AVERAGING OPERATORS ON DECREASING OR POSITIVE FUNCTIONS: EQUIVALENCE AND OPTIMAL BOUNDS

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Abstract. We study the optimal bounds for the Hardy operator S minus the identity, as well as S and its dual operator S^*, on the full range $1 \le p \le \infty$, for the cases of decreasing, positive or general functions (in fact, these two kind of inequalities are equivalent for the appropriate cone of functions). For $1 < p \le 2$, we prove that all these estimates are the same, but for $2 < p < \infty$, they exhibit a completely different behavior.

1. Introduction

Denote by $\mathcal{M}(\mathbb{R}^+)$ the class of all measurable functions on $\mathbb{R}^+ = (0, \infty)$, and for $f \in \mathcal{M}(\mathbb{R}^+)$, let us consider the classical Hardy averaging operator S and its dual S^* (see, e.g., [6]):

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad \text{and} \qquad S^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

In what follows, we will always assume that $S(|f|)(1) < \infty$ and $S^*(|f|)(1) < \infty$. These are natural conditions so that $S(f)$ and $S^*(f)$ are well defined. Moreover, such functions satisfy that $S^*(|f|) \in L^1(0,1)$, which is an important property we will use later on. In fact,

$$\int_0^1 S^*(|f|)(t)\,dt = \int_0^1 |f(t)|\,dt + \int_1^\infty |f(t)|\frac{dt}{t}. \tag{1}$$

An important fact about S and S^*, which is a consequence of the classical Hardy inequalities, is that these operators are bounded in $L^p(\mathbb{R}^+)$

$$\|Sf\|_p \le p'\|f\|_p, \quad \|S^*f\|_p \le p\|f\|_p \tag{2}$$

and $p' = p/(p-1)$ and p are, respectively, the best constants. Furthermore, their norms are comparable (see [7, pp. 240, 244]),

$$\frac{1}{p'}\|Sf\|_p \le \|S^*f\|_p \le p\|Sf\|_p, \text{ for } 1 < p < \infty. \tag{3}$$

However, the optimal constants in (3) were actually found in [9] very recently:

Theorem 1.1. [9, Theorem 1.1] *Let $f \in \mathcal{M}(\mathbb{R}^+)$ be a positive function and assume $1 < p < \infty$. Then,*

$$(p-1)\|Sf\|_p \le \|S^*f\|_p \le (p-1)^{1/p}\|Sf\|_p, \tag{4}$$

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if $1 < p \leq 2$, *and*

$$(p-1)^{1/p}\|Sf\|_p \leq \|S^*f\|_p \leq (p-1)\|Sf\|_p, \tag{5}$$

if $2 \leq p < \infty$. *Moreover, the constants in* (4) *and* (5) *are the best possible.*

In recent years, extensions of Hardy's inequalities, for other operators, have been considered restricted to the cone of decreasing functions. In particular, sharp estimates for $S - \mathrm{Id}$ and $f\downarrow$ were proved in [5], on the range $2 \leq p < \infty$ (see also [10] for a previous result when $p = 2, 3, 4, \dots$). The best constants in the full interval $1 < p < \infty$, again for decreasing functions, were finally characterized in [9]. One of the main techniques used in [9] is to show that the boundedness for $S - \mathrm{Id}$ on decreasing functions is equivalent to estimates of the form $\|Sf\|_p \lesssim \|S^*f\|_p$, for positive functions $f \geq 0$. Inequalities of restricted type have been also obtained for all indexes $1 < p < \infty$ [11].

To complete this framework, it is interesting to also recall that, motivated by the well-known Iwaniec's conjecture [8] regarding the norm of the Beurling operator B on $L^p(\mathbb{C})$ (see [1, 4] for more information), and using the fact that, for radial functions $F(z) = f(|z|^2)$, the following equality holds

$$BF(z) = \frac{\overline{z}}{z}(S - \mathrm{Id})f(|z|^2), \quad z \in \mathbb{C},$$

then the norm of B on such radial functions is equivalent to the norm of $S - \mathrm{Id}$ for general functions f. In this case, optimal constants are only known if $1 < p \leq 2$ (see [2]).

In this work, we will describe the best constants in all known cases, for the inequalities involving $S - \mathrm{Id}$, S and S^*, in all three different cones of functions (general f, positive $f \geq 0$ or decreasing $f\downarrow$). For the estimates still open, we will give the best bounds available.

The following result, based on the arguments given in [9] for decreasing and positive functions, is an important tool we will use to prove the equivalence of the optimal constants for general functions. In what follows, we will assume X and Y to be lattice quasi-Banach spaces over $\mathbb{R}^+$ (e.g., L^p, $1 \leq p \leq \infty$ or $L^{1,\infty}$).

Lemma 1.2. *Let X and Y be two lattice quasi-Banach spaces over $\mathbb{R}^+$. If for every function f, the following inequality holds*

$$\|Sf - f\|_Y \leq C_{X,Y}\|f\|_X,$$

then

$$\|Sf\|_Y \leq C_{X,Y}\|S^*f\|_X.$$

Proof. Let f be a general function and let $g(x) = S^*f(x)$. Then, as a consequence of (1), it is easy to see that we can use Fubini's theorem, and hence

$$Sg(x) - g(x) = S(S^*f)(x) - S^*f(x) = Sf(x) + S^*f(x) - S^*f(x) = Sf(x).$$

Thus, by hypothesis,

$$\|Sf\|_Y = \|Sg - g\|_Y \leq C_{X,Y}\|g\|_X = C_{X,Y}\|S^*f\|_X.$$

□

The following table summarizes our main goals. For a given norm (or quasi-norm) $\|\cdot\|$ we are going to describe the best known constants for each inequality on the first column, and for all three different cones of the first row.

Norm: $\|\cdot\|$	$f\downarrow$	$f \geq 0$	f
$\|Sf - f\| \lesssim \|f\|$	A	B	C
$\|Sf\| \lesssim \|S^*f\|$	D	E	F

Table 1. Best constants for different cones of functions.

Observe that we easily get the inequalities $A \leq B \leq C$; $D \leq E \leq F$, and, using Lemma 1.2, we also have that $F \leq C$. Moreover, in all cases we are going to also obtain that $A = E$.

In Section 2 we review the optimal constants on Table 1, for the range $1 < p < 2$ and observe that they are all equal to the value $1/(p-1)$. Section 3 deals with the range $2 \leq p < \infty$. Here we show that the constants behave more wildly (they are all different, except for the trivial cases). Our main results are the following: In Theorem 3.2 we find the sharp estimate for $S - \mathrm{Id}$, on the cone of positive functions $f \geq 0$ and in Theorem 3.3 we also obtain the best bound relating S and S^*, for f decreasing, whenever $p = 2, 3, 4, \dots$ In Section 4, we study what happens at the endpoints $p = 1$ and $p = \infty$. In Section 5 we make some final remarks about the constants C, D and F, which are still unknown in its full generality, for the range $2 < p < \infty$.

