Stability implies constancy for fully autonomous reaction–diffusion–equations on finite metric graphs

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In memoriam Karl–Peter Hadeler 1936–2017.

Abstract

We show that there are no stable stationary nonconstant solutions of the evolution problem (1) for fully autonomous reaction–diffusion–equations on the edges of a finite metric graph G under continuity and Kirchhoff flow transition conditions at the vertices.

$$(1)\qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty)), & \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) & \text{on the edges } k_j, \\ (K) \quad \sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i,t) = 0 & \text{at the vertices } v_i. \end{cases}$$

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1 Introduction

In a fundamental paper of 1979 H. Matano [16] showed that autonomous reaction–diffusion equations involving the Laplacian under Neumann boundary conditions in a convex $\mathcal{C}^3$–domain cannot possess stable (spatially) nonconstant stationary solutions. The present paper deals with the non existence of stable nonconstant stationary solutions of reaction–diffusion–equations on the edges of a metric graph and natural transition conditions at the vertices. The parabolic problem in question reads

$$(1)\qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty)), \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j \qquad\qquad \text{for } 1 \le j \le N, \end{cases}$$

where $\{k_j \mid 1 \le j \le N\}$ denotes the edge set of the metric graph G and where the subscript K stands for the validity of the Kirchhoff condition

$$(2)\qquad \sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 \quad \text{for} \quad 1 \le i \le n$$

at each vertex v_i of the graph. Note that we do not require the coefficients in (2) to be consistent with the diffusion coefficient, i.e. we do not impose $c_{ij} = 1$, but only the dissipativity assuring condition that all $c_{ij} > 0$. At the ramification nodes we impose the *continuity condition*

$$(3)\qquad \forall v_i \in V_{\mathrm{r}} \ : \ k_j \cap k_s = \{v_i\} \implies u_j(v_i) = u_s(v_i),$$

that clearly is contained in the condition $u \in \mathcal{C}(G)$. Throughout, the constant coefficients and nonlinearities are assumed to satisfy

$$(4)\qquad f \in \mathcal{C}^1(\mathbb{R}), \quad c_{ij} > 0.$$

Now we can state the

Main Theorem. *On any finite connected metric graph all stable stationary solutions of Problem* (1) *are constant.*

In 2001 E. Yanagida [19] published a list of five exceptional graphs that do not allow stable nonconstant stationary solutions in the consistent case of (1), see Section 5. Moreover, he established some fundamental instability tools, as the instability criterion in the presence of two different critical points in one edge. In 2015 the authors [9] showed that the same exclusion result for any metric graph with sufficiently small edge lengths, as well as for any metric graph for the cubic balanced case $f(u) = u - u^3$, for $f(u) = \eta \ \sin(u)$ and for some other nonlinearities. Other recent instability criteria, also for dynamical Kirchhoff conditions, can be found in [11].

At first glance, the main result might seem to be surprising in view of the existence of stable nonconstant stationary solutions on non convex domains in higher dimensions, see [13, 16] and the references therein. But, as it stands, with respect to the autonomous semilinear parabolic flows defined by (1), finite metric graphs behave like one–dimensional objects, i.e. like intervals of the real line. Clearly, Problem (1) can be regarded as an abstract interaction problem on a suitable interval in the sense of Gramsch and Ali Mehmeti, see e.g. [3, 7], where the node transition conditions can be read as equivalent Cauchy conditions of order 0 and 1 and identifying conditions at interior points and on the boundary of the interval. However, tame deviations from the autonomous character of the differential equations as different diffusion coefficients or edge dependent nonlinearities can lead to the existence of nonconstant stable stationary solutions, see Section 8.

The present paper is organized as follows. After some stability prerequisites and graph theoretical preliminaries in Section 2, Section 3 presents the exclusion of stable nonconstant stationary solutions on paths and circuits. The basic cutting technique that is a crucial tool in showing the main theorem, is established in Section 4 for more general reaction–diffusion–problems and states, for short, that if the metric graph G allows the stationary solution u and if it is cut at some point $p = \pi_j(x_j)$ with $\partial_j u_j(x_j) = 0$, then stability of the corresponding solution on the resulting graph $\widetilde{G}$ is equivalent with the one of u on G. As another extension technique the edge doubling is presented at the end of this section. In Section 5 we present some elementary cases of graphs that exclude stable nonconstant stationary solutions. In Section 6 the proof of the main theorem will be given, first for trees and then for a general finite metric graph where the first case is a key tool of a recurrence procedure using the corank of the graph. In Section 7 some energy and localization results for stationary solutions in the consistent case are treated, based on the Hamiltonian system associated to the edge differential equations. Finally, Section 8 presents some examples of stable nonconstant stationary solutions of non autonomous problems as edge dependent diffusion coefficients or edge dependent nonlinearities, as well as under different node transition conditions.

2 Metric graphs and stability

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \#\{k \in E \mid v \in k\}$. Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, connected and finite with

$$n = \#V, \qquad N = \#E.$$

The vertices will be numbered by $v_1, \dots, v_n$, the respective valencies by $\gamma_1, \dots, \gamma_n$, and the edges by $k_1, \dots, k_N$. The *boundary vertices* $V_{\mathrm{b}} = \{v_i \in V \mid \gamma_i = 1\}$ will be distinguished from the *ramification nodes* $V_{\mathrm{r}} = \{v_i \in V \mid \gamma_i \geq 2\}$ and the *essential ramification nodes* $V_{\mathrm{ess}} = \{v_i \in V \mid \gamma_i \geq 3\}$. By definition, a *circuit* is a connected and regular graph of valency 2. A *path* is a connected graph with two distinct vertices of valency 1 while the other vertices are all of valency 2. By definition, a *viaduct* π in Γ is a path in Γ joining two distinct vertices v and $\tilde{v}$ such that there is no other walk in Γ joining v and $\tilde{v}$ having a vertex in the set $V(\pi)\backslash\{v,\tilde{v}\}$. For further graph theoretical terminology we refer to [12, 18].

Moreover, we consider each graph as a *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$ and the edge set consists in a collection of Jordan curves

$$E(\Gamma) = \{\pi_j : [0, \ell_j] \to \mathbb{R}^m \mid 1 \leq j \leq N\}$$

with the following properties: Each support $k_j = \pi_j([0, \ell_j])$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $k_j \neq k_h$ satisfy $k_j \cap k_h \subset V(\Gamma)$ and $\#(k_j \cap k_h) \leq 1$. The arc length parameter of an edge k_j is denoted by x_j. Unless otherwise stated, we identify the graph $\Gamma = (V, E, \in)$ with its associated *metric graph*, *network* or *quantum graph*

$$G = \bigcup_{j=1}^{N} \pi_j([0, \ell_j]),$$

especially each edge π_j with its support k_j. Throughout it will be assumed that all $\pi_j \in \mathcal{C}^2([0, \ell_j], \mathbb{R}^m)$. Thus, endowed with the induced topology G is a connected and compact space in $\mathbb{R}^m$. Throughout, we shall denote the total graph length by

$$L = L(\Gamma) = \sum_{j=1}^{N} \ell_j.$$

The orientation of the graph Γ is given by the *incidence matrix* $\mathcal{D}(\Gamma) = (d_{ik})_{n\times N}$ with

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

For a function $u : G \to \mathbb{C}$ we set $u_j := u \circ \pi_j : [0, \ell_j] \to \mathbb{C}$ and use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial x_j} u_j(x_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc. and}$$

$$\int_G u\, dx := \sum_{j=1}^{N} \int_0^{\ell_j} u_j(x_j)\, dx_j.$$

Endowed with a usual product norm we set

$$L^p(G) = \prod_{j=1}^{N} L^p(0, \ell_j) \quad \text{and} \quad H^k(G) = \prod_{j=1}^{N} H^k(0, \ell_j)$$

for $p \in [1, \infty]$ and $k \in \mathbb{N}$, respectively. The validity of the Kirchhoff law (2) in a function space will be indicated by the subscript K. In particular, for $m \in \mathbb{N}^*$ we set

$$\mathcal{C}^m_K(G) = \{u \in \mathcal{C}(G) \mid \forall j \in \{1, \ldots, N\} : u_j \in \mathcal{C}^m([0, \ell_j]), u \text{ satisfies } (2)\}.$$

Closing this section we recall Lyapunov's notion of stability associated to the following reaction - diffusion - problem

$$
(5) \qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}^{2,1}_K(G \times (0,\infty)), \\ \partial_t u_j = \partial_j \left(a_j(x_j)\, \partial_j u_j\right) + f_j(u_j) \quad \text{on } k_j \qquad \text{for } 1 \le j \le N, \\ (K) \quad \sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 \qquad\qquad \text{for} \quad 1 \le i \le n, \end{cases}
$$

under the hypotheses

$$
(6) \qquad f_j \in \mathcal{C}^1(\mathbb{R}), \quad a_j \in \mathcal{C}^1\left([0,\ell_j]\right), \quad a_j > 0, \quad c_{ij} > 0.
$$

Conceivably, a stationary solution w of Problem (5) is called *stable* if for each $\epsilon > 0$, there exists a $\delta > 0$ such that, for each initial data $u_0 \in \mathcal{C}(G)$ with $\|u_0 - w\|_{\infty,G} < \delta$ the solution u of (5) exists in $[0,\infty)$ and satisfies

$$
\forall t > 0 \ : \|u(\cdot,t) - w\|_{\infty,G} < \epsilon.
$$

Several instability criteria have been established in [9], among which we cite the following.

Lemma 2.1 *If a stationary solution $u \in \mathcal{C}^2_K(G)$ of (5) under consistent Kirchhoff conditions satisfies either*

$$
\int_G f'(u) dx > 0 \quad \textit{or} \quad \int_G f'(u)\, u^2 dx > \int_G f(u)\, u dx,
$$

then u is unstable.

Moreover, let us cite the following elementary result from [9].

Lemma 2.2 *A stationary solution $u \in \mathcal{C}^2_K(G)$ of (5) with consistent Kirchhoff conditions satisfies*

$$
\int_G f(u) dx = 0 \quad \textit{and} \quad \int_G f(u)\, u^{2k+1}\, dx \ge 0
$$

for all $k \in \mathbb{N}$. If, in addition, u is nonconstant, then the last inequality is strict.

On all finite metric graphs it can happen that there are no stationary solutions at all. E.g. for $f \equiv 1$, there is no such solution $u \in \mathcal{C}^2_K(G)$ of (1) with a consistent Kirchhoff law, since such a solution would have to fulfill $\int_G \partial^2 u \ dx = 0$ by the Kirchhoff flow condition (2).

