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THE EFFECTIVE RESISTANCE OF EXTENDED OR CONTRACTED NETWORKS

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Abstract. In this paper we introduce new effective resistances on a given network, associated with a positive value and two weights on the vertex set and investigate under which conditions they determine a distance. We prove that this property is closely related with superharmonicity. Moreover, we analyze the behavior of these distances under the usual network transformations, specially the so–called *star–mesh* transformation. We also compute the effective resistance for an *extended network*; that is the network obtained from the former one by joining a new vertex, and then study the effect of the *contraction of this new network*; that is we apply a star-mesh transformation with center in the joined vertex.

1. Introduction and Preliminaries

The effective resistance on a given a network is a distance on it, intrinsically associated with the combinatorial Laplacian. This means that to compute the effective resistance, all vertices are equally considered and the only parameter really significant is the weight on each edge, its *conductance*. Unlike the standard geodesic distance, defined as the length of the shortest or less resistive path between vertices, the resistance distance takes into account all the paths between vertices. It results that this distance is very sensitive to small changes in the conductances and hence allows us to discriminate between networks with similar structure.

It is possible to define effective resistances that, in addition to the conductance, take also into account a positive value on each vertex; that is a weight on the vertex set. In addition, when the weight is fixed we can define a one-parametric family of effective resistances. Each one of these generalized effective resistances also determine a distance on the network, and is defined through linear operators more general that the combinatorial Laplacian, namely positive semidefinite Schrödinger operators, where the parameter is the lowest eigenvalue and the weight function is the associated eigenfunction, see [4, 7]. The Schrödinger operators are defined as the combinatorial Laplacian plus a potential, being this potential the element that identifies both, the weight and the parameter. In particular, when the weight is constant; that is, when it does not discriminate between vertices, and the value of the parameter is 0, the potential is null and hence the corresponding Schrödinger operator coincides with the combinatorial Laplacian and we recover the standard resistance distance. The main properties of these effective resistances where analyzed in [1], by using techniques from Discrete Potential Theory, and also in [7], where we show that in the case of constant weight these distances coincide with the so-called *adjusted forest metrics* introduced by P. Chebotarev and E. Shamis at the late 90's, see [10]. In the above mentioned works, we can find that each one of this one-parametric family of

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resistance distances is associated with a family of irreducible and symmetric M-matrices with the same off-diagonal entries. The idea of associating an effective resistance to an M-matrix goes back to M. Fiedler, who studied the case of diagonally dominant M-matrices, see [12]. The adjusted forest metrics corresponds to the particular case in which the diagonal excess is constant and the general case was solved in [4].

In this paper we introduce new families of effective resistances on a given network. More specifically fixed a positive definite Schrödinger operator; or equivalently a network plus a potential q, we define effective resistances with respect to this potential and another weight on the vertex set. We newly analyze theses resistances under the light of Discrete Potential Theory and prove that they determine a distance on the network when the additional weight is q-superharmonic. Moreover, we also analyze the behavior of these distances under the usual network transformations, specially the so–called *star–mesh* transformation. We also compute the effective resistance for an *extended network*; that is the network obtained from the former one by joining a new vertex, and then study the effect of the *contraction of this new network*; that is we apply a star-mesh transformation with center at the joined vertex.

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, vertex x is adjacent to vertex y, $x \sim y$, iff $c(x, y) > 0$. Definitely, a finite network is entirely characterized by its vertex set and its conductance function and hence it can be represented as $\Gamma = (V, c)$. If $c(x, y) > 0$, then the value $r(x, y) = c(x, y)^{-1}$ is usually called *resistance between x and y*. In the sequel, we consider the finite network $\Gamma = (V, c)$ fixed. A connected network $\widehat{\Gamma} = (\widehat{V}, \hat{c})$ is called *host network of* Γ when $V \subset \widehat{V}$, $V \neq \widehat{V}$ and $\hat{c} = c$ on $V \times V$. In this case we also say that Γ *is embedded into* $\widehat{\Gamma}$.

