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Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Green functions on product networks

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ARTICLE INFO

Article history:
Received 29 September 2017
Received in revised form 27 September 2018
Accepted 1 October 2018
Available online xxxx

Keywords:
Discrete potential theory
Schrödinger operators
Green functions
Product networks
Spectra
Separation of variables

ABSTRACT

We aim here at determining the Green function for general Schrödinger operators on product networks. The first step consists in expressing Schrödinger operators on a product network as sum of appropriate Schrödinger operators on each factor network. Hence, we apply the philosophy of the separation of variables method in PDE, to express the Green function for the Schrödinger operator on a product network using Green functions on one of the factors and the eigenvalues and eigenfunctions of some Schrödinger operator on the other factor network. We emphasize that our method only needs the knowledge of eigenvalues and eigenfunctions of one of the factors, whereas other previous works need the spectral information of both factors. We apply our results to compute the Green function of $P_m \times S_h$, where P_m is a Path with m vertices and S_h is a Star network with $h+1$ vertices.

1. Introduction

Green's functions on a network are closely related with self-adjoint boundary value problems for Schrödinger operators. Although there exists a very interesting variety of such a boundary value problems, see for instance [3], we restrict ourselves here to analyze either the *Dirichlet Problem* or the *Poisson equation*. As mentioned in a recent paper by A. Gilbert *et alt.*: "The idea of discrete Green's functions has, implicitly or explicitly, a long history arising in many important problems and fields such as the study of inverses of tri-diagonal matrices, potential theory, the study of Schrödinger operators on graphs, and the graph-theoretic analog of Poisson's equation. Additionally, Green's function methods have yielded interesting results in many areas including the properties of random walks, chip-firing games, analysis of online communities, machine learning algorithms and load balancing in networks.", see [15] and references therein.

In spite of its importance, only few explicit expressions for Green's functions associated with Schrödinger operators on very structured networks, are known. The most common technique to get these expressions consists in using the spectral decomposition property. So, in general, determining Green's functions is a very difficult task. Another strategy to compute these elements is to split the network into small and structured pieces and then to express the eigenvalues and eigenfunctions in terms of those corresponding to each piece. Since composite networks as join, corona or cluster have been studied in a very general setting, see [1,5] and also [16] for graphs, we analyze here the case of *cartesian product* of networks. As we show, our treatment is the discrete version of the *Fourier Method*, also called, *Separation of Variables Method*. We first prove that when we consider weights that are tensor product of weights, then the corresponding Schrödinger operators are expressed in separated variables and hence the Fourier Method fits accurately.

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https://doi.org/10.1016/j.dam.2018.10.004

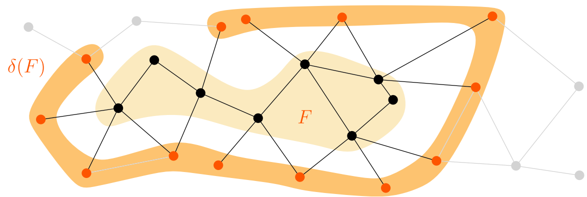

Fig. 1. A vertex set F (•) and its boundary $\delta(F)$ (•).

This class of problems have been also studied by F. Chung, R. Ellis and S.T. Yau, see [11,13,14], considering the normalized Laplacian. However, since in general the normalized Laplacian of a product network cannot be described in separated variables involving the normalized Laplacian of the factor networks, in the above referred works the authors must consider only cartesian product of regular networks, that is also a regular network. We remark that in this case, the problem is reduced to the analysis of the combinatorial Laplacian, since for regular networks the normalized Laplacian is a multiple of the combinatorial one. We treat here with Schrödinger operators on the product network without any assumption on the regularity of each factor network, but under the hypothesis that the potential is related with tensor product of weights. We emphasize that the application of the separation of variable method only requires the knowledge of eigenvalues and eigenfunctions of one of the factors. So, applying our results we can get explicit expressions for Green's functions in a wide range of product networks.

A finite network $\Gamma = (V, c)$, consists of a finite set V, called *vertex set* and a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$, called *conductance*, satisfying that $c(x, x) = 0$ for any $x \in V$. Two vertices $x, y \in V$ are *adjacent* iff $c(x, y) > 0$.

We always assume that Γ is *connected*; that is, that for any pair of different vertices $x, y \in V$, there exist $m \in \mathbb{N}^*$ and $x_0, \dots, x_m \in V$ such that $x = x_0$, $y = x_m$ and $\prod_{j=0}^{m-1} c(x_j, x_{j+1}) > 0$.

In what follows $\mathcal{C}(V) = \mathcal{C}(V;\mathbb{R})$ and $\mathcal{C}(V;\mathbb{C})$ stand respectively for the spaces of real and complex functions defined on the vertex set V. Given $v \in \mathcal{C}(V;\mathbb{C})$, $\bar{v}$ denotes its conjugate and then, $\langle u, v\rangle = \sum_{x\in V} u(x)\bar{v}(x)$ determines an inner product on $\mathcal{C}(V;\mathbb{C})$, whose associated norm is denoted by $\|\cdot\|$. Therefore, $\|u\| = \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}}$ for any $u \in \mathcal{C}(V;\mathbb{C})$. Clearly, this inner product induces the standard one on $\mathcal{C}(V)$. Given $u \in \mathcal{C}(V,\mathbb{C})$, $u^{\perp}$ denotes the subspace of $\mathcal{C}(V,\mathbb{C})$ orthogonal to u.

A real-valued function $\omega \in \mathcal{C}(V)$ is called *weight* if $\omega(x) > 0$ for any $x \in V$ and in addition $\|\omega\| = 1$. The sets of weights on V is denoted by $\Omega(V)$ or simply by Ω when it does not lead to confusion. Clearly the weight ν defined as $\nu(x) = |V|^{-\frac{1}{2}}$, $x \in V$, is the unique constant weight on V.

For any $x \in V$, ε_x is the *Dirac function* at x. Clearly $\varepsilon_x \in \mathcal{C}(V)$ for any $x \in V$. Moreover, κ denotes the (generalized) *degree of* Γ; that is, the function defined as $\kappa(x) = \sum_{y\in V} c(x, y)$, for any $x \in V$. The network is called *regular* when its degree is a constant function. The *volume* of the network Γ is $\mathsf{v} = \sum_{x\in V}\kappa(x)$. Since Γ is connected $\mathsf{v}^{-\frac{1}{2}}\sqrt{\kappa}$ is a weight, called the *volume weight*.

Given $F \subset V$ a nonempty subset, F^c denotes its complementary and $\mathcal{C}(F)$ and $\mathcal{C}(F;\mathbb{C})$ are the subspaces of real and complex functions vanishing on F^c. It is clear that $\mathcal{C}(F)$ and $\mathcal{C}(F;\mathbb{C})$ can be identified respectively with the space of real or complex functions defined on F. Moreover, the set

$$\delta(F) = \left\{z \in F^c : c(z, y) > 0 \ \text{ for some } \ y \in F\right\}$$

is called the *boundary of* F and then, $\bar{F} = F \cup \delta(F)$ is the *closure* F, see Fig. 1. Clearly, $\delta(F) = \emptyset$, or equivalently $F = \bar{F}$, iff $F = V$.

Analogously, given $F \subset V$, each function $K\colon F \times F \longrightarrow \mathbb{R}$ can be identified as a function on $V \times V$ vanishing outside of $F \times F$. The above function is called *symmetric* if it satisfies that $K(x, y) = K(y, x)$ for any $x, y \in F$. Clearly if we consider K extended by 0 on $(F \times F)^c$, then K is symmetric on $V \times V$ iff it is symmetric on $F \times F$.

The *combinatorial Laplacian* of Γ, or simply the *Laplacian of* Γ, is the linear operator $\mathcal{L}\colon \mathcal{C}(V;\mathbb{C}) \longrightarrow \mathcal{C}(V;\mathbb{C})$ that assigns to any $u \in \mathcal{C}(V;\mathbb{C})$ the function $\mathcal{L}(u)$ defined as

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x, y)\left(u(x) - u(y)\right), \quad x \in V.$$

More generally, given $q \in \mathcal{C}(V;\mathbb{C})$, the *Schrödinger* operator with *potential* q, see [7], is $\mathcal{L}_q\colon \mathcal{C}(V;\mathbb{C}) \longrightarrow \mathcal{C}(V;\mathbb{C})$ defined as $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$ for any $u \in \mathcal{C}(V;\mathbb{C})$. The Schrödinger operator whose potential is the conjugate of q; that is, $\mathcal{L}_{\bar{q}}$, is called *the adjoint of* $\mathcal{L}_q$ since it satisfies that $\langle \mathcal{L}_q(u), v\rangle = \langle u, \mathcal{L}_{\bar{q}}(v)\rangle$ for any $u, v \in \mathcal{C}(V;\mathbb{C})$.

