EQUIVARIANT CLASSIFICATION OF b^m-SYMPLECTIC SURFACES AND NAMBU STRUCTURES

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ABSTRACT. In this paper we extend the classification scheme in [S] for b^m-symplectic surfaces, and more generally, b^m-Nambu structures to the equivariant setting. When the compact group is the group of deck-transformations of an orientable covering, this yields the classification of these objects in non-orientable manifolds. The paper also includes recipes to construct b^m-symplectic structures on surfaces. Feasibility of such constructions depends on orientability and on the colorability of an associated graph. We recast the strategy used in [Mt] to classify stable Nambu structures of top degree on orientable manifolds to classify b^m-Nambu structures (not necessarily oriented) using the language of b^m-cohomology. The paper ends up with an equivariant classification theorem of b^m-Nambu structures of top degree.

1. INTRODUCTION

The topological classification of compact surfaces is determined by orientability and genus. The geometrical classification of symplectic surfaces was established by Moser [M]. Moser proved that any two compact symplectic surfaces with symplectic forms lying on the same De Rham cohomology class are equivalent in the sense that there exists a diffeomorphism taking one symplectic structure to the other.

Poisson structures show up naturally in this scenario as a generalization of symplectic structures where the non-degeneracy condition is violated. The first examples of Poisson structures are Symplectic manifolds and manifolds with the zero Poisson structure. In-between these two extreme examples there is a wide variety of Poisson manifolds. Poisson structures with dense symplectic leaves and controlled singularities have been the object of study of several recent articles (see for instance [GMP], [GMP2], [GMPS], [GL], [MO2]). The classification of these objects in dimension 2 is given by a suitable cohomological condition. In the extreme case of symplectic manifolds this cohomology is de Rham cohomology and this classification was already known to Moser [M]. For orientable b-symplectic manifolds, the classification can be formulated in terms of b-cohomology (see [GMP]) which reinterprets former classification invariants by Radko [R].

It is possible to consider other classes of Poisson manifolds with simple singularities like b^m-symplectic manifolds [S] or more general singularities [MS] by relaxing the transversality condition for b-symplectic manifolds. These structures have relevance in mechanics: Most of the examples are found naturally in the study of celestial mechanics (see [KM], [DKM], [KMS]). In the same way, b^m-symplectic structures are classified in terms of b^m-cohomology [S]. The recent papers [Ca],

[MO2], [FMM] have renewed the interest on the non-orientable counterparts of these structures.

In this article we focus our attention on two kind of objects: The first one is b^m-symplectic manifolds and the second class is a generalization of stable Nambu structures which we call b^m-Nambu structures of top degree. Those objects coincide in dimension 2. For them we prove Radko-type equivariant classification. When the group considered is the group of deck-transformations of an orientable covering, this yields the classification of non-orientable compact surfaces in the b^m-case. Such a classification was missing in the literature.

Organization of the paper: In Section 2 we include the necessary preliminaries of b-structures. In Section 3 we present some examples of b^m-symplectic surfaces on orientable and non-orientable manifolds. In Section 4 we present an equivariant b^m-Moser theorem and use it to classify non-orientable b^m-symplectic surfaces. In Section 5 we give explicit constructions of b^m-symplectic structures with prescribed critical set depending on orientability and colorability of an associated graph. In Section 6, b^m-Nambu structures are classified using the equivariant techniques.

2. Preliminaries

The context of this paper is the so called b-Poisson or b-symplectic geometry. First we present an overview of the basic concepts of this field.

Recall that a **Poisson manifold** is a pair (M, Π) where Π is a bivector field satisfying the condition $[\Pi, \Pi] = 0$ where $[\cdot, \cdot]$ stands for the Schouten bracket.

A class of Poisson manifolds are b-Poisson manifolds introduced and studied in [GMP] and [GMP2]. The main feature of these manifolds is that they are symplectic away from a hypersurface Z and determine a regular Poisson structure on Z. The singularity associated to this structure behaves reasonably well. Assume we consider a symplectic form on $M \backslash Z$ whose dual Poisson structure vanishes along Z in the following controlled way.

Definition 2.1 (Guillemin-Miranda-Pires ([GMP], [GMP2])). Let (M^{2n}, Π) be an (oriented)[1] Poisson manifold. If the map

$$p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(TM)$$

is transverse to the zero section, then Π is called a b-**Poisson structure** on M. The hypersurface $Z = \{p \in M | (\Pi(p))^n = 0\}$ is the **critical hypersurface** of Π. The pair (M, Π) is called a b-**Poisson manifold**.

The condition explained above gives really powerful information about the Poisson structure (from now on we will call it b-Poisson). In particular, Weinstein's splitting theorem [We] can be rewritten in a very simple way b-Poisson structures. Actually it looks more like a Darboux-type result like is shown in [GMP2].

Theorem 2.2 (b-Darboux theorem, [GMP2]). *Let (M, Π) be a b-Poisson manifold. Then, on a neighbourhood of a point $p \in Z$ in the critical surface, there exist coordinates $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that the critical hypersurface is given by $z = 0$ and*

$$\omega = \sum_{i=2}^{n} x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

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[1]it is also possible to consider non-oriented Poisson manifolds [Ca] and [GL]

2.1. **Radko's classification of b-symplectic surfaces.** We recall Radko's classification invariants for b-symplectic surfaces [R].

Definition 2.3. The **Liouville volume** is the well-defined limit:

$$V(\Pi) := \lim_{\varepsilon \to 0} \int_{|h| > \varepsilon} \omega.$$

This limit exists and is independent of the choice of the defining function h of Z (see [R] for the proof).