2. Sharp estimates for $1 < p < 2$

Sharp estimates for the norm of the Hardy operator minus the identity in the cone of decreasing functions, on the range $1 < p < 2$, were proved in [9]. The main techniques used in that paper consist of establishing the equivalence of optimal constants between L^p norms of the functions Sf and S^*f, $f \geq 0$, and L^p norms of the functions $S\varphi - \varphi$ and φ, where φ is a decreasing function on $\mathbb{R}^+$. The result is the following:

Theorem 2.1. [9, Theorems 1.1 and 1.2] *Let* $1 < p < 2$.

(i) If $f \geq 0$ *on* $\mathbb{R}^+$, *then the inequality*

$$\|Sf\|_p \leq \frac{1}{p-1}\|S^*f\|_p, \tag{6}$$

is sharp.

(ii) If f *is decreasing on* $\mathbb{R}^+$, *then the inequality*

$$\|Sf - f\|_p \leq \frac{1}{p-1}\|f\|_p \tag{7}$$

is sharp.

Remark 2.2. The optimality of (6) was proved by considering the family of decreasing functions $g_\varepsilon(x) = x^{\varepsilon - 1/p}\chi_{(0,1)}(x)$, for $0 < \varepsilon \leq 1/p$. Therefore, we also obtain that the best constant in (6), for decreasing functions, is equal to $1/(p-1)$. Thus, in the notation of Table 1, we have that $A = D = E = 1/(p-1)$.

Estimate (7), for general functions $f \in L^p(\mathbb{R}^+)$, and $1 < p < 2$, was established in [2]. The proof makes use of very subtle arguments and is more involved than the case of decreasing functions.

Theorem 2.3. [2, Theorem 4.1] *If $1 < p < 2$, and $f \in L^p(\mathbb{R}^+)$, then the inequality*

$$\|Sf - f\|_p \leq \frac{1}{p-1}\|f\|_p \tag{8}$$

is sharp.

Remark 2.4. We observe that using (7) and (8), we have that $\|Sf - f\|_p \leq \frac{1}{p-1}\|f\|_p$ is also optimal for $f \geq 0$. Thus, in the notation of Table 1, we also have that $A = B = C = D = E = 1/(p-1)$.

Corollary 2.5. *Let $1 < p < 2$, then, for any $f \in L^p(\mathbb{R}^+)$ the following inequality is optimal,*

$$\|Sf\|_p \leq \frac{1}{p-1}\|S^* f\|_p.$$

Proof. It is a direct consequence of Theorems 2.1 and 2.3, and Lemma 1.2. ☐

Thus, we have finally proved that on the range $1 < p < 2$ all constants are the same: $A = B = C = D = E = F = 1/(p-1)$.

3. Sharp estimates for $2 \leq p < \infty$

In [5], as a particular case of the sharp constants obtained for the norm of the operator $S - \mathrm{Id}$, acting on the cone of decreasing functions in weighted L^p spaces, we were able to prove that for any $p \geq 2$ and $f \downarrow$, then

$$\|Sf - f\|_p \leq (p-1)^{-1/p}\|f\|_p,$$

and the constant $A = (p-1)^{-1/p}$ is optimal. For $p \geq 2$ a natural number, this was previously obtained in [10]. Again, using (5), it also holds that $E = (p-1)^{-1/p}$; i.e., for any positive function $f \geq 0$

$$\|Sf\|_p \leq (p-1)^{-1/p}\|S^* f\|_p. \tag{9}$$

For the proof of our main result on the norm of $S - \mathrm{Id}$ acting on the cone of positive functions in $L^p(\mathbb{R}^+)$, the following lemma will be useful.

Lemma 3.1. *If $p \geq 2$ and $0 < t \leq s$, then*

(i)

$$f_1(s,t) = s^p\left[\left(1 - \frac{t}{s}\right)^p - 1 + \frac{p\,t}{s}\right] - t^p \geq 0$$

and

(ii)

$$f_2(s,t) = pt^{p-1}(t-s) - t^p + s^p\left[1 - \left(1 - \frac{t}{s}\right)^p\right] \geq 0.$$

Proof. For the proof of part *(i)*, by homogeneity, it is enough to consider f_1 as a function of the variable $0 \le r = t/s \le 1$ and define

$$g_1(r) = f_1(1, r) = (1 - r)^p - 1 + pr - r^p.$$

Then, $g_1(0) = 0$ and g_1 is increasing in $[0, 1]$, since its derivative

$$g_1'(r) = p(1 - r^{p-1} - (1 - r)^{p-1}) \ge 0,$$

because $p \ge 2$ and $0 \le r \le 1$.

Similarly, to prove *(ii)*, let us write f_2 in terms of the variable $0 \le r = t/s \le 1$ and define

$$g_2(r) = f_2(1, r) = (p - 1)r^p - pr^{p-1} - (1 - r)^p + 1.$$

In this case, $g_2(0) = g_2(1) = 0$ and its derivative

$$g_2'(r) = p(p - 1)r^{p-2}(r - 1) + p(1 - r)^{p-1}$$

has a unique zero at the point $r_0 = 1/(1+(p-1)^{1/(p-2)})$, which is a local maximum, since $p \ge 2$. As a consequence, g_2 is increasing in $[0, r_0)$ and decreasing in $(r_0, 1]$, so $g_2(r) \ge 0$, for $r \in [0, 1]$. □

It is well-known that the operator $S-\mathrm{Id}$ is an isometry in $L^2(\mathbb{R}^+)$ (see [6]). Hence, $A = B = C = 1$, if $p = 2$. Concerning $p > 2$, we have the following result:

Theorem 3.2. *If $2 < p < \infty$, and f is a positive function in $L^p(\mathbb{R}^+)$, then the following sharp estimate holds*

$$\|Sf - f\|_p \le \|f\|_p.$$

Proof. By a density argument, we can restrict ourselves to the case of f a positive function of compact support that agrees with a polynomial in its support. Then, $Sf - f$ is also a continuous function on $\mathbb{R}^+$, except possibly at one point where it would have a jump discontinuity, has a finite number of changes of sign, and no zeros outside the support of f.

