ON THE UNIQUENESS AND ANALYTICITY IN VISCOELASTICITY WITH DOUBLE POROSITY

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Abstract: In this paper we analyze the system of equations that models the behaviour of materials with a double porous structure. We introduce dissipation mechanisms in both structures. We show existence, uniqueness and analyticity for the solutions of the system. As consequences, exponential stability and impossibility of localization for the solutions are obtained.

Keywords: viscoelasticity, double porosity, uniqueness, analyticity, exponential decay.

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1. Introduction

Materials with double porosity have been the aim of study of many recent works. The first contributions about this theory were written at the beginning of the 60's and are due to Barenblatt *et al.* [1, 2]. Some other authors have also contributed to develop this theory (without trying to be exhaustive, we refer to [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). The now called double porosity model allows a material to have two porous structures: the macro porosity, which is connected with the pores of the body of the material, and the micro porosity, which is connected with the fissures in the skeleton of the material. For this reason, this model is very interesting in different fields as geophysics or mechanics of bones [2, 6, 7, 14]. As Straughan [13] noticed, 'a good example of this may be seen in [15] where they show a pile of rocks, but the rocks themselves are full of fissures (or cracks), and the macro porosity degrades over a period of ten years leaving a pile of finer material characteristic of the micro porous structure'. The previous contributions are based on the use of Darcy's law. The usual formulation of the theory involves the displacement, the pressure associated with the pores and the pressure associated with the fissures [9, 12, 13, 16].

Nunziato and Cowin [17] proposed a theory for the behavior of porous solids. They established that the skeletal is elastic and the interstices are void of material. Following these ideas, Iesan and Quintanilla [18] set another theory for elastic materials with double porosity, which is currently being studied [19, 20, 21]. As our work is also related with the study of the solutions of porous-elastic materials, it is worth recalling several contributions about this topic [22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

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In this paper we consider that the skeletal is viscoelastic and, at the same time, we introduce viscosity effects in both porous structures. This new setting gives rise to a dissipative system and it is not based on the law of Darcy. We will prove that the solutions of this system are generated by an analytic semigroup. Therefore, some regularity properties to the solutions arise. For instance, their orbits are analytical functions with respect to the time. That means that we can recover the solutions in terms of the time derivatives of the function in a fixed point (it is important to know the thermomechanical situations where such kind of regularity holds). Exponential stability and impossibility of localization of the solutions are also direct consequences of the analyticity.

We want to highlight the mathematical similarities between the equations for porous-elastic materials and those for microstretch materials. In fact, the equations that we analyze in this paper can also be viewed as equations describing a mixture of microstretch materials when their macroscopic structures coincide.

The plan of the paper is as follows. In Section 2 we recall the basic equations of the theory we are dealing with and the conditions that we assume over the constitutive coefficients. Sections 3, 4 and 5 are devoted, respectively, to prove the uniqueness, existence and analyticity of the solutions of the proposed system of equations. Finally, in Section 6 we state the conclusions.

2. Basic Equations

We consider a body that at time t_0 occupies a bounded region B of the Euclidean three-dimensional space. As usual, the configuration of the body at time t_0 is taken as the reference configuration, and the motion of the body is referred to the reference configuration from a fixed system of rectangular Cartesian axes. Each particle of the body is identified with its position, which in the reference configuration is given by x_j, for $j = 1, 2, 3$. At time t, the position of the particle will be denoted by

$$u_i = u_i(x_j, t), \quad (x_j) \in B, t \in I,$$

where $I = (0, t)$ is a given interval of time.

We denote by φ the volume fraction field corresponding to pores, and by ψ the volume fraction field corresponding to fissures.

Following the ideas proposed by Iesan and Quintanilla [18], we know that, in the absence of supply terms, the evolution equations are given by

$$\begin{aligned} \rho \ddot{u}_i &= t_{ji,j} \\ k_1 \ddot{\varphi} &= \sigma_{j,j} + \xi \\ k_2 \ddot{\psi} &= s_{j,j} + \zeta \end{aligned} \tag{2.1}$$

In the above equations, t_{ji} is the stress tensor, σ_j and s_j are the equilibrated stress vectors, and ξ and ζ are the equilibrated body forces. Constants ρ, k_1 and k_2 represent, respectively, the mass density and the products of the mass density by the equilibrated inertia.

Notice that, from a mathematical point of view, the kind of materials that we study here can be considered as a sub-case of porous viscoelastic mixtures [32]. Therefore, we can take the equations from there.

The constitutive equations that we consider in this paper are given by

$$
\begin{aligned}
t_{ij} &= C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ij}\psi + C^*_{ijrs}\dot{e}_{rs}\\
\sigma_i &= \alpha_{ij}\varphi_{,j} + b_{ij}\psi_{,j} + \alpha^*_{ij}\dot{\varphi}_{,j} + N_{ij}\dot{\psi}_{,j}\\
s_i &= b_{ji}\varphi_{,j} + \gamma_{ij}\psi_{,j} + M_{ij}\dot{\varphi}_{,j} + \gamma^*_{ij}\dot{\psi}_{,j}\\
\xi &= -B_{ij}e_{ij} - \alpha_1\varphi - \alpha_3\psi\\
\zeta &= -D_{ij}e_{ij} - \alpha_3\varphi - \alpha_2\psi
\end{aligned} \tag{2.2}
$$

Here, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor, C_{ijrs}, B_{ij}, D_{ij}, α_{ij}, b_{ij}, γ_{ij}, α_1, α_2, α_3, C^*_{ijrs}, α^*_{ij}, N_{ij}, M_{ij} and γ^*_{ij} are tensors of different order satisfying the following symmetries:

$$
\begin{aligned}
C_{ijrs} = C_{rsij} = C_{jirs}, \quad \alpha_{ij} = \alpha_{ji}, \quad \alpha^*_{ij} = \alpha^*_{ji}, \quad B_{ij} = B_{ji},\\
C^*_{ijrs} = C^*_{rsij} = C^*_{jirs}, \quad \gamma_{ij} = \gamma_{ji}, \quad \gamma^*_{ij} = \gamma^*_{ji}, \quad D_{ij} = D_{ji}.
\end{aligned}
$$

