists of 5 chapters, including this introduction. The following chapter treats the existing literature on the topic. Next, in chapter three, the main ingredients of the model are explained and in chapter four the model is solved, together with a simple numerical example. Finally, chapter 5 concludes the work.

[1] More information abouut the Tomtom Congestion Index can be found in https://www.tomtom.com/en_gb/traffic-news/

2 Literature Review

The treatment of congestion as an endogenous problem traces back until Vickrey (1969). In his work, Vickrey assumed a single bottleneck with finite capacity through which a number of commuters must cross during the morning rush hour. Commuters, who behave strategically, anticipate that congestion will arise and they shift to an earlier or later departure time in order to reduce their travel time. In this way, the duration of the rush-hour depends not only on the capacity of the network but also on the strategic behaviour made by the users when trying to anticipate their travel time.

The Morning Commute Model was later systematized and generalized to account for different features by a great number of researchers. In Arnott et al. (1988), the basic model is extended to account for users with different values of times, in Arnott et al. (1993) the extension allows to account for elastic demand while in Smith (1984) the assumption that all commuters wish to arrive at the same time is relaxed. Arnott et al. (1990) use the model to derive an optimal coarse toll (a uniform toll applied only during a period of time). For a review of the vast literature that uses the Morning Commute Model as a keystone see Small (2015).

To the best of the author's knowledge, Daganzo (2007) is the first work that theorizes the existence of a Macro Fundamental Diagram (MFD), a function that relates the average density of vehicles to the average flow of vehicles at the network level provided that some assumptions about stationary and uniformity are met. This has important consequences since it allows to related the rate at which trips are ended, the *capacity* of the network, with the state variables (density and flow) at the network level. The work of Geroliminis and Daganzo (2008) proof the empirical existence of a MFD for the first time in the city of Yokohoma, Japan. Since then, the MFD for different cities have been found through simulation (Geroliminis and Daganzo (2007), Gonzales et al. (2011)).

With the work of Daganzo and his associates in hands, a new extension of the Morning Commute Model seemed readily available. The basic model could be extended from a simple bottleneck to a network level analysis provided that the fix capacity assumption was replaced by an MFD. The first contribution in that direction was the one made by Geroliminis and Levinson (2009). However, this first contribution relied on the assumption that travel time was a function of the accumulation at the exiting moment and not of the whole past, as intuition seems to suggest.

The works of Arnott (2013) and Fosgerau and Small (2013) where the first ones that extended the Morning Commute Model with an MFD and a definition of travel time that accounts for all past density states of the network. However, these two works are also the first ones to recognize the intractability of the problem and they had to rely on other simplifying assumptions to reach their results. On the one hand, Arnott (2013) assumes that each commuter has a particular probability of exiting the network independently of the distance travelled. On the other hand, Fosgerau and Small (2013) assumes the MFD to be a piecewise constant function implying that the network has only two relevant exit rates, a congested and an uncongested one. These simplifying assumptions allow the authors to get analytical solutions at expense of realistic features. To the date, the only complete extension of the Morning Commuting Model with an MFD that has an analytical solution is the one developed in Arnott et al. (2016). In this work, the authors assume that all commuters travel the same distance but the only equilibrium solution they find consists in mass departures and mass arrivals, a feature that does not seem very realistic.

This works contributes to the literature by developping a closed-form solution of the Morning Commute Model extended by means of an MFD. The main difference between our work and that found in the recent literature is that we will drop out the assumption that all users travel the same distance and this will allow us to find a continuous solution.

3 The Model

3.1 Preliminary definitions

Let $E(t)$ be the cumulative number of commuters who have entered the system by time t. This function is defined as

$$\begin{aligned} E\colon \mathbb{R} &\to \mathbb{R} \\ t \mapsto u = E(t) &= \int_{\underline{t}'}^{t} e(\tau)d\tau \end{aligned} \tag{1}$$

where $e(t)$ is the system entry rate at time t and $\underline{t}'$ is the first time a commuter enters the system.

The derivative of function (1) can be found by applying the Fundamental Theorem of Calculus to get

$$\frac{dE(t)}{dt} = e(t) \tag{2}$$

At the same time, the inverse of function $E(t)$ can be defined as

$$\begin{aligned} E^{-1}\colon \mathbb{R} &\to \mathbb{R} \\ u &\mapsto t = E^{-1}(u) \end{aligned} \tag{3}$$

and one can interpret function $E^{-1}(u)$ as a function that returns by what time t a cumulative number of users u has entered the system.

The derivative of function (3) can be found by remembering that the derivative of an inverse function is the inverse of the derivative of the original function

$$\frac{dE^{-1}(u)}{du} = \frac{1}{e(t)} \tag{4}$$

In computing the derivative of function (3) we must be aware of the fact that it will not exist whenever $e(t) = 0$ and this happens to be the case at the end of the rush-hour when there are no more commuters entering the network[2].

Similarly, let $X(t)$ be the cumulative number of commuters who have exited the system by time t. This function is defined as

$$\begin{aligned} X\colon \mathbb{R} &\to \mathbb{R} \\ t \mapsto u = X(t) &= \int_{\underline{t}}^{t} x(\tau)d\tau \end{aligned} \tag{5}$$

where $x(t)$ is the system exit rate at time t and $\underline{t}$ is the first time a commuter exits the system.

The derivative of function $X(t)$ can be found by applying the Fundamental Theorem of Calculus to get

$$\frac{dX(t)}{dt} = x(t) \tag{6}$$

The inverse of function $X(t)$ can be defined as

$$\begin{aligned} X^{-1}\colon \mathbb{R} &\to \mathbb{R} \\ u &\mapsto t = X^{-1}(u) \end{aligned} \tag{7}$$

[2]For a more rigorous explanation of this result, the reader can check Apostol (1967) where in chapter 6 the derivative of inverse functions is treated in detail.

and one can interpret function $X^{-1}(u)$ as a function that returns by what time t a cumulative number of users u has exited the system. Reasoning as above, its derivative can be found as

$$\frac{dX^{-1}(u)}{du} = \frac{1}{x(t)} \tag{8}$$

Notice that this derivative is not defined for $t \in [\underline{t}', \underline{t}]$ because at that time $x(t) = 0$.