Definition 2.4. Let (M, Π) be a Poisson manifold, Ω a volume form on it and denote by X_f denote the Hamiltonian vector field of a smooth function $f : M \to \mathbb{R}$. The **modular vector field** (X^Ω) is the derivation that acts on any function f on M in the following way:

$$f \mapsto \frac{\mathcal{L}_{u_f} \Omega}{\Omega}.$$

The **modular period** of Π around γ a connected component of $Z(\Pi)$ is

$$T_\gamma(\Pi) := \text{period of } X^\Omega|_\gamma.$$

Radko [R] proved that these invariants together with the topology of the critical set completely classify b-symplectic surfaces:

Theorem 2.5 (**Radko** [R]). *Two Poisson structures Π, Π' are globally equivalent if and only if the following invariants coincide:*

- *the Liouville volume,*
- *the topology of Z, and*
- *the modular periods on each connected component of Z.*

2.2. **Arnold's A_m singularities.** Arnold classified the singularities of functions using a hierarchy (confer [AR1], [AR2]). This classification of singularities of functions automatically gives the local classification of Poisson structures.

Metatheorem 2.1 (**Arnold**). *Any local classification of a singularity of a smooth function gives a local classification of a Poisson structure in dimension two.*

In what follows, we concentrate on the singularities A_m, where $f(x) = x^m$ and $\Pi = x^m \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. We also extend this study to higher dimensions.

A Poisson structure on an even dimension manifold M^{2n} that locally can be expressed in this form:

$$\Pi = x_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n},$$

is called a b^m-**Poisson structure**. The vanishing set of Π^n, Z, is called **critical set** of Π.

2.3. b^m-**Poisson manifolds via forms.** It is possible to work with b^m-Poisson manifolds using the language of forms. Denote by Z the critical set of the b^m-Poisson structure. We develop a concept which allows to extend the symplectic structure from $M \backslash Z$ to the whole manifold M. This singular form will be called a "b^m-symplectic" form on M.

Let us start with the simple case of b-Poisson structures: A b-**manifold** is a pair (M, Z) of a n-dimensional manifold M and a hypersurface $Z \subset M$. A b-**vector**

field on a b-manifold (M, Z) is a vector field tangent to the hypersurface Z at every point $p \in Z$. If x is a local defining function for the hypersurface Z on some open set $U \subset M$ and consider local coordinates $(x, x_1, \ldots, x_{n-1})$ on U, then the set of b-vector fields on U is a free $C^\infty(M)$-module with basis

$$\{x\frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}.$$

By the Serre-Swan theorem [Sw] this modules defines a vector bundle such that sections of the bundle are precisely b-vector fields. This vector bundle is called the **b-tangent bundle** and we use the notation ${}^bTM$ to refer to it. By duality we define its dual, the **b-cotangent bundle** ${}^bT^*M$.

We define a b-form to be a section of the b-cotangent bundle. In the same way we can define a **b-de Rham k-form** to be a section of the bundle $\Lambda^k({}^bT^*M)$. We denote the set of b-forms as ${}^b\Omega^k(M)$.

The space of de Rham k-forms sits inside this space in a natural way; having fixed a local defining function of Z, f, every b-de Rham k-form can be decomposed as[2]

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{1}$$

This decomposition yields an extension of the differential of the De Rham complex d to ${}^b\Omega(M)$ as follows:

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

So we may refer to the complex of b-forms as the **b-de Rham complex**.

The cohomology associated to this complex is called **b-cohomology** and it is denoted by ${}^bH^*(M)$.

A special class of closed 2-forms of this complex are b-symplectic forms as defined in [GMP2],

Definition 2.6. Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is **b-symplectic** if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all point $p \in M$.

We will also denote this condition of maximality of the rank as **non-degeneracy**.

Instead of working with b-Poisson structures we can dualize them and work with b-forms. As it was proved in [GMP2] using this duality there is a bijection between b-symplectic forms and b-Poisson structures.

The following theorem relates b^m-cohomology with De Rham cohomology by a simple formula:

Theorem 2.7 (Mazzeo-Melrose). *The b-cohomology groups of M^{2n} satisfy*

$$ {}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

Remark 2.8. Observe in particular that the first cohomology grup is always non zero.

In [GMP2], it was shown that b-cohomology of a b-symplectic manifold is indeed isomorphic to its Poisson cohomology. The modular class of the vector field

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[2]For non-orientable manifolds, we may replace f by an adapted distance function [MO2].

transverse to the symplectic foliation in Z gives the component in $H^1(Z)$ of the decomposition under the Mazzeo-Melrose isomorphism of Theorem 2.7 and it is a Poisson invariant of the manifold. Several classification theorems can be interpreted using this theorem.

For instance as it was proved in [GMP2] Radko's theorem can merely be stated as a Moser-type theorem:

Theorem 2.9 (**Radko's theorem in b-cohomological language,** [GMP2])**.** *Let S be a compact orientable surface and and let ω_0 and ω_1 be two b-symplectic forms on (M, Z) defining the same b-cohomology class (i.e.,$[\omega_0] = [\omega_1]$) then, there exists a diffeomorphisms $\phi : M \to M$, such that $\phi^* \omega_1 = \omega_0$.*

In the same way, we may consider b^m-Poisson structures and define the b^m-tangent and b^m-cotangent bundles and consider the complex of b^m-forms (for details please consult [S]).

The Mazzeo-Melrose theorem for b^m-forms becomes:

Theorem 2.10 (b^m**-Mazzeo-Melrose** [S])**.**

$$^{b^m} H^p(M) \cong H^p(M) \oplus \left(H^{p-1}(Z) \right)^m \tag{2}$$

Among the set of 2-forms a distinguished class is the following,

Definition 2.11. A **symplectic b^m-manifold** is a b^m-manifold (M, Z) with a closed b^m-two-form ω which is non-degenerate.

It can be proved that b^m-symplectic structures are in one-to-one correspondence with b^m-Poisson structures.

2.4. **Classification of b^m-surfaces.** In this section we single out some classification results (local and global) for b^m-symplectic manifolds. The results contained in this subsection are proved either in [S] or [GMW].

We assume that locally the defining function for the critical set Z is given by $x = 0$. We recall from [S].