Substituting the constitutive equations into the evolution equations, we obtain the system that we want to analyze.

$$
\left\{
\begin{aligned}
\rho\ddot{u}_i &= \left(C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ij}\psi + C^*_{ijrs}\dot{e}_{rs}\right)_{,j}\\
k_1\ddot{\varphi} &= \left(\alpha_{ij}\varphi_{,j} + b_{ij}\psi_{,j} + \alpha^*_{ij}\dot{\varphi}_{,j} + N_{ij}\dot{\psi}_{,j}\right)_{,i} - B_{ij}e_{ij} - \alpha_1\varphi - \alpha_3\psi\\
k_2\ddot{\psi} &= \left(b_{ji}\varphi_{,j} + \gamma_{ij}\psi_{,j} + M_{ij}\dot{\varphi}_{,j} + \gamma^*_{ij}\dot{\psi}_{,j}\right)_{,i} - D_{ij}e_{ij} - \alpha_3\varphi - \alpha_2\psi
\end{aligned}
\right. \tag{2.3}
$$

The internal energy of the system is given by the function defined by

$$
2U = C_{ijrs}e_{ij}e_{rs} + 2B_{ij}e_{ij}\varphi + 2D_{ij}e_{ij}\psi + \alpha_{ij}\varphi_{,i}\varphi_{,j} + 2b_{ij}\varphi_{,i}\psi_{,j} + \gamma_{ij}\psi_{,i}\psi_{,j} + \alpha_1\varphi^2 + \alpha_2\psi^2 + 2\alpha_3\varphi\psi,
$$

and the dissipation by

$$
D = C^*_{ijrs}\dot{e}_{ij}\dot{e}_{rs} + \alpha^*_{ij}\dot{\varphi}_{,i}\dot{\varphi}_{,j} + (N_{ji} + M_{ij})\dot{\varphi}_{,i}\dot{\psi}_{,j} + \gamma^*_{ij}\dot{\psi}_{,i}\dot{\psi}_{,j}.
$$

If we work with isotropic and homogeneous materials, then the constitutive equations reduce to

$$
\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + d\delta_{ij}\varphi + d\delta_{ij}\psi + \lambda^*\dot{e}_{rr}\delta_{ij} + 2\mu^*\dot{e}_{ij}\\
\sigma_i &= \alpha\varphi_{,i} + b_1\psi_{,i} + \alpha^*\dot{\varphi}_{,i} + N\dot{\psi}_{,i}\\
\tau_i &= b_1\varphi_{,i} + \gamma\psi_{,i} + M\dot{\varphi}_{,i} + \gamma^*\dot{\psi}_{,i}\\
\xi &= -be_{ii} - \alpha_1\varphi - \alpha_3\psi\\
\zeta &= -de_{ii} - \alpha_3\varphi - \alpha_2\psi
\end{aligned} \tag{2.4}
$$

Now λ, μ, b, d, λ^*, μ^*, α, b_1, α^*, N, γ, M, γ^*, α_1, α_2 and α_3 are real numbers. The internal energy and the dissipation become, respectively,

$$
2U = \lambda e_{ii}e_{rr} + 2\mu e_{ij}e_{ij} + 2be_{ii}\varphi + 2de_{ii}\psi + \alpha\varphi_{,i}\varphi_{,i} + 2b_1\varphi_{,i}\psi_{,i} + \gamma\psi_{,i}\psi_{,i} + \alpha_1\varphi^2 + \alpha_2\psi^2 + 2\alpha_3\varphi\psi,
$$

and

$$
D = \lambda^*\dot{e}_{ii}\dot{e}_{jj} + 2\mu^*\dot{e}_{ij}\dot{e}_{ij} + \alpha^*\dot{\varphi}_{,i}\dot{\varphi}_{,i} + (N + M)\dot{\varphi}_{,i}\dot{\psi}_{,i} + \gamma^*\dot{\psi}_{,i}\dot{\psi}_{,i}.
$$

The following inequalities (see [18]) are the necessary and sufficient conditions for the internal energy U to be positive:

$$
\begin{aligned}
\mu > 0, 3\lambda + 2\mu > 0, \alpha_2 > 0, \alpha_1\alpha_2 - \alpha_3^2 > 0, \alpha > 0, \alpha\gamma > b_1^2,\\
(3\lambda + 2\mu)(\alpha_1\alpha_2 - \alpha_3^2) > 3(\alpha_1 d^2 + \alpha_2 b^2 - 2\alpha_3 bd).
\end{aligned} \tag{2.5}
$$

The conditions to guarantee the positivity of the dissipation D are (see [18]):

$$
\mu^* > 0, 3\lambda^* + 2\mu^* > 0, \alpha^* > 0, 4\alpha^*\gamma^* > (N + M)^2. \tag{2.6}
$$

We want to study the problem determined by the evolution equations (2.1) and the constitutive equations (2.2) on the bounded region B. To set a well posed problem, we need to impose boundary and initial conditions.

As boundary conditions we assume that

$$u_i(\mathbf{x},t)=\hat{u}_i, \varphi(\mathbf{x},t)=\hat{\varphi} \text{ and } \psi(\mathbf{x},t)=\hat{\psi} \text{ for } \mathbf{x}\in\partial B\times(0,t), \tag{2.7}$$

where $\hat{u}_i$, $\hat{\varphi}$ and $\hat{\psi}$ are prescribed functions.

As initial conditions we assume that

$$\begin{array}{ll} u_i(\mathbf{x},0)=u_i^0(\mathbf{x}), & \dot{u}_i(\mathbf{x},0)=v_i^0(\mathbf{x}), \\ \varphi(\mathbf{x},0)=\varphi^0(\mathbf{x}), & \dot{\varphi}(\mathbf{x},0)=\nu^0(\mathbf{x}) \\ \psi(\mathbf{x},0)=\psi^0(\mathbf{x}), & \dot{\psi}(\mathbf{x},0)=\tau^0(\mathbf{x}). \end{array} \tag{2.8}$$

where u_i^0, v_i^0, φ^0, ψ^0, ν^0 and τ^0 are also prescribed functions.