Then, let us consider $I_j = (a_{j-1}, a_j)$, $1 \le j \le N$ be a finite partition of $\mathbb{R}^+$ with $a_0 = 0$, $a_N = \infty$, in such a way that the sign of $Sf - f$ is constant in the interval I_j and it changes between two consecutive intervals. Let J be the collection of indexes $1 \le j \le N$ such that $Sf - f > 0$ on I_j, and $J' = \{1, ..., N\} \setminus J$; that is, $Sf - f < 0$ on I_j, $j \in J'$.

For $\alpha \in \mathbb{R}$, we will use the following representation in power series of the function $(1 + t)^\alpha$:

$$(1 + t)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} t^k, \ |t| < 1, \tag{10}$$

where, for every positive integer k,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k)}{k!},$$

and $\binom{\alpha}{0} = 1$.

In every interval $I_j = (a_{j-1}, a_j)$, with $j \in J$, since $-1 < (f - Sf)/Sf < 0$, using the binomial identity (10), we have

$$\begin{aligned}\int_{I_j} f^p(x)\,dx &= \int_{I_j} \left(\frac{f(x) - Sf(x)}{Sf(x)} + 1\right)^p (Sf(x))^p\,dx\\ &= \int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)}\right)^k (Sf(x))^p\,dx.\end{aligned}$$

Then, if we denote by $u(t) = Sf(t)$, we have that $f(t) = (tu(t))' = tu'(t) + u(t)$, and hence this last integral is equal to

$$\begin{aligned}\int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{tu'(t)}{u(t)}\right)^k (u(t))^p\,dt &= \int_{I_j} \left(u^p(t) + ptu^{p-1}(t)u'(t)\right) dt\\ &\quad + \int_{I_j} \sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{tu'(t)}{u(t)}\right)^k (u(t))^p\,dt.\end{aligned}$$

Integrating the first term of the right-hand side, we obtain

$$\int_{I_j} \left(u^p(t) + ptu^{p-1}(t)u'(t)\right) dt = a_j u^p(a_j) - a_{j-1}u^p(a_{j-1}), \tag{11}$$

where we understand that if $j = N$, and hence $a_N = \infty$, then $a_N u^p(a_N) = \lim_{t\to\infty} tu^p(t) = 0$. Therefore, with the original notation, we have obtained that, for every interval $I_j = (a_{j-1}, a_j)$, such that $j \in J$,

$$\begin{aligned}\int_{I_j} \left(f^p(x) - (Sf(x) - f(x))^p\right) dx &= a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p\\ &\quad + \int_{I_j} \sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)}\right)^k (Sf(x))^p\,dx \qquad (12)\\ &\quad - \int_{I_j} (Sf(x) - f(x))^p\,dx.\end{aligned}$$

Similarly, in every interval $I_j = (a_{j-1}, a_j)$, with $j \in \tilde{J}$, since $-1 < -Sf/f < 0$, using again the binomial identity (10), we obtain

$$\begin{aligned}\int_{I_j} (f(x) - Sf(x))^p\,dx &= \int_{I_j} \left(\frac{-Sf(x)}{f(x)} + 1\right)^p (f(x))^p\,dx\\ &= \int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)}\right)^k (f(x))^p\,dx.\end{aligned}$$

Therefore, using (11) expressed in terms of $u(t) = Sf(t)$ and $f(t) = (tu(t))' = tu'(t) + u(t)$, we obtain that for every interval $I_j = (a_{j-1}, a_j)$, such that $j \in \tilde{J}$,

$$\begin{aligned}\int_{I_j} \left(f^p(x) - (f(x) - Sf(x))^p\right) dx = a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p \\ + \int_{I_j} (p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1}\, dx \\ - \int_{I_j} \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)}\right)^k (f(x))^p\, dx.\end{aligned} \tag{13}$$

Summing up the equalities proved in (12) for every $j \in J$, and in (13) for every $j \in \tilde{J}$,

(14)

$$\begin{aligned}\int_0^{\infty} \left(f^p(x) - |f(x) - Sf(x)|^p\right) dx = \sum_{j=1}^{N} \left(a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p\right) \\ + \sum_{j\in J} \int_{I_j} \left[\sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)}\right)^k (Sf(x))^p - (Sf(x) - f(x))^p\, dx\right] \\ + \sum_{j\in \tilde{J}} \int_{I_j} \left[(p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1} - \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)}\right)^k (f(x))^p\, dx\right].\end{aligned}$$

In order to see that this last expression is positive, we look at the three terms of the right-hand side that can be simplified as follows. The first one,

$$\sum_{j=1}^{N} \left(a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p\right) = \lim_{r\to\infty} rSf(r)^p - \lim_{r\to 0^+} rSf(r)^p = 0,$$

due to the fact that f has compact support. For the second one, we observe that, since $j \in J$,

$$\begin{aligned}&\sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)}\right)^k (Sf(x))^p - (Sf(x) - f(x))^p \\ &=(Sf(x))^p \left[\left(1 - \frac{Sf(x) - f(x)}{Sf(x)}\right)^p - 1 + p\frac{Sf(x) - f(x)}{Sf(x)}\right] - (Sf(x) - f(x))^p\end{aligned}$$

and this last function is positive, due to Lemma 3.1 *(i)* for $s = Sf(x) > 0$ and $t = Sf(x) - f(x) > 0$, where $0 < t < s$.

Finally the third one, since $j \in \tilde{J}$, then

$$\begin{aligned}&(p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1} - \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)}\right)^k (f(x))^p \\ &=p(Sf(x))^{p-1}(Sf(x) - f(x)) - (Sf(x))^p + (f(x))^p \left[1 - \left(1 - \frac{Sf(x)}{f(x)}\right)^p\right]\end{aligned}$$

and this last function is positive, due to Lemma 3.1 *(ii)* for $s = f(x) > 0$ and $t = Sf(x) > 0$, since $0 < t < s$. These observations allow us to conclude from (14) that

$$\|Sf - f\|_p \leq \|f\|_p. \tag{15}$$

Finally, in order to check that this last estimate is sharp, let us take $f_r(x) = \chi_{[1,r+1)}(x)$, with $r > 0$. For this function f_r,

$$(Sf_r - f_r)(x) = -\frac{1}{x}\chi_{[1,1+r)}(x) + \frac{r}{x}\chi_{(r+1,\infty)}(x).$$

Thus

$$\lim_{r\to 0^+} \frac{\|Sf_r - f_r\|_p^p}{\|f_r\|_p^p} = \lim_{r\to 0^+} \frac{1}{p-1} \frac{(r+1)^{p-1} - 1 + r^p}{r(r+1)^{p-1}} = 1.$$

This shows that the constant 1 in the right-hand side of (15) is optimal. □

Equivalently, Theorem 3.2 proves that, for every $p \geq 2$, we have that $B = 1$. Continuing with the study of best estimates, we observe that, for $p = 2$, and any general function f, using the equalities $SS^* = S^*S = S + S^*$, we can prove

$$\|Sf\|_2 = \|S^*f\|_2.$$

In particular, this holds for f decreasing. Now, for other values of $p \geq 2$ and f decreasing, we have the following theorem concerning the constant D.