3 Paths and circuits

In this section we establish the exclusion of stable nonconstant stationary solutions on paths and circuits. Let $u \in \mathcal{C}^2_K(G)$ be a stationary solution of the autonomous and consistent problem

$$
(7) \qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}^{2,1}_K(G \times (0,\infty)), & \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j & \text{for } 1 \le j \le N, \\ (K) \quad \displaystyle\sum_{j=1}^{N} d_{ij} \partial_j u_j(v_i, \cdot) = 0 & \text{for} \quad 1 \le i \le n. \end{cases}
$$

Recall that the Rayleigh quotient of the linearized elliptic operator at u for (7) is given by

$$
R(\varphi; u) = \frac{\displaystyle\int_G (\partial\varphi)^2 - f'(u)\varphi^2 \, dx}{\displaystyle\int_G \varphi^2 \, dx},
$$

and that its admissible functions φ are just given by the elements of $\mathcal{C}(G) \cap H^1(G)$. Set $\psi_j = \partial_j u_j$. On each edge k_j, ψ_j satisfies the linearized equation

$$
(8) \qquad \partial_j^2 \psi_j + f'(u_j)\psi_j = 0 \quad \text{in} \quad [0, \ell_j]
$$

by standard regularity arguments. Now let Π be a path with N edges and vertices numbered by $1, \ldots, N+1$ such that v_i and v_{i+1} are adjacent, and such that $V_\text{b} = \{v_1, v_{N+1}\}$ forms the set of boundary vertices. The incidence is given by

$$
(9) \qquad d_{ij} = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}
$$

Lemma 3.1 *Suppose that $u \in \mathcal{C}^2_K(\Pi)$ is a nonconstant stationary solution of* (7) *on the path Π. Then u is unstable, more precisely*

$$
\lambda_0(u) = \min\left\{ R(\varphi; u) \mid \varphi \in \mathcal{C}(\Pi) \cap H^1(\Pi) \right\} < 0.
$$

Proof. Here, in fact, $\psi = (\psi_j)_{N\times 1}$ defines an admissible function for the Rayleigh quotient since ψ constitutes a function belonging to $\mathcal{C}^1[0, L]$ by the consistent Kirchhoff condition in (7), by (3) and by the edge differential equations. Thus, denoting the numerator of $R(\varphi; u)$ by $\mathcal{N}(\varphi)$, we obtain by (8)

$$
\begin{aligned}
0 &= \sum_{j=1}^{N} \int_0^{\ell_j} \left(\partial_j^2 \psi_j + f'(u_j)\psi_j \right) \psi_j dx_j \\
&= -\sum_{j=1}^{N} \int_0^{\ell_j} (\partial_j \psi_j)^2 - f'(u_j)\psi_j^2 \, dx_j + \sum_{j=1}^{N} \left[\partial_j \psi_j \ \psi_j \right]_0^{\ell_j} = -\mathcal{N}(\psi),
\end{aligned}
$$

since (3) and $f(u(v_i)) = -\partial_j\psi_j(v_i)$ imply

$$\sum_{j=1}^{N}\big[\partial_j\psi_j\ \psi_j\big]_0^{\ell_j} = -\sum_{i=1}^{n} f(u(v_i))\sum_{j=1}^{N} d_{ij}\psi_j(v_i) = 0. \tag{10}$$

Thus, $R(\psi;u) = 0$ and, in turn, $\lambda_0(u) \le 0$. If $\lambda_0(u) < 0$, the instability follows from [9, 19]. It remains to exclude the case $\lambda_0(u) = 0$. But, in that case, ψ is not only admissible, but an eigenfunction belonging to the simple eigenvalue $\lambda_0 = 0$ and has a sign, say $\psi_j(x_j) > 0$ at every point of $\Pi\setminus\{v_1, v_{N+1}\}$. By a well–known property of unrestricted minimizers of R in $\mathcal{C}(\Pi)\cap H^1(\Pi) = H^1[0,L]$, see e.g. [1, 2, 7, 16, 17], the function ψ satisfies the Kirchhoff condition in (7), in particular

$$\psi(v_1) = \psi(v_{N+1}) = \partial_1\psi_1(v_1) = \partial_N\psi(v_{N+1}) = 0.$$

But as a solution of the second order linear ODE $\partial_j^2\psi_j + f'(u_j)\psi_j = 0$ on each edge k_j, ψ_1 vanishes identically by uniqueness, as well as all the other ψ_j do by connectedness of Π, by the Kirchhoff condition and by (3). Thus, by continuity, u must be constant, which leads to a contradiction. ∎

Now we consider the general Kirchhoff law (2) on the path. As on all trees due to its homogeneous character, it can be written in the form

$$\sum_{j=1}^{N} d_{ij}\ c_j\ \partial_j u_j(v_i) = 0 \quad \text{for} \quad 1 \le i \le n \tag{11}$$

with positive constants c_j on each edge, that leads to a self–adjoint resolvent by using a suitable scalar product and to the Rayleigh quotient

$$R(\varphi;u;c) = \frac{\displaystyle\sum_{j=1}^{N} c_j \int_0^{\ell_j} (\partial\varphi_j)^2 - f'(u_j)\varphi_j^2\ dx_j}{\displaystyle\sum_{j=1}^{N} c_j \int_0^{\ell_j} \varphi_j^2\ dx_j} \tag{12}$$

with $c = (c_j)_{N\times 1}$. Note that (11) is also the Kirchhoff law considered by Yanagida [19], as well as in [8] in order to reduce the eigenvalue asymptotics on trees to a consistent case. In fact, on the path Π, the form (11) is readily achieved by setting

$$c_1 = 1, \quad c_{j+1} = \frac{c_{j+1,j+1}c_j}{c_{j+1,j}} \quad \text{for} \quad 2 \le j \le N-1. \tag{13}$$

The same technique applies on arbitrary trees and shows that for equilibria the linearized stability under (2) and (11) are equivalent. Set

$$\lambda_0(u;c) = \min\left\{R(\varphi;u;c)\mid 0 \ne \varphi \in H^1(G)\cap\mathcal{C}(G)\right\}.$$

As 0 is the minimal eigenvalue of the Laplacian $-\left(\partial_j^2\right)_{N\times 1}$, and as the only harmonic functions on a finite metric graph are the constant ones, an equilibrium u_0 of (1) on a tree leads to

$$\lambda_0(u_0; c) = -f'(u_0)$$

for all positive coefficients c.

Theorem 3.2 *Let Π be a path. Then any stationary nonconstant solution $u \in \mathcal{C}_K^2(\Pi)$ of the problem*

$$\text{(14)}\qquad \begin{cases} u \in \mathcal{C}(\Pi \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(\Pi \times (0,\infty)), & \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j & \text{for } 1 \le j \le N, \\ (K): \quad \displaystyle\sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 & \text{for } 1 \le i \le n, \end{cases}$$

is unstable.

Proof. The assertion is true on an interval by Matano's result [16, 19]. Thus, we can assume that $N \ge 2$. Suppose that $u \in \mathcal{C}_K^2(\Pi)$ is a stable nonconstant stationary solution of (14) on Π. Distinguish the following three cases.

If $\partial_j u_j(x_j) = 0$ for some $x_j \in [0, \ell_j]$ with $\pi_j(x_j) \notin V_b$ and $k_j \cap V_b \neq \emptyset$, then u would be unstable by [9, Lemma 4.4] or Yanagida's Two Points Lemma [19], which is impossible.

If $\partial_j u_j(x_j) = 0$ for some $x_j \in [0, \ell_j]$ with $k_j \cap V_b = \emptyset$, then Π splits up at $\pi_j(x_j)$ into two paths having less than N edges and on which the restrictions of u would be unstable by recurrence, which is impossible.

Thus, we are lead to the final case

$$\partial_j u_j(x_j) = 0 \iff \pi_j(x_j) \in V_{\mathrm{b}} = \{v_1, v_{N+1}\}\,.$$

Write (K) in the form (11) and set $\psi_j = c_j \partial_j u_j$ on each edge, where it solves $\partial_j^2 \psi_j + f'(u_j)\psi_j = 0$. By (11), the ψ_j define a function

$$\psi \in \mathcal{C}^1(\Pi) \quad \text{with} \quad \psi(v_1) = \psi_1(0) = \psi_N(\ell_N) = \psi(v_{N+1}) = 0.$$

If $\partial_1 \psi_1(0) = 0$, then ψ_1 vanishes identically on k_1, and u has to be constant on k_1. Omitting this edge, the restriction of u to the remaining edges has to be unstable by recurrence, which is impossible. The same argument applies to $\partial_N \psi_N(\ell_N) = 0$. Thus, we can assume that

$$\text{(15)}\qquad \partial_1 \psi_1(0) \neq 0 \quad \text{and} \quad \partial_N \psi_N(\ell_N) \neq 0.$$

Denoting the numerator of $R\left(\psi; u; (c_j{}^{-1})\right)$ by $\mathcal{N}\left(\psi; u; (c_j{}^{-1})\right)$ we get

$$\begin{aligned} 0 &= \sum_{j=1}^{N} \frac{1}{c_j} \int_0^{\ell_j} \left(\partial_j^2 \psi_j + f'(u_j)\psi_j\right) \psi_j dx_j \\ &= -\sum_{j=1}^{N} \frac{1}{c_j} \int_0^{\ell_j} (\partial_j \psi_j)^2 - f'(u_j)\psi_j^2 \, dx_j + \sum_{j=1}^{N} c_j{}^{-1} \left[\partial_j \psi_j \ \psi_j\right]_0^{\ell_j} = -\mathcal{N}\left(\psi; u; \left(c_j{}^{-1}\right)\right), \end{aligned}$$

since (3) and $c_j f(u(v_i)) = -\partial_j \psi_j(v_i)$ imply

$$\sum_{j=1}^{N} c_j{}^{-1} \left[\partial_j \psi_j \ \psi_j\right]_0^{\ell_j} = -\sum_{i=1}^{n} f(u(v_i)) \sum_{j=1}^{N} d_{ij} \psi_j(v_i) = 0.$$

Thus,

$$R\left(\psi; u; \left(c_j{}^{-1}\right)\right) = 0.$$

We conclude that the edge Schrödinger operators $\partial_j^2 + f'(u_j)$ in $H^1(\Pi) \cap \mathcal{C}(\Pi)$ under the Kirchhoff condition

$$\sum_{j=1}^{N} d_{ij} c_j{}^{-1} \partial_j \varphi_j(v_i) = 0. \tag{16}$$

lead to a Rayleigh quotient satisfying

$$\lambda_0\left(u; \left(c_j{}^{-1}\right)\right) = \min_{\varphi \in H^1(\Pi) \cap \mathcal{C}(\Pi)} R\left(\varphi; u; \left(c_j{}^{-1}\right)\right) \le 0. \tag{17}$$

Clearly, ψ does not fulfill (16). But, if $\lambda_0 = 0$, then ψ is a minimizer of $R\left(\cdot; u; (c_j{}^{-1})\right)$ and fulfills (16), in particular the Neumann condition at v_1 and v_{N+1}, which is impossible by (15). Thus, we can assume that there is $\eta \in \mathcal{C}^2(\Pi)$ satisfying (16) such that

$$R\left(\eta; u; \left(c_j{}^{-1}\right)\right) = \lambda_0 < 0 \quad \text{and} \quad \eta > 0 \text{ in } \Pi. \tag{18}$$

This means that the zero solution is unstable for the edge operators $\partial_j^2 + f'(u_j)$ in $H^1(\Pi) \cap \mathcal{C}(\Pi)$ under (16). But all the norms

$$\|\varphi\|_{c, L^\infty(\Pi)} = \sum_{j=1}^{N} c_j \|\varphi_j\|_{L^\infty[0,\ell_j]}$$

are equivalent in $\mathcal{C}(\Pi)$, which implies that the zero solution is unstable for $\partial_j^2 + f'(u_j)$ in $H^1(\Pi) \cap \mathcal{C}(\Pi)$ under any dissipative Kirchhoff law too, in particular under (K). Using the same argument as in [19, Lemma 2.3] in order to show that negative value of the Rayleigh quotient lead to instability, we conclude that u is unstable. ∎

On a circuit ζ the derivatives of a nonconstant stationary solution $u \in \mathcal{C}_K^2(\zeta)$ have to vanish at least twice. By cutting ζ at one of these points, we are led to a path to which the foregoing results can be applied. Thus we can state the

Corollary 3.3 *Let ζ be a circuit and $u \in \mathcal{C}_K^2(\zeta)$ be a nonconstant stationary solution of the problem*

$$\begin{cases} u \in \mathcal{C}(\zeta \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(\zeta \times (0,\infty)), \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j \qquad \text{for } 1 \le j \le N, \\ (K): \quad \displaystyle\sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 \qquad \text{for } 1 \le i \le n. \end{cases} \tag{19}$$

Then u is unstable.