3. Uniqueness of solutions

In this section we will see the uniqueness of the solutions supposing only that the dissipation is positive definite (nothing is said about the internal energy). In fact, we will prove that the only solution with null initial conditions and null boundary conditions is the null solution. This fact will prove the uniqueness.

Let us write the null Dirichlet boundary conditions that we assume:

$$u_i(\mathbf{x},t)=\varphi(\mathbf{x},t)=\psi(\mathbf{x},t)=0 \text{ for } \mathbf{x}\in\partial B\times(0,t), \tag{3.1}$$

We also suppose in this section that ρ, k_1 and k_2 are positive and that D is positive definite. This last assertion says that the inequality

$$D\geq C(\dot{e}_{ij}\dot{e}_{ij}+\dot{\varphi}_{,i}\dot{\varphi}_{,i}+\dot{\psi}_{,i}\dot{\psi}_{,i})$$

holds for an appropriate positive constant C.

The conservation of the energy, when homogeneous Dirichlet boundary conditions and null initial conditions are assumed, gives

$$E(t)=\int_B\big(\rho\dot{u}_i\dot{u}_i+k_1|\dot{\varphi}|^2+k_2|\dot{\psi}|^2+2U\big)dV+2\int_0^t\int_B D\,dV\,ds=E(0)=0. \tag{3.2}$$

The last equality comes from the assumption that the initial conditions vanish.

From the above expression, we introduce the following notation:

$$\begin{aligned} I_1&=\textstyle\int_B\big(\rho\dot{u}_i\dot{u}_i+k_1|\dot{\varphi}|^2+k_2|\dot{\psi}|^2\big)dV,\\ I_2&=\textstyle\int_B 2U\,dV,\\ I_3&=2\textstyle\int_0^t\int_B D\,dV\,ds, \end{aligned}$$

and we define the function $J(t)=I_1+I_3$. Obviously, $J(t)=-I_2$.

Therefore,

$$J(t)\leq C^*\left(\int_0^t\int_B\big(e_{ij}e_{ij}+\varphi_{,i}\varphi_{,i}+\psi_{,i}\psi_{,i}+\varphi^2+\psi^2\big)dV\,ds\right)^{\frac{1}{2}}I_3^{\frac{1}{2}},$$

where C^* is a calculable positive constant depending on the constitutive coefficients and the first eigenvalue of the fixed membrane problem (see [33]),

$$\Delta\Phi + \lambda\Phi = 0 \text{ on } B \text{ and } \Phi = 0 \text{ on } \partial B.$$

The next step in our analysis uses the Poincaré-type inequality that says that the following estimate

$$\int_0^t f^2(s)ds \le \frac{4t^2}{\pi^2}\int_0^t (\dot{f}(s))^2 ds$$

holds for every function $f \in W^{1,2}(I)$ such that $f(0) = 0$.

Therefore, in our case, there exists a positive constant C_1 such that

$$\int_0^t\int_B \big(e_{ij}e_{ij} + \varphi_{,i}\varphi_{,i} + \psi_{,i}\psi_{,i} + \varphi^2 + \psi^2\big)dV ds \le t^2\, C_1 \int_0^t\int_B (\dot{e}_{ij}\dot{e}_{ij} + \dot{\varphi}_{,i}\dot{\varphi}_{,i} + \dot{\psi}_{,i}\dot{\psi}_{,i})dV ds.$$

To obtain the last inequality we have used also the classical Poicaré inequality: if $g \in W_0^{1,2}(B)$ is a function that vanishes in the boundary of B then there exists $C_2 > 0$ such that

$$\int_B g^2(\mathbf{x})\, dV \le C_2 \int_B g_{,i}(\mathbf{x}) g_{,i}(\mathbf{x})\, dV.$$

From the positivity of the dissipation function, it can be seen that

$$J(t) \le C^{**} t I_3 \le C^{**} t J(t), \tag{3.3}$$

where C^{**} is a positive constant that can be calculated in terms of the constitutive coefficients and the first eigenvalue of the fixed membrane problem. From (3.3), it follows that $(1 - C^{**}t)J(t) \le 0$. If we consider $t_0 = (2C^{**})^{-1}$, then we find that $J(t)$ vanishes in the interval $[0, t_0]$. From the definition of $J(t)$, and taking into account again the positivity of the dissipation D, it follows that $\dot{u}_i = \dot{\varphi} = \dot{\psi} = 0$ for every $t \le t_0$. Thus, we have proved that the problem has only the null solution in the interval $[0, t_0]$. Repeating this argument in $[nt_0, (n+1)t_0]$, for $n \in \mathbb{N}$, we obtain that the only solution is the null solution.

Theorem 3.1. *Let us assume that $\rho > 0$, $k_1 > 0$, $k_2 > 0$ and that D is positive definite. Then the Dirichlet initial boundary value problem has at most one solution.*

4. Existence of solution

In this section we use the results of the semigroup of linear operators theory to obtain an existence theorem. We restrict our attention to the homogeneous version of the Dirichlet boundary conditions (3.1).

In the remaining part of the paper we assume that D is positive definite and that the internal energy U is also positive. That means that there exists a positive constant K such that

$$U \ge K\big(e_{ij}e_{ij} + \varphi_{,i}\varphi_{,i} + \psi_{,i}\psi_{,i} + \varphi^2 + \psi^2\big).$$

We assume again that ρ, k_1 and k_2 are positive.

We now transform the boundary-initial value problem defined by system (2.3), boundary conditions (3.1) and initial conditions (2.8) into an abstract problem on a suitable Hilbert space.

We denote

$$\mathcal{H} = \{\mathbf{U} = (\mathbf{u}, \mathbf{v}, \varphi, \nu, \psi, \tau) : u_i \in W_0^{1,2}(B), v_i \in L^2(B), \varphi, \psi \in W_0^{1,2}(B), \nu, \tau \in L^2(B)\},$$

where $W_0^{1,2}(B)$ and $L^2(B)$ are the usual Sobolev spaces, which take values in the complex field.