Theorem 3.3. *Let $p \geq 2$ be a natural number. Then, for any decreasing function $f \in L^p(\mathbb{R}^+)$, the following inequality is sharp*

$$\|Sf\|_p \leq \left(\frac{p'}{p!}\right)^{1/p} \|S^*f\|_p.$$

Proof. By using the density of simple and decreasing functions in the corresponding cone in $L^p(\mathbb{R}^+)$, we can restrict ourselves to prove the inequality for functions

$$f_N(x) = \sum_{k=1}^{N} b_k \chi_{(0,a_k)}(x),$$

with $b_j \geq 0$, $j = 1, \dots, N$, and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$. Easy calculations show that

$$Sf_N(x) = \sum_{j=1}^{N} b_j \left(\chi_{(0,a_j)}(x) + \frac{a_j}{x}\chi_{(a_j,\infty)}(x)\right)$$

and

$$S^*f_N(x) = \sum_{j=1}^{N} b_j \log\left(\frac{a_j}{x}\right)\chi_{(0,a_j)}(x).$$

Thus, our goal is to check the following sharp inequality, for every function f_N

$$\int_0^\infty (Sf_N)^p(x)\,dx \leq \frac{p'}{p!}\int_0^\infty (S^*f_N)^p(x)\,dx. \tag{16}$$

Looking at both sides of (16) as a homogeneous polynomial of degree N in the variables $b_1, \cdots, b_N$, we observe that it is enough to prove the estimate for every term $\prod_{j=1}^N b_j^{\alpha_j}$ associated to the multi-index $\bar{\alpha} = (\alpha_j)_{j=1}^N$, with $0 \leq \alpha_j \leq p$ and

$\sum_{i=1}^p \alpha_i = p$. In order to do that, notice that for a fixed multi-index $\bar\alpha = (\alpha_j)_{j=1}^N$, the terms on the left-hand side of (16) correspond to the following integral

$$\int_0^\infty \prod_{j=1}^N \left(\chi_{(0,a_j)}(x) + \left(\frac{a_j}{x}\right)^{\alpha_j} \chi_{(a_j,\infty)} \right) dx, \tag{17}$$

and, similarly, the right-hand side for the corresponding term of the same multi-index is equal to

$$\int_0^\infty \prod_{j=1}^N \log^{\alpha_j}\left(\frac{a_j}{x}\right) \ \chi_{(0,a_j)}^{\alpha_j}(x)\, dx = \int_0^{a_{j_0}} \prod_{j=j_0}^N \log^{\alpha_j}\left(\frac{a_j}{x}\right) dx,$$

where $j_0 = \min\{j :\ \alpha_j \neq 0\}$.

To find an explicit expression of the integral in (17), we first introduce the following notation. For $\bar\alpha = (\alpha_1, \cdots, \alpha_N)$ a fixed multi-index, with $0 \le \alpha_j \le p$ and $\sum_{i=1}^p \alpha_i = p$, let us define $A_j = \sum_{i=1}^j \alpha_i$, for any $1 \le j \le N$. Then, it is clear that $0 \le A_1 \le \cdots \le A_N = p$, and (17) is equal to

$$a_1 + \sum_{j=1}^N \int_{a_j}^{a_{j+1}} \frac{\prod_{k=1}^j a_k^{\alpha_k}}{x^{A_j}}\, dx, \tag{18}$$

where $a_0 = 0$ and $a_{N+1} = \infty$. As before, let j_0 be the smallest index j such that $\alpha_j \neq 0$. We will distinguish the following two cases:

If $\alpha_{j_0} = 1$, let us define j_1 as the first index such that $A_{j_1} \ge 2$. Hence, $A_k = 1$, for all $j_0 \le k \le j_1 - 1$ and $A_k \ge 2$, for $j_1 \le k \le N$. Note that for any $j < j_0$, since $\alpha_j = 0$, then $a_j^{\alpha_j} = 1$. Therefore, (18) is equal to

$$a_{j_0} + a_{j_0} \sum_{j=j_0}^{j_1-1} \log\left(\frac{a_{j+1}}{a_j}\right) + \sum_{j=j_1}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k}\right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1},$$

taking into account that for $j = N$, we have that $A_N = p > 1$ and $a_{N+1}^{1-A_N} = 0$.

On the other hand, if $\alpha_{j_0} \ge 2$, there are no terms with $A_j = 1$ and the integral in (18) is equal to

$$a_{j_0} + \sum_{j=j_0}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k}\right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1}.$$

As a consequence, in order to obtain the estimate (16), we will prove that for every multi-index $\bar\alpha$, with $\alpha_{j_0} = 1$, the following estimate is sharp

$$\begin{aligned} &a_{j_0} + a_{j_0} \sum_{j=j_0}^{j_1-1} \log\left(\frac{a_{j+1}}{a_j}\right) + \sum_{j=j_1}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k}\right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} \\ \le &\frac{p'}{p!} \int_0^{a_{j_0}} \prod_{j=j_0}^N \log^{\alpha_j}\left(\frac{a_j}{x}\right) dx = \frac{p'}{p!} a_{j_0} \int_0^1 \prod_{j=j_0}^N \log^{\alpha_j}\left(\frac{a_j}{a_{j_0} x}\right) dx. \end{aligned} \tag{19}$$

A second kind of bounds will consist in proving that the following inequality is optimal and holds for every multi-index $\bar{\alpha}$, with $\alpha_{j_0} \geq 2$,

$$a_{j_0} + \sum_{j=j_0}^{N} \left(\prod_{k=j_0}^{j} a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} \leq \frac{p'}{p!} a_{j_0} \int_0^1 \prod_{j=j_0}^{N} \log^{\alpha_j} \left(\frac{a_j}{a_{j_0} x} \right) dx. \tag{20}$$

First of all, let us prove (20). Since in the expression to be estimated, all the variables a_j involved correspond to $j_0 \leq j \leq N$, we can assume, without loss of generality, that $j_0 = 1$. We adopt the notation $\alpha_1 = \alpha$ and separate in (20) the term with $j = j_0 = 1$. Let us consider, then, the following function Φ of N variables $(a_1, \cdots, a_N)$ restricted to the set $0 < a_1 \leq \cdots \leq a_N$,

$$\Phi(a_1, \cdots, a_N) = a_1 + \frac{a_1^{\alpha}}{\alpha - 1}(a_1^{1-\alpha} - a_2^{1-\alpha}) \\ + \sum_{j=2}^{N} \left(\prod_{k=2}^{j} a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} - \frac{p'}{p!} a_1 \int_0^1 \log^{\alpha}\left(\frac{1}{x}\right) \prod_{j=2}^{N} \log^{\alpha_j} \left(\frac{a_j}{a_1 x} \right) dx.$$

With this notation, (20) is equivalent to proving that $\Phi(a_1, \cdots, a_N) \leq 0$, on the set $0 < a_1 \leq \cdots \leq a_N$.