Definition 2.12. A **Laurent Series** of a closed b^m-form ω is a decomposition of ω in a tubular neighbourhood U of Z of the form

$$\omega = \frac{dx}{x^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\alpha_i) x^i \right) + \beta \tag{3}$$

with $\pi : U \to Z$ the projection of the tubular neighborhood onto Z and α_i a closed smooth De Rham form on Z and β a De Rham form on M.

In [S] it is proved that in a neighborhood of Z, every closed b^m-form ω can be written in a Laurent form of type (3) having fixed a (semi)local defining function.

The critical hypersurface Z is a regular Poisson submanifold of (M^{2n}, Π) which is foliated by symplectic leaves of dimension $2n - 2$. As it happened for b-Poisson structures, we can find a Poisson vector field v on Z transverse to the symplectic foliation.

We recall from [GMW] the following,

Proposition 2.13 (**Guillemin, Miranda, Weitsman,** [GMW]). *Given a symplectic b^m-structure with b^m-symplectic form ω, the closed 1-form α_0 in the Laurent decomposition*

$$\omega = \frac{dx}{x^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\alpha_i)x^i \right) + \beta$$

defines the codimension-one symplectic foliation $\mathcal{F}$ of the regular Poisson structure induced by the dual b^m-Poisson structure on the critical hypersurface Z. In addition there exists a Poisson vector field v on Z transverse to this foliation.

The symplectic foliation on the critical set Z is easy to describe as it describes a mapping torus [GMW].

The Moser path method also works in the setting of b^m-symplectic structures and can be found in [GMP] and [S] (for general b^m-symplectic forms):

Theorem 2.14 (**Moser's theorem**). *Let ω_0, ω_1 be two b^m-symplectic forms on (M^{2n}, Z) with Z compact and $\omega_0|_Z = \omega_1|_Z$, then there are neighborhoods U_0, U_1 of Z and a b^m-symplectomorphism $\varphi : (U_0, Z, \omega_0) \to (U_1, Z, \omega_1)$ such that $\varphi|_Z = Id$.*

Two consequences of the path method to b^m-forms are the following:

- A local description of a b^m-symplectic manifold:

  Theorem 2.15 (b^m-**Darboux theorem,** [GMW]). *Let ω be a b^m-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $x_1 = 0$ and*

  $$\omega = \frac{dx_1}{x_1^m} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

- A global classification of b^m-symplectic surfaces à la Radko in terms of b^m-cohomology classes:

  Theorem 2.16 (**Scott,** [S]). *Let ω_0 and ω_1 be two b^m-symplectic forms on a compact connected b^m-surface (M, Z). Then, the following conditions are equivalent:*
  - *Their b^m-cohomology classes coincide $[\omega_0] = [\omega_1]$,*
  - *The surfaces are globally b^m-symplectomorphic,*
  - *the Liouville volumes of ω_0 and ω_1 and the numbers*

    $$\int_\gamma \alpha_i$$

    for all connected components $\gamma \subseteq Z$ and all $1 \le i \le m$ coincide (where α_i are the terms appearing in the Laurent decomposition of the two b^m-forms).

## 3. Toy examples of b^m-symplectic surfaces

In this section we describe some examples of orientable and non-orientable b^m-symplectic surfaces.

- b^n-**symplectic structures on the sphere:** Consider the sphere S^2 with the b^m-symplectic form $\omega = \frac{1}{h^m} dh \wedge d\theta$. Where h stands for the height and θ for the angular coordinate. Observe that with this form has the equator as the critical set Z.

- b^n-**symplectic structures on the torus:** Consider $\mathbb{T}^2$ as quotient of the plane ($\mathbb{T}^2 = \{(x,y) \in (\mathbb{R}/\mathbb{Z})^2\}$). And consider in it the b^m-symplectic manifold on $\mathbb{R}^2$ given by $\omega = \frac{dx}{(\sin 2\pi y)^n} \wedge dy$. The action of $\mathbb{Z}^2$ leaves this form invariant and thus this b^m-form descends to the quotient. Observe that this b^m form defines $Z = \{y \in \{0, \frac{1}{2}\}\}$.
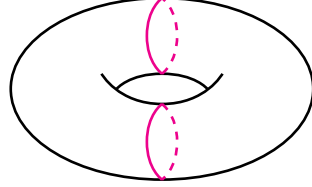
FIGURE 1. Example: b^{2k}-symplectic structure in the torus.

- **A** b^{2k+1}-**symplectic structure on the projective Space:** Consider the previous example of b^m-symplectic form on the sphere. Then, take the quotient by the action of the group $\mathbb{Z}/2\mathbb{Z} \times S^2 \to S^2$ such that the image of $(1, x)$ is (x) and the image of $(-1, x)$ is $(-x)$. Observe that ω is invariant by the action ($\omega(x) = \omega(-x)$) for $m = 2k + 1$, and hence one obtains a b^{2k+1}-symplectic form in $\mathbb{P}^2_\mathbb{R}$ with critical set $\overline{Z}$, that is the equator modulo the antipodal identification (thus diffeomorphic to S^1).
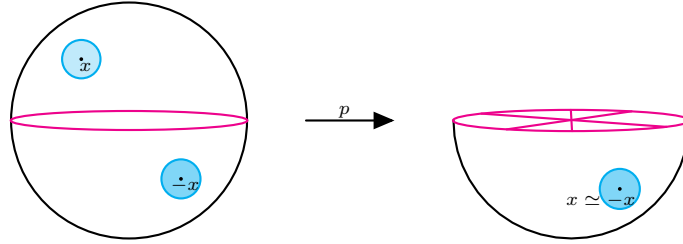
FIGURE 2. The symplectic structure on the sphere S^2 that vanishes at the equator induces a symplectic structure on the projective space $\mathbb{P}^2_\mathbb{R}$.