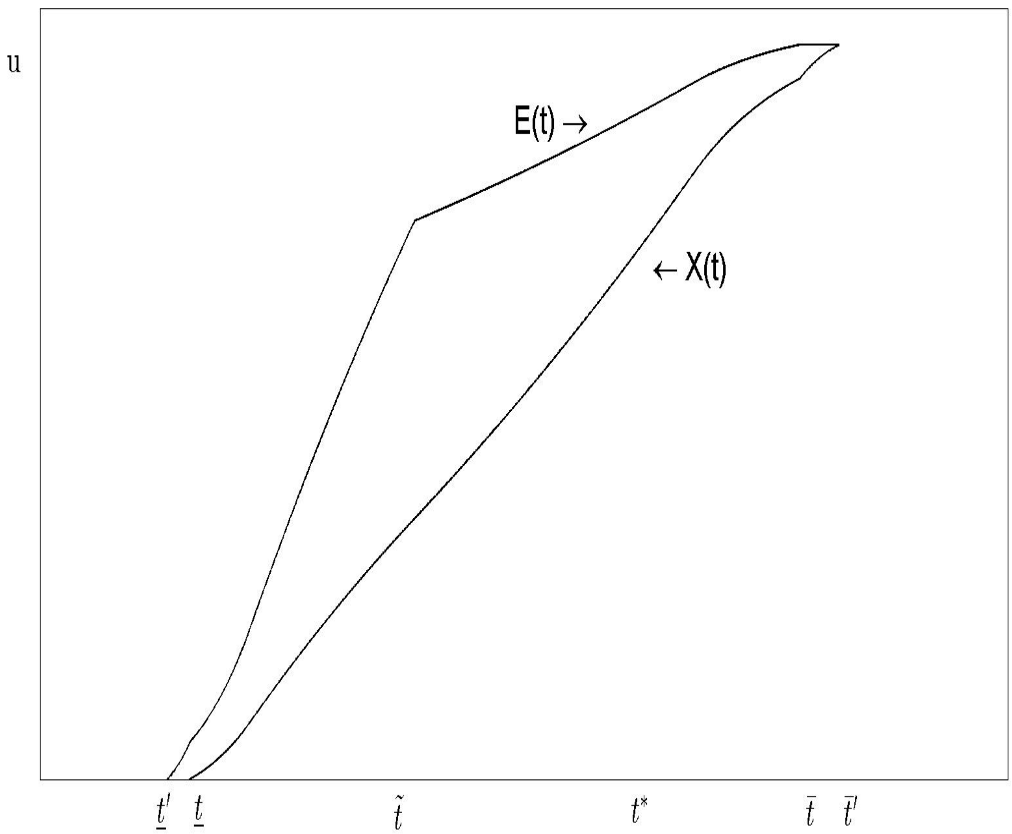

Figure 1: Cumulative arrivals and departures functions.

Next, let $n(t)$ be the number of commuters in the system at time t

$$\begin{aligned} n\colon \mathbb{R} &\to \mathbb{R} \\ t &\mapsto n = n(t) = E(t) - X(t) \end{aligned} \tag{9}$$

and let $T(u)$ be the travel time when a cumulative number of users u have entered the system

$$\begin{aligned} T\colon \mathbb{R} &\to \mathbb{R} \\ u &\mapsto T = T(u) = X^{-1}(u) - E^{-1}(u) \end{aligned} \tag{10}$$

At time t, the cumulative number of users u that have entered they system is equal to the number of users in the system at time t plus the number of users who have exited the system by time t. Hence

$$\begin{aligned} u\colon \mathbb{R} &\to \mathbb{R} \\ t &\mapsto u = u(t) = n(t) + X(t) \end{aligned} \tag{11}$$

Therefore, travel time T can be written as a function of time t if the composition of functions $T(u)$ and $u(t)$ is considered

$$\begin{aligned}(T \circ u)(t) &\colon \mathbb{R} \to \mathbb{R} \\ t &\mapsto T = T(u(t)) = T(t)\end{aligned} \tag{12}$$

Notice that to keep notation as simple as possible, we are using the symbol T as a *variable* and as a *function*, both in terms of variable t and in terms of variable u.

3.2 The cost function

Once we have rigorously defined the basic functions we will be dealing with, let us turn our attention to the cost function that commuters will seek to minimize. Imagine a downtown area populated by a fixed number N of individuals. Every morning these individuals wish to travel from their homes to their jobs. They are allowed to independently choose their departure time t but they all want to arrive at work at the same time t^*. The road network has limited capacity and hence congestion delays regularly arise during the rush hour. As a consequence, when choosing their departure time t commuters can either enter the (congested) network to arrive at work on time at expense of a high travel time or they can incur a schedule delay (either early or late) in order to ease their travel time.

Let $\tilde{t}$ be the departure time for which a commuter arrives at work on time. Then, by definition it follows that

$$\tilde{t} + T(\tilde{t}) = t^* \tag{13}$$

A commuter that departs at $t < \tilde{t}$ is early by an amount $t^* - t - T(t)$ while a commuter that departs at $t > \tilde{t}$ is late by an amount $t + T(t) - t^*$. Following the literature (see Hendrickson and Kocur (1981), Arnott et al. (1990) or Geroliminis and Levinson (2009)), we will assume that the cost function $C(t)$ for all commuters is a linear function of both the travel time and the schedule delay

$$C(t) = \begin{cases} \alpha T(t) + \beta(t^* - t - T(t)) & t < \tilde{t} \\ \alpha T(t) + \gamma(t + T(t) - t^*) & t > \tilde{t} \end{cases} \tag{14}$$

where α is the shadow value of travel time in congested conditions, β is the shadow value of an early schedule delay and γ is the shadow value of a late schedule delay. This function is commonly known in the literature as the $\alpha - \beta - \gamma$ cost function[3].

It is commonly assumed that $\gamma > \alpha > \beta$ reflecting the fact that the marginal cost of being late at work is higher that the cost of being stuck in traffic and the marginal cost of being stuck in traffic is higher than the marginal cost of arriving early at work. This assumption is critical for the existence of an equilibrium in the original model (Arnott et al. (1990)) but there is empirical evidence that suggests that it is indeed a valid assumption (Small (1982))

3.3 The Macro Fundamental Diagram (MFD)

We further assume that network is governed by a Macroscopic Fundamental Diagram (MFD). An MFD is a function that relates the average density of vehicles k to the average flow of vehicles q at the network level provided that some assumptions about stationary and uniformity are met. This has as an important consequence that it is possible to relate the rate at which trips are ended, the *capacity* of the network, with the state variables (density and flow) at the network level. The intuition behind this idea is simple: when the number of vehicles in the network increases, a queue that builds up in a particular street may eventually

[3] Examples of different cost function can be found in Smith (1984) where a general convex and differentiable penalty function is used.

block the adjacent streets, reducing the flow of vehicles in different directions and hence decreasing the rate at which trips can be completed.

The derivation here is identical to that in Daganzo (2007). Consider a homogeneous looping road that has has length L. We can assume that the average trip distance has length l and that system demand is uniformly distributed along the link. Under these assumptions, there exists a unimodal fundamental diagram $q = q(k)$ that captures the relationship between flow and average density. Daganzo notices that total distance travelled per unit time can be expressed in two ways: as the product of the exit rate x and the average distance travelled by exiting vehicle and as the sum of te distance travelled by the vehicles in the system at any given time

$$xl = nv = Lkv = lq(n/L) \tag{15}$$

and hence

$$x = \frac{L}{l} q(n/L) \tag{16}$$

which shows that the exit rate of the network is just the flow scaled by a factor L/l. This function is known in the literature as the Network Exit Funtion (NEF) (Gonzales and Daganzo (2012)).