Proof. Suppose that u is stable on ζ. By compactness, there is $\pi_j(x_j) = p \in \zeta$ such that

$$\partial_j u_j(x_j) = 0.$$

Cut the circuit at p into two boundary vertices $\tilde{v}_1$ and $\tilde{v}_{N+1}$ and get a path $\tilde{\Pi}$ whose quantities stemming from ζ will be denoted by a tilde. In particular, $\tilde{u} \in \mathcal{C}_K^2(\tilde{\Pi})$ defines a nonconstant stationary solution of (19) on $\tilde{\Pi}$ with the distinguished property

$$\tilde{u} \in \mathcal{F} := \left\{ w \in \mathcal{C}(\tilde{\Pi}) \mid \tilde{w}\,(\tilde{v}_1) = \tilde{w}\,(\tilde{v}_{N+1}) \right\}.$$

Clearly, $\mathcal{C}(\zeta)$ can be identified with $\mathcal{F}$. By Theorem 3.2, $\tilde{u}$ is unstable on $\tilde{\Pi}$. Thus, there is some $\tilde{\varepsilon}_0 > 0$ such that for each $\delta > 0$ there is an initial data $\tilde{w}_0 \in \mathcal{C}(\tilde{\Pi})$ with $\|\tilde{w}_0 - \tilde{u}\|_{\infty,\tilde{\Pi}} < \delta$ and

$$\|\tilde{w}(\cdot, t_0) - \tilde{u}\|_{\infty,\tilde{\Pi}} \geq \tilde{\varepsilon}_0$$

for some $t_0 > 0$ where $\tilde{w}$ denotes the solution of (19) with initial condition $\tilde{w}_0$ on $\tilde{\Pi}$.

For each $0 < \varepsilon < \tilde{\varepsilon}_0$ there is some $\delta > 0$ such that the solution $z \in \mathcal{C}(\zeta \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(\zeta \times (0,\infty))$ of (19) with initial condition $\|z_0 - u\|_{\infty,\zeta} < \delta$ satisfies

$$\forall t > 0 \ : \|z(\cdot,t) - u\|_{\infty,\zeta} < \epsilon.$$

But for each $t > 0$, $\widetilde{z(\cdot,t)}$ belongs to $\mathcal{F}$, and $\tilde{z}$ is the solution of (19) on $\tilde{\Gamma}$ with initial condition $\tilde{z}_0 = \widetilde{z(\cdot,0)}$. This leads to the contradiction

$$\forall t > 0 \ : \|\tilde{z}(\cdot,t) - \tilde{u}\|_{\infty,\tilde{\Pi}} < \varepsilon < \tilde{\varepsilon}_0.$$

■

4 Stability surgery

In this section we establish some basic facts and results about stationary stable or unstable classical solutions of the more general reaction - diffusion - problem

$$\text{(5)} \qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty)), \\ \partial_t u_j = \partial_j \left(a_j(x_j)\, \partial_j u_j\right) + f_j(u_j) \quad \text{on } k_j \quad \text{for } 1 \leq j \leq N, \\ (K) \quad \displaystyle\sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 \qquad\qquad \text{for} \quad 1 \leq i \leq n, \end{cases}$$

under the hypotheses (6). In particular we compare the stability on a metric graph Γ with the one on a modified graph $\tilde{\Gamma}$ obtained by cutting an edge at a zero of the derivative of a stationary solution. In [6, 7], a general Osgood type L^∞–estimate for nonlinear parabolic problems on metric graphs has been shown. For the reader's convenience and for the proofs of Theorems 4.2 and 6.1, we cite it here in the setting of the special case (5).

Theorem 4.1 *[6, 7] Suppose $T \geq 0$ and let $u \in \mathcal{C}(G \times [0,T] \cap \mathcal{C}_K^{2,1}(G \times (0,T])$ be a solution of Problem* (5) *subject to the Osgood condition*

$$\exists b_1, b_2 \geq 0 \, \forall j \in \{1, ..., N\} \, \forall z \in \mathbb{R} : \, z f_j(z) \leq b_1 z^2 + b_2.$$

Then, the following estimate holds:

$$\|u\|_{\infty, G\times[0,T]} \leq \inf_{\lambda > b_1} \left(e^{\lambda T} \max \left\{ \|u(\cdot, 0)\|_{\infty, G} \, , \sqrt{\frac{b_2}{\lambda - b_1}} \right\} \right).$$

The basic reduction tool for establishing instability is the following surgery technique.

Theorem 4.2 *Suppose that $u \in \mathcal{C}_K^2(G)$ is a stationary solution of* (5) *such that $\partial_j u_j(x_j)$ vanishes at $p = \pi_j(x_j)$ on some edge k_j. Cut the graph Γ at p and get a new possibly non connected graph $\tilde{\Gamma}$ on which u defines a stationary solution $\tilde{u}$ of* (5) *on $\tilde{\Gamma}$ that belongs to $\mathcal{C}_{\tilde{K}}^2(\tilde{G})$. Here the Kirchhoff conditions $\tilde{K}$ extend the old one (K) by adding the Neumann condition at the new vertex or at the two new vertices. In the first case note that $p = v_i$ leads to $\tilde{d}_{ij} = 0$.*

Then $\tilde{u}$ is stable on $\tilde{\Gamma}$ if and only if u is stable on Γ.

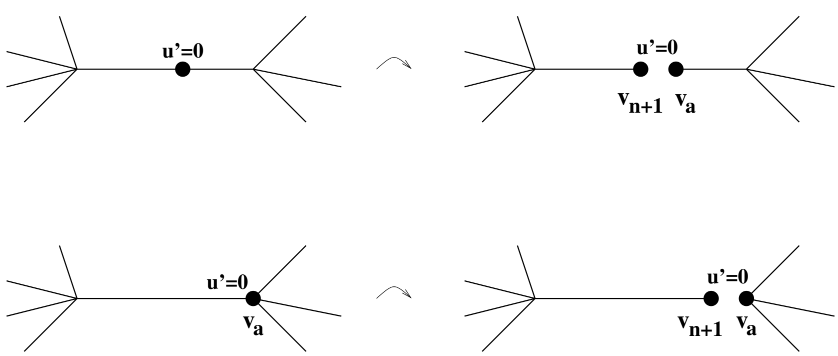

Figure 1: Cutting at p with $\partial u(p) = 0$.

Proof. First, we consider the case where there is some constant $b_1 \geq 0$ such that

(20)
$$\sup_j f_j' \leq b_1.$$

The cutting of the edge k_j defines one or two new vertices in $\tilde{\Gamma}$ that will be denoted by $v_{n+1} \in V_{\mathrm{b}}(\tilde{\Gamma}) \backslash V(\Gamma)$ and $v_a \in \left(V_{\mathrm{b}}(\tilde{\Gamma}) \backslash V(\Gamma) \right) \, \dot{\cup} \, \left(V(\tilde{\Gamma}) \cap V(\Gamma) \right)$. Note that $v_a \in \left(V_{\mathrm{b}}(\tilde{\Gamma}) \backslash V(\Gamma) \right)$ corresponds to $x_j \in (0, \ell_j)$ and to a new boundary vertex, while $v_a \in \left(V(\tilde{\Gamma}) \cap V(\Gamma) \right)$ means that $p \in V(\Gamma)$. Set

$$F = \left\{ \varphi \in \mathcal{C}_K^1(G) \mid \partial_j \varphi_j(p) = 0 \right\},$$

$$\tilde{C} = \left\{ \psi \in \mathcal{C}(\tilde{G}) \mid \psi(v_{n+1}) = \psi(v_a) \right\},$$

and for $\varphi \in \mathcal{C}(G)$ and identifying $G \backslash \{v_{n+1}, v_a\}$ with $\tilde{G} \backslash \{v_{n+1}, v_a\}$,

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in G \backslash \{v_{n+1}, v_a\}, \\ \varphi(p) & \text{if } x \in \{v_{n+1}, v_a\}. \end{cases}$$

Then

$$\iota = (\varphi \mapsto \tilde{\varphi})$$

is an isometric isomorphism with respect to both L^∞–norms between $\mathcal{C}(G)$ and $\tilde{C} = \iota(\mathcal{C}(G))$. Moreover, $\overline{\iota(F)} = \iota(\mathcal{C}(G))$, since $\overline{F} = \mathcal{C}(G)$ by the Stone–Weierstrass–Theorem. This shows immediately that if u is unstable on Γ, then $\tilde{u}$ is unstable on $\tilde{\Gamma}$.

Now, suppose that $\tilde{u}$ is unstable on $\tilde{G}$. Thus, there is some $\tilde{\varepsilon}_0 > 0$ such that for each $\delta > 0$ there is an initial data $\tilde{w}_0 \in \mathcal{C}(\tilde{G})$ with $\|\tilde{w}_0 - \tilde{u}\|_{\infty,\tilde{G}} < \delta$ and $\|\tilde{w}(\cdot, t_0) - \tilde{u}\|_{\infty,\tilde{G}} \geq \tilde{\varepsilon}_0$ for some $t_0 > 0$ where $\tilde{w}$ denotes the solution of (5) with initial condition $\tilde{w}_0$ on $\tilde{G}$. Set $\varepsilon_0 = \frac{1}{2}\tilde{\varepsilon}_0$, and choose δ such that

$$0 < \delta \leq \frac{\varepsilon_0}{2} e^{-b_1 t_0}.$$

If $\tilde{w}_0 \in \overline{\iota(F)} = \tilde{C}$, then, by uniqueness, $w := \iota^{-1}(\tilde{w})$ is the solution of (5) with initial condition w_0 on G that satisfies $\|w_0 - u\|_{\infty,G} < \delta$ and $\|w(\cdot, t_0) - u\|_{\infty,G} \geq \tilde{\varepsilon}_0 > \varepsilon_0$.

Next, suppose that $\tilde{w}_0 \notin \overline{\iota(F)}$. As $\tilde{u}$ belongs to $\iota(F)$, and as $\|\tilde{w}_0 - \tilde{u}\|_{\infty,\tilde{G}} < \delta$, we can nevertheless choose $\tilde{z}_0 \in \mathcal{C}(\tilde{G}) \cap \overline{\iota(F)}$ with $|\text{supp}(\tilde{w}_0 - \tilde{z}_0)|$ sufficiently small such that $\|\tilde{z}_0 - \tilde{w}_0\|_{\infty,\tilde{G}} < \delta$. Let $\tilde{z}$ denote the solution of (5) on $\tilde{G}$ with initial condition $\tilde{z}_0$ and z be the solution of (5) on G with initial condition $z_0 = \iota^{-1}(\tilde{z}_0)$. Again, by uniqueness, $z = \iota^{-1}(\tilde{z})$.