- **A** b^{2k+1}-**symplectic structure on a Klein Bottle:** Consider the Torus with the structure given in the previous example. It is known that the torus $S^1 \times S^1$ is homeomorphic to $[0,1] \times [0,1]/\approx$, where $\approx$ identifies the edges of the square by $(x, 0) \approx (x, 1)$ and $(0, y) \approx (1, y)$. We also define the Klein bottle to be $K = [0,1] \times [0,1]/\sim$, where $\sim$ identifies the edges of the square by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

  For the torus, we have an explicit continuous surjection

$$\pi : [0, 1] \times [0, 1] \to S^1 \times S^1 : (x, y) \mapsto \left(e^{i\pi x}, e^{i\pi y}\right)$$

using the standard identification of S^1 with the unit circle in the complex plane. Note that we now have:

$$(x_1, y_1) \approx (x_2, y_2) \iff \pi(x_1, y_1) = \pi(x_2, y_2).$$

This is equivalent to saying that π induces a well-defined homeomorphism $([0, 1] \times [0, 1]/ \approx) \to S^1 \times S^1$.

Consider the following map:

$$\phi : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] : (x,y) \mapsto \begin{cases} (2x, y) & \text{if } x \leq \frac{1}{2}, \\ (2(1-x), y) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Composing this map with the projection $\pi_\sim : [0,1] \times [0,1] \rightarrow K$, we get the map $\psi = \pi_\sim \circ \phi$ from $[0,1] \times [0,1]$ to K. We now claim that ψ induces a double cover from $[0,1] \times [0,1]/\approx$ to K. To show this, it is enough to see that:

$$|(\psi^{-1}(\{\vec{x}\})/\approx)| = 2$$

for each $\vec{x} \in K$. In other words, each element of K has exactly two pre-images in $[0,1] \times [0,1]$, up to equivalence under $\approx$. Consider $(x,y) \in [0,1] \times [0,1]/\sim$, then the only pre-images of (x,y) are $(x/2, y)$ and $(1 - x/2, y)$ and these points will never be equal. The group $\mathbb{Z}/2\mathbb{Z}$ acts on (x,y) in the following way: $\text{Id} \cdot (x,y) = (x,y)$ and $\text{-Id} \cdot (x,y) = (1-x, y)$.

Thus, the b^m-symplectic form $\omega = \frac{dx}{(\sin 2\pi x)^n} \wedge dy$ induces a b^m-symplectic structure in T if ω is invariant by the action of the group. We can check that $-id$ leaves the form invariant. Let $\rho_{-\text{id}}$ denote morphism induced by the action of $-\text{id}$.

$$\rho^*_{-\text{id}}\omega = \rho^*_{-\text{id}} \left(\frac{dx}{(\sin 2\pi x)^n} \wedge y \right) = \frac{d(1-x)}{(\sin(2\pi - 2\pi x))^n} \wedge dy = \frac{-dx}{(-1)^n \sin 2\pi x} \wedge dy.$$

Thus, ω is invariant if m is odd, and in this case we have constructed an example of b^m-symplectic structure in the Klein Bottle.

Remark 3.1. The previous examples only exhibit b^{2k+1}-symplectic structures on non-orientable surfaces. As we will see in section 5 only orientable surfaces can admit b^{2n}-symplectic structures.

4. Equivariant classification of b^m-surfaces. Non-orientable b^m-surfaces.

In this section we give an equivariant Moser theorem for b^m-symplectic manifolds. This yields the classification of non-orientable surfaces thus extending the classification theorems of Radko and Scott for non-orientable surfaces.

A classification of b-symplectic surfaces à la Moser was obtained in [GMP]. The classification of b-symplectic surfaces is given by the class in b-cohomology determined by the b-symplectic structures. This result is the same kind of result as the one of Moser for the global classification of symplectic surfaces using De Rham cohomology classes. The class in b-cohomology determines, in its turn, the period of the modular vector field along the curves which define the singular locus.

We now extend the classification result for manifolds admitting a group action which leaves the b^m-symplectic structure invariant.

Theorem 4.1 (Equivariant b^m-Moser theorem for surfaces). *Suppose that M is a compact surface, let Z be a union of non-intersecting curves and let ω_0 and ω_1 be two b^m-symplectic structures on (M, Z) which are invariant under the action of a compact Lie group $\rho : G \times M \longrightarrow M$ and defining the same b^m-cohomology class, $[\omega_0] = [\omega_1]$. Then, there exists an equivariant diffeomorphism $\gamma_1 : M \rightarrow M$, such that ϕ_1 leaves Z invariant and satisfies $\gamma_1^*\omega_1 = \omega_0$.*

Proof. Denote by $\rho : G \times M \longrightarrow M$ the group action and denote by ρ_g the induced diffeomorphism for a fixed $g \in G$. i.e., $\rho_g(x) := \rho(g, x)$. Consider the linear family of b^m-forms $\omega_t = t\omega_1 + (1 - t)\omega_0$. Observe that since the manifold is a surface, the fact that ω_0 and ω_1 are non-degenerate b^m-forms (thus non-vanishing sections of $\Lambda^2(^bT^*(M))$) implies that the linear path is non-degenerate too. We will prove that there exists a family $\gamma_s : \mathcal{U} \to M$, with $0 \le s \le 1$ such that

$$\gamma_s^* \omega_s = \omega_0. \tag{4}$$

This is equivalent to the following equation,

$$\mathcal{L}_{X_s} \omega_s = \omega_0 - \omega_1, \tag{5}$$

where $X_s = \frac{d\gamma_s}{ds} \circ \gamma_s^{-1}$.

Since the cohomology class of both forms coincide, $\omega_1 - \omega_0 = d\alpha$ for α a b^m-form of degree 1.

Therefore equation (5) becomes

$$\iota_{X_s} \omega_s = -\alpha. \tag{6}$$

This equation has a unique solution X_s because ω_s is b^m-symplectic and therefore it is non-degenerate. Furthermore, the solution is a b^m-vector field. From this solution we will construct an equivariant solution such that its t-dependent flow gives an equivariant diffeomorphism.