This result relies on a lot of simplifying assumptions but there is strong evidence that suggests it can be extrapolated to real networks. An MFD has been constructed by simulation for the city of San Francisco (Geroliminis and Daganzo (2007)) and Nairobi (Gonzales et al. (2011)). In (Geroliminis and Daganzo (2008)), the MFD for the city of Yokohama has been empirically found.

In our case, we will assume that our network can be modelled by means of a 2-step linear piecewise relationship

$$q(k) = \begin{cases} \frac{\mu}{k_{opt}} k & 0 \le k \le k_{opt} \\ \frac{\mu}{\Omega - k_{opt}} (\Omega - k) & k_{opt} \le k \le \Omega \end{cases} \tag{17}$$

where μ is the maximum flow and k_{opt} is the density of vehicles associated to that maximum flow.

Using equation (16), we can relate the exit rate of the network to the number of users as

$$x(n) = \begin{cases} \frac{L}{l} \frac{\mu}{n_{opt}} n & 0 \le n \le n_{opt} \\ \frac{L}{l} \frac{\mu}{\Omega L - k_{opt}} (\Omega L - n) & n_{opt} \le n \le \Omega L \end{cases} \tag{18}$$

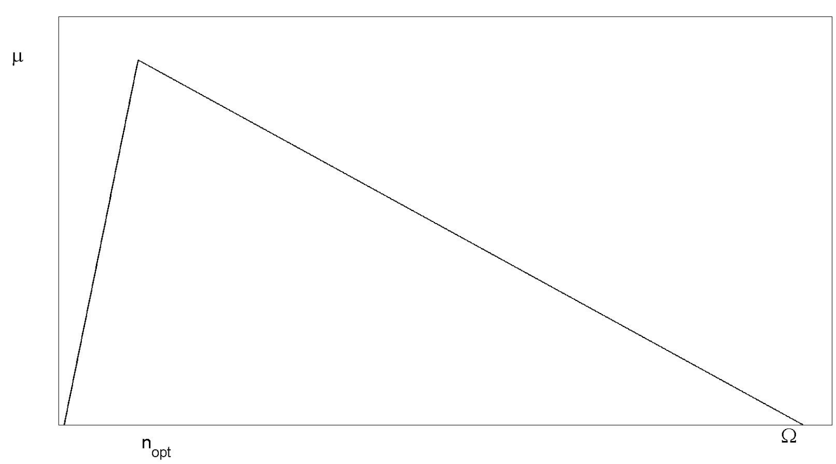

Figure 2: A 2-step piecewise MFD.

4 An analytical solution

To start solving the model, consider first taking the derivative of function (12) with respect to t. Since function (12) depends on t thorough variable u, the chain rule can be applied to yield

$$\frac{dT}{dt} = \frac{dT}{du}\frac{du}{dt} \tag{19}$$

Function $u(t)$ is defined in (11). Hence, its derivative can be readily obtained as

$$\frac{du}{dt} = \frac{dn}{dt} + x(n(t)) \tag{20}$$

where the exit rate term x is written in terms of n to clearly emphasize that the macro fundamental diagram assumption holds.

Similarly, function $T(u)$ is defined in (10) and its derivative can be obtained as

$$\frac{dT}{dt} = \frac{dX^{-1}(u)}{dt} - \frac{dE^{-1}(u)}{dt} = \frac{1}{x(n(t))} - \frac{1}{e(t)} = \frac{e(t) - x(n(t))}{x(n(t))e(t)} \tag{21}$$

Notice that this result is only valid for $t \in [\underline{t}, \bar{t}]$ because the derivative of the inverse functions are only valid in this domain.

Next we can plug equations (20) and (21) in equation (19) to get

$$x(n(t))e(t)\frac{dT}{dt} = (e(t) - x(n(t)))\left(\frac{dn}{dt} + x(n(t))\right) \tag{22}$$

The derivative of function (9) is

$$\frac{dn}{dt} = e(t) - x(n(t)) \tag{23}$$

and this last equation allows us to rewrite (22) only as a function of variables n and t

$$\left(\frac{dn}{dt}\right)^2 + x(n(t))\left(1 - \frac{dT}{dt}\right)\frac{dn}{dt} - x(n(t))^2\frac{dT}{dt} = 0 \tag{24}$$

The final step would consist in simplifying the previous result to a regular ODE by means of the well-known quadratic formula using the derivative of n with respect to t as the unknown

$$\frac{dn}{dt} = \frac{-x(n(t)) + \sqrt{x^2(n(t))\left(1 - \frac{dT}{dt}\right)^2 + 4x^(n(t))\frac{dT}{dt}}}{2} = \frac{dT}{dt}x(n(t)) \tag{25}$$

This last expression is an ODE that can be solved by separation of variables since the term $\frac{dT}{dt}$ will, in the most general case, be only a function of t and the term $x(n(t))$ is only a function of n, due to the MFD assumption[4].

Remember that in equation (18) we have already specified a shape for the MFD and therefore the only element that we are missing to actually be able to solve the ODE in (25) is an specification to the derivative of travel time T with respect to time. This will be done in the next section by imposing the equilibrium condition that all users will experiment the same travel time costs.

[4]The other solution that appears when applying the quadratic formula, $\frac{dn}{dt} = -x(n(t))$, is believed not to have any physical meaning for the problem we are dealing with.

4.1 The equilibrium condition

An equilibrium arises when every commuter has no incentive to unilaterally deviate to a different departure time. As a consequence, in an equilibrium all commuters must have the same cost independently of their departure time

$$C(t) = c \quad \forall t \tag{26}$$

As explained above, the cost function is a piecewise function and we must therefore proceed by cases to impose the equilibrium condition.

Consider first the case $t < \tilde{t}$. Taking derivatives at both sides of (26) yields

$$(\alpha - \beta)\frac{dT}{dt} - \beta = 0 \tag{27}$$

rearranging we get the following ordinary differential equation

$$\frac{dT}{dt} = \frac{\beta}{\alpha - \beta} \tag{28}$$

that can be readily solved with the initial condition $T(\underline{t}') = \underline{T}$

$$T = \frac{\beta}{\alpha - \beta}(t - \underline{t}') + \underline{T} \tag{29}$$

which shows that travel time increases[5] linearly from time $\underline{t}'$ to time $\tilde{t}$.

Following the same steps, for $t > \tilde{t}$ we must solve

$$\frac{dT}{dt} = -\frac{\gamma}{\alpha + \gamma} \tag{30}$$

whose solution is

$$T = -\frac{\gamma}{\alpha + \gamma}(t - \bar{t}) + \bar{T} \tag{31}$$

where the initial condition $T(\bar{t}) = \bar{T}$ was used to pin down a solution.