For a given nonempty subset $F \subset V$ and a given potential $q \in \mathcal{C}(V;\mathbb{C})$ we consider the following *Boundary Value Problem*: *Given* $f \in \mathcal{C}(F;\mathbb{C})$ *and* $g \in \mathcal{C}(\delta(F);\mathbb{C})$, *find* $u \in \mathcal{C}(\bar{F};\mathbb{C})$ *such that*

$$\mathcal{L}_q(u) = f \ \text{ on } F, \quad u = g \ \text{ on } \delta(F). \tag{1}$$

When $F \neq V$, this problem is known as *Dirichlet Problem on* F, whereas when $F = V$ it is called *Poisson equation on* V. In this last case the data g has no sense, since then $\delta(F) = \emptyset$.

When $F \neq V$, each Dirichlet problem on F is equivalent to a semihomogeneous Dirichlet problem. Specifically, $u \in \mathcal{C}(\bar{F};\mathbb{C})$ is a solution of Problem (1) iff $v = u - g$ is a solution of the Dirichlet problem

$$\mathcal{L}_q(u) = f - \mathcal{L}(g) \text{ on } F, \quad u = 0 \text{ on } \delta(F). \tag{2}$$

Therefore, to analyze the existence and uniqueness of solution of the boundary value problem for any $f \in \mathcal{C}(F;\mathbb{C})$ is equivalent to analyze the same topics for the following problem:

$$\text{Given } f \in \mathcal{C}(F;\mathbb{C}), \text{ find } u \in \mathcal{C}(F;\mathbb{C}) \text{ such that } \mathcal{L}_q(u) = f \text{ on } F. \tag{3}$$

This formulation encompasses both, Dirichlet problems and Poisson equations; the last ones appear when $F = V$.

Notice that Schrödinger operators with real-valued potential; that is, $\mathcal{L}_q$ with $q \in \mathcal{C}(V)$ are also endomorphisms on $\mathcal{C}(V)$ and moreover, they are self-adjoint; since $\langle \mathcal{L}_q(u), v\rangle = \langle u, \mathcal{L}_q(v)\rangle$ for any $u, v \in \mathcal{C}(V;\mathbb{C})$. In particular, $\sum_{x\in F} \mathcal{L}_q(u)(x)v(x) = \sum_{x\in F} \mathcal{L}_q(v)(x)u(x)$ for any $u, v \in \mathcal{C}(F;\mathbb{C})$; that means that Problem (3) is self-adjoint on $\mathcal{C}(F;\mathbb{C})$ when the potential is real-valued.

This work is mainly concerned with Schrödinger operators with real-valued potentials and for this reason we usually consider only real-valued functions; that is, the space $\mathcal{C}(V)$. Moreover, in this case we also consider the *Energy for the potential* q that is the quadratic form $\mathcal{E}_q : \mathcal{C}(V) \longrightarrow \mathbb{R}$ that assigns to any $u \in \mathcal{C}(V)$ the value

$$\mathcal{E}_q(u) = \langle \mathcal{L}_q(u), u\rangle = \frac{1}{2}\sum_{x,y\in V} c(x,y)\big(u(x)-u(y)\big)^2 + \sum_{x\in V} q(x)u(x)^2.$$

2. Real-valued potentials and Doob transforms

For any weight $\omega \in \Omega$, we call the function $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$ *Doob potential associated with* ω. Therefore,

$$q_\omega(x) = -\kappa(x) + \omega(x)^{-1}\sum_{y\in V} c(x,y)\omega(y) > -\kappa(x), \quad \text{for any } x \in V.$$

Given two weights $\sigma, \omega \in \Omega$, then $q_\sigma \geq q_\omega$ iff $q_\sigma = q_\omega$ and this happens iff $\sigma = \omega$, see [4, Lemma 2.1]. In particular, $q_\sigma = 0$ iff σ is constant and hence, q_σ takes positive and negative values when $\sigma \in \Omega$ is not constant. Notice that $\langle \sigma, q_\sigma\rangle = -\langle 1, \mathcal{L}(\sigma)\rangle = 0$.

Although at first glance Doob transforms could seem a bit strange and Doob potentials a very specific kind of potentials, they play a main role among real-valued potentials. In fact, as a consequence of the *Perron–Frobenius Theory*, given a real-valued potential $q \in \mathcal{C}(V)$ there exist a unique unitary weight $\omega \in \Omega$ and a unique real value $\lambda \in \mathbb{R}$ such that $q = q_\omega + \lambda$, see [2]. The following result involving Doob potentials has been strongly used by the authors, see for instance [2,4].

Proposition 2.1 (*Doob Transform*)**.** *Let a real-valued potential* q *and consider* $\omega \in \Omega$ *and* $\lambda \in \mathbb{R}$ *such that* $q = q_\omega + \lambda$*. Then, if* $F \subset V$ *is a non-empty subset, for any* $u \in \mathcal{C}(F)$ *we have that*

$$\mathcal{L}_q(u)(x) = \frac{1}{\omega(x)}\sum_{y\in\bar{F}} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right) + \lambda\, u(x), \quad x \in F$$
$$\mathcal{E}_q(u) = \frac{1}{2}\sum_{x,y\in\bar{F}} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right)^2 + \lambda\|u\|^2.$$

As we will show, these expressions have interesting consequences in the treatment of the boundary value problems we have raised.

We first remark that the well-known normalized Laplacian introduced in 1996 by F. Chung and R. Langlands, see [9–11], is nothing else but a Schrödinger operator on an appropriate network. The *normalized Laplacian for the network* $\Gamma = (V, c)$ is the operator $\mathscr{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $u \in \mathcal{C}(V)$ the function

$$\mathscr{L}(u)(x) = \frac{1}{\sqrt{\kappa(x)}}\sum_{x,y\in V} c(x,y)\left(\frac{u(x)}{\sqrt{\kappa(x)}} - \frac{u(y)}{\sqrt{\kappa(y)}}\right), \quad x \in V.$$

Therefore, if $\mathcal{L}$ is the combinatorial Laplacian of Γ and $\mathcal{T} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ is given by $\mathcal{T}(u) = \sqrt{\kappa}\, u$, we get that $\mathscr{L} = \mathcal{T}^{-1} \circ \mathcal{L} \circ \mathcal{T}^{-1}$.

Moreover, we consider now the network $\widehat{\Gamma} = (V, \hat{c})$, where $\hat{c}(x,y) = \dfrac{c(x,y)}{\sqrt{\kappa(x)\kappa(y)}}$ for any $x, y \in V$ and $\hat{\mathcal{L}}$ its combinatorial Laplacian. Notice that any pair of vertices $x, y \in V$ are adjacent in Γ iff they are adjacent in $\widehat{\Gamma}$, so the graphs subjacent to Γ and to $\widehat{\Gamma}$ are the same. Choosing the *volume weight of* Γ, $\omega = \mathrm{v}^{-\frac{1}{2}}\sqrt{\kappa}$, from the expression for $\hat{\mathcal{L}}_{q_\omega}$ obtained after the Doob Transform associated with ω, for any $u \in \mathcal{C}(V)$ and any $x \in V$, we have

$$\hat{\mathcal{L}}_{q_\omega}(u)(x) = \frac{\sqrt{\mathrm{v}}}{\sqrt{\kappa(x)}}\sum_{y\in V}\frac{1}{\mathrm{v}}\hat{c}(x,y)\sqrt{\kappa(x)}\sqrt{\kappa(y)}\left(\frac{\sqrt{\mathrm{v}}u(x)}{\sqrt{\kappa(x)}} - \frac{\sqrt{\mathrm{v}}u(y)}{\sqrt{\kappa(y)}}\right) = \mathscr{L}(u)(x),$$

and hence, the normalized Laplacian $\mathscr{L}$ on Γ is equivalent to the Schrödinger operator $\hat{\mathcal{L}}_{q_\omega}$ on $\widehat{\Gamma}$.

On the other hand, from the expression for the energy obtained after the Doob Transform, we have that

$$\min_{\substack{u\in\mathcal{C}(F)\\ \|u\|=1}} \left\{\mathcal{E}_q(u)\right\} \geq \lambda \tag{4}$$

and the equality holds iff $F = V$. In this case, $\mathcal{E}_q$ attains its minimum at $u = \pm\omega$. Therefore, the Schrödinger operator $\mathcal{L}_q$ is positive semidefinite on $\mathcal{C}(V)$; that is, its energy is non-negative, iff $\lambda \geq 0$ and positive definite on $\mathcal{C}(V)$ iff $\lambda > 0$. In addition, when $\lambda \geq 0$, the Schrödinger operator $\mathcal{L}_q$ is positive definite on $\mathcal{C}(F)$ for any proper subset F.

The variational characterization of the solutions for the boundary value problems (3) is described in the following result, see [4, Proposition 3.5] for its proof.