Since the forms ω_0 and ω_1 are G-invariant, we can find a G-invariant primitive $\tilde{\alpha}$ by averaging with respect to a Haar measure the initial form α: $\tilde{\alpha} = \int_G \rho_g^*(\alpha)d\mu$ and therefore the invariant vector field,

$$X_s^G = \int_G \rho_{g_*}(X_s)d\mu$$

is a solution of the equation,

$$\iota_{X_s^G} \omega_s = -\tilde{\alpha}, \tag{7}$$

We can get an equivariant γ_s^G by integrating X_s^G. This family satisfies $\gamma_t^{G^*} \omega_t = \omega_0$ and it is equivariant.

Also observe that since X_t^G is tangent to Z this diffeomorphism preserves Z.

$\square$

Recall that considering a non-orientable manifold is equivalent to considering a pair $(\tilde{M}, \rho)$ where $\tilde{M}$ stands for the orientable covering and ρ is the action given by deck-transformations on the orientable covering by the discrete group $G = \mathbb{Z}/2\mathbb{Z}$. This point of view is very convenient for classification issues because equivariant mappings on the orientable covering yield actual diffeomorphism on the non-orientable manifolds. We adopt this point of view to provide a classification theorem for non-orientable b^m-surfaces in cohomological terms.

Corollary 4.2. *Let S be a non-orientable compact surface and let ω_1 and ω_2 be two b^m-symplectic forms on S. Assume $[\omega_1] = [\omega_2]$ in b^m-cohomology then (S, ω_1) is equivalent to (S, ω_2). That is to say, there exists a diffeomorphism $\varphi : S \to S$ such that $\varphi^* \omega_2 = \omega_1$.*

Proof. Consider $p : \tilde{S} \to S$ a covering map, and $\tilde{S}$ the orientation double cover. Since $[\omega_1] = [\omega_2]$ then $[p^*(\omega_1)] = [p^*(\omega_2)]$. Because $\tilde{S}$ is orientable we may use the b^m-Moser theorem in order to guarantee the existence of a symplectomorphism $\tilde{\varphi} : (\tilde{S}, p^*(\omega_1)) \to (\tilde{S}, p^*(\omega_2))$. Consider the following diagram:

$$
\begin{array}{ccc}
(\tilde{S}, p^*(\omega_1)) & \xrightarrow{\ \ \tilde{\varphi}\ \ } & (\tilde{S}, p^*(\omega_2)) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
(S, \omega_1) & \xrightarrow{\ \ \varphi\ \ } & (S, \omega_2)
\end{array}
\tag{8}
$$

We want to prove that there exists a symplectomorphism φ making the diagram commute. By the universal property of the quotient applied to $p \circ \tilde{\varphi}$, there exists a unique φ making the diagram commute if and only if the images by $p \circ \tilde{\varphi}$ of the points identified at the quotient are the same. This is equivalent to asking that the images of p by $\tilde{\varphi}$ are sent to the orbit of p. And this is true because $\tilde{\varphi}(gp) = g\tilde{\varphi}(p)$ as a consequence of Theorem 4.1 (in this case the Lie group acting on the manifold is the discrete group $\mathbb{Z}/2\mathbb{Z}$). It is possible to apply this theorem since the symplectomorphism between $p^*\omega_1$ and $p^*\omega_2$ given by b^m-Moser theorem, yields a family of forms with invariant b^m-cohomology class. $\square$

A similar equivariant Moser b^m theorem as Theorem 4.1 holds for higher dimensions. In that case we need to require that the linear path ω_t is a path of b^m-symplectic structures which is not true in general. The proof follows the same lines as Theorem 4.1. Indeed it was already proved for b-symplectic manifolds (see Theorem 8 in [GMPS]).

Theorem 4.3 (**Equivariant b^m-Moser theorem**). *Let M be a compact manifold and let Z be a smooth hypersurface. Consider ω_t for $0 \leq t \leq 1$, a smooth family of b^m-symplectic forms on (M, Z) such that the b^m-cohomology class $[\omega_t]$ does not depend on t.*

Assume that the family of b^m-symplectic structures is invariant by the action of a compact Lie group G on M, then, there exists a family of equivariant diffeomorphisms $\phi_t : M \to M$, with $0 \leq t \leq 1$ such that ϕ_t leaves Z invariant and satisfies $\phi_t^*\omega_t = \omega_0$.

5. Constructions and classification of b^m-symplectic structures

In this section we describe constructions of b^m-symplectic structures on surfaces. We start associating a graph to a pair (M, Z) where Z is the prescribed critical set for a b^m-symplectic structure.

5.1. b^m-Graphs.

Definition 5.1. We define the associated graph to a b^m-manifold (M, Z), as the graph with set of vertices given by the connected components of $S \setminus Z$ and with edges connecting two vertices when the connected components associated to them U_1 and U_2, $\partial U_1 \cap \partial U_1 \neq \{0\}$.

Remark 5.2. The associated graph to a b^m-manifold can contain loops and double edges (that is an edge can connect the same vertex). Observe also that for any loop,

there is no change of sign of the Liouville volume whenever we cross the connected component of Z associated to it.
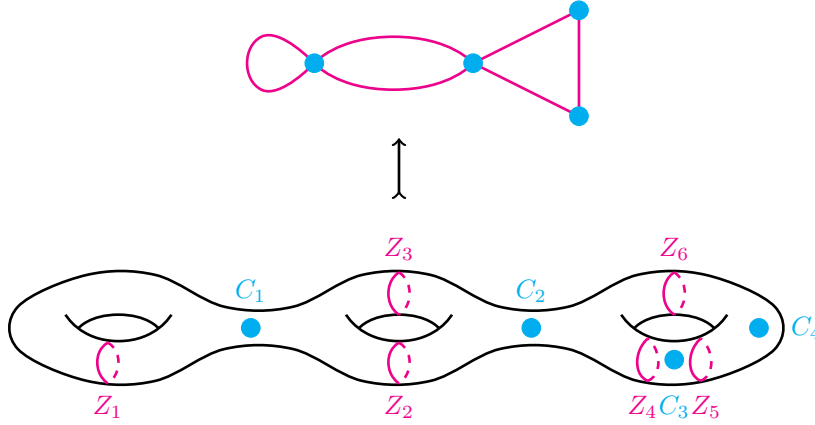