For any $x \in V$, the value $\kappa(x) = \sum\limits_{y \in V} c(x, y)$ is called *total conductance at x* or *(weighted) degree of x*. Since Γ has not any isolated vertex, we obtain that $\kappa(x) > 0$ for any $x \in V$.

Given $F \subset V$, its *boundary* and its *closure* are the sets $\delta(F) = \{x \in V : c(x, y) > 0$ for some $y \in F\}$ and $\bar{F} = F \cup \delta(F)$, respectively. Clearly $\delta(F) \subset V \setminus F$ and F is a proper subset iff $\delta(F) \neq \emptyset$.

Given $x, y, z \in V$, we say that *z separates x and y* iff the set $V \setminus \{z\}$ is not connected and x and y belong to different connected components. Equivalently, z separates x and y iff any path joined x and y passes through z.

If $P = \big\{x = x_1 \sim x_2 \sim \cdots \sim x_k = y\big\}$ is a path joining vertices x and y, its *length* is the value $\ell_c(P) = \sum\limits_{i=1}^{k-1} r(x_i, x_{i+1})$. The *geodesic distance between two vertices x and y* is defined as the length of the less resistive path joining them; that is,

$$d_c(x, y) = \min\Big\{\ell_c(P) : P \text{ is a path from } x \text{ to } y\Big\}.$$

The function d_c determines a distance on the network that fulfills the property the triangular inequality is an equality when the central node separates the two others; that is, $d_c(x, y) = d_c(x, z) + d_c(z, y)$ if every path from x to y passes through z. In general a distance on a network, say d, is called *graph geodetic* if $d(x, y) = d(x, z) + d(z, y)$ when z separates x and y, see [11, page 278]. When the triangular inequality becomes an equality iff every path from x to y passes through z; the distance is called *cutpoint additive*, see [9], although sometimes is also named geodetic, see [10]. To avoid missunderstandings, here we use this terms in the sense of [11], so differentiating between geodetic and cutpoint additive distance.

In the sequel, $\mathcal{C}(V)$ denotes the sets of real functions on V and given $u \in \mathcal{C}(V)$, we define the values $||u||_1 = \sum\limits_{x \in V} |u(x)|$ and $||u||_2 = \Big(\sum\limits_{x \in V} u(x)^2\Big)^{\frac{1}{2}}$. If for $u, v \in \mathcal{C}(V)$ we consider $\langle u, v\rangle = \sum\limits_{x \in V} u(x)v(x)$, then $\langle \cdot, \cdot \rangle$ determines an inner product which associated norm is $|| \cdot ||_2$. Given $u \in \mathcal{C}(V)$ we denote by $u^\perp$ the subspace of $\mathcal{C}(V)$ orthogonal to u.

The constant function that takes the value 1 at each vertex is denoted by $\mathsf{1}$, whereas for any $x \in V$, ε_x is the function that takes the value 1 at x and 0 otherwise.

Given $u \in \mathcal{C}(V)$ its *support* is $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$ and hence, $\mathsf{supp}(u) = \emptyset$ iff $u = 0$. Given $F \subset V$ a non empty subset, $\mathcal{C}(F)$ is the subspace of real functions vanishing on $F^c = V \setminus F$ the complementary set of F and hence, $u \in \mathcal{C}(F)$ iff $\mathsf{supp}(u) \subset F$. In the sequel we identify $\mathcal{C}(F)$ with the set of functions $u\colon F \longrightarrow \mathbb{R}$. Analogously, each function $h\colon F \times F \longrightarrow \mathbb{R}$ is identified with $h\colon V \times V \longrightarrow \mathbb{R}$ satisfying $h(x,y) = 0$ when $(x,y) \notin F \times F$. If $h\colon V \times V \longrightarrow \mathbb{R}$ its *trace* is the value $\mathsf{tr}(h) = \sum\limits_{x \in V} h(x,x)$.

Given $u, v \in \mathcal{C}(V)$ we define $u \otimes v\colon V \times V \longrightarrow \mathbb{R}$ as $(u \otimes v)(x,y) = u(x)v(y)$ for any $x, y \in V$.