Proposition 2.2 (*Dirichlet Principle*). *Let $F \subset V$ be a non empty subset, $\omega \in \Omega$, $\lambda \geq 0$ and the potential $q = q_\omega + \lambda$. Given $f \in \mathcal{C}(F)$ consider the quadratic functional $\mathcal{J} : \mathcal{C}(V) \longrightarrow \mathbb{R}$ given by*

$$\mathcal{J}(u) = \mathcal{E}_q(u) - 2\langle f, u\rangle.$$

Then $u \in \mathcal{C}(F)$ satisfies that $\mathcal{L}_q(u) = f$ on F iff it minimizes $\mathcal{J}$ on $\mathcal{C}(F)$. Moreover $\mathcal{J}$ has a unique minimum except when $F = V$ and $\lambda = 0$ simultaneously. In this case $\mathcal{J}$ has a minimum iff $f \in \omega^\perp$ and moreover, there exists a unique minimum belonging to $\omega^\perp$.

3. Green functions, eigenvalues and eigenfunctions

In this section we consider fixed the finite and connected network $\Gamma = (V, c)$, a weight $\omega \in \Omega$, a non-negative value $\lambda \geq 0$, the real-valued potential $q = q_\omega + \lambda$ and its corresponding Schrödinger operator $\mathcal{L}_q$. Under these hypotheses, for any proper subset $F \subset V$ and any $f \in \mathcal{C}(F)$, Dirichlet Problem (3) has a unique solution; that is, there exists a unique $u \in \mathcal{C}(F)$ such that $\mathcal{L}_q(u) = f$ on F. Moreover, when $\lambda > 0$ for any $f \in \mathcal{C}(V)$, Poisson equation (3) has a unique solution; that is, there exists a unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f$ on V.

When either $F \subset V$ is a proper subset or $\lambda > 0$, the *Green function of F for the potential q* is $G_q^F : F \times F \longrightarrow \mathbb{R}$ such that for any $y \in F$, $G_q^F(\cdot, y)$ is the unique solution of the Dirichlet Problem $\mathcal{L}_q(u) = \varepsilon_y$ on F, $u = 0$ en $\delta(F)$, when F is proper, or the Poisson equation $\mathcal{L}_q(u) = \varepsilon_y$ on V when $F = V$ but $\lambda > 0$.

The *Green operator of F for the potential q* is $\mathcal{G}_q^F : \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ defined for any $f \in \mathcal{C}(F)$ as $\mathcal{G}_q^F(f)(x) = \sum_{y\in F} G_q^F(x, y)f(y)$, $x \in F$. Then $\mathcal{G}_q^F$ is self-adjoint and for any $f \in \mathcal{C}(F)$, the function $u = \mathcal{G}_q^F(f) \in \mathcal{C}(F)$ satisfies that $\mathcal{L}_q(u) = f$ on F. Since $\mathcal{G}_q^F$ is a self-adjoint operator, then G_q^F is a symmetric function and the Minimum Principle also implies that $0 < \omega(y)G_q^F(x, y) < \omega(x)G_q^F(y, y)$ for any $x, y \in F$, see for instance [4].

When $\lambda = 0$, then $q = q_\omega$ and the Poisson equation $\mathcal{L}_q(u) = f$ on V is solvable only if $f \in \omega^\perp$ and in this case, there exists a unique solution belonging to $\omega^\perp$. The *Green function of V for the potential q* is $G_q^V : V \times V \longrightarrow \mathbb{R}$ such that for any $y \in V$, $G_q^V(\cdot, y)$ is the unique solution of the Poisson equation $\mathcal{L}_q(u) = \varepsilon_y - \omega(y)\omega$ belonging to $\omega^\perp$.

The *Green operator of V for the potential q* is $\mathcal{G}_q^V : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ defined for any $f \in \mathcal{C}(V)$ as $\mathcal{G}_q^V(f)(x) = \sum_{y\in V} G_q^V(x, y)f(y)$, $x \in V$. Then for any $f \in \mathcal{C}(V)$, $\mathcal{G}_q^V(f) = \mathcal{G}_q^V(f - \langle\omega, f\rangle\,\omega)$, $\mathcal{G}_q^V$ is self-adjoint and the function $u = \mathcal{G}_q^V(f) \in \mathcal{C}(V)$ is the unique function in $\omega^\perp$ satisfying that $\mathcal{L}_q(u) = f - \langle\omega, f\rangle\,\omega$. Newly, the self-adjointness of $\mathcal{G}_q^V$ implies that G_q^V is a symmetric function and the Minimum Principle also implies that $\omega(y)G_q^V(x, y) < \omega(x)G_q^V(y, y)$ for any $x, y \in V$, see newly [4].

We remark that the existence and uniqueness of solution for the boundary value problem (3) means that $\mathcal{L}_q$ is an automorphism of $\mathcal{C}(F)$ and hence, $\mathcal{G}_q^F$ is its inverse. Moreover, when $\lambda = 0$, then $\mathcal{L}_q$ is an automorphism of $\omega^\perp$ whose inverse can be extended to $\mathcal{C}(V)$ by considering for any $f \in \mathcal{C}(V)$ its orthogonal component with respect to ω. This extension is precisely $\mathcal{G}_q^V$ and clearly is singular, since $\mathcal{G}_q^V(\omega) = 0$.

On the other hand, if we label the vertices of Γ, say $V = \{x_1, \ldots, x_n\}$ where $n = |V|$, then each endomorphism of $\mathcal{C}(F)$ can be interpreted as a matrix of order $|F|$. So $\mathcal{L}_q$ is identified with the matrix L_q^V whose diagonal entries are $\kappa(x_j) + q(x_j)$ and whose off-diagonal entries are $-c(x_i, x_j)$, $i, j = 1, \ldots, n$. Moreover if for a proper subset $F \subset V$, we interpret $\mathcal{L}_q$ as an endomorphism of $\mathcal{C}(F)$, then it can be identified with the matrix L_q^F obtained from L_q^V by deleting the rows and the columns corresponding to the vertices in F^c. Notice that, as the potential is real-valued, all the above matrices are real-valued and symmetric.

We also denote by G_q^F the matrix identified with the Green operator $\mathcal{G}_q^F$ defined above. With these identifications, $\mathsf{G}_q^F = \left(\mathsf{L}_q^F\right)^{-1}$ when either F is a proper subset of V or $\lambda > 0$. Moreover, when $\lambda = 0$, then $\mathsf{G}_q^V = \left(\mathsf{L}_q^V\right)^{\#}$, the *Group Inverse of L_q^V*. Since the group inverse coincides with the inverse when the matrix is invertible, we have that $\mathsf{G}_q^F = \left(\mathsf{L}_q^F\right)^{\#}$ for any non-empty subset $F \subset V$ and any $\lambda \geq 0$.

Given a non-empty subset $F \subset V$, an *eigenvalue of the boundary problem* (3) is $z \in \mathbb{C}$ such that the Schrödinder operator $\mathcal{L}_{q-z}$ is singular on $\mathcal{C}(F; \mathbb{C})$. Equivalently, $z \in \mathbb{C}$ is an eigenvalue of the boundary problem (3) if there exists $u \in \mathcal{C}(F; \mathbb{C})$ non-null and such that $\mathcal{L}_q(u) = z\,u$ on F. Each $u \in \mathcal{C}(F; \mathbb{C})$ satisfying the above identity is called *eigenfunction of the boundary problem* (3) *associated with z*.

Since q is a real-valued potential, any eigenvalue must be real. This claim follows by taking into account that if $u \in \mathcal{C}(F; \mathbb{C})$ is non null and satisfies that $\mathcal{L}_q(u) = z\,u$ on F, then

$$z\|u\|^2 = \langle\mathcal{L}_q(u), u\rangle = \langle u, \mathcal{L}_{\bar{q}}(u)\rangle = \langle u, \mathcal{L}_q(u)\rangle = \bar{z}\|u\|^2$$

which implies $z = \bar{z}$; that is $z \in \mathbb{R}$. On the other hand, if $u, v \in \mathcal{C}(F, \mathbb{C})$ are eigenfunctions corresponding to z and $\hat{z}$ respectively, then

$$z\langle u, v\rangle = \langle \mathcal{L}_q(u), v\rangle = \langle u, \mathcal{L}_q(v)\rangle = \hat{z}\langle, u, v\rangle$$

which implies that if $z \neq \hat{z}$, then $\langle u, v\rangle = 0$. In particular, if $u \in \mathcal{C}(F, \mathbb{C})$ is an eigenfunction corresponding to $z \in \mathbb{R}$, then u is a real-valued function; that is, $u \in \mathcal{C}(F)$.