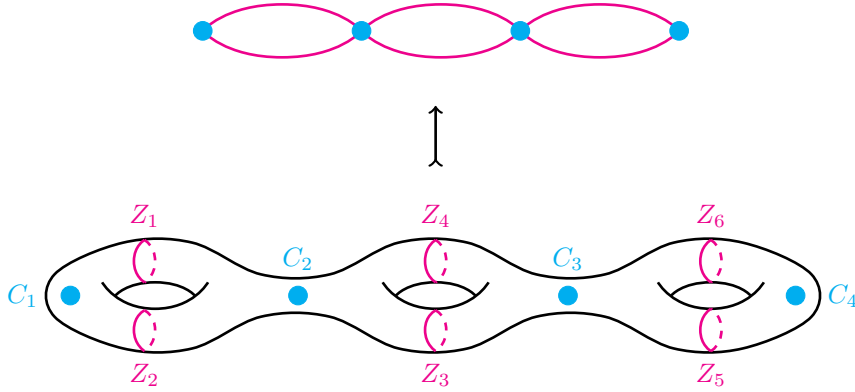FIGURE 3. Example of a non-colorable associated graph.

FIGURE 4. Example of a colorable associated graph.

Definition 5.3. A **2-coloring of a graph** is a labeling of the graphs of the vertices such that no two vertices sharing the same edge have the same label.

Not every graph admits a 2-coloring.

Definition 5.4. A graph is called **2-colorable** if it admits a 2-coloring.

5.2. b^m-surfaces with m even. We start by proving that only orientable manifolds admit b^{2k}-symplectic structures. This is quite expected since for $m = 2k$, $\omega \wedge \omega$ is a positive section of $(\Lambda^2)^b T^*(M)$.

Theorem 5.5. *If a compact surface admits a b^{2k}-symplectic structure then it is orientable.*

Proof. Assume M is a non-orientable surface. The proof consists in building a collar of b^{2k}-Darboux neighborhoods with compatible orientations in a neighborhood of each connected component of Z. Pick $(\tilde{M}, \tilde{Z})$ a 2:1 orientable covering of the b^{2k}-surface (M, Z), with $\tau : \mathbb{Z}/2\mathbb{Z} \times M \to M$ the deck transformation. For each point

$p \in \tilde{Z}$ we can find a b^{2k}-Darboux neighborhood U_p (by shrinking the neighborhood if necessary) which does not contain other points identified by τ. Thus $U_p \approx \pi(U_p) =: V_p$, and $\omega = \frac{1}{x^{2k}} dx \wedge dy$. Being ω of this type, it defines an orientation on $V_p \setminus \pi(Z \cup U_p)$. Since $\tilde{Z}$ is compact we can take a covering for such neighborhoods to define a collar V of compatible orientations. Furthermore we can assume this covering to
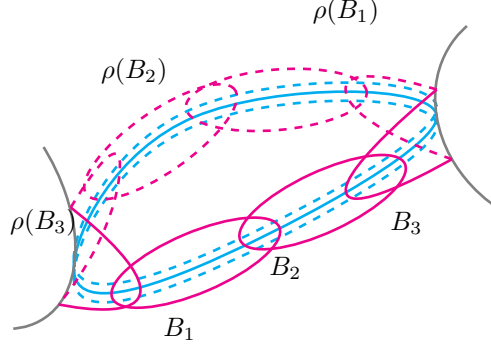

FIGURE 5. A collar of compatible neighborhoods.

be symmetric, that is that for each U_p the image $\tau(U_p)$ is also one of the subsets of the covering. This covering will be compatible with the orientation because τ preserves ω. The compatible orientations and the symmetric coverings descend to (M, Z), thus defining an orientation in (M, Z). Thus, we have an orientation in $V \setminus \tilde{Z}$. Then, by perturbing ω in V we obtain a symplectic structure on V $\tilde{\omega}$ and thus an orientation in V. Now, using the standard techniques of Radko [R] these can be glued to define an orientation via the symplectic form $\tilde{\omega}$ on the whole M, In the case Z has more than one connected component we may proceed in the same way by isolating collar neighborhoods of each component. This proves that M is oriented.

$\square$

Theorem 5.6. *Given a pair (S, Z) there exists a b^m-symplectic structure with critical set Z whenever:*

- *$m = 2k$,*
- *$m = 2k + 1$ if only if the associated graph is 2-colorable.*

For the proof we will need the following,

Lemma 5.7 (**Weinstein normal form theorem**). *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω), then there exists a neighborhood of L, U_L which is symplectomorphic to a neighborhood of the zero section of the cotangent bundle of T^*L endowed with the symplectic form $-d\lambda$ with λ the Liouville one form.*

Proof. (of Theorem 5.6)

Let $C_1, \ldots, C_r$ the connected components of $S \setminus Z$, let $Z_1, \ldots, Z_s$ the connected components of Z and let $\mathcal{U}(Z_1), \ldots, \mathcal{U}(Z_s)$ tubular neighborhoods of the connected components.

We can prove this using a 3-step proof.

- **Step 1: Using Weinstein normal form theorem:** By virtue of lemma 5.7, each tubular neighborhood $\mathcal{U}(Z_i)$ can be identified with a zero section of the cotangent bundle of Z_i. Now replace, the cotangent bundle of Z_i by the b^k-cotangent bundle of Z_i. The neighborhood of the zero section of the b^k-cotangent bundle has a b^k-symplectic structure $\omega_{\mathcal{U}(Z_i)}$.
- **Step 2: Constructing compatible orientation using the graph** We assign a couple of signs to each tubular neighbourhood using the sign of the Liouville volume of $\omega_{\mathcal{U}(Z_i)}$. Note that the sign does not change for k even, and it does change for odd k. Observe that we can apply Moser's trick to glue two rings that share some C_j if and only if the sign of the two ring match on this component. In terms of coloring graphs, this condition translates into one vertex only having one color. In other words the condition of adjacent signs matching determines the color of a vertex.