Since $\frac{dT}{dt}$ does not exist at time $\tilde{t}$ [6], we must further impose continuity at this point in order to guarantee the existence an equilibrium

$$\frac{\beta}{\alpha - \beta}(\tilde{t} - \underline{t}) + \underline{T} = -\frac{\gamma}{\alpha + \gamma}(\tilde{t} - \bar{t}) + \bar{T} \tag{32}$$

Therefore, travel time follows a piece-wise linear function between time $\underline{t}$ and time $\bar{t}$

$$T(t) = \begin{cases} \frac{\beta}{\alpha - \beta}(t - \underline{t}') + \underline{T} & t \leq \tilde{t} \\ -\frac{\gamma}{\alpha + \gamma}(t - \bar{t}) + \bar{T} & t \geq \tilde{t} \end{cases} \tag{33}$$

4.2 The solution for the number of users n

Consider first the situation $t < \tilde{t}$. Initially, we can assume that the network will work at its uncongested branch. Under these assumptions, we can find the evolution of n with respect to t by solving the following ODE

$$\frac{dn}{dt} = \frac{L}{l}\frac{\mu}{n_{opt}}\frac{\beta}{\alpha - \beta}n \tag{34}$$

[5] The assumption $\alpha > \beta$ is thus critical for guaranteeing that travel time initially increases.

[6] Given our assumption that $\gamma > \alpha > \beta$, the derivative from the left is different to the derivative from the right at time $\tilde{t}$.

which has been derived by pluging equations (18) and (28) in equation (25).

This ODE can readily be solved by separation of variables to yield

$$n = \underline{n}u(t) \qquad (35)$$
$$with \quad u(t) = exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t})\right)$$

where the initial condition $n(\underline{t}) = \underline{n}$ is used to pin down a solution.

Since we have assumed that initially the network works at its uncongested branch, this result is only valid up to $n = n_{opt}$. Hence, in order to find the moment at which the MFD reaches its congested branch we should solve for the condition $n(\underline{t}_c) = n_{opt}$. Solving for this condition yields as a result

$$\underline{t}_c = \underline{t} + \frac{l}{L}\frac{n_{opt}}{\mu}\frac{\alpha-\beta}{\beta}ln\left(\frac{n_{opt}}{\underline{n}}\right) \qquad (36)$$

Next, we can solve the ODE in (25) when the network works in its congested branch. This time, its specification is

$$\frac{dn}{dt} = \frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\beta}{\alpha-\beta}(\Omega L - n) \qquad (37)$$

and it can also be solved by separation of variables to yield

$$n = \frac{\Omega L(w(t)-1) + n_{opt}}{w(t)} \qquad (38)$$
$$with \quad w(t) = exp\left(\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t}) - \frac{n_{opt}}{\Omega L - n_{opt}}ln\left(\frac{n_{opt}}{\underline{n}}\right)\right)$$

In this case, the condition $n(\underline{t}_c) = n_{opt}$ was used to pin down a solution and the result in (36) was used to rewrite it in terms of $\underline{t}$.

The discharging part of the problem (i.e. when $t > \tilde{t}$) is solved similarly. For the sake of brevity, we will only give the results and its derivation is left as an exercise to the reader. The solution for the uncongested branch of the MFD is

$$n = \bar{n}z(t) \qquad (39)$$
$$with \quad z(t) = exp\left(-\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t})\right)$$

while the solution for the congested branch is

$$n = \frac{\Omega L(v(t)-1) + n_{opt}}{v(t)} \qquad (40)$$
$$with \quad v(t) = exp\left(-\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t}) - \frac{n_{opt}}{\Omega L - n_{opt}}ln\left(\frac{n_{opt}}{\bar{n}}\right)\right)$$

and the moment $\bar{t}_c$ at which the transition between the congested and the uncongested regim occurs is defined as

$$\bar{t}_c = \bar{t} - \frac{l}{L}\frac{n_{opt}}{\mu}\frac{\alpha+\gamma}{\gamma}ln\left(\frac{n_{opt}}{\bar{n}}\right) \qquad (41)$$

Finally, it seems a natural condition to impose continuity between the charging part of the problem and the discharging part of the problem. Physically this means that there will not be neither a mass arrival nor a mass departure of users from the system at the moment of transition between the uncongested branch and the congested branch of the MFD

$$\frac{\Omega L(w(\tilde{t})-1) + n_{opt}}{w(\tilde{t})} = \frac{\Omega L(v(\tilde{t})-1) + n_{opt}}{v(\tilde{t})}$$
$$w(\tilde{t}) = v(\tilde{t}) \qquad (42)$$

4.3 Cumulative departures

Now that we know the functions that govern the evolution of the number of users n in the system, an explicit function for the cumulative arrivals and departures can be found.

The function of cumulative departures from the system $X(t)$ is defined in (5). Since the number of vehicles in the system n is a piecewise function, the calculation of $X(t)$ must also be done in a piecewise manner.

- Between $\underline{t}$ and $\underline{t}_c$

$$X(t) = \int_{\underline{t}}^{t} \frac{L}{l}\frac{\mu}{n_{opt}} n dt = \int_{\underline{t}}^{t} \frac{L}{l}\frac{\mu}{n_{opt}} \underline{n} exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t})\right) dt =$$
$$= \frac{\alpha-\beta}{\beta}\underline{n}(u(t)-1) \quad with \quad u(t) = \exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t})\right) \tag{43}$$

This result is only valid for $t \in [\underline{t}, \underline{t}_c]$. Hence, at the end of this period the cumulative departures are

$$X(\underline{t}_c) = \frac{\alpha-\beta}{\beta}\underline{n}(u(\underline{t}_c)-1) = \frac{\alpha-\beta}{\beta}(n_{opt}-\underline{n}) \tag{44}$$

- Between $\underline{t}_c$ and $\tilde{t}$

$$X(t) - X(\underline{t}_c) = \int_{\underline{t}_c}^{t} \frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}(\Omega L - n)dt = \int_{\underline{t}_c}^{t} \frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\left(\Omega L - \frac{\Omega L(w-1)+n_{opt}}{w}\right) dt =$$
$$= \int_{\underline{t}_c}^{t} \frac{L}{l}\frac{\mu}{w}dt = \frac{\alpha-\beta}{\beta}\left(\frac{w(t)-1}{w(t)}\right)(\Omega L - n_{opt}) \tag{45}$$
$$with \quad w(t) = exp\left(\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t}) - \frac{n_{opt}}{\Omega L - n_{opt}} ln\left(\frac{n_{opt}}{\underline{n}}\right)\right)$$

So far we have computed the increment in departures between the initial time $\underline{t}_c$ and any time t. In order to have the cumulative departure function we must add the departures during the previous period

$$X(t) = \frac{\alpha-\beta}{\beta}\left(\frac{w(t)-1}{w(t)}\right)(\Omega L - n_{opt}) + \frac{\alpha-\beta}{\beta}(n_{opt}-\underline{n}) =$$
$$= \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(t)-1}{w(t)}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \tag{46}$$