Dividing all the expression above by $a_1 > 0$, and renaming the variables as $r_j = a_1/a_j$, $1 \leq j \leq N$, our problem is equivalent to consider the function

$$\Psi(1, r_2, \cdots, r_N) = \frac{\alpha}{\alpha - 1} - \frac{r_2^{\alpha-1}}{\alpha - 1} \\ + \sum_{j=2}^{N} \left(\prod_{k=2}^{j} r_k^{-\alpha_k} \right) \frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j - 1} - \frac{p'}{p!} \int_0^1 \log^{\alpha}\left(\frac{1}{x}\right) \prod_{j=2}^{N} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) dx,$$

restricted to the set S where $0 < r_N \leq \cdots \leq r_2 \leq 1 = r_1$ and with the assumption $r_{N+1} = 0$. Also, without loss of generality, we assume $\alpha_N \geq 1$, otherwise there is no dependence of the variable r_N. We will prove that on S the function attains its maximum at the point $1 = r_2 = \cdots = r_N$. Since

$$\Psi(1, \cdots, 1) = 1 + \frac{1}{p-1} - \frac{p'}{p!} \int_0^1 \log^{p}\left(\frac{1}{x}\right) dx = p' - \frac{p'}{p!}\Gamma(p+1) = 0,$$

this would prove that the inequality (20) is sharp. We are going to show that Ψ is an increasing function with respect to the variable r_N on the set S. Computing its partial derivative, we obtain

$$\frac{\partial \Psi}{\partial r_N}(1, r_2, \cdots, r_n) = -\frac{\alpha_N}{p-1} \prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-2} \\ + \frac{p'}{p!} \frac{\alpha_N}{r_N} \int_0^1 \log^{\alpha}\left(\frac{1}{x}\right) \prod_{j=2}^{N-1} \log^{\alpha_j}\left(\frac{1}{r_j x}\right) \log^{\alpha_N - 1}\left(\frac{1}{r_N x}\right) dx.$$

Therefore, in order to prove that this derivative is positive on the set S, we must justify that the following inequality holds on S,

$$\prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-1} \leq \frac{1}{(p-1)!} \int_0^1 \log^{\alpha}\left(\frac{1}{x}\right) \prod_{j=2}^{N-1} \log^{\alpha_j}\left(\frac{1}{r_j x}\right) \log^{\alpha_N - 1}\left(\frac{1}{r_N x}\right) dx.$$

But, since $r_N \le r_k \le 1$, for $k \ge N-1$,

$$\begin{aligned}\prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-1} &= \prod_{k=2}^{N-1}\left(\frac{r_N}{r_k}\right)^{\alpha_k} \le 1\\ &= \frac{1}{(p-1)!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{N-1}\log^{\alpha_j}\left(\frac{1}{x}\right)\log^{\alpha_N-1}\left(\frac{1}{x}\right)dx\\ &\le \frac{1}{(p-1)!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{N-1}\log^{\alpha_j}\left(\frac{1}{r_jx}\right)\log^{\alpha_N-1}\left(\frac{1}{r_Nx}\right)dx,\end{aligned}$$

and we are done. In this way, we will put $r_N = r_{N-1}$ and consider the function of $N-2$ variables $\Psi(1, r_2, \cdots, r_{N-1}, r_{N-1})$ on the set $0 < r_{N-1} \le \cdots \le r_2$, which will also define an increasing function on this set with respect the variable r_{N-1}, and then we proceed recursively by considering, for any $2 \le k \le N-1$, the function of $k-1$ variables

$$\begin{aligned}\Psi(1, r_2, \cdots, r_k, \cdots, r_k) &= \frac{\alpha}{\alpha-1} - \frac{r_2^{\alpha-1}}{\alpha-1} + \sum_{j=2}^{k-1}\left(\prod_{i=2}^{j} r_i^{-\alpha_i}\right)\frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j-1}\\ &+ \frac{\prod_{i=2}^{k-1} r_i^{-\alpha_i}\prod_{i=k}^{N} r_k^{-\alpha_i}}{A_N-1} r_k^{A_N-1} - \frac{p'}{p!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{k-1}\log^{\alpha_j}\left(\frac{1}{r_jx}\right)\log^{p-A_{k-1}}\left(\frac{1}{r_kx}\right)dx,\end{aligned}$$

and its partial derivative with respect to the last variable r_k is equal to

$$\begin{aligned}&-\frac{A_{k-1}-p}{p-1}\prod_{i=2}^{k-1} r_i^{-\alpha_i} r_k^{A_{k-1}-2}\\ &+\frac{p'}{p!}\frac{p-A_{k-1}}{r_k}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{k-1}\log^{\alpha_j}\left(\frac{1}{r_jx}\right)\log^{p-A_{k-1}-1}\left(\frac{1}{r_kx}\right)dx.\end{aligned}$$

Similarly, as in the previous case, we have that this derivative is positive on the set $0 < r_k \le \cdots \le r_2 \le 1$ due to the chain of inequalities

$$\begin{aligned}\prod_{i=2}^{k-1} r_i^{-\alpha_i} r_k^{A_{k-1}-1} &= \prod_{k=2}^{k-1}\left(\frac{r_k}{r_i}\right)^{\alpha_i} \le 1 = \frac{1}{(p-1)!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\log^{p-\alpha-1}\left(\frac{1}{x}\right)dx\\ &\le \frac{1}{(p-1)!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{k-1}\log^{\alpha_j}\left(\frac{1}{r_jx}\right)\log^{p-A_{k-1}-1}\left(\frac{1}{r_kx}\right)dx.\end{aligned}$$

By this procedure, we can finally reduce the number of the variables until we just consider, on the set $0 < r_2 \le 1$, the function

$$\Psi(1, r_2, \cdots, r_2) = \frac{\alpha - r_2^{\alpha-1}}{\alpha-1} + \frac{\prod_{i=2}^{N} r_2^{-\alpha_i} r_2^{p-1}}{p-1} - \frac{p'}{p!}\int_0^1 \log^\alpha\left(\frac{1}{x}\right)\prod_{j=2}^{N}\log^{\alpha_j}\left(\frac{1}{r_2x}\right)dx.$$

As before, we can prove that its derivative in $r_2 \in (0,1]$ is also positive and, therefore, on the set $0 < r_N \le \cdots \le r_2 \le 1$

$$\Psi(1, r_2, \cdots, r_N) \le \Psi(1, \cdots, 1) = 0,$$

which ends the proof of (20).