  Now, let us consider separately the odd an even cases:
  - For b^{2k} the color of adjacent vertices must coincide. And hence we have no additional constraint on the topology of the graph.
  - In the b^{2k+1} case the sign of two adjacent vertices must be different. Then, we have to impose the associated graph to be 2-colorable.
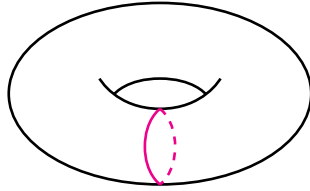

FIGURE 6. Pair (M, Z) for which there is no b^{2k+1} symplectic structure.

- **Step 3:** Now we may glue back this neighborhood to $S \setminus \mathcal{U}_L$ in such a way that the symplectic structures fit on the boundary, by means of the Moser's path method.
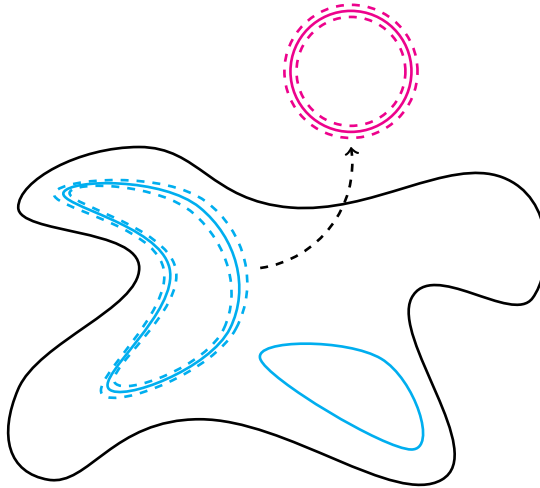

FIGURE 7. Applying the Weinstein normal form theorem for Lagrangian submanifolds, replacing the cotangent bundle by the b-cotangent bundle, and then applying the Moser's path method inductively on the connected components of Z.

$\square$

Remark 5.8. Another way to construct b^{2k}-structures on a surface is to use decomposition theorem as connected sum of b^{2k}-spheres (3) and b^{2k}-torus (3). The drawback of this construction is that it is harder to adapt having fixed a prescribed Z.
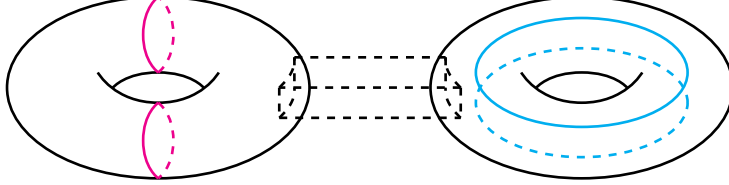

FIGURE 8. The connected sum of two b^{2k}-tori with different b^{2k}-symplectic forms.

Theorem 5.9. *Any pair (S, Z) with S a non-orientable surface and Z an hypersurface without self-intersections admits a b^{2k+1}-symplectic structure with Z as its associated set if and only if the graph of the covering $(\tilde{S}, \tilde{Z})$ is 2-colorable.*

Proof. Apply 5.6 to the covering $(\tilde{S}, \tilde{Z})$ a b^{2k+1}-symplectic structure, and then apply the equivariant b^m-Moser theorem 4.1 to the associated deck transformation to conclude the proof. $\square$

6. CONSTRUCTIONS AND CLASSIFICATION OF b^m-NAMBU STRUCTURES

Let us start defining a b^m-Nambu structure of top degree,

Definition 6.1. A b^m-Nambu structure of top degree on a pair (M^n, Z) with Z a smooth hypersurface is given by a smooth n-multivector field Θ such that exists a local system of coordinates for which

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$$

and Z is locally defined by $x_1 = 0$.

Dualizing the local expression of the Nambu structure we obtain the form

$$\Theta = \frac{1}{x_1^m} dx_1 \wedge \ldots \wedge dx_n$$

(which is not a smooth de Rham form), but it is a b^m-form of degree n defined on a b^m-manifold. As it is done in [GMP2], we can check that this dual form is non-degenerate. So we may define a b^m-Nambu form as follows.

Mimicking the same condition as for b^m-symplectic forms we can talk about non-degenerate b^m-forms of top degree. This means that seen as a section of $\Lambda^n(^bT^*M)$ the form does not vanish.

Notation: We will denote by Λ the Nambu multivectorfield and by Θ its dual.

Definition 6.2. A b^m-Nambu form is a non-degenerate b^m-form of top degree.

6.1. **Examples.** We first include a collection of motivating examples, and then prove an equivariant classification theorem.

- b^m-**symplectic surfaces:** Any b^m-symplectic surface is a b^m-Nambu manifold with Nambu structure of top degree.
- b^m-**symplectic manifolds as b^{nm}-Nambu manifolds:** Let (M^{2n}, ω) be a b^m-symplectic manifold, then $(M^{2n}, \omega \wedge \cdots \wedge \omega)$ is automatically b^{nm}-Nambu.
- **Orientable manifolds:** Let (M^n, Ω) be any orientable manifold (with Ω a volume form) and let f be a regular global function on M^d, then $(1/f^m)\omega$ defines a b^m-Nambu structure of top degree.

  Deciding whether any Nambu structure can be written in this way is equivalent to deciding whether an hypersurface can be globally described as the vanishing set of a smooth function.
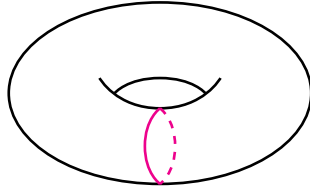

FIGURE 9. Example of a b^{2k}-symplectic torus for which the critical set Z does not admit a global defining function f.

- **Spheres:** In [Mt] the example $(S^n, \sqcup_i S_i^{(n-1)})$ was given a special importance because of the Schoenflies Theorem, that imposes the associated graph to be a tree. The nice feature of this example is that $O(n)$ acts on the b^m-manifold $(S^n, S^{(n-1)})$, and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type $(G_1/G_2, G_2/G_3)$ with G_2 and G_3 with codimension 1 in G_1 and G_2 respectively.