This result is only valid for $t \in [\underline{t}_c, \tilde{t}]$. Hence, at the end of this period the cumulative departures are

$$X(\tilde{t}) = \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \tag{47}$$

- Between $\tilde{t}$ and $\bar{t}_c$

$$X(t) - X(\tilde{t}) = \int_{\tilde{t}}^{t} \frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}(\Omega L - n)dt = \int_{\tilde{t}}^{t} \frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\left(\Omega L - \frac{\Omega L(v-1)+n_{opt}}{v}\right) dt =$$
$$= \int_{\tilde{t}}^{t} \frac{L}{l}\frac{\mu}{v}dt = \frac{\alpha+\gamma}{\gamma}\left(\frac{v(\tilde{t})-v(t)}{v(\tilde{t})v(t)}\right)(\Omega L - n_{opt}) \tag{48}$$
$$with \quad v(t) = exp\left(-\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t}) - \frac{n_{opt}}{\Omega L - n_{opt}} ln\left(\frac{n_{opt}}{\bar{n}}\right)\right)$$

Similarly as before, the previous result is only the increment between time $\tilde{t}$ and any time t. The cumulative departure function is obtained as

$$X(t) = \frac{\alpha+\gamma}{\gamma}\left(\frac{v(\tilde{t})-v(t)}{v(\tilde{t})v(t)}\right)(\Omega L - n_{opt}) + \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \tag{49}$$

This result is only valid for $t \in [\tilde{t}, \bar{t}_c]$. Hence, at the end of this period the cumulative departures are

$$X(\bar{t}_c) = \frac{\alpha+\gamma}{\gamma}\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \tag{50}$$

- Between $\bar{t}_c$ and $\bar{t}$

$$\begin{aligned} X(t) - X(\bar{t}_c) &= \int_{\bar{t}_c}^{t} \frac{L}{l}\frac{\mu}{n_{opt}} n dt = \int_{\bar{t}_c}^{t} \frac{L}{l}\frac{\mu}{n_{opt}} \bar{n} \exp\left(-\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t})\right) dt = \\ &= \frac{\alpha+\gamma}{\gamma}(n_{opt} - \bar{n}z(t)) \quad where \quad z(t) = \exp\left(-\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t})\right) \end{aligned} \tag{51}$$

Once again, the previous result is only the increment between time $\bar{t}_c$ and any time t. The cumulative departure function is obtained as

$$\begin{aligned} X(t) =& \frac{\alpha+\gamma}{\gamma}(n_{opt} - \bar{n}z(t)) + \frac{\alpha+\gamma}{\gamma}\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + \\ &+ \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) = \\ =& \frac{\alpha+\gamma}{\gamma}\left(\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}z(t)\right) + \\ &+ \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \end{aligned} \tag{52}$$

This result is only valid for $t \in [\bar{t}_c, \bar{t}]$. Hence, at the end of this period the cumulative departures are

$$\begin{aligned} X(\bar{t}) =& \frac{\alpha+\gamma}{\gamma}\left(\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}\right) + \\ &+ \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) \end{aligned} \tag{53}$$

4.4 Cumulative arrivals

Once the cumulative departures from the system have been computed, it is easy to find the cumulative entries to the system thanks to equation (9).

- Between $\underline{t}$ and $\underline{t}_c$

$$\begin{aligned} E(t) &= X(t) + n(t) = \frac{\alpha-\beta}{\beta}\underline{n}(u(t)-1) + \underline{n}u(t) = \\ &= \frac{\alpha}{\beta}\underline{n}(u(t)-1) + \underline{n} \quad where \quad u(t) = \exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t})\right) \end{aligned} \tag{54}$$

- Between $\underline{t}_c$ and $\tilde{t}$

$$\begin{aligned} E(t) &= X(t) + n(t) = \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(t)-1}{w(t)}\right) + n_{opt} - \underline{n}\right) + \frac{\Omega L(w(t)-1) + n_{opt}}{w} = \\ &= \frac{\alpha}{\beta}\left(\left(\frac{w(t)-1}{w(t)}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \underline{n} \\ &where \quad w(t) = exp\left(\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\beta}{\alpha-\beta}(t-\underline{t}) - \frac{n_{opt}}{\Omega L - n_{opt}} ln\left(\frac{n_{opt}}{\underline{n}}\right)\right) \end{aligned} \tag{55}$$

- Between $\tilde{t}$ and $\bar{t}_c$

$$
\begin{aligned}
E(t) = X(t) + n(t) &= \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \\
&+ \frac{\alpha+\gamma}{\gamma}\left(\frac{v(\tilde{t})-v(t)}{v(\tilde{t})v(t)}\right)(\Omega L - n_{opt}) + \frac{\Omega L(v-1)+n_{opt}}{v} = \\
&= \frac{\alpha}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \frac{\alpha}{\gamma}(\Omega L - n_{opt})\left(\frac{v(\tilde{t})-v(t)}{v(\tilde{t})v(t)}\right) + \underline{n} \\
&where \quad v(t) = exp\left(-\frac{L}{l}\frac{\mu}{\Omega L - n_{opt}}\frac{\gamma}{\alpha+\gamma}(t-\bar{t}) - \frac{n_{opt}}{\Omega L - n_{opt}} ln\left(\frac{n_{opt}}{\bar{n}}\right)\right)
\end{aligned} \tag{56}
$$

- Between $\bar{t}_c$ and $\bar{t}$

$$
\begin{aligned}
E(t) = X(t) + n(t) &= \frac{\alpha+\gamma}{\gamma}\left(\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}z(t)\right) + \\
&+ \frac{\alpha-\beta}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \bar{n}z(t) = \\
&= \frac{\alpha}{\gamma}\left(\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}z(t)\right) + \\
&+ \frac{\alpha}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \underline{n}
\end{aligned} \tag{57}
$$

4.5 Solving for the initial ($\underline{T}$) and final ($\bar{T}$) travel time

At this moment of the derivation, we have found analytical expressions for the number of vehicles in the system n and both for the cumulative departures from the system X and arrivals to the system E. However, this analytical expressions are only valid for $t \in [\underline{t}, \bar{t}]$ because the ODE in (25) is only properly defined in that part of the domain of our problem. Hence, we are missing the analytical expressions whenever $t \in [\underline{t}', \underline{t}]$ and $t \in [\bar{t}, \bar{t}']$.

Notice that whenever $t \in (\underline{t}', \underline{t})$, then $t + T(t) \in (\underline{t}, \tilde{t})$ since $\underline{T}$ is defined as $\underline{T} = \underline{t} - \underline{t}'$. By the conservation of users we have that

$$
E(t) = X(t + T(t)) \quad \forall t \in [\underline{t}', \underline{t}] \tag{58}
$$

and we can therefore know the cumulative arrivals to the system during the period $t \in [\underline{t}', \underline{t}]$ because we already now the cumulative departures from the system during period $[\underline{t}, \tilde{t}]$.