Now we claim

$$(*) \qquad \|\tilde{z}(\cdot, t_0) - \tilde{w}(\cdot, t_0)\|_{\infty,\tilde{G}} = \|z(\cdot, t_0) - w(\cdot, t_0)\|_{\infty,G} < \frac{1}{4}\tilde{\varepsilon}_0.$$

Proof. $(*)$ The difference $d = z - w \in \mathcal{C}(G \times [0, t_0]) \cap \mathcal{C}_K^{2,1}(G \times (0, t_0])$ satisfies on each edge k_j the equation

$$\partial_t d_j = a_j \, \partial_j^2 d_j + f_j(z_j) - f_j(w_j) = a_j \, \partial_j^2 d_j + f_j' \left(\lambda z_j + (1 - \lambda) w_j\right) d_j$$

with a function $\lambda : \mathbb{R} \to [0, 1]$. Since the coefficient of the linear term d_j is bounded from above by b_1, the Osgood type a priori estimate 4.1 with $b_2 = 0$ yields

$$\|d(\cdot, t_0)\|_{\infty,\tilde{T}} \leq e^{b_1 t_0} \max_{\tilde{T}} |d(\cdot, 0)| = e^{b_1 t_0} \|z_0 - w_0\|_{\infty,G} < \frac{1}{4}\tilde{\varepsilon}_0.$$

This shows the claim $(*)$. Finally, we conclude

$$\begin{aligned} \|z(\cdot, t_0) - u\|_{\infty,G} &= \|\tilde{z}(\cdot, t_0) - \tilde{u}\|_{\infty,\tilde{G}} \\ &\geq \|\tilde{u} - \tilde{w}(\cdot, t_0)\|_{\infty,\tilde{G}} - \|\tilde{w}(\cdot, t_0) - \tilde{z}(\cdot, t_0)\|_{\infty,\tilde{G}} \geq \tilde{\varepsilon}_0 - \frac{1}{4}\tilde{\varepsilon}_0 > \varepsilon_0. \end{aligned}$$

Thus, u is unstable on G.

Finally, in order to get rid of (20), i.e. for arbitrary nonlinearities $f_j \in \mathcal{C}^1(\mathbb{R})$, we modify them for the given stationary nonconstant solution $u \in \mathcal{C}^2_K(G)$ of (5) outside $[-2M, 2M]$ with $M := \|u\|_{\infty,G}$ as follows

$$\widetilde{f_j}(z) = \begin{cases} f_j(z) & \text{for } z \in [-2M, 2M], \\ f_j(2M) + (z - 2M) f_j'(2M) & \text{for } z \geq 2M, \\ f_j(-2M) + (z + 2M) f_j'(-2M) & \text{for } z \leq -2M. \end{cases} \tag{21}$$

Thus, u would be a stable nonconstant solution of (28) with the nonlinearities $\widetilde{f_j}$ iff $\tilde{u}$ would be a stable by the results shown first. This concludes the proof of the theorem. ■

As a first application of the surgery theorem 4.2 we generalize Yanagida's Two Points Lemma [9, 19] known in the constant coefficients case under consistent Kirchhoff conditions to the more general problem (5).

Lemma 4.3 *Suppose that $u \in \mathcal{C}^2_K(G)$ is a stationary solution of (5) that is non-constant on some edge k_j. If there are two points on k_j with $0 \leq z_1 < z_2 \leq \ell_j$ such that $\partial_j u_j(z_1) = \partial_j u_j(z_2) = 0$, then u is unstable. If, in addition, the nonlinearity f_j is an odd function, then the same conclusion holds, if there are two points on k_j with $0 \leq z_1 < z_2 \leq \ell_j$ such that $u_j(z_1)\partial_j u_j(z_1) = 0 = u_j(z_2)\partial_j u_j(z_2)$.*

Proof. In the first case, we cut the edge k_j twice and obtain instability on the resulting sole edge as a stationary solution of a Neumann problem by a result by Matano [16]. Then Theorem 4.2 permits to conclude. In the second case, the assertion can be shown exactly in the same way as in the consistent and constant coefficient case shown in [9, Lemma 4.5]. ■

As already pointed out in [9], the hereditary properties of the stability notion with respect to subgraphs are very bad. However, the *edge doubling* is a simple extension technique that permits to conclude for stability from a graph containing the original one. Choose any edge k_j incident to v_i and v_h in Γ and copy all quantities and functions associated to k_j on a new edge k_{N+1} incident to v_i and v_h as well in the new graph $\tilde{\Gamma}$ with $E(\tilde{\Gamma}) = E(\Gamma) \cup \{k_{N+1}\}$ and $V(\tilde{\Gamma}) = V(\Gamma)$ except for the conductivities that are defined by

$$\tilde{c}_{ms} = \begin{cases} \frac{1}{2} c_{is} & \text{if } m \in \{i, h\} \text{ and } s \in \{j, N+1\}, \\ c_{mj} & \text{otherwise,} \end{cases}$$

and lead to the Kirchhoff conditions denoted by $(\tilde{K})$. Then we can state the following

Lemma 4.4 *Suppose that $u \in \mathcal{C}^2_K(G)$ is a stationary solution of (5). Let $\tilde{\Gamma}$ be the graph resulting from an edge doubling described above and $\tilde{u} \in \mathcal{C}^2_{\tilde{K}}(\tilde{G})$ be the corresponding extension of u to $\tilde{G}$ by setting $\tilde{u}_{N+1} = \tilde{u}_j = u_j$. Then $\tilde{u}$ is stable on $\tilde{\Gamma}$ if and only if u is stable on Γ.*

Proof. Each $w \in \mathcal{C}^1_K(G)$ defines uniquely $\tilde{w} \in \mathcal{C}^1_{\tilde{K}}(\tilde{G})$. This leads to the continuous embedding

$$\mathcal{C}^1_K(G) \hookrightarrow \mathcal{C}^1_{\tilde{K}}(\tilde{G}).$$

Thus, identifying $\mathcal{C}(G)$ with $\mathcal{F} := \left\{\tilde{w} \in \mathcal{C}(\tilde{G}) \mid \tilde{w}_{N+1} = \tilde{w}_j\right\}$, the stability of $\tilde{u}$ in $\mathcal{C}(\tilde{G})$ implies its stability in the closed subspace $\mathcal{F}$ and thereby, the stability of u in $\mathcal{C}(G)$.

Conversely, suppose that $\tilde{u}$ is unstable on $\tilde{\Gamma}$, but that u is stable in $\mathcal{C}(G)$. In fact, we can proceed as in the proof of Corollary 3.3. Thus, there is some $\tilde{\varepsilon}_0 > 0$ such that for each $\delta > 0$ there is an initial data $\tilde{w}_0 \in \mathcal{C}(\tilde{G})$ with $\|\tilde{w}_0 - \tilde{u}\|_{\infty,\tilde{G}} < \delta$ and

$$\|\tilde{w}(\cdot, t_0) - \tilde{u}\|_{\infty,\tilde{G}} \geq \tilde{\varepsilon}_0$$

for some $t_0 > 0$ where $\tilde{w}$ denotes the solution of (5) with initial condition $\tilde{w}_0$ on $\tilde{G}$. For each $0 < \varepsilon < \tilde{\varepsilon}_0$ there is some $\delta > 0$ such that the solution z of (5) with initial condition $\|z_0 - u\|_{\infty,G} < \delta$ satisfies

$$\forall t > 0 \ : \|z(\cdot, t) - u\|_{\infty,G} < \varepsilon.$$

But for each $t > 0$, $z(\cdot, t)$ belongs to $\mathcal{F}$, and $\tilde{z}$ is the solution of (5) on $\tilde{\Gamma}$ with initial condition $\tilde{z}_0 = \widetilde{z(\cdot, 0)}$. By construction, this leads to the contradiction

$$\forall t > 0 \ : \|\tilde{z}(\cdot, t) - \tilde{u}\|_{\infty,\tilde{G}} < \varepsilon < \tilde{\varepsilon}_0.$$

■

A non trivial application of the edge doubling is given by the following example.

Example 4.5 Suppose that $N = 2 = n$, and that Γ consists in a loop ζ of length ℓ with ramification node $\{v_1\} = V(\zeta)$ and in an edge k_3 joining v_1 and the boundary vertex v_2.

Suppose that $u \in \mathcal{C}^2_K(G)$ is a stable nonconstant stationary solution of (5). If u vanishes on the whole ζ or on the whole k_3, then the problem is reduced to an interval under Neumann boundary conditions that admits only constant stable stationary solutions.

If v_1 is an extremum of u, then the problem on Γ splits into two interval problems. Thus, by Lemma 4.3 and compactness, the derivative $\partial_j u_j$ vanishes exactly once outside the ramification node v_1 on ζ in Γ, say at $p \in \zeta$, since it cannot vanish in the interior of the edge k_3. The point p cuts ζ into two edges, say k_1 and k_2, on which u_1 and u_2 respectively are strictly monotone by Lemma 4.3. Moreover, u_1 and u_2 fulfill

$$\partial_1 u_1(p) = \partial_2 u_2(p) = 0 \quad \text{and} \quad u_1(p) = u_2(p).$$

By unique solvability of the corresponding Cauchy problems, u_1 and u_2 coincide for $0 \leq x_j \leq \min\{\ell_1, \ell_2\}$, where we have chosen $d_{11} = d_{12} = 1$. But, by continuity at v_1

$$u_1(v_1) = u_2(v_1).$$

Thus, if k_1 and k_2 were of different length, Rolle's Theorem and Lemma 4.3 would lead to instability of u. We conclude that k_1 and k_2 are of the same length

$$\ell_1 = \ell_2 = \frac{\ell}{2}$$

and that

$$\partial_1 u_1 (v_1) = \partial_2 u_2 (v_1) .$$

Choose $d_{13} = -1$ and denote the boundary vertex by v_2. Note that the conductivities on the loop at v_1 are identical, and the original Kirchhoff condition at v_1 reads

$$2c_{11}\partial_1 u_1 (v_1) - c_{13}\partial_3 u_3 (v_1) = 0. \tag{22}$$

Cutting ζ at p and omitting k_2 leads to a path Π of two edges given by k_1 and k_3. It turns out that Γ is just the graph $\tilde{\Pi}$ with k_1 doubled with k_2. Moreover, the restriction of u to Π, say $w \in \mathcal{C}^2_{K_c}(\Pi)$ and $u = \tilde{w}$, constitutes a stable stationary solution on Π and belongs to $\mathcal{C}^2_{K_c}(\Pi)$, where K_c stands for the validity of the inconsistent Kirchhoff law (22) at v_1 and of the Neumann boundary condition at p and v_1. Thus, by Theorem 3.2, w has to be constant, which is absurd.