Let us consider the following operators:

$$
\begin{aligned}
&A_i(\mathbf{u}) = \frac{1}{\rho}\big(C_{ijrs}e_{r,s}\big)_{,j},\\
&A_i^*(\mathbf{v}) = \frac{1}{2\rho}\big(C_{ijrs}^*(v_{r,s}+v_{s,r})\big)_{,j},,\\
&B_i(\varphi) = \frac{1}{\rho}\big(B_{ij}\varphi\big)_{,j},\\
&C_i(\psi) = \frac{1}{\rho}\big(D_{ij}\psi\big)_{,j},\\
&R(\mathbf{u}) = -\frac{1}{k_1}B_{ij}e_{ij},\\
&S(\varphi) = \frac{1}{k_1}[(\alpha_{ij}\varphi_{,j})_{,i}-\alpha_1\varphi],\\
&T(\psi) = \frac{1}{k_1}[(b_{ij}\psi_{,j})_{,i}-\alpha_3\psi],\\
&S^*(\nu) = \frac{1}{k_1}[(\alpha_{ij}^*\nu_{,j})_{,i}],\\
&T^*(\tau) = \frac{1}{k_1}[(N_{ij}\tau_{,j})_{,i}],\\
&V(\mathbf{u}) = -\frac{1}{k_2}D_{ij}e_{ij},\\
&W(\varphi) = \frac{1}{k_2}[(b_{ji}\varphi_{,j})_{,i}-\alpha_3\varphi],\\
&X(\psi) = \frac{1}{k_2}[(\gamma_{ij}\psi_{,j})_{,i}-\alpha_2\psi],\\
&W^*(\nu) = \frac{1}{k_2}[(M_{ij}\nu_{,j})_{,i}],\\
&X^*(\tau) = \frac{1}{k_2}[(\gamma_{ij}^*\tau_{,j})_{,i}].
\end{aligned}
$$

Let $\mathcal{A}$ be the matrix operator defined by

$$
\mathcal{A} = \begin{pmatrix}
\mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{A} & \mathbf{A}^* & \mathbf{B} & \mathbf{0} & \mathbf{C} & 0\\
0 & 0 & 0 & Id & 0 & 0\\
R & 0 & S & S^* & T & T^*\\
0 & 0 & 0 & 0 & 0 & Id\\
V & 0 & W & W^* & X & X^*
\end{pmatrix}, \tag{4.1}
$$

where $\mathbf{A} = (A_i)$, $\mathbf{A}^* = (A_i^*)$, $\mathbf{B} = (B_i)$, $\mathbf{C} = (C_i)$, and $\mathbf{Id}$, Id represent the identity operators in the respective spaces.

The domain of the operator $\mathcal{A}$ is $\mathcal{D} = \{\mathbf{U} \in \mathcal{H} : \mathcal{A}\mathbf{U} \in \mathcal{H}\}$.

It is clear that the subspace

$$
\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\left(W_0^{1,2}\cap W^{2,2}\right)^4 \tag{4.2}
$$

is contained in the domain and it is a dense subset of $\mathcal{H}$.

The initial boundary value problem (2.3), (3.1), (2.8) can be transformed into the following abstract equation in the Hilbert space $\mathcal{H}$,

$$
\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U}(t), \quad \mathbf{U}(0) = \mathbf{U}_0, \tag{4.3}
$$

where

$$
\mathbf{U}_0 = \left(u_i^0, v_i^0, \varphi^0, \nu^0, \psi^0, \tau^0\right). \tag{4.4}
$$

We introduce in $\mathcal{H}$ the inner product

$$
<\mathbf{U},\mathbf{U}^\star> = \int_B \left(\rho v_i\overline{v_i^\star} + k_1\nu\overline{\nu^\star} + k_2\tau\overline{\tau^*} + \mathcal{M}\big[\mathbf{V},\mathbf{V}^\star\big]\right)dV, \tag{4.5}
$$

where $\mathbf{V}=(\mathbf{u},\varphi,\psi)$ and

$$\mathcal{M}\big[\mathbf{V},\mathbf{V}^\star\big]=C_{ijrs}e_{ij}\overline{e^\star_{rs}}++\alpha_{ij}\varphi_{,i}\overline{\varphi^\star_{,j}}+\gamma_{ij}\psi_{,i}\overline{\psi^\star_{,j}}+b_{ij}\big(\varphi_{,i}\overline{\psi^\star_{,j}}+\overline{\varphi^\star_{,j}}\psi_{,j}\big)+$$
$$B_{ij}\big(e_{ij}\overline{\varphi^\star}+\overline{e^\star_{ij}}\varphi\big)+D_{ij}\big(e_{ij}\overline{\psi^\star}+\overline{e^\star_{ij}}\psi\big)+\alpha_1\varphi\overline{\varphi^\star}+\alpha_2\psi\overline{\psi^\star}+\alpha_3\big(\varphi\overline{\psi^\star}+\overline{\varphi^\star}\psi\big).$$

Here, as usual a bar over a variable denotes its complex conjugate.

We note that (4.5) defines a norm which is given by

$$\|\mathbf{U}\|^2_{\mathcal{H}}=\int_B\big(\rho v_i\overline{v_i}+k_1|\nu|^2+k_2|\tau|^2+2\mathcal{W}\big)\,dV,$$

where

$$2\mathcal{W}=C_{ijrs}e_{ij}\overline{e_{rs}}+\alpha_{ij}\varphi_{,i}\overline{\varphi_{,j}}+\gamma_{ij}\psi_{,i}\overline{\psi_{,j}}+b_{ij}\big(\varphi_{,i}\overline{\psi_{,j}}+\overline{\varphi_{,j}}\psi_{,j}\big)+$$
$$B_{ij}\big(e_{ij}\overline{\varphi}+\overline{e_{ij}}\varphi\big)+D_{ij}\big(e_{ij}\overline{\psi}+\overline{e_{ij}}\psi\big)+\alpha_1\varphi\overline{\varphi}+\alpha_2\psi\overline{\psi}+\alpha_3\big(\varphi\overline{\psi}+\overline{\varphi}\psi\big).$$

This norm is equivalent to the usual norm in $\mathcal{H}$.