In order to find the analytical expression we must express the argument in function X as follows

$$
t + T(t) = t + \frac{\beta}{\alpha-\beta}(t - \underline{t}') + \underline{t} - \underline{t}' = \frac{\alpha}{\alpha-\beta}(t - \underline{t}') + \underline{t} \tag{59}
$$

and by plugging it in equation (43) we get

$$
E(t) = \frac{\alpha-\beta}{\beta}\underline{n}\left(\exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\alpha\beta}{(\alpha-\beta)^2}(t - \underline{t}')\right) - 1\right) \tag{60}
$$

Since equation (43) is only valid up to $\underline{t}_c$, equation (60) will only be valid up to a time $\underline{t}_c'$ defined as $\underline{t}_c' + T(\underline{t}_c') = \underline{t}_c$. If one solves for this particular time, we get

$$
\underline{t}_c' = \underline{t}' + \frac{l}{L}\frac{n_{opt}}{\mu}\frac{(\alpha-\beta)^2}{\alpha\beta} ln\left(\frac{n_{opt}}{\underline{n}}\right) \tag{61}
$$

Now, we can plug this result in equation (60) to get the cumulative number of users who have entered the system until the moment the networks switches to its congested state

$$E(\underline{t}_c') = \frac{\alpha - \beta}{\beta}(n_{opt} - \underline{n}) \tag{62}$$

which should not be a surprise since $E(t) = X(t + T(t))$ and $\underline{t}_c'$ is defined as $\underline{t}_c' + T(\underline{t}_c') = \underline{t}_c$.

Next, we can impose $\frac{\alpha-\beta}{\beta}(n_{opt} - \underline{n}) > \underline{n}$, which guarantees that at time $\underline{t}$, the moment when the first commuter arrives at work, the network is still at its uncongested branch[7]. If this condition is satisfied then at time $\underline{t}$ it will be true that

$$\frac{\alpha - \beta}{\beta}\underline{n}\left(\exp\left(\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\alpha\beta}{(\alpha - \beta)^2}\underline{T}\right) - 1\right) = \underline{n} \tag{63}$$

and this fact will allows us to solve for $\underline{T}$ as

$$\underline{T} = \frac{l}{L}\frac{n_{opt}}{\mu}\frac{(\alpha - \beta)^2}{\alpha\beta}\ln\left(\frac{\alpha}{\alpha - \beta}\right) \tag{64}$$

Consider now the period $t \in [\bar{t}, \bar{t}']$, which is conceptually the travel time of the last commuter $\bar{T}$. By the conservation of users, we have that

$$E(t) = X(t + T(t)) \tag{65}$$

and since we now the analytical expression for E we can derive the analytical expression for X.

In order to do so, let us propose the following change of variables

$$\hat{t} = t + T(t) = t - \frac{\gamma}{\alpha + \gamma}(t - \bar{t}) + \bar{t}' - \bar{t} = \left(\frac{\alpha}{\alpha + \gamma}\right)(t - \bar{t}) + \bar{t}' \tag{66}$$

$$t = \frac{\alpha + \gamma}{\alpha}(\hat{t} - \bar{t}') + \bar{t} \tag{67}$$

and we can plug equation (67) in equation (57) to get

$$\begin{aligned} X(\hat{t}) = \; & \frac{\alpha}{\gamma}\left(\left(\frac{v(\tilde{t}) - 1}{v(\hat{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}z(\hat{t})\right) + \\ & + \frac{\alpha}{\beta}\left(\left(\frac{w(\tilde{t}) - 1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \underline{n} \\ & with \quad z(\hat{t}) = \exp\left(-\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\gamma}{\alpha}(\hat{t} - \bar{t}')\right) \end{aligned} \tag{68}$$

If we now equate equation (52) to equation (68) at time $\bar{t}$, we get a condition to find an expression for $\bar{T}$

$$\frac{\alpha + \gamma}{\gamma}\bar{n} = \frac{\alpha}{\gamma}\bar{n}\exp\left(-\frac{L}{l}\frac{\mu}{n_{opt}}\frac{\gamma}{\alpha}\bar{T}\right) \tag{69}$$

$$\bar{T} = \frac{l}{L}\frac{n_{opt}}{\mu}\frac{\alpha}{\gamma}\ln\left(\frac{\alpha + \gamma}{\gamma}\right) \tag{70}$$

[7] We can rewrite this condition as $\underline{n} < \frac{\alpha-\beta}{\alpha}n_{opt}$, which tell us that the boundary condition $\underline{n}$ should be a percentage smaller that the optimal network capacity n_{opt} for this condition to be true.

4.6 Solving for $\underline{t}$ and $\bar{t}$

Until now we have found the cumulative arrivals to the system (equations (60), (54), (55), (56) and (57)) and the cumulative departures from the system (equations (43), (46), (49), (52) and (68)). However, these equations depend on parameters $\underline{t}$ and $\bar{t}$ which have not yet been determined. In order to do so we must impose that a number of commuters N wish to arrive at their destination by the end of the rush-hour

$$E(\bar{t}) = N \tag{71}$$

$$\frac{\alpha}{\gamma}\left(\left(\frac{v(\tilde{t})-1}{v(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \bar{n}\right) + \frac{\alpha}{\beta}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt} - \underline{n}\right) + \underline{n} = N \tag{72}$$

Since $w(\tilde{t}) = v(\tilde{t})$, this expression can be further simplified to

$$\frac{\alpha(\beta+\gamma)}{\beta\gamma}\left(\left(\frac{w(\tilde{t})-1}{w(\tilde{t})}\right)(\Omega L - n_{opt}) + n_{opt}\right) - \frac{\alpha-\beta}{\beta}\underline{n} - \frac{\alpha}{\gamma}\bar{n} = N \tag{73}$$

and we can now solve for $w(\tilde{t})$

$$w(\tilde{t}) = \frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)} \tag{74}$$

Since we know the analytical expression for equation $w(t)$, we can solve for $\underline{t}$

$$\underline{t} = \tilde{t} - \frac{l}{L}\frac{\alpha-\beta}{\beta}\left(\frac{\Omega L - n_{opt}}{\mu}\ln\left(\frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)}\right) + \frac{n_{opt}}{\mu}\ln\left(\frac{n_{opt}}{\underline{n}}\right)\right) \tag{75}$$

Next step consists in expressing the previous results in terms of t^*, which is an exogenous parameter. In order to do so, consider the following equality

$$\underline{t} - \underline{t}' + \tilde{t} - \underline{t} + t^* - \tilde{t} = t^* - \underline{t}' \tag{76}$$

$$\underline{T} + \tilde{t} - \underline{t} = \tilde{t} - \underline{t}' \tag{77}$$

On the other hand, we know that

$$\tilde{t} + \frac{\beta}{\alpha-\beta}(\tilde{t} - \underline{t}') + \underline{T} = t^* \tag{78}$$

and this can be rewritten as

$$\tilde{t} - \underline{t}' = \frac{\alpha-\beta}{\alpha}(t^* - \underline{t}) \tag{79}$$