5 Yanagida graphs

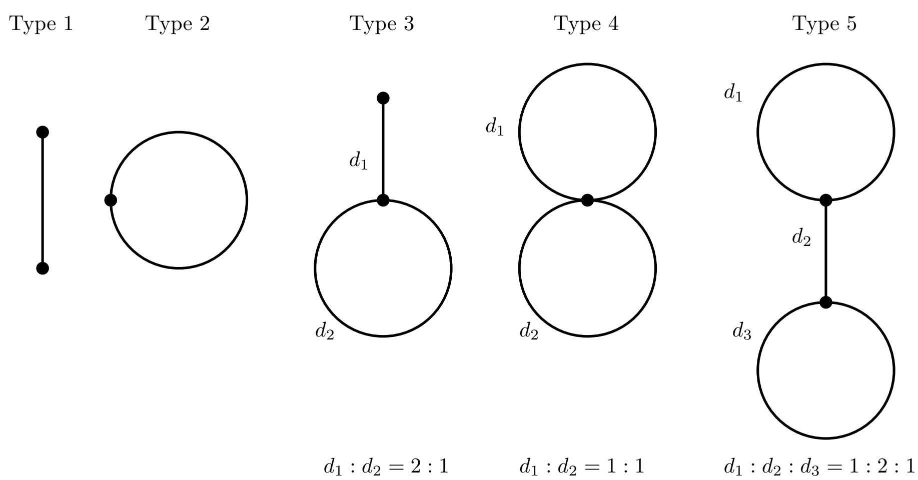

Figure 2: Yanagida's exceptional graphs.

Recall the following result by E. Yanagida from 2001.

Theorem 5.1 *([19]) If Γ is one of the five graphs in Figure 5, then the reaction–diffusion problem*

(23)
$$\begin{cases} u \in \mathcal{C}^2(G \times [0,\infty)) \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) & 1 \le j \le N, \\ \sum_{j=1}^{N} d_{ij}\ d_j\ \partial_j u_j(v_i,\cdot) = 0 & \textit{for} \quad 1 \le i \le n, \end{cases}$$

has no stable stationary nonconstant solution on G. This includes paths and circuits of arbitrary lengths of the same width $d_j > 0$ on all their edges.

An important ingredient of Yanagida's proof was the self–adjoint character of the associated eigenvalue problem of the linearized problem. Note that Example 4.5 restricted to Problem (23), but without restriction on the d_j, yields another exceptional graph on which no stable stationary nonconstant solution can exist. In this argument only Theorem 3.2, Example 4.5 and the classical Two Points Lemma [9, 19] interfere. As for paths and circuits in Section 3, we can add the following exceptional graphs.

Theorem 5.2 *The reaction–diffusion problem*

(24)
$$\begin{cases} u \in \mathcal{C}^{2,1}(G \times [0,\infty)) \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) & 1 \le j \le N, \\ \sum_{j=1}^{N} d_{ij}\ \partial_j u_j(v_i,\cdot) = 0 & \textit{for} \quad 1 \le i \le n, \end{cases}$$

does not admit any stable stationary nonconstant solution, if the graph Γ is a generalized Yanagida graph with arbitrary edge lengths of Type 3, 4, or 5, i.e.

- *(3) either Γ contains exactly one boundary vertex and exactly one essential ramification node, the latter being of valency* 3,
- *(4) or $V_{\mathrm{b}} = \emptyset$ and $V_{\mathrm{ess}} = \{v_1\}$ and $\gamma_1 = 4$,*
- *(5) or $V_{\mathrm{b}} = \emptyset$ and Γ contains exactly two essential ramification nodes, the latter being both of valency* 3 *and being joined by an edge or a viaduct.*

Proof. Throughout, let us suppose that $u \in \mathcal{C}_K^2(G)$ is a stable nonconstant stationary solution of (24). If u is constant on some edge k_j with $d_{ij} \neq 0 \neq d_{hj}$, then we can identify v_i and v_h and omit k_j. Thus, we can assume that u is nonconstant on each edge of Γ.

Case (3)

By hypothesis, Γ is unicyclic. If some $\partial_j u_j$ vanishes at v_1, the graph reduces to a sole path or splits into a circuit and a path. Working with $\mathcal{C}^1$–solutions, in particular at nodes of valency 2, we can assume w.l.o.g. that the circuit ζ is a loop of length ℓ and that the path joining v_1 and the boundary vertex v_2 is just an edge, say k_3.

Exactly as in Example 4.5 and using the notations given there, cut ζ at $p \in \zeta\backslash\{v_1\}$ into two edges k_1 and k_2 with

$$\partial_1 u_1(p) = \partial_2 u_2(p) = 0, \quad u_1(p) = u_2(p), \quad u_1(v_1) = u_2(v_1), \quad \partial_1 u_1(v_1) = \partial_2 u_2(v_1)$$

and $\ell_1 = \ell_2 = \frac{\ell}{2}$. Then the Kirchhoff law (22) at v_1 reads

$$2\partial_1 u_1(v_1) - \partial_3 u_3(v_1) = 0. \tag{25}$$

Cutting ζ at p and omitting k_2 leads to a path Π of two edges given by k_1 and k_3. Then the restriction $\tilde{u}$ of u to Π is a stationary solution of the differential equations on the edges of Π and belonging to $\mathcal{C}^2_{K_c}(\Pi)$, where K_c stands for the validity of the inconsistent Kirchhoff law (25) at v_1 and of the Neumann boundary condition at p and v_1. Denoting by Σ the star formed by k_1, k_2 and k_3, respective identifying leads to the embeddings

$$\iota : \mathcal{C}^1_{K_c}(\Pi) \hookrightarrow \left\{w \in \mathcal{C}^1_K(\Sigma) \mid w_1(p) = w_2(p)\right\} \hookrightarrow \mathcal{C}^1_K(G), \tag{26}$$

since each $\varphi \in \mathcal{C}^1_{K_c}(\Pi)$ extends uniquely to $\mathcal{C}^1_K(\Sigma)$ and $\mathcal{C}^1_K(\Gamma)$ by even extension with respect to $\pi_1(0) = p = \pi_2(0)$ onto the remaining edge k_2. This leads to stability of the restriction $\tilde{u}$ of u to Π as follows. First, note that if $w \in \mathcal{C}^{2,1}_{K_c}(\Pi \times [0,\infty))$ is the solution on Π with initial condition $w_0 \in \mathcal{C}(\Pi)$, and if $\bar{w} \in \mathcal{C}^{2,1}_K(\Gamma \times [0,\infty))$ is the solution on Γ with initial condition $\bar{w}_0 \in \mathcal{C}(G)$ such that $\bar{w}_0 = \iota(w_0)$, then these solutions coincide by uniqueness of the corresponding flow, i.e.

$$w = \bar{w}\big|_{\Pi} \quad \text{and} \quad \bar{w} = \iota(w).$$

Secondly, for $\varepsilon > 0$ and $\delta > 0$ such that $\|u - z_0\|_{\infty,\Gamma} < \delta$ implies $\|u - z(\cdot,t)\|_{\infty,\Gamma} < \varepsilon$ for all $t > 0$, we conclude for an initial condition $w_0 \in \mathcal{C}(\Pi)$ with $\|\tilde{u} - w_0\|_{\infty,\Pi} < \delta$ that

$$\|\tilde{u} - w(\cdot,t)\|_{\infty,\Pi} = \|u - \bar{w}(\cdot,t)\|_{\infty,\Gamma} < \varepsilon.$$

Thus, the restriction of u to Π is stable there, which is impossible by Theorem 3.2. This achieves the proof in Case (3).

Case (4)

By Lemma 4.3, on each loop or circuit in Γ the derivative $\partial_j u_j$ vanishes exactly once outside the ramification node v_1. As in Case (3), this allows the reduction to two pairs of edges of identical lengths that, in turn, lead to a restriction to a path of two edges even under a consistent Kirchhoff condition at the ramification node. As above, the two evolution flows are compatible with the extension–restriction procedure and permit to conclude with Lemma 3.1, or even with Matano's classical result on an interval.

Case (5)

Again, dealing with $\mathcal{C}^1$–solutions, we can assume w.l.o.g. that the two circuits ζ_1 and ζ_2 are loops of length ℓ_1 and ℓ_2, respectively, and that the viaduct between the nodes v_1 and v_2 of Γ is just an edge denoted by k_3. Assume that $v_i \in V(\zeta_i)$.

If $\partial_j u_j$ vanishes at some point on k_3, then we are led to two graphs of Case 3, or one of this type and a loop. Thus, it remains to consider the case where the derivative $\partial_j u_j$ vanishes exactly once on each ζ_i outside v_i. As in Case (3), this allows the reduction to two pairs of edges of identical lengths that, in turn, lead to a restriction to a path of 3 edges under an inconsistent Kirchhoff condition at v_1 and v_2 as in (25). As above, the two evolution flows are compatible and permit to conclude with Theorem 3.2. ■

Figure 5 presents some graphs fulfilling the conditions of Theorem 5.2, that are not in Yanagida's list. Note that the embedding (26) yields compatibility of solutions for the extension–reduction–procedure related to appropriate subgraphs. However, in general graphs, such embeddings compatible with the different involved flows do not seem to be available. They would strongly simplify subgraph reduction techniques in showing instability. Therefore, the surgery techniques from Section 4 will be applied for the general case, rather than the ones above.

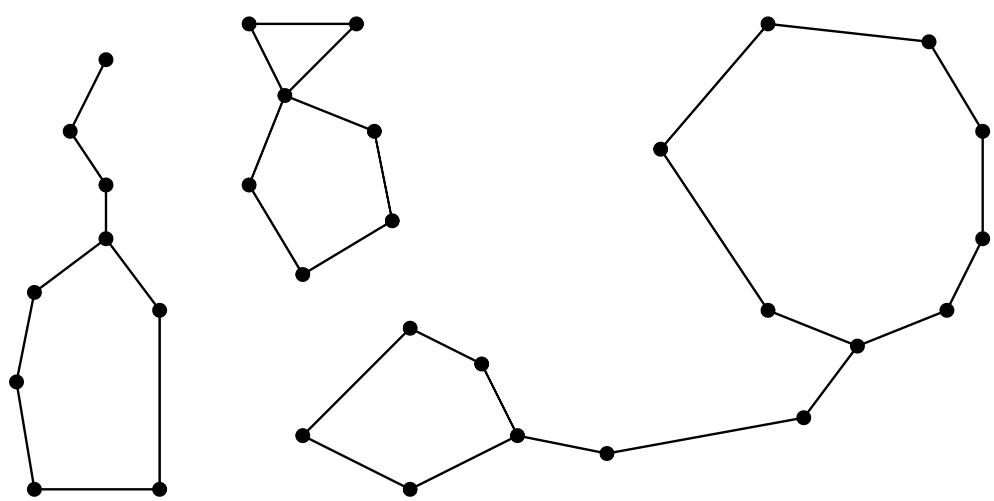

Figure 3: More "exceptional" graphs by Theorem 5.2.

6 Proof of the main result

In this section we shall use the notation

$$\Delta_{ij} = d_{ij}\partial_j u_j(v_i) \tag{27}$$

for the outer normal derivative of a function u on the edge k_j at the vertex $v_i \in k_j$. First, we consider Problem (1) on a tree T, i.e.

$$(28) \qquad \begin{cases} u \in \mathcal{C}(T \times [0,\infty)) \cap \mathcal{C}^{2,1}_K(T \times (0,\infty)), & \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j & \text{for } 1 \le j \le N, \\ (K): \quad \sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, \cdot) = 0 & \text{for } 1 \le i \le n, \end{cases}$$

under Condition (4). Theorem 3.2 settles the case of a tree without essential ramification nodes and forms part of the following

Theorem 6.1 *On any finite metric tree T there is no stable stationary nonconstant solution of Problem* (28)*.*

Proof. Let $u \in \mathcal{C}^2_K(T)$ be a stable stationary nonconstant solution of (28). For a sole edge or a path this is impossible due to Theorem 3.2. Thus we can suppose that $N > 2$ and that $\#V_{\text{ess}} \ge 1$, and reason by recurrence on N. Modifying f for the given solution $u \in \mathcal{C}^2_K(T)$ of (28) outside $[-2M, 2M]$ with $M := \|u\|_{\infty,T}$ as in (21) for the proof of Theorem 4.2, we can assume w.l.o.g. that

$$(29) \qquad f' \le b_1$$

with some constant $b_1 \ge 0$. Moreover, recall that under a dissipative Kirchhoff condition, i.e. all $c_{ij} > 0$, as the third condition in (28), differentiable functions at an extremum in a vertex behave like at an interior point, and all incident derivatives have to vanish there.