If $z \in \mathbb{C}$ is not an eigenvalue of the boundary value problem (3), then $\mathcal{L}_{q-z}$ is an automorphism of $\mathcal{C}(F; \mathbb{C})$ and then we denote by $\mathcal{G}^F_{q-z}$ its inverse. Moreover, if $G^F_{q-z}: F \times F \longrightarrow \mathbb{R}$ is given for any $y \in V$ as $G^F_{q-z}(\cdot, y)$, the unique solution of the equation $\mathcal{L}_q(u) = \varepsilon_y$ on F, then $\mathcal{G}^F_{q-z}(f)(x) = \sum_{y\in F} G^F_{q-z}(x, y)f(y)$, for any $f \in \mathcal{C}(F, \mathbb{C})$ and any $x \in F$.

The following result is the discrete version of the well-known *Spectral Theorem*. Its proof follows the standard reasoning involving the minimization of the energy, so we have Inequality (4) into account.

Theorem 3.1 (*Spectral Theorem*). *For any non-empty subset $F \subset V$, there exist real values $\mu^F_1 \le \cdots \le \mu^F_{|F|}$ and an orthonormal basis $\{v^F_j\}_{j=1}^{|F|} \subset \mathcal{C}(F)$ satisfying the following properties:*

(i) $\mathcal{L}_q(v^F_j) = \mu^F_j v^F_j$ on F, $j = 1, \dots, |F|$. Moreover, if $z \in \mathbb{R}$ is an eigenvalue of the boundary value problem (3), then $z = \mu^F_j$ for some $j = 1, \dots, |F|$.
(ii) $\lambda \le \mu^F_1 < \mu^F_2$ and $v^F_1(x) > 0$ for any $x \in F$. Moreover, $\mu^F_1 = \lambda$ iff $F = V$ and then $v^F_1 = \omega$. In particular, $\mu^F_1 > 0$, except when $F = V$ and $\lambda = 0$, simultaneously.
(iii) For any $u \in \mathcal{C}(F; \mathbb{C})$ then $\mathcal{L}_q(u)(x) = \sum_{j=1}^{|F|} \mu^F_j \langle u, v^F_j\rangle v^F_j(x)$ for any $x \in F$.

As a very nice consequence of the Spectral Theorem, we can obtain the expression of $G^F_q(x, y)$, the Green function of F for the potential q, in terms of eigenvalues and eigenfunctions of $\mathcal{L}_q$. Prior to do this, for any $a \in \mathbb{C}$ we define $a^\# = \begin{cases} a^{-1}, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$.

Using the same notation as in Theorem 3.1, we get the following result.

Theorem 3.2 (*Mercer Theorem*). *Given a non-empty subset $F \subset V$, then*

$$G^F_q(x, y) = \sum_{j=1}^{|F|} (\mu^F_j)^\# v^F_j(x) v^F_j(y), \quad x, y \in V.$$

Moreover, if $z \in \mathbb{C} \setminus \left\{\mu^F_1 \le \cdots \le \mu^F_{|F|}\right\}$, then

$$G^F_{q-z}(x, y) = \sum_{j=1}^{|F|} (\mu^F_j - z)^{-1} v^F_j(x) v^F_j(y), \quad x, y \in V.$$

4. Schrödinger operators on product networks

In this section we prove that Schrödinger operators on product network can be expressed in separated variables and hence we can obtain a discrete version of the separation of variables method.

Let us consider two different connected networks (Γ_1, c_1) and (Γ_2, c_2) with vertex sets V_1 and V_2.

We define the *product network* as the network $\Gamma = \Gamma_1 \times \Gamma_2 = (V, c)$ where $V = V_1 \times V_2$ and the conductance is given by

$$c\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} c_1(x_1, x_2), & \text{if } y_1 = y_2, \\ c_2(y_1, y_2), & \text{if } x_1 = x_2, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Clearly $\Gamma_1 \times \Gamma_2$ is also connected.

Given $u \in \mathcal{C}(V_1 \times V_2)$ for any $(x, y) \in V_1 \times V_2$, $u_y \in \mathcal{C}(V_1)$, $u^x \in \mathcal{C}(V_2)$ denote the functions defined as $u_y(z) = u(z, y)$ for any $z \in V_1$ and by $u^x(z) = u(x, z)$ for any $z \in V_2$.

Given $u \in \mathcal{C}(V_1)$ and $v \in \mathcal{C}(V_2)$ the *tensor product of u and v* is $u \otimes v \in \mathcal{C}(V_1 \times V_2)$ defined as $(u \otimes v)(x, y) = u(x)v(y)$ for any $(x, y) \in V_1 \times V_2$. Notice that given two weights $\omega_i \in \Omega(V_i)$, $i = 1, 2$, then $\omega_1 \otimes \omega_2 \in \Omega(V_1 \times V_2)$. Moreover, given $x \in V_1$ and $y \in V_2$ we have $\varepsilon_{(x,y)} = \varepsilon_x \otimes \varepsilon_y$.

We denote by $\mathcal{L}^i$ the combinatorial Laplacian of the network Γ_i, $i = 1, 2$ and by $\mathcal{L}$ the combinatorial Laplacian of the product network $\Gamma_1 \times \Gamma_2$. The following result establishes that the combinatorial Laplacian of a product network can be expressed in separable variables when it operates on a tensor product function. This property justifies the name of *separation of variables* for the technique to solve boundary value problems on product networks.

Proposition 4.1. *Given $u_i \in \mathcal{C}(V_i)$, $i = 1, 2$ then*

$$\mathcal{L}(u_1 \otimes u_2) = \mathcal{L}^1(u_1) \otimes u_2 + u_1 \otimes \mathcal{L}^2(u_2).$$

In particular, if $\omega_i \in \Omega(V_i)$, $i = 1, 2$, then $q_{\omega_1 \otimes \omega_2} = q_{\omega_1} + q_{\omega_2}$ and hence, for any $u \in \mathcal{C}(V_1 \times V_2)$ we have

$$\mathcal{L}_{q_{\omega_1 \otimes \omega_2}}(u)(x, y) = \mathcal{L}^1_{q_{\omega_1}}(u_y)(x) + \mathcal{L}^2_{q_{\omega_2}}(u^x)(y), \quad (x, y) \in V_1 \times V_2.$$

Proof. Given $u \in \mathcal{C}(V_1 \times V_2)$ for any $(x, y) \in V_1 \times V_2$ we have that

$$\begin{aligned}
\mathcal{L}(u)(x, y) &= \sum_{\substack{z \in V_1 \\ w \in V_2}} c\big((x, y), (z, w)\big)\big(u(x, y) - u(z, w)\big) \\
&= \sum_{z \in V_1} c_1(x, z)\big(u(x, y) - u(z, y)\big) + \sum_{w \in V_2} c_2(y, w)\big(u(x, y) - u(x, w)\big) \\
&= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y).
\end{aligned}$$

On the other hand, since $(u_1 \otimes u_2)_y = u_1 u_2(y)$ and $(u_1 \otimes u_2)^x = u_1(x) u_2$ we obtain that

$$\mathcal{L}(u_1 \otimes u_2)(x, y) = u_2(y)\mathcal{L}^1(u_1)(x) + u_1(x)\mathcal{L}^2(u_2)(y).$$

In particular $\mathcal{L}(\omega_1 \otimes \omega_2) = \mathcal{L}^1(\omega_1) \otimes \omega_2 + \omega_1 \otimes \mathcal{L}^2(\omega_2)$ and hence,

$$q_{\omega_1 \otimes \omega_2} = -(\omega_1 \otimes \omega_2)^{-1}\mathcal{L}(\omega_1 \otimes \omega_2) = -\omega_1^{-1}\mathcal{L}^1(\omega_1) - \omega_2^{-1}\mathcal{L}^2(\omega_2) = q_{\omega_1} + q_{\omega_2}.$$

From all above identities we finally obtain that

$$\begin{aligned}
\mathcal{L}_{q_{\omega_1 \otimes \omega_2}}(u)(x, y) &= \mathcal{L}(u)(x, y) + q_{\omega_1 \otimes \omega_2}(x, y)u(x, y) \\
&= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y) + \big(q_{\omega_1}(x) + q_{\omega_2}(y)\big)u(x, y) \\
&= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y) + q_{\omega_1}(x)u_y(x) + q_{\omega_2}(y)u^x(y) \\
&= \mathcal{L}^1_{q_{\omega_1}}(u_y)(x) + \mathcal{L}^2_{q_{\omega_2}(y)}(u^x)(y). \quad \Box
\end{aligned}$$

5. Boundary value problems on product networks

As in the preceding section, we consider connected networks (Γ_i, c_i) with vertex set V_i and combinatorial Laplacian $\mathcal{L}^i$, $i = 1, 2$. Then, we also consider the product network $\Gamma_1 \times \Gamma_2$ and its combinatorial Laplacian $\mathcal{L}$.