6.2. b^m-**Nambu structures of top degree and orientability.** As we did in the case of b^{2k}-symplectic structure we can prove the following theorem:

Theorem 6.3. *A compact n-dimensional manifold M admitting a b^{2k}-Nambu structure is orientable.*

Proof. The proof consists in building a collar of b^m-Darboux charts for the b^{2k}-Nambu structure (such that in local coordinates the Nambu structure can be written as $x_1^{2k} \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$) with compatible orientations in a neighborhood of each connected component of Z.

Consider a 2:1 orientable covering $(\tilde{M}, \tilde{Z})$ of the manifold and denote by $\tau : \mathbb{Z}/2\mathbb{Z} \times M \to M$ the deck transformation. For each point $p \in \tilde{Z}$ we provide a Darboux neighborhood U_p which does not contain other points identified by τ. Thus $U_p \approx \pi(U_p) =: V_p$, and $\Theta = \frac{1}{x^{2k}} dx_1 \wedge \ldots \wedge dx_n$. This form defines an orientation on $V_p \setminus \pi(Z \cup U_p)$. Take a symmetric covering of such neighborhoods to define a collar of Z with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to (M, Z), thus defining an orientation in (M, Z). Thus, we have an orientation in $V \setminus \tilde{Z}$. Then, by perturbing ω in V we obtain a volume form on V, $\tilde{\omega}$, and thus an orientation in

V. Now, these can be glued to define an orientation via the volume form $\tilde{\Theta}$ on the whole M proving that M is oriented. $\square$

6.3. Classification of b^m-Nambu structures of top degree and b-cohomology.
We present the definitions contained in [Mt] of modular period attached to the connected component of an orientable Nambu structure using the language of b^m-forms.

Let Θ be the dual to the multivectorfield Λ defining a Nambu structure.

From the general decomposition of b^m-forms as it was set in equation 1 we may write.

$$\Theta = \Theta_0 \wedge \frac{df}{f}, \text{ with } \Theta_0 \in \Omega^{n-1}. \tag{9}$$

This decomposition is valid in a neighborhood of Z whenever the defining function is well-defined. Otherwise for non-orientable manifolds a similar decomposition can be proved by replacing the defining function f by an adapted distance (see [MO2]).

With this language in mind, the the **modular $(n-1)$-vector field** in [Mt] of Θ along Z is the dual of the form Θ_0 in the decomposition above which is indeed the **modular $(n-1)$-form** along Z in [Mt].

Recall from [Mt] in our language:

Definition 6.4. The **modular period** T_Λ^Z of the component Z of the zero locus of Λ is

$$T_\Lambda^Z \approx \int_Z \Theta_0 > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of Z up to isotopy as it was proved in [Mt].

The following theorem gives a classification of b^m-Nambu structures.

We will prove a more general result for non-necessarily orientable manifolds admitting a Nambu structure of top degree. We do it using the dual Nambu forms associated to the multivectorfields.

Theorem 6.5. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof. We will apply the same techniques from [M] with the only difference that we work with b^m-volume forms instead of volume forms.

Since Θ_0 and Θ_1 are non-degenerate b^m-forms both of them are multiple of a volume form and thus the linear path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of non-degenerate b^m-forms.

Since Θ_0 and Θ_1 determine the same cohomology class:

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$.

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta \tag{10}$$

Observe that since β is a b^m-form and Θ_t is non-degenerate. The vector field X_t is a b-vector field. Let ϕ_t be the t-dependent flow integrating X_t.

The ϕ_t gives the desired diffeomorphism $\phi_t : M \to M$, leaving Z invariant (since X_t is tangent to Z) and $\phi_t^*\Theta_t = \Theta_0$. $\square$

In particular we recover the classification of b-Nambu structures of top degree in [Mt]:

Theorem 6.6 (**Classification of b-Nambu structures of top degree,** [Mt]). *A generic b Nambu structure Θ is determined, up to orientation preserving diffeomorphism, by the following three invariants: the the diffeomorphism type of the oriented pair (M, Z), the modular periods and the regularized Liouville volume.*

By 2.10,

$$^bH^n(M) \cong H^n(M) \oplus H^{n-1}(Z)$$

The first term on the right is the Liouville volume image by the De Rham theorem, as it was done in in [GMPS] for b-symplectic forms the second term collects the periods of the modular vector field. So if the three invariants coincide then they determine the same b-cohomology class.

In other words, the statement in [Mt] is equivalent to the following theorem in the language of b-cohomology.

Theorem 6.7. *Let Θ_1 and Θ_2 be two b-Nambu forms on an orientable manifold M. If $[\Theta_1] = [\Theta_2]$ in b-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_2$.*

This global Moser theorem for b^m-Nambu structures admits an equivariant version,

Theorem 6.8. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n and let $\rho : G \times M \longrightarrow M$ be a compact Lie group action preserving both b^m-forms. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists an equivariant diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof. As in the former proof, write

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$. Observe that the path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of invariant b^m-forms.

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta \tag{11}$$

Since Θ_t is invariant we can find an invariant $\tilde{\beta}$.(For instance take $\tilde{\beta} = \int_G \rho_g^*(\beta)d\mu$ with μ a de Haar measure on G and ρ_g the induced diffeomorphism $\rho_g(x) := \rho(g, x)$.)

Now replace β by $\tilde{\beta}$ to obtain,

$$\iota_{X_t^G}\Theta_t = -\tilde{\beta} \tag{12}$$

with $X_t^G = \int_G \rho_{g*} X_t d\mu$. The vector field X_t^G is an invariant b-vector field. Its flow ϕ_t^G preserves the action and $\phi_t^{G*}\Theta_t = \Theta_0$.

$\square$

Playing the equivariant b^m-Moser trick as we did in section 4 we obtain,

Corollary 6.9. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

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