If $\partial_j u_j(x_j) = 0$ for some $x_j \in [0, \ell_j]$ with $\pi_j(x_j) \notin V_b$ and $k_j \cap V_b \neq \emptyset$, then u is unstable by Lemma 4.3.

If $\partial_j u_j(x_j) = 0$ for some $x_j \in [0, \ell_j]$ with $k_j \cap V_b = \emptyset$, then T splits up at $\pi_j(x_j)$ into two trees having less than N edges and on which the restrictions of u are stable. This is impossible by recurrence.

Thus, we conclude that $\partial_j u_j(x_j) \neq 0$ except at the boundary vertices V_{b} at which clearly $\partial_j u_j(v_i) = 0$. It follows that on each edge k_j, the function u_j is strictly monotone.

Choose any essential ramification node v_i. Then there are at least two incident edges, say k_1 and k_2, satisfying

$$\Delta_{i1} \Delta_{i2} = d_{i1} \partial_1 u_1(v_i)\, d_{i2} \partial_2 u_2(v_i) > 0.$$

Remove k_1 from T and get two disjoint subtrees of T. Let $\tilde{T}$ denote the one containing v_i and set

$$\tilde{c}_{mj} = \begin{cases} c_{i2} + c_{i1} \frac{\Delta_{i1}}{\Delta_{i2}} & \text{if } m = i \text{ and } j = 2, \\ c_{mj} & \text{otherwise.} \end{cases}$$

Then the restriction of u to $\tilde{T}$ belongs to $\mathcal{C}^2_{\tilde{K}}(\tilde{T})$ with the dissipative Kirchhoff law

$$(\widetilde{K}) \qquad \sum_j d_{ij}\tilde{c}_{ij}\partial_j u_j(v_i) = 0 \quad (1 \le i \le n(\widetilde{T}))$$

and constitutes a stationary nonconstant solution of (28) on $\tilde{T}$ that has to be unstable by recurrence. Thus, there is some $\varepsilon_0 > 0$ such that for each $\delta > 0$ there is an initial data $\tilde{w}_0 \in \mathcal{C}(\tilde{T})$ with $\|\tilde{w}_0 - u\|_{\infty,\tilde{T}} < \delta$ and $\|\tilde{w}(\cdot,t_0) - u\|_{\infty,\tilde{T}} \ge \varepsilon_0$ for some $t_0 > 0$ where $\tilde{w}$ denotes the solution of (28) with initial condition $\tilde{w}_0$ on $\tilde{T}$.

Next, we modify and extend $\tilde{w}_0$ to a function w_0 on T as follows. Choose δ_0 sufficiently small such that for all $0 < \delta \le \delta_0 \le \frac{\varepsilon_0}{4}e^{-b_1 t_0}$, there exists $w_0 \in \mathcal{C}(\tilde{T})$ that coincides with u in a small neighborhood of v_i and fulfills the constraints

$$\|w_0 - u\|_{\infty,\tilde{T}} < \delta, \quad \|w_0 - \tilde{w}_0\|_{\infty,\tilde{T}} < \frac{\varepsilon_0}{2}e^{-b_1 t_0}.$$

Then define w_0 outside $\tilde{T}$ on T simply by u. Evidently,

$$\|w_0 - u\|_{\infty,T} = \|w_0 - u\|_{\infty,\tilde{T}} < \delta.$$

Moreover, for the solution w of (28) with the initial data w_0 on T we claim

$$(*) \qquad \|w(\cdot,t_0) - \tilde{w}(\cdot,t_0)\|_{\infty,\tilde{T}} < \frac{\varepsilon_0}{2}.$$

Proof. $(*)$ The difference $z = w - \tilde{w} \in \mathcal{C}(\tilde{T} \times [0,t_0]) \cap \mathcal{C}^{2,1}_{\tilde{K}}(\tilde{T} \times (0,t_0])$ satisfies on each edge k_j the equation

$$\partial_t z_j = \partial_j^2 z_j + f_j(w_j) - f_j(\tilde{w}_j) = \partial_j^2 z_j + f_j'\left(\lambda w_j + (1-\lambda)\tilde{w}_j\right) z_j$$

with a function $\lambda : \mathbb{R} \to [0,1]$. Since the coefficient of the linear term z_j is bounded from above by b_1, the Osgood type a priori estimate 4.1 with $b_2 = 0$ yields

$$\|z(\cdot,t_0)\|_{\infty,\tilde{T}} \le e^{b_1 t_0} \max_{\tilde{T}} |z(\cdot,0)| = e^{b_1 t_0}\|w_0 - \tilde{w}_0\|_{\infty,\tilde{T}} < \frac{\varepsilon_0}{2}.$$

This shows the claim $(*)$.

By stability of u on T, there is a $\delta \in (0,\delta_0]$ such that the initial data w_0 from above with $\|w_0 - u\|_{\infty,\tilde{T}} = \|w_0 - u\|_{\infty,T} < \delta$ leads to the solution satisfying

$$\|w(\cdot,t) - u\|_{\infty,T} < \frac{\varepsilon_0}{2}$$

for all $t > 0$. On $\tilde{T}$ evaluated at $t_0 > 0$, this leads to

$$\begin{aligned}\varepsilon_0 &\le \|\tilde{w}(\cdot,t_0) - u\|_{\infty,\tilde{T}} \le \|\tilde{w}(\cdot,t_0) - w(\cdot,t_0)\|_{\infty,\tilde{T}} + \|w(\cdot,t_0) - u\|_{\infty,\tilde{T}} \\ &\le \|\tilde{w}(\cdot,t_0) - w(\cdot,t_0)\|_{\infty,\tilde{T}} + \|w(\cdot,t_0) - u\|_{\infty,T} < \varepsilon_0,\end{aligned}$$

which is absurd. ∎

In order to achieve the proof of the general case we need a technical combinatorial lemma for graphs with circuits.

Lemma 6.2 *Let Γ be a finite graph that contains circuits of lengths at least 2. Let the set of real numbers $\{\Delta_{ij} \mid 1 \le i \le n, 1 \le j \le N\}$ satisfy the following properties:*

(a) $\forall v_i \in V_\mathrm{r}\ \forall j \in \{1, \ldots, N\} : (d_{ij} \neq 0 \iff \Delta_{ij} \neq 0)$.

(b) $\forall v_i \in V_\mathrm{r} : \sum_{j=1}^{N} \Delta_{ij} = 0$.

(c) $\forall j \in \{1, \ldots, N\}\ \forall v_i, v_h \in V_\mathrm{r} : (v_i, v_h \in k_j \implies \Delta_{ij}\Delta_{hj} < 0)$.

(d) Each circuit ζ in Γ contains a pair of edges k_j, k_s incident to $v_i \in V(\zeta)$ such that

$$\Delta_{ij}\Delta_{is} > 0.$$

Then there exists an edge k_j with ramification nodes $v_i, v_h \in k_j$ or a viaduct π with endpoints v_i, v_h such that there are two edges k_r and k_s with

$$\Delta_{ij}\Delta_{is} > 0 \quad \text{and} \quad \Delta_{hj}\Delta_{hr} > 0.$$

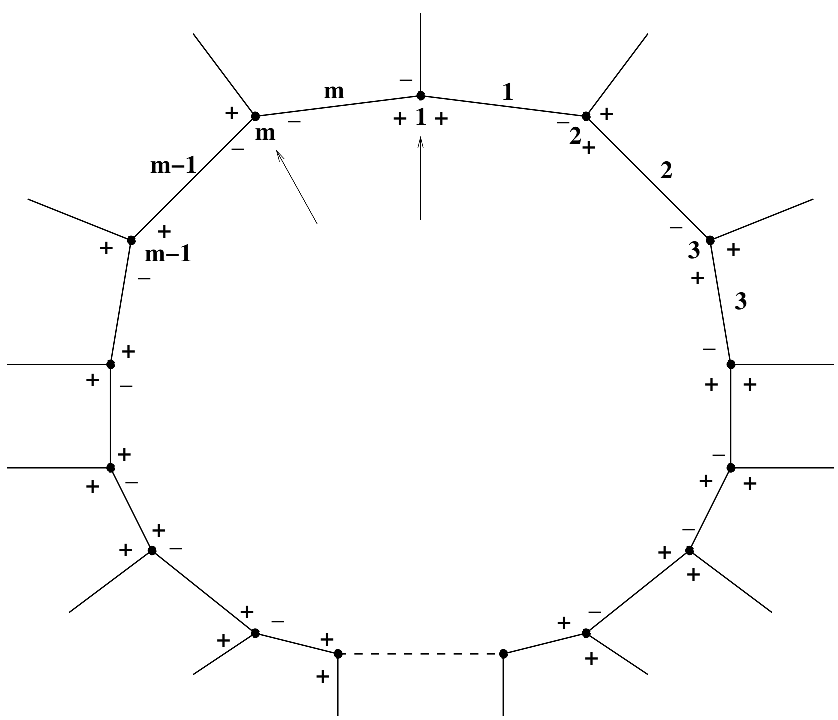

Figure 4: Proof of Lemma 6.2. The indicated signs are those of the Δ_{ij}. The two thin arrows indicate the nodes v_m and v_1 fulfilling the assertion.

Proof. Note first that, by (c) and (d), each circuit ζ in Γ contains a pair of edges k_j, k_s incident to $v_i \in V(\zeta)$ such that $\Delta_{ij} > 0$ and $\Delta_{is} > 0$ or a pair of edges k_l, k_t incident to $v_m \in V(\zeta)$ such that $\Delta_{ml} < 0$ and $\Delta_{mt} < 0$.

Let ζ be a circuit of length m with $V(\zeta) = \{v_1, \ldots, v_m\}$ ordered by the relations $d_{ii} = -1$ and $d_{i+1,i} = 1$ with indices to be taken mod m. Suppose that

$$\Delta_{11}\Delta_{1m} > 0.$$

W.l.o.g. assume that $\Delta_{11} > 0$ and $\Delta_{1m} > 0$ and that each viaduct is replaced by an edge of corresponding length. Then each

$$\gamma(v_i; \Gamma) \geq 3.$$

By (c), $\Delta_{21} < 0$. If $\Delta_{22} < 0$ or $\Delta_{2j} < 0$ for some incident edge outside ζ, then the assertion is shown. Thus, we can assume

$$\Delta_{22} > 0, \quad \Delta_{2j} > 0, \quad \text{and} \quad \Delta_{32} < 0.$$

It follows recursively, that for $\Delta_{i,i-1} < 0$, if $\Delta_{ii} < 0$ or $\Delta_{ij} < 0$ for some incident edge outside ζ, then the assertion is shown with v_i and v_{i-1}. Thus, we can assume

$$\Delta_{ii} > 0, \quad \Delta_{ij} > 0, \quad \text{and} \quad \Delta_{i+1,i} < 0$$

with some incident edge k_j outside ζ. If no pair v_i and v_{i-1} for $i \leq m-1$ has been found yet as asserted, then, finally, v_{m-1} and v_m (as well as v_m and v_1) will fulfill the requirements since then

$$\Delta_{m-1,m-1} > 0, \quad \Delta_{m-1,j} > 0, \quad \Delta_{m-1,m} < 0 \quad \text{and} \quad \Delta_{m,m} < 0$$

with some incident edge k_j outside ζ. This permits to conclude. ∎

Note that the assertion of Lemma 6.2 does not hold on trees as simple examples readily display. Now, we can show the exclusion result in the general case.