The boundary value problems we analyze in $\Gamma_1 \times \Gamma_2$, refer to subsets that are also expressed as cartesian products. So, given non empty subsets $F_i \subset V_i$, $i = 1, 2$ we consider $F = F_1 \times F_2 \subset V_1 \times V_2$. Then, it is satisfied that

$$\delta(F_1 \times F_2) = \big(F_1 \times \delta(F_2)\big) \cup \big(\delta(F_1) \times F_2\big), \tag{6}$$

where we allow $F_i = V_i$ in which case $\delta(F_i) = \emptyset$, $i = 1, 2$.

Given $\omega_i \in \Omega(V_i)$, $i = 1, 2$ and $\lambda \geq 0$, we consider the real-valued potential $q = q_{\omega_1 \otimes \omega_2} + \lambda$. We are interested in studying the boundary value problem (3) on $F = F_1 \times F_2$ and also in computing the corresponding Green function $\mathcal{G}^F_q$. To do this, we first split λ as $\lambda_1 + \lambda_2$ where $\lambda_1, \lambda_2 \geq 0$ and then apply the Spectral Theorem to the two boundary value problems $\mathcal{L}_{q_i}(u_i) = f_i$ on F_i, where $q_i = q_{\omega_i} + \lambda_i$ and $f_i, u_i \in \mathcal{C}(F_i)$, $i = 1, 2$. Specifically, let $\mu_1^{F_i} \leq \cdots \leq \mu_{|F_i|}^{F_i}$ the eigenvalues of the boundary value problem $\mathcal{L}_{q_i}(u_i) = f_i$ on F_i, $i = 1, 2$ and $\{v_j^{F_i}\}_{j=1}^{|F_i|} \subset \mathcal{C}(F_i)$ a corresponding orthonormal system of eigenfunctions.

Remember that always $\mu_1^{F_i}$ is simple and moreover $v_1^{F_i} > 0$ on F_i, $i = 1, 2$. In addition, $\mu_1^{F_i} = \lambda_i$ iff $F_i = V_i$ and then $v_1^{F_i} = \omega_i$. Therefore, $\mu_1^{F_i} > 0$, except when $F_i = V_i$ and $\lambda_i = 0$, simultaneously.

The main result in product networks is that the eigenvalues and the eigenfunctions for the boundary value problem (3) in product subsets, are completely characterized in terms of the eigenvalues and the eigenfunctions of each factor.

Theorem 5.1. *For any $j = 1, \ldots, |F_1|$ and any $k = 1, \ldots, |F_2|$ we have that*

$$\mathcal{L}_q(v_j^{F_1} \otimes v_k^{F_2}) = (\mu_j^{F_1} + \mu_k^{F_2})v_j^{F_1} \otimes v_k^{F_2} \quad \text{on } F_1 \times F_2.$$

Moreover, $\left\{\mu_j^{F_1} + \mu_k^{F_2}\right\}_{\substack{1 \leq j \leq |F_1| \\ 1 \leq k \leq |F_2|}}$ are the eigenvalues of $\mathcal{L}_q$ on $F_1 \times F_2$ and the set $\left\{v_j^{F_1} \otimes v_k^{F_2}\right\}_{\substack{1 \leq j \leq |F_1| \\ 1 \leq k \leq |F_2|}}$ is an orthonormal basis in $\mathcal{C}(F_1 \times F_2)$.

Proof. From Proposition 4.1, on $F_1 \times F_2$ we have

$$\begin{aligned}
\mathcal{L}_q(v_j^{F_1} \otimes v_k^{F_2}) &= \mathcal{L}^1_{q_1}(v_j^{F_1}) \otimes v_k^{F_2} + v_j^{F_1} \otimes \mathcal{L}_{q_2}(v_k^{F_2}) \\
&= \mu_j^{F_1} v_j^{F_1} \otimes v_k^{F_2} + \mu_k^{F_2} v_j^{F_1} \otimes v_k^{F_2} = (\mu_j^{F_1} + \mu_k^{F_2})v_j^{F_1} \otimes v_k^{F_2}.
\end{aligned}$$

Please cite this article in press as: C. Araúz, et al., Green functions on product networks, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.10.004.

Since the system $\left\{v_j^{F_1}\otimes v_k^{F_2}\right\}_{\substack{1\le j\le |F_1|\\ 1\le k\le |F_2|}}\subset \mathcal{C}(F_1\times F_2)$ is orthonormal and $\dim\mathcal{C}(F_1\times F_2)=|F_1|\cdot|F_2|$ we conclude that $\left\{v_j^{F_1}\otimes v_k^{F_2}\right\}_{\substack{1\le j\le |F_1|\\ 1\le k\le |F_2|}}$ is a basis of $\mathcal{C}(F_1\times F_2)$. Moreover, since any eigenfunction corresponding to an eigenvalue, other than $\mu_j^{F_1}+\mu_k^{F_2}$, $j=1,\dots,|F_1|$, $k=1,\dots,|F_2|$, must be orthogonal to the above basis, we conclude that $\left\{\mu_j^{F_1}+\mu_k^{F_2}\right\}_{\substack{1\le j\le |F_1|\\ 1\le k\le |F_2|}}$ determines all eigenvalues. □

Notice that $\mu_1^{F_1}+\mu_1^{F_2}$ is the lowest eigenvalue, it is simple and moreover $v_1^{F_1}\otimes v_1^{F_2}>0$ on $F_1\times F_2$. In addition, since $\mu_j^{F_1}+\mu_k^{F_2}\ge\lambda_1+\lambda_2=\lambda\ge 0$ we have that $\mu_j^{F_1}+\mu_k^{F_2}>0$ except when $F_1=V_1, F_2=V_2$ and $\lambda=0$ simultaneously. Notice that $\lambda=0$ iff $\lambda_1=\lambda_2=0$ and then, if in addition $F_1=V_1, F_2=V_2$, we have $v_1^{F_i}=\omega_i$, $i=1,2$.

In general, the eigenvalues $\mu_j^{F_1}+\mu_k^{F_2}$, $j=1,\dots,|F_1|$, $k=1,\dots,|F_2|$ when $j+k>2$ have multiplicity greater than 1, even if each factor has only simple eigenvalues. For instance this happens in the *square network* $\Gamma\times\Gamma$ for the weight $\omega\otimes\omega$, $\omega\in\Omega$. In this case, given $F\subset V$, each eigenvalue of the boundary value problem (3) on $F\times F$ other than the $2\mu_j^F$, $j=1,\dots,|F|$, has multiplicity 2 at least: Given $j\ne k$, then $v_j^F\otimes v_k^F$ and $v_k^F\otimes v_j^F$ are eigenfunctions corresponding to $\mu_j^F+\mu_k^F$.

The main consequence of Theorem 5.1 is that we can compute the Green function for product networks in terms of the eigenvalues and the eigenfunctions of each factor by applying the Mercer Theorem.

Corollary 5.2. *Under hypothesis of Theorem 5.1, for any $(x_1,y_1),(x_2,y_2)\in V_1\times V_2$, we have that*

$$G_q^{F_1\times F_2}\big((x_1,y_1),(x_2,y_2)\big)=\sum_{j=1}^{|F_1|}\sum_{k=1}^{|F_2|}(\mu_j^{F_1}+\mu_k^{F_2})^{\#}v_j^{F_1}(x_1)v_j^{F_1}(x_2)v_k^{F_2}(y_1)v_k^{F_2}(y_2).$$

The above formula requires the knowledge of eigenvalues and eigenfunctions for the two factors. Therefore, except for structured networks, the application of the above method is very restrictive. F. Chung and S.T. Yau obtained in [11], see also [13,14], a nice formula based in a clever use of the complex integration, that avoids the computation of eigenvalues and eigenvectors and only needs the evaluation of the Green function of each factor, but considering complex-valued potentials. Although the above authors only consider normalized Laplacians on regular networks, and hence combinatorial Laplacians, their technique is easily extensible to positive semidefinite Schrödinger operators.

Lemma 5.3. *Let $a,b\in\mathbb{C}$ and consider γ a smooth and simple curve enclosing a and moreover it leaves $-b$ in its exterior when $a+b\ne 0$. Then,*

$$(a+b)^{\#}=\frac{1}{2\pi i}\int_\gamma\frac{dz}{(a-z)(b+z)}$$

Proof. If $f(z)=\dfrac{1}{(a-z)(b+z)}$, then f has isolated singularities at a and $-b$ and moreover, since $f(z)=\dfrac{1}{a+b}\left[\dfrac{1}{a-z}+\dfrac{1}{b+z}\right]$ when $a+b\ne 0$ and $f(z)=\dfrac{-1}{(z-a)^2}$ when $a+b=0$, the residue of f at a is $(a+b)^{\#}$. Therefore, we obtain the result applying the Residue Theorem, see [12]. □

We can use the above identity, to express *à la* Chung & Yau, the Green function of the boundary value problem (3) in a product set.