We now proceed with estimate (19). As in the previous case, we can assume that $j_0 = 1$, since all the variables a_j involved in (19) are those with $j_0 \le j \le N$. Therefore, we are considering a multi-index $\bar{\alpha}$, with $\alpha_1 = 1$, $A_j = 1$, $1 \le j \le j_1 - 1$, $A_j \ge 2$, for $j_1 \le j \le N$. Then, dividing the inequality by a_1 and defining the variables $r_i = a_1/a_i$, $1 \le i \le N$, we have to prove the following inequality, provided that $0 < r_N \le \cdots \le r_2 \le r_1 = 1$:

$$1+\log\left(\frac{1}{r_{j_1}}\right)+\sum_{j=j_1}^{N}\left(\prod_{k=j_1}^{j} r_k^{-\alpha_k}\right)\frac{r_j^{A_j-1}-r_{j+1}^{A_j-1}}{A_j-1} \le \frac{p'}{p!}\int_0^1 \log\left(\frac{1}{x}\right)\prod_{j=j_1}^{N}\log^{\alpha_j}\left(\frac{1}{r_j x}\right)dx.$$

Note that the assumptions on the multi-index $\bar{\alpha}$ imply that just the variables r_j, with $j_1 \le j \le N$, appear in the above inequality. We are also assuming that $r_{N+1} = 0$. Let us define the following function restricted to the set S such that $0 < r_N \le \cdots \le r_{j_1} \le r_1 = 1$:

$$\begin{aligned}\Psi(1, r_{j_1}, \cdots, r_N) = 1 + \log\left(\frac{1}{r_{j_1}}\right) + \sum_{j=j_1}^{N}\left(\prod_{k=j_1}^{j} r_k^{-\alpha_k}\right)\frac{r_j^{A_j-1}-r_{j+1}^{A_j-1}}{A_j-1}\\ - \frac{p'}{p!}\int_0^1 \log\left(\frac{1}{x}\right)\prod_{j=j_1}^{N}\log^{\alpha_j}\left(\frac{1}{r_j x}\right)dx.\end{aligned}$$

Our goal is to prove that $\Psi(1, r_{j_1}, \cdots, r_N) \le 0$ on S. We proceed in the same recursive way as before, proving that Ψ is an increasing function on S, first in the variable r_N and hence reducing the number of variables, since the role of r_N is assumed by r_{N-1}. The process continues until we are restricted to the study of the function defined on just one variable $0 < r_{j_1} \le 1$:

$$\begin{aligned}\Psi(1, r_{j_1}, \cdots, r_{j_1}) &= 1 + \log\left(\frac{1}{r_{j_1}}\right) + \frac{1}{p-1}\\ &\quad - \frac{p'}{p!}\int_0^1 \log\left(\frac{1}{x}\right)\prod_{j=j_1}^{N}\log^{\alpha_j}\left(\frac{1}{r_{j_1} x}\right)dx\\ &= p' + \log\left(\frac{1}{r_{j_1}}\right) - \frac{p'}{p!}\int_0^1 \log\left(\frac{1}{x}\right)\log^{p-1}\left(\frac{1}{r_{j_1} x}\right)dx.\end{aligned}$$

Its derivative is

$$\begin{aligned}\Psi'(1, r_{j_1}, \cdots, r_{j_1}) &= -\frac{1}{r_{j_1}} + \frac{1}{(p-1)!}\frac{1}{r_{j_1}}\int_0^1 \log\left(\frac{1}{x}\right)\log^{p-2}\left(\frac{1}{r_{j_1} x}\right)dx\\ &\ge -\frac{1}{r_{j_1}} + \frac{1}{(p-1)!}\frac{1}{r_{j_1}}\int_0^1 \log^{p-1}\left(\frac{1}{x}\right)dx = 0.\end{aligned}$$

Since $\Psi(1, \cdots, 1) = 0$, we conclude that the maximum in the set S of the function Ψ is attained at point $(1, \cdots, 1)$, and the sharp estimate (19) is obtained. $\square$

In Section 5 we will make some comments about the optimality of the constant D, on Theorem 3.3, and the remaining cases regarding C and F (see Table 1).

4. Endpoint estimates

We start the study of the endpoint estimates for the case $p = \infty$.

Proposition 4.1. *Let $f \in L^\infty(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) If f is a decreasing function,

$$\|Sf - f\|_\infty \leq \|f\|_\infty.$$

Thus, $A = 1$.

(ii) If f is a positive function,

$$\|Sf - f\|_\infty \leq \|f\|_\infty.$$

Thus, $B = 1$.

(iii) If f is a general function,

$$\|Sf - f\|_\infty \leq 2\|f\|_\infty. \tag{21}$$

Thus, $C = 2$ (see [2, equation (5.5)]*).*

Proof. Let us first prove part *(i)*. Since $|Sf(x) - f(x)| = Sf(x) - f(x) \leq Sf(x)$, then

$$\|Sf - f\|_\infty \leq \|Sf\|_\infty \leq \|f\|_\infty.$$

In order to see that this estimate is sharp, we just consider the decreasing function $f(x) = \chi_{(0,1)}(x)$, for which

$$Sf(x) - f(x) = \frac{1}{x}\chi_{(1,\infty)}(x).$$

Thus,

$$\|Sf - f\|_\infty = \|f\|_\infty = 1.$$

To prove part *(ii)*, we observe that for every positive function f,

$$|Sf(x) - f(x)| \leq \max\{Sf(x), f(x)\}.$$

Then

$$\|Sf - f\|_\infty \leq \max\{\|Sf\|_\infty, \|f\|_\infty\} \leq \|f\|_\infty.$$

Also the sharpness in this last estimate follows by considering the same characteristic function as in part *(i)*.