Theorem 6.3 *There are no stable nonconstant stationary solutions of Problem* (1) *on any finite connected metric graph G.*

Proof. W.l.o.g. by introducing artificial nodes with Kirchhoff conditions leading to continuous differentiability, we can assume that Γ is simple, i.e. Γ has neither loops, nor multiple edges. We shall reason by recurrence on $d := \text{corank}(\Gamma)$. Recall that the *circuit space* $\mathbf{\Pi}(\Gamma)$ of the graph Γ is defined by

$$\mathbf{\Pi}(\Gamma) = \big\langle c \in \ker \mathcal{D}(\Gamma) \mid \text{supp}(c) \text{ is a circuit in } \Gamma \big\rangle$$

and satisfies $\mathbf{\Pi}(\Gamma) = \ker \mathcal{D}(\Gamma)$, see e.g. [12]. Moreover

$$d = \text{corank}(\Gamma) = \dim \ker \mathcal{D}(\Gamma) = \dim \mathbf{\Pi}(\Gamma), \tag{30}$$

that amounts to $N - n + 1$ for connected graphs. For $d = 0$, the assertion is true by Theorem 6.1. Thus, we can suppose

$$d \geq 1.$$

Let $u \in \mathcal{C}_K^2(G)$ be a stable stationary nonconstant solution of (1). Let k_j be an edge such that $\partial_j u_j(x_j) = 0$ for some $x_j \in [0, \ell_j]$ and set $p = \pi_j(x_j)$. First, consider the case that

$$p \notin V_{\mathrm{b}}.$$

1. If k_j is incident to a boundary vertex, then Lemma 4.3 permits to conclude.
2. If k_j is a bridge that is not incident to V_{b}, then cutting at p leads to two disjoint metric subgraphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ of G. If one of them, say $\tilde{\Gamma}_1$, were a tree, then u would have to be unstable by Theorem 4.2 and Theorem 6.1. If both of them contain circuits, then clearly

$$d(\tilde{\Gamma}_1) < d \quad \text{and} \quad d(\tilde{\Gamma}_2) < d.$$

By recurrence, the restrictions of u to $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are unstable, and so is u by Theorem 4.2.

3. If k_j belongs to a circuit, then cutting at p leads to a graph $\tilde{\Gamma}$ of corank $d-1$. By recurrence the restriction of u to $\tilde{\Gamma}$ must be unstable, and so does u by Theorem 4.2.

Thus, it remains to show the assertion in the case

$$\partial_j u_j(x_j) = 0 \iff \pi_j(x_j) \in V_{\mathrm{b}}.$$

Use notation (27). At the endpoints $v_i, v_h \notin V_{\mathrm{b}}$ of an edge k_j or of a viaduct with $d_{ij} d_{hl} \neq 0$ we have

$$\Delta_{ij}\Delta_{hj} < 0$$

or $\Delta_{ij}\Delta_{hl} < 0$, respectively. Thus, w.l.o.g. we can consider any viaduct as an edge in the remaining reasoning. The set of Δ_{ij} clearly satisfies the hypotheses (a)–(c) of Lemma 6.2, while Condition (d) is fulfilled by (6), by the continuity requirement at the nodes and by the strictly monotone character of each u_j. Thus, Lemma 6.2 guarantees the existence of an edge k_1 with ramification nodes $v_i, v_h \in k_1$ such that there are two edges k_r and k_s with

$$\Delta_{i1}\Delta_{ir} > 0 \quad \text{and} \quad \Delta_{h1}\Delta_{hs} > 0, \tag{31}$$

respectively. Then introduce the modified conductivities by

$$\tilde{c}_{mj} = \begin{cases} c_{mj} + c_{m1}\frac{\Delta_{m1}}{\Delta_{mj}} & \text{if } (m,j) = (i,r) \text{ or } (m,j) = (h,s)\,, \\ c_{mj} & \text{otherwise.} \end{cases}$$

Finally, omit the edge k_1 in Γ and proceed on the resulting and possibly non connected graph $\tilde{\Gamma}$ as in the proof of Theorem 6.1. This is possible by the local character of the modifications applied there in the vicinity of k_1. ■

7 Common Hamiltonian edge system

The stationary case of the fully autonomous consistent parabolic problem

$$
(32) \qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty)), \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) \quad \text{on } k_j & \text{for } 1 \le j \le N, \\ (K) \quad \sum_{j=1}^{N} d_{ij} \partial_j u_j(v_i, \cdot) = 0 & \text{for } \quad 1 \le i \le n. \end{cases}
$$

leads to the same first order system (33) defined by f and $' = \partial_j$ on each edge, i.e.

$$
(33) \qquad \begin{cases} u_j' = v_j, \\ v_j' = -f(u_j). \end{cases}
$$

On each edge (33) is Hamiltonian with respect to the function

$$
H(u,v) = \frac{v^2}{2} + \int_0^u f(\eta) d\eta.
$$

Thus, a stationary solution of (32) corresponds to N parts of trajectories in the phase plane of (33) related to each other by the continuity condition (3) and by the Kirchhoff law (K). Let us recall from [9] some properties of stationary solutions of (32).

Proposition 7.1 *Let $u \in \mathcal{C}_K^2(G)$ be a stationary solution of* (32)*. Then*

$$
(34) \qquad \|u\|_{H_0^1(G)}^2 = \int_G f(u) u \, dx.
$$

Moreover, if H_j denotes the Hamiltonian constant of the trajectory belonging to the edge k_j, then for incident edges k_j, k_s at v_i it holds

$$
2\left(H_j - H_s\right) = \left(\partial_j u_j\right)^2 (v_i) - \left(\partial_s u_s\right)^2 (v_i).
$$

Formula (34) applies e.g. in order to exclude stationary nonconstant solutions between -1 and 1 for nonlinearities of the form $f(u) = u^{2k+1} - u^{2m-1}$ with $1 \le m \le k$ or of the form $f(u) = -\sin(\pi u)$. Again, by Lemma 2.2, $\int_G f(u)\, dx = 0$. Thus, f must have zeros and must change sign in $u(G)$. Set $z_{\min} = \min \{z \mid f(z) = 0\}$ and $z_{\max} = \max \{z \mid f(z) = 0\}$.

Lemma 7.2 *Let $u \in \mathcal{C}_K^2(G)$ be a stationary solution of* (32) *and suppose that*

$$
(35) \qquad f(z) > 0 \text{ for } z < z_{\min} \quad \text{and} \quad f(z) < 0 \text{ for } z > z_{\max}.
$$

Then

$$
z_{\min} \le u \le z_{\max}.
$$

If, in addition, u is nonconstant, then

$$
z_{\min} < u < z_{\max}.
$$

In particular, $\|u\|_{L^\infty(G)} \ge \max\{|z_{\min}|, |z_{\max}|\}$ implies that u is one of the equilibria $z_{\min}$ or $z_{\max}$.

Proof. At a point $x_j \in [0, \ell_j]$ where $\pm \|u\|_{L^\infty(G)}$ is attained, we conclude $v_j(x_j) = 0$. But, if u is nonconstant in a neighborhood of x_j, for $u > z_{\max}$ or $u < z_{\min}$, and $v = 0$, it holds $v' > 0$ and $v' < 0$ respectively, i.e. there can never be a maximum or a minimum at $(u, 0)$ respectively. Moreover, if u attains $z_{\max}$ on some edge, then $z_{\max} = \max_G u$. But, by flow uniqueness, there cannot be a nonconstant trajectory arriving at the equilibrium $(z_{\max}, 0)$ on a finite interval in the phase plane. By connectedness, u has to be the constant $z_{\max}$ on G, which is absurd. In the case that u attains $z_{\min}$ on some edge, $z_{\min} = \min_G u$ which shows that u has to be the constant $z_{\min}$ on G with an analogous argument as above. ∎

The lemma applies in particular to the case of the balanced cubic $f(u) = u - u^3$ treated in [9].

Remark 7.3 Without (35) the assertion of the Lemma is no longer true. Clearly, for $f(z) > 0$ in $(-\infty, z_{\min})$ we have $z_{\min} \le u$, while in the case $f(z) < 0$ in $(z_{\max}, \infty)$, we have $u \le z_{\max}$. But in the general case, both conclusions are invalid. E.g. for $f(u) = u^3 - u$, there are nonconstant stationary solutions u defined on a suitable interval corresponding to periodic solutions surrounding the equilibria $(-1, 0)$ and $(1, 0)$ that satisfy $\|u\|_{L^\infty(G)} > 1$. But in all the cases, by Lemma 2.2, a nonconstant stationary solution u satisfies

$$u(G) \cap [z_{\min}, z_{\max}] \ne \emptyset.$$

For a Lyapunov–energy–calculus we introduce

$$\mathcal{E}(u) = \int_G \frac{v^2}{2} - F(u)\,dx \quad \text{with} \quad F(s) = \int_0^s f(\eta)\,d\eta,$$

$$\mathcal{H}(u) = \sum_{j=1}^N \int_0^{\ell_j} H\left(u_j, \partial_j u_j\right)\,dx.$$

Lemma 7.4 *Let $u \in \mathcal{C}(G \times [0, \infty)) \cap \mathcal{C}_K^{2,1}(G \times (0, \infty))$ be a solution of (1).*

(a) Along u the energy decreases:

$$\dot{\mathcal{E}}(u) := \frac{d}{dt}\mathcal{E}(u) = -\int_G (\partial_t u)^2\,dx.$$

(b) If $u \in \mathcal{C}_K^2(G)$ is a stationary solution of (1), then $\mathcal{E}(u) + \mathcal{H}(u) = \|u\|^2_{H_0^1(G)}$ and

$$\mathcal{E}(u) = \int_G \frac{1}{2} f(u) u\,dx - \sum_{j=1}^N \ell_j H_j.$$

In particular, if u is an equilibrium A, then $\mathcal{E}(u) = -LH(u, 0)$.