Proposition 5.4. *In the preceding conditions, for any $(x_1,y_1),(x_2,y_2)\in V_1\times V_2$, we have that*

$$G_q^{F_1\times F_2}\big((x_1,y_1),(x_2,y_2)\big)=\frac{1}{2\pi i}\int_\gamma G_{q_1+z}^{F_1}(x_1,x_2)G_{q_2-z}^{F_2}(y_1,y_2)dz$$

where γ is a smooth and simple curve satisfying the following conditions:

(i) If either $\lambda>0$ or $F_1\times F_2\ne V_1\times V_2$, then γ surrounds $\mu_1^{F_1},\dots,\mu_{|F_1|}^{F_1}$ and leave $-\mu_1^{F_2},\dots,-\mu_{|F_2|}^{F_2}$ in its exterior.
(ii) If $\lambda=0$, $F_1=V_1$ and $F_2=V_2$, then γ surrounds $\mu_1^{F_1},\dots,\mu_{|F_1|}^{F_1}$ and leave $-\mu_2^{F_2},\dots,-\mu_{|F_2|}^{F_2}$ in its exterior.

Proof. From Corollary 5.2 and applying Lemma 5.3, we have that

$$\begin{aligned}G_q^{F_1\times F_2}\big((x_1,y_1),(x_2,y_2)\big)&=\frac{1}{2\pi i}\int_\gamma\left[\sum_{j=1}^{|F_1|}\sum_{k=1}^{|F_2|}\frac{v_j^{F_1}(x_1)v_j^{F_1}(x_2)v_k^{F_2}(y_1)v_k^{F_2}(y_2)}{(\mu_j^{F_1}-z)(\mu_k^{F_2}+z)}\right]dz\\&=\frac{1}{2\pi i}\int_\gamma\left[\sum_{j=1}^{|F_1|}\frac{v_j^{F_1}(x_1)v_j^{F_1}(x_2)}{\mu_j^{F_1}-z}\sum_{k=1}^{|F_2|}\frac{v_k^{F_2}(y_1)v_k^{F_2}(y_2)}{\mu_k^{F_2}+z}\right]dz\end{aligned}$$

and the result follows by applying the second part of Mercer Theorem and taking into account that, according to the definition of γ, any complex value $z\in\mathbb{C}$ lying on the trace of γ, is neither an eigenvalue for the boundary value problem (3) on F_1 nor $-z$ is an eigenvalue for the boundary value problem (3) on F_2.

Although Chung & Yau's method avoids the explicit computation of eigenvalues and eigenfunctions, it requires to calculate an infinite family of Green's functions, depending on a complex parameter, and hence to evaluate a complex integral. We finish this paper showing a technique that mix the two former methods. It only requires the computation of eigenvalues and eigenfunctions for one of the factor networks and also the computation of a finite family of Green's functions corresponding to the other product network. In fact this method is nothing else but the discrete version of the well-known *Separation of Variables Method* to solve boundary value problems in PDE.

The key issue to apply the Separation of Variables Method lies on the use of an appropriate expression for functions in $\mathcal{C}(F_1\times F_2)$. With the above notations, for any given $h\in\mathcal{C}(F_1\times F_2)$, for any $j=1,\dots,|F_1|$ and any $k=1,\dots,|F_2|$ we consider the functions $\alpha_j(h)\in\mathcal{C}(F_2)$ and $\beta_k(h)\in\mathcal{C}(F_1)$ defined as

$$\alpha_j(h)(y)=\langle h_y,v_j^{F_1}\rangle=\sum_{z\in V_1}h(z,y)v_j^{F_1}(z)=\sum_{z\in F_1}h(z,y)v_j^{F_1}(z),\quad y\in V_2,$$
$$\beta_k(h)(x)=\langle h^x,v_k^{F_2}\rangle=\sum_{z\in V_2}h(x,z)v_k^{F_2}(z)=\sum_{z\in F_2}h(x,z)v_k^{F_2}(z),\quad x\in V_1.$$

Lemma 5.5. *For any $h\in\mathcal{C}(F_1\times F_2)$ the following identities hold*

$$h=\sum_{j=1}^{|F_1|}v_j^{F_1}\otimes\alpha_j(h)=\sum_{k=1}^{|F_2|}\beta_k(h)\otimes v_k^{F_2}$$

In particular, if $\hat h\in\mathcal{C}(F_1\times F_2)$, then $h=\hat h$ iff $\alpha_j(h)=\alpha_j(\hat h)$, $j=1,\dots,|F_1|$ or equivalently iff $\beta_k(h)=\beta_k(\hat h)$, $k=1,\dots,|F_2|$.

Proof. Since $\{v_k^{F_2}\}_{1\le k\le|F_2|}$ is an orthonormal basis in $\mathcal{C}(F_2)$, for any $x\in F_1$ we have that $h^x=\sum_{k=1}^{|F_2|}\langle h^x,v_k^{F_2}\rangle v_k^{F_2}=\sum_{k=1}^{|F_2|}\beta(h)(x)v_k^{F_2}$; that is, for any $y\in F_2$ we have

$$h(x,y)=\sum_{k=1}^{|F_2|}\beta_k(h)(x)v_k^{F_2}(y)=\sum_{k=1}^{|F_2|}\big(\beta_k(h)\otimes v_k^{F_2}\big)(x,y).$$

The other identity can be proved in an analogous way. □

Theorem 5.6. *Under the conditions and notations in this section, for $i=1,2$ consider the real-valued potentials $p_k^1=q_1+\mu_k^{F_2}=q_{\omega_1}+\lambda_1+\mu_k^{F_2}\in\mathcal{C}(F_1)$, $k=1,\dots,|F_2|$ and $p_j^2=q_2+\mu_j^{F_1}=q_{\omega_2}+\lambda_2+\mu_j^{F_1}\in\mathcal{C}(F_2)$, $j=1,\dots,|F_1|$. Then,*

$$G_q^{F_1\times F_2}\big((x_1,y_1),(x_2,y_2)\big)=\sum_{k=1}^{|F_2|}G_{p_k^1}^{F_1}(x_1,x_2)v_k^{F_2}(y_1)v_k^{F_2}(y_2)$$
$$=\sum_{j=1}^{|F_1|}G_{p_j^2}^{F_2}(y_1,y_2)v_j^{F_1}(x_1)v_j^{F_1}(x_2).$$

Proof. Because the proof of both identities follows the same reasoning, we only prove the first one. Moreover, we first develop the separation of variables technique in a general setting and then we specify it to the computation of the Green function.

Given $f,u\in\mathcal{C}(F_1\times F_2)$, applying Lemma 5.5 we have

$$f=\sum_{k=1}^{|F_2|}\beta_k(f)\otimes v_k^{F_2}\quad\text{and}\quad u=\sum_{k=1}^{|F_2|}\beta_k(u)\otimes v_k^{F_2}.$$

On the other hand, from Proposition 4.1, we have that

$$\mathcal{L}_q(u)=\sum_{k=1}^{|F_2|}\mathcal{L}_{q_1}^1(\beta_k(u))\otimes v_k^{F_2}+\sum_{k=1}^{|F_2|}\mu_k^{F_2}\beta_k(u)\otimes v_k^{F_2}$$
$$=\sum_{k=1}^{|F_2|}\Big[\mathcal{L}_{q_1}^1(\beta_k(u))+\mu_k^{F_2}\beta_k(u)\Big]\otimes v_k^{F_2}=\sum_{k=1}^{|F_2|}\mathcal{L}_{p_k^1}^1(\beta_k(u))\otimes v_k^{F_2}$$

and hence, $f=\mathcal{L}_q(u)$ on $F_1\times F_2$ iff

$$\mathcal{L}_{p_k^1}^1(\beta_k(u))=\beta_k(f),\quad\text{on } F_1\qquad k=1,\dots,|F_2|.$$

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To analyze the above boundary value problem, we first observe that $p^1_k = q_{\omega_1} + \lambda_1 + \mu^{F_2}_k$, $k = 1, \dots, |F_2|$.

When either $F_1 \times F_2 \neq V_1 \times V_2$ or $\lambda_1 + \lambda_2 = \lambda > 0$, then $\lambda_1 + \mu^{F_2}_k > 0$, $k = 1, \dots, |F_2|$, and hence all the above boundary value problems have a unique solution that is given by

$$\beta_k(u)(x) = \sum_{z\in V_1} G^{F_1}_{p^1_k}(x, z)\beta_k(f)(z).$$

When $F_i = V_i$, $i = 1, 2$ and $\lambda = 0$ then the boundary value problem $\mathcal{L}_q(u) = f$ on $V_1 \times V_2$ is solvable iff $f \in (\omega_1 \otimes \omega_2)^\perp$, and then, there exists a unique solution in $(\omega_1 \otimes \omega_2)^\perp$. In this case $\lambda_1 = \lambda_2 = 0$, $\mu^{F_2}_1 = 0$ and hence $\lambda_1 + \mu^{F_2}_k = \mu^{F_2}_k > 0$, $k = 2, \dots, |F_2|$.