Lemma 4.1. *The operator $\mathcal{A}$ has the property*

$$\Re<\mathcal{A}\mathbf{U},\mathbf{U}>\leq 0, \tag{4.6}$$

for any $\mathbf{U}\in\mathcal{D}$, where the inner product $<.,.>$ is defined in (4.5).

Proof. Let $\mathbf{U}=(\mathbf{u},\mathbf{v},\varphi,\nu,\psi,\tau)\in\mathcal{D}$. Using the divergence theorem and the boundary conditions we see that

$$<\mathcal{A}\mathbf{U},\mathbf{U}>=-\int_B\left(C^\star_{ijrs}v_{i,j}\overline{v}_{r,s}+\alpha^\star_{ij}\nu_{,i}\overline{\nu}_{,j}+\gamma^\star_{ij}\tau_{,i}\overline{\tau}_{,j}+\frac{1}{2}(N_{ji}+M_{ij})(\nu_{,i}\overline{\tau}_{,j}+\overline{\nu}_{,i}\tau_j\right)dV.$$

In view of the assumptions on the tensors we obtain the stated inequality.

Lemma 4.2. *The operator $\mathcal{A}$ satisfies the condition $0\in\varrho(\mathcal{A})$.*

Proof. Let $\mathbf{U}^*=(\mathbf{u}^*,\mathbf{v}^*,\varphi^*,\nu^*,\psi^*,\tau^*)\in\mathcal{H}$. We must show that the equation

$$\mathcal{A}\mathbf{U}=\mathbf{U}^* \tag{4.7}$$

has a solution $\mathbf{U}=(\mathbf{u},\mathbf{v},\varphi,\nu,\psi,\tau)\in\mathcal{D}$. If we write the matrix and we operate, we find the following system:

$$\begin{array}{l}\mathbf{v}=\mathbf{u}^*,\\ \mathbf{A}\mathbf{u}+\mathbf{A}^*\mathbf{v}+\mathbf{B}\varphi+C\psi=\mathbf{v}^*,\\ \nu=\varphi^*,\\ R\mathbf{u}+S\varphi+S^*\nu+T\psi+T^*\tau=\nu^*,\\ \tau=\psi^*,\\ V\mathbf{u}+W\psi+W^*\nu+X\psi+X^*\tau=\tau^*.\end{array} \tag{4.8}$$

Substituting the first, third and fifth equations into the others, we obtain the following system with unknowns $\mathbf{u}$, φ and ψ:

$$\begin{array}{l}\mathbf{A}\mathbf{u}+\mathbf{B}\varphi+\mathbf{C}\psi=\mathbf{v}^*-\mathbf{A}^*\mathbf{u}^*,\\ R\mathbf{u}+S\varphi+T\psi=\nu^*-S^*\varphi^*-T^*\psi^*,\\ V\mathbf{u}+W\varphi+X\psi=\tau^*-W^*\varphi^*-X^*\tau^*.\end{array} \tag{4.9}$$

Notice that

$$(\mathbf{v}^*-\mathbf{A}^*\mathbf{u}^*,\nu^*-S^*\varphi^*-T^*\psi^*,\tau^*-W^*\varphi^*-X^*\tau^*)\in\mathbf{W}^{-1,2}\times W^{-1,2}\times W^{-1,2}.$$

On the other side,

$$\mathcal{B}\big((\mathbf{u},\varphi,\psi),(\mathbf{u}^*,\varphi^*,\psi^*)\big)=\langle(\mathbf{A}\mathbf{u}+\mathbf{B}\varphi+\mathbf{C}\psi\,,\,R\mathbf{u}+S\varphi+T\psi\,,\,V\mathbf{u}+W\varphi+X\psi),(\rho\mathbf{u}^*,k_1\varphi^*,k_2\psi^*)\rangle$$

defines a coercive and bounded bilinear form on $\mathbf{W}_0^{1,2}\times W_0^{1,2}\times W_0^{1,2}$. Hence, the Lax-Milgram theorem implies the existence of a solution to the system of equations (4.9). Thus, equation (4.7) has also a solution. It is clear, from the regularity theory of linear elliptic systems, that $\|U\|_{\mathcal{H}}\le K^*\|U^*\|$ being K^* a positive constant independent of U^*.

Theorem 4.3. *The operator $\mathcal{A}$ generates a semigroup of contractions in $\mathcal{H}$.*

The proof follows from the above lemmas and the Lumer-Phillips corollary to the Hille-Yosida theorem.

Finally, as a consequence, we have the following result.

Theorem 4.4. *Assume that $\mathbf{U}_0\in\mathcal{D}$. Then, there exists a unique solution $\mathbf{U}(t)\in C^1([0,\infty),\mathcal{H})\cap C^0([0,\infty),\mathcal{D})$ to problem (4.3).*

5. Analyticity of solutions

Now we will prove that the semigroup obtained in the previous section is analytic. In order to do so, we use the following result, which can be found in the classical book of Liu and Zheng [34].

Theorem 5.1. *Let us consider $S(t)=e^{\mathcal{A}t}$ a C_0-semigroup of contractions generated by the operator $\mathcal{A}$ in the Hilbert space $\mathcal{H}$. Suppose that $\varrho(\mathcal{A})\supseteq\{i\beta;\beta\in\mathbb{R}\}\equiv i\mathbb{R}$. Then $S(t)$ is analytic if and only if*

$$\overline{\lim}_{|\beta|\to\infty}\|\beta(i\beta\mathcal{I}-\mathcal{A})^{-1}\|<\infty,\quad \beta\in\mathbb{R}$$

holds.