Finally, for the proof of part *(iii)*, we observe that trivially,

$$\|Sf - f\|_\infty \leq \|Sf\|_\infty + \|f\|_\infty \leq 2\|f\|_\infty.$$

The fact that the constant 2 is sharp in the inequality above follows by considering $f(x) = \chi_{(0,1)}(x) - \chi_{(1,2)}(x)$, for which $Sf(x) - f(x) = \frac{2}{x}\chi_{(1,2)}(x)$ and hence

$$\|Sf - f\|_\infty = 2 = 2\|f\|_\infty.$$

□

We now deal with the remaining constants relating S and S^*, in the case $p = \infty$. Observe that since, for any decreasing function f, $\|S^* f\|_\infty = \infty$, then $D = 0$ and we just need to consider the cases of positive and general functions, constants E and F, respectively.

Proposition 4.2. *Let $f \in L^\infty(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) If f is a positive function,

$$\|Sf\|_\infty \le \|S^*f\|_\infty.$$

Thus, $E = 1$.

(ii) If f is a general function,

$$\|Sf\|_\infty \le 2\|S^*f\|_\infty.$$

Thus, $F = 2$.

Proof. Part *(i)* easily follows since, for a fixed $t > 0$,

$$\|S^*f\|_\infty = \int_0^\infty f(s)\,\frac{ds}{s} \ge \int_0^t f(s)\,\frac{ds}{s} \ge Sf(t).$$

The optimality of the inequality is due to the fact that, for $f_r(x) = \chi_{(r,r+1)}(x)$,

$$\|Sf_r\|_\infty = \frac{1}{r+1} \quad \text{and} \quad \|S^*f_r\|_\infty = \log\left(\frac{r+1}{r}\right).$$

Thus

$$\lim_{r\to\infty} \frac{\|Sf_r\|_\infty}{\|S^*f_r\|_\infty} = 1.$$

Part *(ii)* is a consequence of Proposition 4.1 *(iii)* and Lemma 1.2. The sharpness in the inequality follows by considering, for $\varepsilon > 0$,

$$f_\varepsilon(x) = \frac{x}{\varepsilon}\left(\chi_{(1-\varepsilon,1+\varepsilon)}(x) - \chi_{(2-\varepsilon,2)}(x)\right).$$

Then

$$S^*f_\varepsilon(x) = \chi_{(0,1-\varepsilon)}(x) + \frac{1-x}{\varepsilon}\chi_{(1-\varepsilon,1+\varepsilon)}(x) - \chi_{(1+\varepsilon,2-\varepsilon)}(x) + \frac{x-2}{\varepsilon}\chi_{(2-\varepsilon,2)}(x).$$

Thus, for every $\varepsilon > 0$, $\|S^*f_\varepsilon\|_\infty = 1$, and for $1+\varepsilon < x < 2-\varepsilon$, we have that $Sf_\varepsilon(x) = 2/x$. Hence,

$$\|Sf_\varepsilon\|_\infty \ge \frac{2}{1+\varepsilon}$$

and

$$2 = \lim_{\varepsilon\to 0} \frac{2}{1+\varepsilon} \le \liminf_{\varepsilon\to 0+} \frac{\|Sf_\varepsilon\|_\infty}{\|S^*f_\varepsilon\|_\infty}.$$

□

Continuing with the study of the endpoint cases, we now address what happens for $p = 1$. We start with the action of $S - \mathrm{Id}$ on $L^1(\mathbb{R}^+)$ to obtain weak-type estimates. The sharp bound $1/\log 2$, for any $f \in L^1(\mathbb{R}^+)$, was obtained in [2] (see also [3] for further extensions to weak-type L^p estimates, $1 \le p \le 2$). In part *(iii)* below we will give a simpler argument to prove optimality, which will be useful in the proof of Proposition 4.4 *(iii)*.

Proposition 4.3. *Let $f \in L^1(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) If f is a decreasing function,

$$\|Sf - f\|_{1,\infty} \le \|f\|_1.$$

Thus, $A = 1$.

(ii) If f is a positive function,

$$\|Sf - f\|_{1,\infty} \le \|f\|_1.$$

Thus, $B = 1$.

(iii) If f is a general function,

$$\|Sf - f\|_{1,\infty} \le \frac{1}{\log 2}\|f\|_1.$$

Thus, $C = 1/\log 2$ (see [2, Theorem 3.3]).

Proof. To prove parts *(i)* and *(ii)*, let us consider $f \ge 0$ and $t > 0$, and define $\alpha_t = \sup\{x > 0 : \ Sf(x) \ge t\}$. Thus $Sf(x) < t$, if $x > \alpha_t$. Observe that, since $f \in L^1(\mathbb{R}^+)$, $\lim_{x\to\infty} Sf(x) = 0$, and $\alpha_t < \infty$. Also,

$$|Sf(x) - f(x)| \le \max\{f(x), Sf(x)\}.$$

Thus,

$$\begin{aligned}|\{x > 0 : \ |Sf(x) - f(x)| > t\}| &= |\{x \le \alpha_t : \ |Sf(x) - f(x)| > t\}| \\ &\quad + |\{x > \alpha_t : \ |Sf(x) - f(x)| > t\}| \\ &\le \alpha_t + |\{x > \alpha_t : \ f(x) > t\}| \\ &\le \frac{1}{t}\int_0^{\alpha_t} f(x)\,dx + \frac{1}{t}\int_{\alpha_t}^{\infty} f(x)\,dx = \frac{1}{t}\|f\|_1.\end{aligned}$$

Therefore, for every $f \ge 0$

$$\|Sf - f\|_{1,\infty} \le \|f\|_1. \tag{22}$$

Since for the decreasing function $g(x) = \chi_{(0,1)}(x)$, $\|g\|_1 = 1$, $Sg(x) - g(x) = \frac{1}{x}\chi_{(1,\infty)}(x)$, we get

$$\|Sg - g\|_{1,\infty} = \sup_{0<\lambda\le 1} \lambda\frac{1-\lambda}{\lambda} = 1.$$

This proves that (22) is optimal, for both the cone of decreasing functions and the cone of positive functions.

Part *(iii)* is a consequence of [2, Theorem 3.3]. The optimality of the constant can be obtained by considering, for any $k \in \mathbb{N}$, the family of functions defined as

$$g_k(x) = k\chi_{(0,1/k)}(x) - \log x\, \chi_{(1,2)}(x). \tag{23}$$

Moreover, for this function $\|g_k\|_1 = 2\log 2$, and

$$Sg_k(x) - g_k(x) = \frac{1}{x}\chi_{(1/k,1)}(x) + \chi_{(1,2)}(x) + \frac{2-\log 4}{x}\chi_{(2,\infty)}(x).$$

Hence,

$$|\{x > 0 : \ |Sg_k(x) - g_k(x)| \ge 1\}| = 2 - \frac{1}{k}.$$

We obtain as a consequence,

$$\sup_f \frac{\|Sf - f\|_{1,\infty}}{\|f\|_1} \ge \lim_{k\to\infty} \frac{\|Sg_k - g_k\|_{1,\infty}}{\|g_k\|_1} \ge \lim_{k\to\infty} \frac{2-1/k}{2\log 2} = \frac{1}{\log 2}, \tag{24}$$

and, therefore, the constant $1/\log 2$ is sharp. □

Proposition 4.4. *Let $f \in L^1(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) If f is a decreasing function,

$$\|Sf\|_{1,\infty} \le \|S^*f\|_1.$$

Thus, $D = 1$.