Moreover, $\langle \beta_1(f), \omega_1\rangle = \langle f, \omega_1 \otimes \omega_2\rangle$ since $v^{F_2}_1 = \omega_2$ and hence we have that $f \in (\omega_1 \otimes \omega_2)^\perp$ iff $\beta_1(f) \in \omega_1^\perp$. So, under the above conditions, if $f \in (\omega_1 \otimes \omega_2)^\perp$ all the boundary value problems $\mathcal{L}^1_{p^1_k}(\beta_k(u)) = \beta_k(f)$ on F_1, $k = 2, \dots, |F_2|$ have

$$\beta_k(u)(x) = \sum_{z\in V_1} G^{F_1}_{p^1_k}(x, z)\beta_k(f)(z)$$

as unique solution, whereas

$$\beta_1(u)(x) = \sum_{z\in V_1} G^{F_1}_{p^1_1}(x, y)\beta_1(f)(z)$$

is the unique solution of the boundary value problem $\mathcal{L}^1_{p^1_1}(\beta_1(u)) = \beta_1(f)$ belonging to $\omega_1^\perp$.

Consider now $x_2 \in F_1$ and $y_2 \in F_2$. Then $f = \varepsilon_{(x_2,y_2)} = \varepsilon_{x_2} \otimes \varepsilon_{y_2}$ and hence,

$$\beta_k(f) = v^{F_2}_k(y_2)\varepsilon_{x_2}, \quad k = 1, \dots, |F_2|.$$

When either $F_1 \times F_2 \neq V_1 \times V_2$ or $\lambda > 0$, for any $k = 1, \dots, |F_2|$, we have

$$\beta_k(u)(x) = \sum_{z\in V_1} G^{F_1}_{p^1_k}(x, z)\beta_k(f)(z) = G^{F_1}_{p^1_k}(x, x_2)v^{F_2}_k(y_2), \quad x \in V_1,$$

which implies that

$$\begin{aligned} G^{F_1\times F_2}_q\big((x_1, y_1), (x_2, y_2)\big) &= \sum_{k=1}^{|F_2|} \beta_k(u)(x_1)v^{F_2}_k(y_1) \\ &= \sum_{k=1}^{|F_2|} G^{F_1}_{p^1_k}(x_1, x_2)v^{F_2}_k(y_2)v^{F_2}_k(y_1). \end{aligned}$$

When $F_i = V_i$, $i = 1, 2$ and $\lambda = 0$, since $v^{F_i}_1 = \omega_i$, we consider now $f = \varepsilon_{(x_2,y_2)} - \omega_1(x_2)\omega_2(y_2)(\omega_1 \otimes \omega_2) = \varepsilon_{x_2} \otimes \varepsilon_{y_2} - \omega_1(x_2)\omega_2(y_2)(\omega_1 \otimes \omega_2)$. Then,

$$\beta_1(f) = \omega_2(y_2)\Big[\varepsilon_{x_2} - \omega_1(x_2)\omega_1\Big], \quad \beta_k(f) = v^{F_2}_k(y_2)\varepsilon_{x_2}, \quad k = 2, \dots, |F_2|$$

which implies that

$$\beta_k(u)(x) = \sum_{z\in V_1} G^{F_1}_{p^1_k}(x, z)\beta_k(f)(z) = G^{F_1}_{p^1_k}(x, x_2)v^{F_2}_k(y_2), \quad x \in V_1$$

where we have taken into account that $p_1 = q_{\omega_1}$ and that

$$\sum_{z\in V_1} G^{F_1}_{p^1_1}(x, z)\omega_1(z) = \mathcal{G}^{F_1}_{q_{\omega_1}}(\omega_1) = 0.$$

Therefore, the result follows as in the above case. □

6. The Green function of $P_m \times S_h$

As an illustration of the main result of the paper, we compute the Green function of a positive semidefinite Schrödinger of the cartesian product of two networks: a path P_m and a Star network S_h. Notice that none of the factors is regular as it was required in previous works, see for instance [14]. In this case, we are going to apply Theorem 5.6 by using the eigenvalues and eigenfunctions of the path and the Green function of the Star. The product network $P_5 \times S_3$ is depicted in Fig. 3 and its factors in Fig. 2.

Consider P_m the path on m vertices labeled $V_1 = \{x_1, \dots, x_m\}$ and constant conductance $c(x_i, x_{i+1}) = c$ for $i = 1, \dots, m-1$ and $c > 0$. It is known that, the eigenvalues and eigenfunctions of a path P_m for $\mathcal{L}$ are, see [6],

$$\mu_j = 2c(1 - r_j)$$

Please cite this article in press as: C. Araúz, et al., Green functions on product networks, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.10.004.

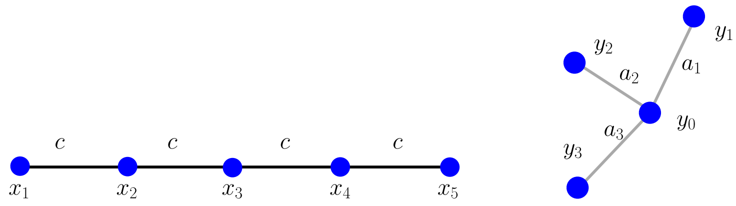

Fig. 2. A path P_5 (left) and a Star S_3 (right).

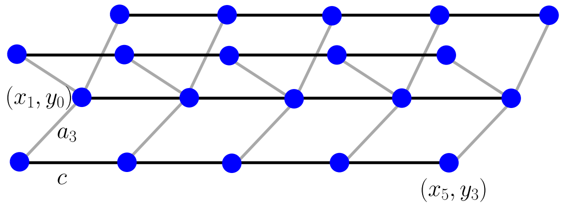

Fig. 3. The product network $P_5 \times S_3$.

where $r_j = \cos\left(\dfrac{j\pi}{m}\right)$, $0 \le j \le m-1$, and the orthonormal eigenfunctions are

$$u_0(x_k) = \frac{1}{\sqrt{m}}, \quad u_j(x_k) = \sqrt{\frac{2}{m}} \cos\left(\frac{(2k-1)j\pi}{2m}\right), \quad k = 1, \dots, m.$$

The *Star network* S_h has $h+1$ vertices, $V_2 = \{y_0, y_1, \dots, y_h\}$, and non null conductances $a_i = c(y_i, y_0) > 0$, $i = 1, \dots, h$. Moreover, let $\omega_i = \omega(y_i)$, $i = 0, \dots, h$ be a weight on S_h. In addition, given $\lambda_2 \ge 0$, and the potential $q_2 = q_\omega + \lambda_2$ we also consider the corresponding positive semi-definite Schrödinger operator $\mathcal{L}_{q_2}$. For the sake of simplicity we consider the following value

$$Q(\lambda_2, \omega) = \sum_{j=1}^{h} \frac{\omega_j^3}{\lambda_2 \omega_j + a_j \omega_0}.$$

In particular, for $\lambda_2 = 0$, $Q(\omega) = \dfrac{1}{\omega_0} \sum_{j=1}^h \dfrac{\omega_j^3}{a_j}$.

It is known that if $\lambda_2 > 0$, see [8, Corollary 5.4], the Green function $G_{q_2}^{V_2}(y_k, y_s)$ is given by

$$\begin{aligned}
G_{q_2}^{V_2}(y_0, y_0) &= \frac{\omega_0^2}{\lambda_2\big[1 - \lambda_2 Q(\lambda_2, \omega)\big]}, \\
G_{q_2}^{V_2}(y_0, y_i) &= \frac{a_i \omega_i \omega_0^2}{\lambda_2\big[1 - \lambda_2 Q(\lambda_2, \omega)\big]\big[\lambda_2 \omega_i + a_i \omega_0\big]}, \\
G_{q_2}^{V_2}(y_k, y_i) &= \frac{a_i a_k \omega_i \omega_k \omega_0^2}{\lambda_2\big[1 - \lambda_2 Q(\lambda_2, \omega)\big]\big[\lambda_2 \omega_i + a_i \omega_0\big]\big[\lambda_2 \omega_k + a_k \omega_0\big]}, \\
G_{q_2}^{V_2}(y_i, y_i) &= \frac{a_i^2 \omega_i^2 \omega_0^2}{\lambda_2\big[1 - \lambda_2 Q(\lambda_2, \omega)\big]\big[\lambda_2 \omega_i + a_i \omega_0\big]^2} + \frac{\omega_i}{\lambda_2 \omega_i + a_i \omega_0},
\end{aligned} \tag{7}$$

where $i, k = 1, \dots, h$ and $k \ne i$. Moreover, for $\lambda_2 = 0$, the Green function of the Star, with respect to ω is given by

$$\begin{aligned}
G_{q_\omega}^{V_2}(y_0, y_0) &= \omega_0^2 Q(\omega), \qquad G_{q_\omega}^{V_2}(y_0, y_i) = \omega_i \left[\omega_0 Q(\omega) - \frac{\omega_i}{a_i}\right], \\
G_{q_\omega}^{V_2}(y_k, y_i) &= \frac{\omega_i \omega_k}{\omega_0} \left[\omega_0 Q(\omega) - \frac{\omega_i}{a_i} - \frac{\omega_k}{a_k}\right], \\
G_{q_\omega}^{V_2}(y_i, y_i) &= \frac{\omega_i^2}{\omega_0} \left[\omega_0 Q(\omega) - \frac{2\omega_i}{a_i}\right] + \frac{\omega_i}{a_i \omega_0},
\end{aligned} \tag{8}$$

where $i, k = 1, \dots, h$ and $k \ne i$.