To apply Theorem 5.1 to our situation, we need to consider the resolvent equation

$$\lambda\mathbf{U}-\mathcal{A}\mathbf{U}=\mathbf{F},$$

where $\mathbf{U}=(\mathbf{u},\mathbf{v},\varphi,\nu,\psi,\tau)$ and $\mathbf{F}=(\mathbf{f}_1,\mathbf{f}_2,f_3,f_4,f_5)$. We shall take $\lambda=i\delta$, with $\delta\in\mathbb{R}$. Therefore, our equation becomes

$$\begin{aligned}&i\delta\mathbf{u}-\mathbf{v}=\mathbf{f}_1,\\&i\delta\mathbf{v}-\mathbf{A}\mathbf{u}-\mathbf{A}^*\mathbf{v}-\mathbf{B}\varphi-\mathbf{C}\psi=\mathbf{f}_2,\\&i\delta\varphi-\nu=f_3,\\&i\delta\nu-R\mathbf{u}-S\varphi-S^*\nu-T\psi-T^*\tau=f_4,\\&i\delta\psi-\tau=f_5,\\&i\delta\tau-V\mathbf{u}-W\varphi-W^*\nu-X\psi-X^*\tau=f_6.\end{aligned}\tag{5.1}$$

Lemma 5.2. *The imaginary axis is contained in the resolvent of $\mathcal{A}$. That is: $i\mathbb{R}\subset\varrho(\mathcal{A})$*

Proof. Since $0\in\varrho(\mathcal{A})$, following the arguments of Liu and Zheng (see [34], page 25), we assume that the imaginary axis is not contained in the resolvent set. Then there exists a real number $h\neq0$ with $\|\mathcal{A}^{-1}\|^{-1}\le|h|<\infty$ such that the set $\{i\lambda,|\lambda|<|h|\}$ is contained in $\varrho(\mathcal{A})$ and $\sup\{\|(i\lambda I-\mathcal{A})^{-1}\|,|\lambda|<|h|\}=\infty$. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n\to h$, $|\lambda_n|<|h|$, and a sequence of unit norm vectors $\mathbf{U}_n=(\mathbf{u}_n,\mathbf{v}_n,\varphi_n,\nu_n,\psi_n,\tau_n)$ in the domain of the operator $\mathcal{A}$ such that

$$\|(i\lambda_nI-\mathcal{A})\mathbf{U}_n\|\to0.$$

Notice that, if we consider $(i\lambda_n I - \mathcal{A})\mathbf{U}_n$, we obtain a system like the one considered in (5.1) changing δ by λ_n and where $\mathbf{f}_{1n} \to 0$ in $\mathbf{W}_0^{1,2}(B)$, $\mathbf{f}_{2n} \to 0$ in $\mathbf{L}^2(B)$, $f_{3n}, f_{5n} \to 0$ in $W_0^{1,2}(B)$ and $f_{4n}, f_{6n} \to 0$ in $L^2(B)$.

If we multiply $(i\lambda_n I - \mathcal{A})\mathbf{U}_n$ by $\mathbf{U}_n$ and take its real part we obtain that

$$\mathbf{v}_n \to 0 \text{ in } \mathbf{W}_0^{1,2}(B), \nu_n, \tau_n \to 0 \text{ in } W_0^{1,2}(B).$$

It then follows that

$$\mathbf{u}_n \to 0 \text{ in } \mathbf{W}_0^{1,2}(B) \text{ and } \varphi_n, \psi_n \to 0 \text{ in } W_0^{1,2}(B).$$

These two facts contradict the assumption that $\mathbf{U}_n$ has unit norm and the proof is complete.

Lemma 5.3. *There exists a positive constant C such that for any $\mathbf{F} \in \mathcal{H}$*

$$\int_B (f_{ij}\overline{f}_{ij} + \nu_{,j}\overline{\nu}_{,j} + \tau_j\overline{\tau}_{,j})dV \leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}},$$

where

$$f_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

Proof. If we multiply the first equation of (5.1) by $-\mathbf{A}\overline{\mathbf{u}} - \mathbf{B}\overline{\varphi} - \mathbf{C}\overline{\psi}$, the second one by $\overline{\mathbf{v}}$, the third one by $-R\overline{\mathbf{u}} - S\overline{\varphi} - T\overline{\psi}$, the fourth one by $\overline{\nu}$, the fifth by $-V\overline{\mathbf{u}} - W\overline{\varphi} - X\overline{\psi}$, the last one by $\overline{\tau}$, and we add all the results we obtain at the left-hand side the following expression:

(5.2)

$$\begin{aligned} i\delta\Big(&- \langle \mathbf{u}, \mathbf{A}\mathbf{u} + \mathbf{B}\varphi + \mathbf{C}\psi\rangle - \langle \varphi, R\mathbf{u} + S\varphi + T\psi\rangle - \langle \psi, V\mathbf{u} + W\varphi + X\psi\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \nu, \nu\rangle + \langle \tau, \tau\rangle\Big) \\ &+\langle \mathbf{v}, \mathbf{A}\mathbf{u} + \mathbf{B}\varphi + \mathbf{C}\psi\rangle - \langle \mathbf{A}\mathbf{u} + \mathbf{B}\varphi + \mathbf{C}\psi, \mathbf{v}\rangle + \langle \nu, R\mathbf{u} + S\varphi + T\psi\rangle - \langle R\mathbf{u} - S\varphi + T\psi, \nu\rangle \\ &+\langle \tau, V\mathbf{u} + W\varphi + X\psi\rangle - \langle V\mathbf{u} + W\varphi + X\psi, \tau\rangle \\ &-\langle \mathbf{A}^*\mathbf{v}, \mathbf{v}\rangle - \langle S^*\nu + T^*\tau, \nu\rangle - \langle W^*\nu + X^*\tau, \tau\rangle. \end{aligned}$$

Notice that the first line of (5.2) is an imaginary number. The second and third lines are also imaginary because they are the sum of three differences of conjugate complex numbers. Finally, recalling the definition of the operators $\mathbf{A}^*$, S^*, T^*, W^* and X^*, from the fourth line we obtain the dissipation, which is positive.