We can plug equations (79) in equation (77) and rewrite it as follows

$$\underline{t} = t^* - \frac{\alpha}{\alpha-\beta}(\underline{T} + \tilde{t} - \underline{t}) \tag{80}$$

Notice that we already have an analytical expression for $\underline{T}$ in (64) and for $\tilde{t} - \underline{t}$ in (75). Therefore, getting an analytical expression for $\underline{t}$ is straightward

$$\begin{aligned}\underline{t} =& t^* - \frac{l}{L}\frac{n_{opt}}{\mu}\left(\frac{\alpha-\beta}{\beta}\ln\left(\frac{\alpha}{\alpha-\beta}\right) + \frac{\alpha}{\beta}\ln\left(\frac{n_{opt}}{\underline{n}}\right)\right) + \\ &- \frac{l}{L}\frac{\Omega L - n_{opt}}{\mu}\frac{\alpha}{\beta}\ln\left(\frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)}\right)\end{aligned} \tag{81}$$

From equation (72), a similar result to equation (81) can be derived in terms of $v(\tilde{t})$

$$v(\tilde{t}) = \frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)} \tag{82}$$

and this allows us to solve for $\bar{t}$

$$\bar{t} = \tilde{t} + \frac{l}{L}\frac{\alpha+\gamma}{\gamma}\left(\frac{\Omega L - n_{opt}}{\mu}\ln\left(\frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)}\right) + \frac{n_{opt}}{\mu}\ln\left(\frac{n_{opt}}{\bar{n}}\right)\right) \tag{83}$$

Next, consider equation (13) and rewrite it as follows

$$\tilde{t} - \frac{\gamma}{\alpha+\gamma}(\tilde{t} - \bar{t}) + \bar{t}' - \bar{t} = t^* \tag{84}$$

$$\bar{t}' = t^* + \frac{\alpha}{\alpha+\gamma}(\bar{t} - \tilde{t}) \tag{85}$$

by pluggint equation (83) in we get

$$\bar{t}' = t^* + \frac{l}{L}\frac{n_{opt}}{\mu}\frac{\alpha}{\gamma}\ln\left(\frac{n_{opt}}{\bar{n}}\right) + \frac{l}{L}\frac{\Omega L - n_{opt}}{\mu}\frac{\alpha}{\gamma}\ln\left(\frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)}\right) \tag{86}$$

and since $\bar{t} = \bar{t}' - \bar{T}$

$$\begin{aligned}\bar{t} = t^* + \frac{l}{L}\frac{n_{opt}}{\mu}\frac{\alpha}{\gamma}\left(\ln\left(\frac{\alpha+\gamma}{\alpha}\right) + \ln\left(\frac{n_{opt}}{\bar{n}}\right)\right) + \\ + \frac{l}{L}\frac{\Omega L - n_{opt}}{\mu}\frac{\alpha}{\gamma}\ln\left(\frac{\Omega L - n_{opt}}{\Omega L - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\left(N + \frac{\alpha-\beta}{\beta}\underline{n} + \frac{\alpha}{\gamma}\bar{n}\right)}\right)\end{aligned} \tag{87}$$

The last step that remains to be done is to impose that users travelling in the charging part of the problem have the same generalized costs that users travelling in the charging part of the problem. This can be done in several ways but the most straightforward is to remember equation (32), a constraint that guarantees the continuity of the travel time function at point $\tilde{t}$ and hence it also guarantees the existence of an equilibrium[8].

Equation (32) reads

$$\frac{\beta}{\alpha-\beta}(\tilde{t} - \underline{t}) + \underline{T} = -\frac{\gamma}{\alpha+\gamma}(\tilde{t} - \bar{t}) + \bar{T} \tag{88}$$

and it can be rewritten in terms of our main results as

$$\frac{\beta}{\alpha-\beta}(\tilde{t} - \underline{t}) + \frac{\alpha}{\alpha-\beta} + \underline{T} = -\frac{\gamma}{\alpha+\gamma}(\tilde{t} - \bar{t}') + \bar{T} \tag{89}$$

We can now plug equations (64), (70), (75) and (86) to get

$$\bar{n} = \underline{n}\left(\frac{\alpha+\gamma}{\gamma}\right)^{\frac{\alpha}{\gamma}}\left(\frac{\alpha}{\alpha-\beta}\right)^{1-\frac{\alpha}{\beta}} \tag{90}$$

which tells us the exact relationship that must exist between $\underline{n}$ and $\bar{n}$ in order to have an equilibrium.

[8] An alternative ways would be to equate the travel time cost of the first commuter to that of the last commuter. It can be proven that both conditions are equivalent but the algebra is slightly cumbersome and it has been omitted.

4.7 Discussion and limitations

Despite all the efforts made, this model has important limitations that must be clearly explained for the sake of intellectual honesty.

An attentive reader may have seen that we have said very little about the distance that commuters must travelled from home to work. In section 3.3, following the assumptions of Geroliminis and Daganzo (2008), we have assumed that the average trip distance is l and this value has been taken as an exogenous parameter, similar to the different values of time or the number of commuters. However in other papers of the literature, such as in Arnott et al. (2016), the assumption of a common distance travelled by all commuters plays a crucial role in deriving the properties of the model. The assumption of a common distance travelled by all commuters can be written as

$$l = \int_t^{t+T(t)} v(k(\tau))d\tau \tag{91}$$

and taking derivatives at both sides yields

$$v(k(t)) = v(k(t+T(t)))\left(1 + \frac{dT}{dt}\right) \tag{92}$$

which means that there exists a well-defined relationship between the density of vehicles in the network when a commuter enters and leaves.

In our case, we have derived the evolution of the number of vehicles n independently of this assumption. This means that we can plug equations (35), (38), (40), (39) in integral (91) and solve it to find a $function$ that tells us the distance that each commuter will travel. Therefore, our model implies that each commuter will travel a different distance depending on the velocity that he encounters throughout his trip. In principle, there is nothing wrong with this result but the problem arises when we remember that we had assumed an average trip distance l and there is absolutely nothing that guarantees that the two results (the one coming from our original assumption and the one resulting from this distance function) coincide. This is an important flaw of the model because it implies that not all the pieces match, or at least, not as well as they should match.

Another important drawback that must be emphasized is the interpretation of the boundary condition $\underline{n}$. Remember that this boundary condition imposes a constraint on the amount of people in the network at moment $\underline{t}$, the instant when the first commuter arrives to work. While this boundary condition is absolutely necessary mathematically, it seems difficult to provide a justification for its value based on real life experience. This fact gets special significance because the whole idea of the work is to derive a model of endogenous congestion and the dependence of the results on a rather arbitrary parameter poses a serious thread to this objective.