We are now going to obtain the Green function of the product network $P_m \times S_h$ for the Schrödinger operator $\mathcal{L}_q$, where $q = q_{1\otimes\omega} + \lambda$, $\lambda > 0$. Moreover, we consider that $\lambda_1 = 0$ and $\lambda = \lambda_2$. Under the conditions and notations of Section 5 we get the following result, where for the sake of simplicity we denote $G_q^{V_1\times V_2}$ by G_q.

Theorem 6.1. *For $\lambda > 0$, and the real-valued potentials $p_j = q_\omega + \lambda_j \in \mathcal{C}(V(S_h))$, where $\lambda_j = \lambda + 4c\sin^2\left(\frac{j\pi}{2m}\right)$, $j = 0,\dots,m-1$, the Green function for $P_m \times S_h$ is*

$$G_q\big((x_i,y_0),(x_k,y_0)\big) = \widehat{Q}(\lambda,\omega) + 2\sum_{j=1}^{m-1}\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right),$$

$$G_q\big((x_i,y_0),(x_k,y_s)\big) = \frac{a_s\omega_s\widehat{Q}(\lambda,\omega)}{\lambda\omega_s + a_s\omega_0} + 2a_s\omega_s\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\lambda_j\omega_s + a_s\omega_0},$$

$$G_q\big((x_i,y_r),(x_k,y_s)\big) = \frac{a_ra_s\omega_r\omega_s\widehat{Q}(\lambda,\omega)}{\big[\lambda\omega_r + a_r\omega_0\big]\big[\lambda\omega_s + a_s\omega_0\big]} + 2a_ra_s\omega_r\omega_s\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\big[\lambda_j\omega_r + a_r\omega_0\big]\big[\lambda_j\omega_s + a_s\omega_0\big]},$$

$$G_q\big((x_i,y_r),(x_k,y_r)\big) = \frac{a_r^2\omega_r^2\widehat{Q}(\lambda,\omega)}{\big[\lambda\omega_r + a_r\omega_0\big]^2} + \frac{\omega_r}{m\big[\lambda\omega_r + a_r\omega_0\big]} + 2a_r^2\omega_r^2\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\big[\lambda_j\omega_r + a_r\omega_0\big]^2} + \frac{2\omega_r}{m}\sum_{j=1}^{m-1}\frac{\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\lambda_j\omega_r + a_r\omega_0},$$

for any $i,k = 1,\dots,m$, $r,s = 1,\dots,h$ and $r \neq s$, where $\widehat{Q}(\lambda,\omega) = \dfrac{\omega_0^2}{m\lambda\big[1-\lambda Q(\lambda,\omega)\big]}$.

Proof. From Theorem 5.6, we get that

$$G_q\big((x_i,y_r),(x_k,y_s)\big) = \sum_{j=0}^{m-1}G_{p_j}^{V_2}(y_r,y_s)u_j(x_i)u_j(x_k) = \frac{1}{m}G_{p_0}^{V_2}(y_r,y_s) + \frac{2}{m}\sum_{j=1}^{m-1}G_{p_j}^{V_2}(y_r,y_s)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right).$$

The result follows by substituting the values for $G_{p_j}^{V_2}(y_r,y_s)$ given in (7). □

Finally, we study the case $\lambda = 0$.

Theorem 6.2. *For $\lambda = 0$, and the real-valued potentials $p_j = q_\omega + \lambda_j \in \mathcal{C}(V(S_h))$, where $\lambda_j = 4c\sin^2\left(\frac{j\pi}{2m}\right)$, $j = 0,\dots,m-1$, the Green function for $P_m \times S_h$ is*

$$G_{q_\omega}\big((x_i,y_0),(x_k,y_0)\big) = \frac{\omega_0^2}{m}Q(\omega) + 2\sum_{j=1}^{m-1}\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right),$$

$$G_{q_\omega}\big((x_i,y_0),(x_k,y_s)\big) = \frac{\omega_s}{m}\left[\omega_0Q(\omega) - \frac{\omega_s}{a_s}\right] + 2a_s\omega_s\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\lambda_j\omega_s + a_s\omega_0},$$

$$G_{q_\omega}\big((x_i,y_r),(x_k,y_s)\big)=\frac{\omega_r\omega_s}{m\omega_0}\left[\omega_0Q(\omega)-\frac{\omega_r}{a_r}-\frac{\omega_s}{a_s}\right] + 2a_ra_s\omega_r\omega_s\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\big[\lambda_j\omega_r+a_r\omega_0\big]\big[\lambda_j\omega_s+a_s\omega_0\big]},$$

$$G_{q_\omega}\big((x_i,y_r),(x_k,y_r)\big)=\frac{\omega_r^2}{m\omega_0}\left[\omega_0Q(\omega)-\frac{2\omega_r}{a_r}\right]+\frac{\omega_r}{ma_r\omega_0} + 2a_r^2\omega_r^2\sum_{j=1}^{m-1}\frac{\widehat{Q}(\lambda_j,\omega)\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\big[\lambda_j\omega_r+a_r\omega_0\big]^2} + \frac{2\omega_r}{m}\sum_{j=1}^{m-1}\frac{\cos\left(\frac{(2i-1)j\pi}{2m}\right)\cos\left(\frac{(2k-1)j\pi}{2m}\right)}{\lambda_j\omega_r+a_r\omega_0},$$

for any $i,k=1,\dots,m$, $r,s=1,\dots,h$ *and* $r\neq s$.

As far as the authors knowledge neither the eigenvalues nor the eigenfunction of a general Schrödinger operator for the Star network are known. Therefore, the above developments are the only way of obtaining an explicit expression of the Green function of the product network $P_m\times S_h$. Even in the most simple case; that is, constant weight and conductances in the Star network, the orthonormal basis of eigenfunctions is still quite intricate.

For $\omega=\dfrac{1}{\sqrt{h+1}}$ and $c(y_0,y_i)=a>0$ for $i=1,\dots,h$, the eigenvalues for $\mathcal{L}$ are,

$$\mu_1=0,\ \mu_j=a,\ \mu_{h+1}=a(h+1),$$

$j=2,\dots,h$. The associated eigenfunctions v_k are:

(i) When $\mu_1=0$, the eigenfunction is $v_1=\dfrac{1}{\sqrt{h+1}}$.

(ii) For $\mu_j=a$, $j=2,\dots,h$, the eigenfunction is

$$v_j(y_k)=\frac{1}{\sqrt{j(j-1)}}\begin{cases}1, & k=2,\dots,j,\\ 1-j, & k=j+1,\\ 0, & \text{otherwise.}\end{cases}$$

(iii) For the eigenvalue $a(h+1)$ the eigenfunction is,

$$v_{h+1}(y_0)=\sqrt{\frac{h}{h+1}},\quad v_{h+1}(y_k)=-\frac{1}{\sqrt{(h+1)h}}$$

for any $k=1,\dots,h$.

In addition, to apply Theorem 5.6 we have to take into account the Green function of the Path. If $\lambda_1>0$, see [6], the Green function $G_{\lambda_1}^{V_1}$ is given by

$$G_{\lambda_1}^{V_1}(x_i,x_k)=\frac{V_{i-1}(q)V_{m-k}(q)}{\lambda_1U_{m-1}(q)}, \tag{9}$$

where $q=1+\dfrac{\lambda_1}{2c}$ and $G_{\lambda_1}^{V_1}(x_i,x_k)=G_{\lambda_1}^{V_1}(x_k,x_i)$ for any $1\le i\le k\le m$.

When $\lambda_1=0$, the expression for the Green function of the path with constant conductance is

$$G^{V_1}(x_i,x_k)=\frac{3i(i-1)+3(m-k)(m+1-k)+1-m^2}{6mc}, \tag{10}$$

and $G^{V_1}(x_i,x_k)=G^{V_1}(x_k,x_i)$ for any $1\le i\le k\le m$.

So, in this context it would be possible to get an explicit expression of G_q but it would include a lot of cases, according with vertices y_i, and it would have an awful appearance.

Acknowledgment

This work has been supported by the Spanish Research Council, Spain, under projects MTM2014-60450-R and MTM2017-85996-R.

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