(ii) If f is a positive function,

$$\|Sf\|_{1,\infty} \le \|S^*f\|_1.$$

Thus, $E = 1$.

(iii) If f is a general function,

$$\|Sf\|_{1,\infty} \le \frac{1}{\log 2}\|S^*f\|_1.$$

Thus, $F = 1/\log 2$.

Proof. First of all we observe that, if $f \ge 0$, Fubini's theorem implies

$$\|S^*f\|_1 = \|f\|_1.$$

Hence, sharp weak-type estimates of parts *(i)* and *(ii)* follow from the corresponding well-known weak-type optimal estimate for the Hardy operator,

$$\|Sf\|_{1,\infty} \le \|S^*f\|_1 = \|f\|_1.$$

As before, part *(iii)* is a consequence of Proposition 4.3 *(iii)* and Lemma 1.2. For the optimality, if g_k is as in (23), given $j,k \in \mathbb{N}$, we choose $g_{j,k}$ such that $g_{j,k} \to g_k$ in $L^1(\mathbb{R}^+)$ and there exists $f_{j,k}$ satisfying $S^*(f_{j,k}) = g_{j,k}$ (observe that, necessarily, $g_{j,k}$ is a continuous function). For example, it suffices to consider

$$f_{j,k}(x) = jkx\,\chi_{(1/k,1/k+1/j)}(x) + \chi_{(1,2)}(x) - jx\log 2\,\chi_{(2,2+1/j)}(x).$$

Thus, $Sf_{j,k} = Sg_{j,k} - g_{j,k}$, $S^*f_{j,k} = g_{j,k}$ and using (24)

$$F = \sup_f \frac{\|Sf\|_{1,\infty}}{\|S^*f\|_1} \ge \lim_{k\to\infty}\lim_{j\to\infty}\frac{\|Sg_{j,k} - g_{j,k}\|_{1,\infty}}{\|g_{j,k}\|_1} = \lim_{k\to\infty}\frac{\|Sg_k - g_k\|_{1,\infty}}{\|g_k\|_1} \ge \frac{1}{\log 2}.$$

□

5. Further comments

In previous sections we have obtained all sharp constants of Table 1, $A,\dots,F$, for the range $1 < p \le 2$ and the endpoints $p = 1,\infty$. For the remaining case $2 < p < \infty$, we do know the values of A, B, and E. Let us review what can we say for the other cases:

Remark 5.1. Theorem 3.3 shows that for $p \ge 2$, an integer exponent, and f a decreasing function in $L^p(\mathbb{R}^+)$, the following inequality is sharp

$$\|Sf\|_p \le \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}\|S^*f\|_p; \tag{25}$$

i.e., $D = \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}$, if $p = 2,3,4,\dots$ Now, checking the inequality (25) with $f(x) = \chi_{(0,1)}(x)$, we also get that $D \ge \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}$, for all values of $p \ge 2$. Hence, it is natural to conjecture that the estimate (25) holds and is sharp, for any $p \ge 2$.

Up to now, the best we can say for a general $p \geq 2$ is the following: recall that if D is the sharp constant in the inequality,

$$\|Sf\|_p \leq D\|S^*f\|_p,$$

valid for any decreasing f, then from the estimate (9) and the previous observation, we have

$$\left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} \leq D \leq \frac{1}{(p-1)^{1/p}}.$$

In order to give a better bound, we observe that for every decreasing function f, $r > 1$, and $x > 0$

$$\int_x^\infty \frac{f(t)}{t}\, dx \geq f(rx) \log r.$$

Thus,

$$\|S^*f\|_p^p \geq \log^p r \int_0^\infty f^p(rx)\, dx = \frac{\log^p r}{r}\|f\|_p^p.$$

Optimizing this last inequality, we observe that since the maximum of $g(r) = \log^p r/r$ is attained at $r = e^p$, the following inequality holds

$$\|S^*f\|_p \geq \frac{p}{e}\|f\|_p.$$

From here, using (2), it follows that for every decreasing function,

$$\frac{\|Sf\|_p}{\|S^*f\|_p} \leq \frac{p'\|f\|_p}{\frac{p}{e}\|f\|_p} = \frac{e}{p-1}.$$

Therefore,

$$\left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} \leq D \leq \min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right). \tag{26}$$

Finally, observe that

$$\lim_{p\to 2^+}(p-1)\left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} = \lim_{p\to 2^+}(p-1)\min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right) = 1$$

and

$$\lim_{p\to \infty}(p-1)\left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} = \lim_{p\to \infty}(p-1)\min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right) = e.$$

Hence, the inequalities in (26) are asymptotically optimal, for $p \geq 2$.

Remark 5.2. For general functions on the range $2 < p < \infty$ (i.e., the constants C and F), we do not really have any conjecture about the possible values (see also [2]). However, using Lemma 1.2 we can prove that $F \leq C$.

Regarding C, if we interpolate the isometric property with $p = 2$ and (21) for $p = \infty$, and evaluating at $f(x) = \chi_{(0,1)}(x) - \chi_{(1,2)}(x)$, we obtain that

$$2^{1-2/p} \geq C \geq \left(\frac{2^{p-1}-1}{p-1}\right)^{1/p} > 1 = B, \quad p > 2.$$

Finally, observe that

$$\lim_{p\to 2+} 2^{1-2/p} = \lim_{p\to 2+}\left(\frac{2^{p-1}-1}{p-1}\right)^{1/p} = 1$$

and

$$\lim_{p\to\infty} 2^{1-2/p} = \lim_{p\to\infty}\left(\frac{2^{p-1}-1}{p-1}\right)^{1/p} = 2.$$

Using these estimates and as we did in the proof of Proposition 4.4 *(iii)*, we can also obtain a similar result for F:

$$2^{1-2/p} \geq C \geq F \geq \left(\frac{2^{p-1}-1}{p-1}\right)^{1/p} > 1 > \frac{1}{(p-1)^{1/p}} = E, \quad p > 2.$$

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