4.8 A numerical example

Consider a downtown area populated by $N = 100.000$ commuters who all wish to arrive at $t^* = 9$ h after travelling a distance of $l = 5$ km. All commuters have the same values of time ($\alpha = 10$ €/h, $\beta = 5$ €/h and $\gamma = 15$ €/h). The network of the downtown area is 500 km long and can be modelled by means of a 2-step linear pieacewise MFD, whose maximum flow is $\mu = 1250$ veh/h. The network works at free flow conditions until it reaches a density of $n_{opt} = 25$ veh/km. At a density $\Omega = 250$ veh/km the network reaches gridlock. At the moment the first commuter arrives at work, there are $\underline{n} = 5000$ vehicles in the network.

According to our model, the rush-hour would start at 8.20 h and it would end at 9.24 h. The first commuter would arrive at work at 8.23 h and she would thus experiment a travel time of approximately 2 minutes. On the other hand, the last commuter would leave home at 9.18 h and she would experiment a delay of approximately 4 minutes. The commuter departing from home at 8.58 h would experiment the greatest travel time, approximately 25 min.

In figure 3 we can see the evolution of the number of vehicles and the velocity in the network during the rush-hour. The number of vehicles in the network monotonically increases until a maximum of approximately 40.000 vehicles at 8.58 h, after that time the number of vehicles in the network monotonically decreases. Notice that velocity has a slightly different behaviour since it only starts decreasing after the number of vehicles in the network increases above $n_{opt} = 12.500$ veh which happens at 8.32 h. This behaviour is entirely dependant on the particular functional form that we have chosen for the MFD. In our formulation, velocity is held constant during the free flow stage and only after a particular threshold it starts decreasing. At the moment where the network is packed the most, velocity drops to less than 20 km/h and after that it monotonically increases to 50 km/h, when the network reaches free-flow conditions again.

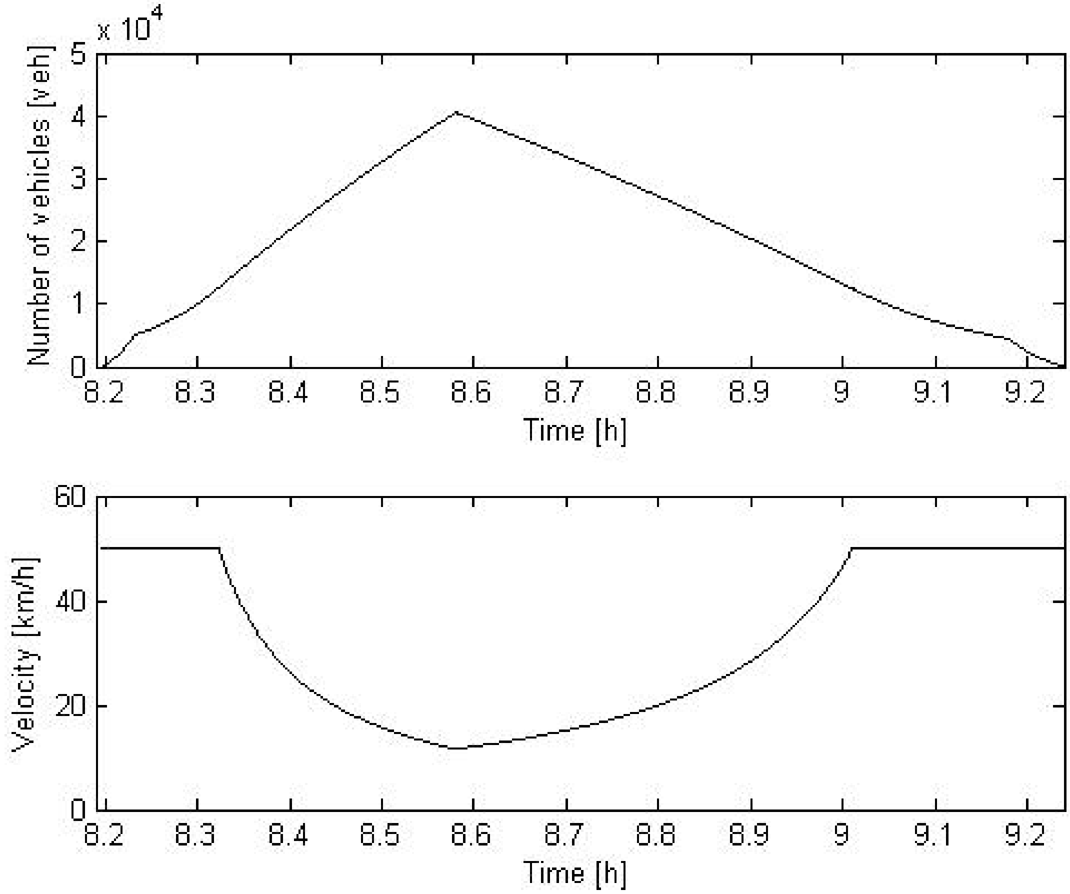

Figure 3: Number of vehicles and velocity during the rush-hour.

5 Conclusions

The work developed here contributes to the literature by developing a continuous solution of the Vickrey Morning Commute Model extended by means of a Macropscopic Fundamental Diagram. Several previous works (Fosgerau and Small (2013), Arnott (2013), Arnott et al. (2016)) have recognized the difficulty of this task and we believe that an small yet relevant contribution has been achieved.

The usefulness of this model lies in the fact that it treat congestion endogenously. Hence, its application can be very useful to try to predict the behaviour of users when different policies that try to deal with congestion are implemented. Following the lines of Arnott et al. (1990), this model could be the first step to derive an optimal toll that tries to minimize congestion. In fact, this was the original idea of this work but due to the difficulty of the problem we are dealing with a more modest goal had to be set. However, we must be aware that in order for the model to become a policy maker useful tool, it should be extended to make it closer to reality. Several extensions are both possible and necessary: to allow the user to choose between private transport o transit, to account for different kinds of users with different values of times, to have a distribution of desired arrival time, etc. Furthermore, what is even more important, the model should be tested again the available data to see whether it is able to predict reasonable results.

To summarize, let us outline the structure of the model. Without any kind of doubts, the key result is equation (25) that reads $\frac{dn}{dt} = \frac{dT}{dt}x(n)$. This equation is at the core of the model because it relates the evolution of the number of vehicles in the network to the two fundamental ingredients of the model: the MFD and the notion of equilibrium of travel time costs. Once this ODE has been solved, the only thing that remained to be done to solve the model is to impose that a fix number of users N commute from home to work. In order to do that, we first had to compute the cumulative arrivals and departure curves taking into account the piecewise nature of the problem.

Despite the appeal of the model, we must be aware of the limitations that it has. As discussed, the main limitation is that we are assuming an average trip distance l that is disconnected from the distance function $l(t)$ that results from solving the model. This implies that these two numbers will not generally coincide creating an important flaw in the model. This undesired feature should be further investigated to see whether it is possible to correct it or, in case it is not possible, to see whether it is significant enough to invalidate the model.

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