Proof. As for (a), we can follow a standard density argument using the Kirchhoff and the continuity condition:

$$\begin{aligned}\dot{\mathcal{E}}(u) &= \int_G \partial_{xt}u\partial_x u - f(u)\partial_t u\,dx \\ &= \underbrace{\sum_j [\partial_t u \partial_x u]_0^{\ell_j}}_{=0} - \int_G \partial_t u \underbrace{\left(\partial_x^2 u - f(u)\right)}_{\partial_t u}\,dx = -\int_G (\partial_t u)^2\,dx.\end{aligned}$$

As for (b), both assertions follow readily with the definitions and from Proposition 7.1. ∎

In order to apply Lasalle's principle, we have to impose an additional condition to f. E.g. under the hypothesis

$$\{z \in \mathbb{R} \mid F(z) \geq 0\} \text{ bounded,} \tag{36}$$

we obtain with $M := \max_{\mathbb{R}} F^+ < \infty$ that

$$\mathcal{E}(u) = \int_G \frac{v^2}{2} - F(u)\,dx \geq -\int_G F(u)\,dx \geq -ML.$$

This enables the application of Lasalle's Principle [4] in order to conclude the following

Corollary 7.5 *Under Condition* (36) *the solutions* $u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty))$ *of* (32) *tend to stationary solutions as* $t \to \infty$ *with respect to* $\|\cdot\|_{L^\infty(G)}$*, since their* ω*–limits belong to the set of functions satisfying* $\dot{\mathcal{E}}(u) = 0$.

We apply the preceding results to the attractivity properties of the equilibria for a nonlinearity f subject to the following conditions.

$$\begin{cases} f \in \mathcal{C}^1(\mathbb{R}), \quad -\infty < A < B < C < \infty, \quad f(A) = f(B) = f(C) = 0, \\ f'(A) < 0, \quad f'(C) < 0, \quad f'(B) > 0, \\ f > 0 \text{ in } (-\infty, A) \cup (B, C), \quad f < 0 \text{ in } (A, B) \cup (C, \infty). \end{cases} \tag{37}$$

They clearly include the case of a cubic $f(x) = \alpha(A-x)(B-x)(C-x)$ with $\alpha > 0$. Moreover, Condition (36) is satisfied with

$$M := \max_{\mathbb{R}} F^+ = \max\{F(A), F(C)\},$$

and Corollary 7.5 applies. Now we can state the following results about the flow defined by Problem (32) subject to Condition (37).

Theorem 7.6 *(a) The equilibrium A is a local attractor, whose domain of attraction satisfies*

$$\mathcal{D}(A) = \left\{u_0 \in \mathcal{C}(G) \mid \lim_{t\to\infty} \int_G u(\cdot,t)dx = LA\right\} \supset \mathcal{C}\left(G; (-\infty, B]\right) \setminus \{B\}.$$

(b) The equilibrium C is a local attractor, whose domain of attraction satisfies

$$\mathcal{D}(C) = \left\{ u_0 \in \mathcal{C}(G) \mid \lim_{t\to\infty} \int_G u(\cdot,t)dx = LC \right\} \supset \mathcal{C}(G;[B,\infty)) \setminus \{B\}.$$

(c) Any nonconstant stationary solution $w \in \mathcal{C}^2_K(G)$ satisfies $A < w < C$ and $\mathcal{E}(w) \geq -ML$ and takes values in (A,B) and values in (B,C).

(d) A stationary solution $w \in \mathcal{C}^2_K(G)$ satisfies $\mathcal{E}(w) = -LM$ iff w is an equilibrium of minimal energy.

Proof. The claimed inclusions in (a) and (b) follow with [9, Theorem 4.1]. By continuity, Corollary 7.5 ensures that the solutions $u \in \mathcal{C}(G\times[0,\infty)) \cap \mathcal{C}^{2,1}_K(G\times(0,\infty))$ of (32) with initial data u_0 belonging to $\mathcal{D}(A)$ or to $\mathcal{D}(C)$ satisfy $\lim_{t\to\infty}\int_G u(\cdot,t)dx = A\int_G dx = LA$ and $\lim_{t\to\infty}\int_G u(\cdot,t)dx = LC$, respectively. It follows that $\mathcal{D}(A)$ and $\mathcal{D}(C)$ belong to the sets in the middle. But according to Lemma 7.2, a nonconstant stationary solution w satisfies

$$-LA < \int_G w dx < LC.$$

Thus, for u_0 belonging to one of the sets in the middle, its solution has to be attracted by the equilibrium A or C, respectively.

The assertion (c) follows readily by Lemma 7.2 and (a) and (b). As for (d), $\mathcal{E}(w) = -LM$ implies that $\|w\|^2_{H^1_0(G)} \leq \int_G (F(w) - M)\, dx \leq 0$, which permits to conclude. ∎

8 The non autonomous case and other transition conditions

The smallest example of the existence of a stable nonconstant stationary solution in presence of reaction terms depending on the edges is the following one.

Example 8.1 Let Γ be the path of length 2 with $\ell_1 = \ell_2 = 1$ and the orientation and labeling given by (9). Define $w \in \mathcal{C}^2_K(G)$ by

$$w_1(x) = \frac{1}{2} - \frac{x^2}{2}, \qquad w_2(x) = \frac{x^2}{2} - x.$$

Then $\partial_1^2 w_1 + 1 = 0$ and $\partial_2^2 w_2 - 1 = 0$ in $[0,1]$, and u is stable with respect to the flow generated by the edge evolution equations $\partial_t u_j = \partial_j^2 u_j - (-1)^j$ in $\mathcal{C}^0(G)$ (or $L^2(G)$). This follows from the fact that for any solution $u \in \mathcal{C}(G\times[0,\infty)) \cap \mathcal{C}^{2,1}_K(G\times(0,\infty))$, the difference $\delta = u - w$ solves the heat equation $\partial_t \delta_j = \partial_j^2 \delta_j$ on each edge k_j. But the minimal eigenvalue of the Laplacian $-\left(\partial_j^2\right)_{N\times 1}$ under (2) and (3) is 0. Thus, eigenfunction expansion and Dirichlet's Theorem yield

$$\|u(\cdot,t) - w\|_{\infty,G} \leq \text{const.}\ \|u(\cdot,0) - w\|_{\infty,G}.$$

Example 8.2 If we admit homogeneous Dirichlet boundary conditions at some vertices, stable nonconstant stationary solution occur already on an interval. On $[0,\ell]$ consider the parabolic problem

$$
(38) \qquad \begin{cases} u \in \mathcal{C}([0,\ell]\times[0,\infty)) \cap \mathcal{C}^{2,1}([0,\ell]\times(0,\infty)) \\ \partial_t u = u'' + 1 \quad \text{ on } [0,\ell] \\ u(0,\cdot) = 0 = u'(\ell,\cdot) \end{cases}
$$

Then

$$
w(x) = \ell x - \frac{x^2}{2}
$$

defines a global attractor for the solutions of (38) in $\mathcal{C}^0[0,\ell]$ (or $L^2(0,\ell)$), since for any solution $u \in \mathcal{C}([0,\ell]\times[0,\infty)) \cap \mathcal{C}^{2,1}([0,\ell]\times(0,\infty))$, the difference $\delta = u - w$ solves again the heat equation on $[0,\ell]$. But the minimal eigenvalue of the Laplacian on $[0,\ell]$ under $u(0) = 0 = u'(\ell)$ amounts to

$$
\lambda_1 = \frac{\pi^2}{4\ell^2}
$$

and leads via eigenfunction expansion and Dirichlet's Theorem to the conclusion

$$
\|u(\cdot,t) - w\|_{\infty,G} \leq \text{const. } e^{-\lambda_1 t} \|u(\cdot,0) - w\|_{\infty,G}\,.
$$

Example 8.3 If the nonlinearity depends on the edges and on x_j, but not on u, then either there is no stationary solution or there is a unique stable one, that can be nonconstant. In detail and generalizing Example 8.1, for given $f_j \in \mathcal{C}[0,\ell_j]$ we consider the inhomogeneous heat flow problem

$$
(39) \qquad \begin{cases} u \in \mathcal{C}(G\times[0,\infty)) \cap \mathcal{C}_K^{2,1}(G\times(0,\infty)), \\ \partial_t u_j = \partial_j^2 u_j + f_j(x_j) \quad \text{on } k_j & \text{for } 1 \leq j \leq N, \\ (K) \quad \displaystyle\sum_{j=1}^N d_{ij}\partial_j u_j(v_i,t) = 0 & \text{for } \quad 1 \leq i \leq n. \end{cases}
$$

If $\int_G f(x)\,dx \neq 0$, then there is no stationary solution $w \in \mathcal{C}_K^2(G)$ of (39), since this would lead to

$$
0 = \sum_{j=1}^N \int_0^{\ell_j} \partial_j^2 w_j(x_j)\,dx_j = -\sum_{j=1}^N \int_0^{\ell_j} f_j(x_j)\,dx_j = \int_G f(x)\,dx.
$$

But if f fulfills $\int_G f(x)\,dx = 0$, then a unique stationary solution $w \in \mathcal{C}_K^2(G)$ of (39) can be obtained as follows. Introduce

$$
F_j(x_j) = \int_0^{x_j} f_j(\xi)\,d\xi
$$

Using (3) and (K), it readily follows as in [5, 7, 10, 14, 15], that there exist unique coefficients b_j and c_j for $1 \leq j \leq N$ such that $w \in \mathcal{C}_K^2(G)$ defined by

$$
w_j(x_j) = c_j + b_j x_j - \int_0^{x_j} F_j(s_j)\,ds_j
$$

is the unique stationary solution of (39). As in Example 8.1, w is seen to be stable, since for every solution u of (39), the difference $u - w$ solves the heat equation in $\mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty))$.

Example 8.4 If the diffusion coefficients are allowed to be different, then again stable nonconstant stationary solutions can occur. The example follows a refinement of Matano's type of counterexamples for non convex domains established by Cònsul and Solà–Morales [13]. Consider the path Π with 3 edges using the numbering (9) and choosing the edge lengths and diffusion coefficients a_j to be

$$\ell_1 = \ell_3 = 1, \quad \ell_2 = \delta > 0, \quad a_1 = a_3 = 1, \quad a_2 = \varepsilon > 0$$

with δ and ε sufficiently small to be determined later. As common nonlinearity we choose $f(u) = u - u^3$. Using the double well potential $G(u) = \frac{1}{4}(1-u^2)^2$ and the modified energy

$$E(u) = \sum_{j=1}^{3} \int_0^{\ell_j} \frac{a_j}{2} (\partial_j u_j)^2 + G(u) dx_j,$$

it can be shown that for $\varepsilon = \delta^2$, there exists a minimizer w of E in

$$\left\{ u \in \mathcal{C}(\Pi) \cap H^1(\Pi) \middle| \int_0^{\ell_1} u_1 \; dx_1 \leq 0, \int_0^{\ell_3} u_3 \; dx_3 \geq 0, \, -1 \leq u \leq 1 \right\}$$

that is close to -1 on k_1 and close to 1 on k_3. Moreover, w is stable.

Remark 8.5 The anti–Kirchhoff condition: If we replace the transition conditions (2) and (3) by their orthogonal condition, the famous *anti–Kirchhoff condition*

$$\sum_{v_i \in k_j} u_j(v_i, t) = 0 \quad \text{for} \quad 1 \leq i \leq n, \tag{40}$$

$$k_j \cap k_s = \{v_i\} \implies d_{ij}\partial_j u_j(v_i,t) = d_{is}\partial_s u_s(v_i,t) \quad \text{for} \quad 1 \leq i \leq n, \tag{41}$$

then the stability results change dramatically. E.g., there are no exceptional graphs, since for any finite metric graph, there is a suitable nonlinearity f such that there is a stable nonconstant stationary solution $(u_j)_{N\times 1}$ governed by the edge equations $\partial_j^2 u_j + f(u_j) = 0$. We refer to [10] for the details.

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