Moreover, the norm of the right-hand side is bounded by $C_1\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}$, for a positive constant C_1.

Lemma 5.4. *There exists a positive constant C such that for any $\mathbf{F} \in \mathcal{H}$*

$$|\delta|\|\mathbf{U}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}} \quad \forall \delta \in \mathbb{R},$$

where $\mathbf{U}$ is the solution to problem (5.1).

Proof. Now we multiply the first equation of (5.1) by $-i(\mathbf{A}\overline{\mathbf{u}} + \mathbf{B}\overline{\varphi}) + \mathbf{C}\overline{\psi}$, the second one by $i\overline{\mathbf{v}}$, the third one by $-i(R\overline{\mathbf{u}} + S\overline{\varphi} + T\overline{\psi})$, the fourth one by $i\overline{\nu}$, the fifth by $-i(V\overline{\mathbf{u}} + W\overline{\varphi} + X\overline{\psi})$, the last one by $\overline{\tau}$, and we add all the results. Notice that if $\langle x, y\rangle = \Re\langle x, y\rangle + i\Im\langle x, y\rangle$, then $\langle x, iy\rangle = \Im\langle x, y\rangle - i\Re\langle x, y\rangle$. That means that now the real part of the sum of the left-hand side is just the imaginary part of (5.2).

Notice that

(5.3)
$$-i\Big(\langle \mathbf{v}, \mathbf{A}\mathbf{u}+\mathbf{B}\varphi+\mathbf{C}\psi\rangle - \langle \mathbf{A}\mathbf{u}+\mathbf{B}\varphi+\mathbf{C}\psi, \mathbf{v}\rangle + \langle \nu, R\mathbf{u}+S\varphi+T\psi\rangle - \langle R\mathbf{u}+S\varphi+T\psi, \nu\rangle$$
$$+\langle \tau, V\mathbf{u}+W\varphi+X\psi\rangle - \langle V\mathbf{u}+W\varphi+X\psi, \tau\rangle\Big) = -2\Im\mathcal{M}[(\mathbf{v},\nu,\tau),(\mathbf{u},\varphi,\psi)],$$

and, hence, it is real.

Taking into account the first line of (5.2), now we obtain another real part, which is $\delta\|\mathbf{U}\|^2_{\mathcal{H}}$. Therefore, a positive constant C_2 can be found such that

$$2\Im\mathcal{M}[(\mathbf{v},\nu,\tau),(\mathbf{u},\varphi,\psi)] \le C_2\|\mathbf{U}\|^{1/2}_{\mathcal{H}}\|\mathbf{F}\|^{1/2}_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}},$$

and, hence,

$$|\delta|\|\mathbf{U}\|^2_{\mathcal{H}} \le 2\Im\mathcal{M}[(\mathbf{v},\nu,\tau),(\mathbf{u},\varphi,\psi)] + C_3\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}. \tag{5.4}$$

As a consequence, we obtain

$$|\delta|\|\mathbf{U}\|^2_{\mathcal{H}} \le C_4\|\mathbf{U}\|^{3/2}_{\mathcal{H}}\|\mathbf{F}\|^{1/2}_{\mathcal{H}} + C_5\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

From this inequality we get that

$$|\delta|\|\mathbf{U}\|_{\mathcal{H}} \le C_6\|\mathbf{F}\|_{\mathcal{H}},$$

where $C_6 > 0$ and δ is sufficiently greater.

Theorem 5.5. *The semigroup generated by the operator $\mathcal{A}$ is analytic.*

Proof. Since $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup, $\mathbb{R}_+ \in \varrho(\mathcal{A})$ and as $0 \in \varrho(\mathcal{A})$, we have $i\mathbb{R} \subset \varrho(\mathcal{A})$. From Lemma 5.4 we have

$$\|\delta(i\delta\mathcal{I}-\mathcal{A})^{-1}\mathbf{F}\|_{\mathcal{H}} = |\delta|\|\mathbf{U}\|_{\mathcal{H}} \le C\|\mathbf{F}\|_{\mathcal{H}}.$$

Then,

$$\overline{\lim}_{|\delta|\to\infty}\|\delta(i\delta\mathcal{I}-\mathcal{A})^{-1})\| < \infty.$$

One important consequence of the analyticity of the semigroup is the stability of the solutions.

Corollary 5.6. *System (2.3) is exponentially stable. That is, there exist two positive constants M and ω such that $\|\mathbf{U}(t)\|_{\mathcal{H}} \le Me^{-\omega t}\|\mathbf{U}(0)\|_{\mathcal{H}}$.*

Another consequence of the analyticity of the solutions is the impossibility of localization. That means that the only solution that can be identically zero after a finite period of time is the null solution.

Corollary 5.7. *Let $(\mathbf{u},\varphi,\psi)$ be a solution to system (2.3) with boundary conditions (2.7) and initial conditions (2.8) such that $\mathbf{u}=\varphi=\psi\equiv 0$ after a finite time $t_0 > 0$. Then, $\mathbf{u}=\varphi=\psi\equiv 0$ for every $t \ge 0$.*

6. Conclusions

In this paper we have investigated several qualitative properties for the system of equations that models the behaviour of viscoelastic materials with double porosity when we assume viscous dissipation on each porous structure. First we have proved the uniqueness of solutions whenever the dissipation is strictly positive. Later we have shown their analyticity when the internal energy and the dissipation are both positive. Exponential stability and impossibility of localization of the solutions are consequences of the analitycity.

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References

[1] G. I. Barenblatt, I. P. Zheltov and I. N. Kockina, Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks (Strata), Prikl.Mat. Mekh. (English Translation), vol. 24, pp.1286–1303, 1960.

[2] G. I. Barenblatt and I. P. Zheltov, On the Basic Equations of Seepage of Homogeneous Liquids in Fissured Rock, Akad. Nauk SSSR (English Translation), vol. 132, 545–548, 1960.

[3] R. Wilson and E. Aifantis, On the Theory of Consolidation with Double Porosity, Int. J. Eng. Sci., vol. 20, pp. 1009–1035, 1982.

[4] N. Khalili and S. Valliappan, Unified Theory of Flow and Deformation in Double Porous Media, Eur. J. Mech. A, Solids, vol. 15, pp. 321–336, 1996.

[5] I. Masters, W.K.S. Pao and R.W. Lewis, Coupling Temperature to a Double Porosity Model of Deformable Porous Media, Int. J. Numer. Methods Eng., vol. 49, pp. 421–438, 2000.

[6] J.G. Berryman and H.F. Wang, Elastic Wave Propagation and Attenuation in a Double - porosity Dual - permeability Medium, Int. J. Rock Mechanics and Mining Sciences, vol. 37, pp. 63-78,2000.

[7] N. Khalili and A.P.S. Selvadurai, A Fully Coupled Constitutive Model for Thermohydro- mechanical Analysis in Elastic Media with Double Porosity, Geophys. Res. Lett., vol. 30, Art. No. 2268, 2003.

[8] S.R. Pride and J.G. Berryman, Linear Dynamics of Double-porosity Dual-permeability Materials-I. Phys. Rev. E, vol. 68, 036603, 2003.

[9] Y. Zhao and M. Chen, Fully Coupled Dual-porosity Model for Anisotropic Formations. Int. J. Rock Mech. Min. Sci., vol. 43, pp. 1128–1133, 2006.

[10] M. Svanadze, Dynamical Problems of the Theory of Elasticity for Solids with Double Porosity, Proceedings in Applied Mathematics and Mechanics, vol. 10, pp. 309–310, 2010.

[11] A. Ainouz, Homogenized double porosity models for poro-elastic media with interfacial ow barrier, Mathematica Bohemica, vol. 136, pp. 357–365, 2011.

[12] M. Svanadze, Plane waves and boundary value problems in the theory of elasticity for solids with double porosity. Acta Applicandae Mathematicae, vol. 122, pp. 461–471, 2012.

[13] B. Straughan, Stability and Uniqueness in Double Porosity Elasticity, Int. J. Eng. Sci., vol. 65, pp. 1–8, 2013.

[14] S.C. Cowin, Bone Poroelasticity, J. Biomech., vol. 32, pp. 217–238, 1999.

[15] D. Masin, V. Herbstova and J. Bohac, Properties of double porosity clayfills and suitable constitutive models. `http://web.natur.cuni.cz/uhigug/masin/download/mhb_16ICSMGE_Osaka.pdf`,2102.

[16] B. Straughan, Mathematical Aspects of MultiPorosity Continua, Springer, 2017.

[17] J. W. Nunziato and S.C. Cowin, A Nonlinear Theory of Elastic Materials with Voids, Arch. Rat. Mech. Anal., vol. 72, pp. 175–201, 1979.

[18] D. Ieşan and R. Quintanilla, On a Theory of Thermoelastic Materials with a Double Porosity Structure, Journal of Thermal Stresses, vol. 37, pp. 1017–1036, 2014.

[19] D. Ieşan, Thermoelastic Models of Continua. Kluwer Academic, Dordrecht, 2004.

[20] M. Svanadze, Boundary value problems of steady vibrations in the theory of thermoelasticity for materials with a double porosity structure. Archives of Mechanics, vol. 69, pp. 347–370, 2017.

[21] B. Straughan, Mathematical Aspects of Multi-Porosity Elasticity, in: Advances in Mechanics and Applied Mathematics. To be published.

[22] S.C. Cowin and J.W. Nunziato, Linear Elastic Materials with Voids, J. Elasticity, vol. 13, pp. 125–147, 1983.

[23] S.C. Cowin and P. Puri, The Classical Pressure Vessel Problems for Linear Elastic Materials with Voids, J. Elasticity, vol. 13, pp. 157–163, 1983.

[24] A. Bedford and D.S. Drumheller, Recent Advances: Theories of Immiscible and Structured Mixtures, Int. J. Eng. Sci., vol. 21, pp. 863–960, 1983.

[25] M. Ciarletta and B. Straughan, Thermo-poroacoustic Acceleration Waves in Elastic Materials with Voids, J. Math. Anal. Appl., vol. 333, pp. 142–150, 2007.

[26] A. Magaña and R. Quintanilla, On the Decay of in Porous-elasticity with Quasistatic Microvoids, J. Math. Anal. Appl. ,vol. 331, pp. 617–630, 2007.

[27] J.E. Muñoz-Rivera and R. Quintanilla, On the Time Polynomial Decay in Elastic Solids with Voids, J Math. Anal. Appl., vol. 338, pp. 1296–1309, 2008.

[28] P.X. Pamplona, J.E. Muñoz-Rivera and R. Quintanilla, Stabilization in Elastic Solids with Voids, J. Math. Anal. Appl., vol. 350, pp. 37–49, 2009.

[29] D. Ieşan and R. Quintanilla, On Thermoelastic Bodies with Inner Structure and Microtemperatures, J. Math. Anal. Appl., vol. 354, pp. 12–23, 2009.

[30] M.C. Leseduarte, A. Magaña and R. Quintanilla, On the time Decay of Solutions in Porous-thermo-elasticity of Type II. Disc. Cont. Dyn. Syst. Series B, vol. 13, pp. 375–391, 2010.

[31] P.X. Pamplona, J.E. Muñoz-Rivera and R. Quintanilla, Analyticity in Porousthermoelasticity with Microtemperatures.J. Math. Anal. Appl., vol.394, pp. 645–655, 2012.

[32] D. Ieşan and R. Quintanilla, *A Theory of Porous Thermoviscoelastic Mixtures*, Journal of Thermal Stresses, vol. 30, pp. 693–714, 2007.

[33] V.G. Sigillito, Explicit a priori inequalities with applications to boundary value problems. Pitman Publishing, London, 1977.

[34] Z. Liu and S. Zheng, Semigroups associated to dissipative systems, in: Research Notes in Mathematics, vol. 398, Chapman & Hall/CRC Boca Raton, FL, 1999.