If $u \in \mathcal{C}(V)$ and $F \subset V$, the notation $u \geq 0$ on F or $u > 0$ on F means that $u(x) \geq 0$ or $u(x) > 0$ respectively, for any $x \in F$. If $u \in \mathcal{C}(V)$ and moreover $u > 0$ on V, then u is called a *weight*. We denote by Ω the set of *unitary weights*; that is, $\Omega = \{u \in \mathcal{C}(V) : u > 0 \text{ on } V \text{ and } ||u||_2 = 1\}$.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big), \quad x \in V. \tag{1}$$

It is well-known, that $\langle \mathcal{L}(u), v\rangle = \langle u, \mathcal{L}(v)\rangle$ and that

$$\langle u, \mathcal{L}(u)\rangle = \frac{1}{2}\sum_{x,y \in V} c(x,y)\big(u(x) - u(y)\big)^2 \geq 0, \quad \text{for any } u, v \in \mathcal{C}(V);$$

that is, the Laplacian is a self-adjoint and positive semidefinite operator. Moreover, since Γ is connected, $\mathcal{L}(u) = 0$ iff u is a constant function.

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, see for instance [2, 3]. A function $u \in \mathcal{C}(V)$ is called *q–superharmonic* when $\mathcal{L}_q(u) \geq 0$ on V and *q–harmonic* when $\mathcal{L}_q(u) = 0$ on V.

2. Admissible potentials and Doob Transforms

A potential $q \in \mathcal{C}(V)$ is called *admissible* for the network Γ iff the associated Schrödinger operator $\mathcal{L}_q$ is positive semidefinite. Our next objective is to provide an useful characterization of those potentials that are admissible. To do this, we need to introduce some concepts and techniques.

If $\omega \in \mathcal{C}(V)$ is a weight, then the potential $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$ is called *the potential determined by* ω. Notice that when $\sigma = a\omega$ then $q_\sigma = q_\omega$. The *Doob Transform (with respect to ω)* consists in the identity

$$\mathcal{L}_{q_\omega}(u)(x) = \frac{1}{\omega(x)}\sum_{y \in V} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right), \quad x \in V, \quad u \in \mathcal{C}(V), \tag{2}$$

which, in particular, implies that

$$\langle u, \mathcal{L}_{q_\omega}(u)\rangle = \frac{1}{2}\sum_{x,y \in V} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right)^2,$$

and hence that $\mathcal{L}_{q_\omega}$ is positive semidefinite. Moreover, we have the following property that generalizes the corresponding one for the combinatorial Laplacian:

$$\mathcal{L}_{q_\omega}(u) = 0 \ \text{ iff } \ u = a\omega, \quad a \in \mathbb{R}. \tag{3}$$

In particular, we conclude that $q_\sigma = q_\omega$ iff $\sigma = a\omega$ for some $a > 0$.

We have just proved that any potential determined by a weight is admissible. Moreover, we have a complete characterization of this fact as a consequence of the properties of symmetric M–matrices, see for instance [5, Chapter 6] and also [2, Proposition 3.3].

Proposition 2.1. *The map*

$$\begin{aligned} \mathfrak{q}\colon \Omega\times\mathbb{R} &\longrightarrow \mathcal{C}(V)\\ (\omega,\lambda) &\longrightarrow q_\omega+\lambda \end{aligned}$$

is bijective and in particular, $0=\mathfrak{q}(\omega,\lambda)$ iff $\lambda=0$ and moreover ω is constant. In addition, if $q=\mathfrak{q}(\omega,\lambda)$ then λ is the lowest eigenvalue of $\mathcal{L}_q$; it is simple and $\mathcal{L}_q(\omega)=\lambda\omega$. Therefore, $q=\mathfrak{q}(\omega,\lambda)$ is admissible iff $\lambda\ge 0$.

We are interested in the relation between the positive definiteness of the Schrödinger operator $\mathcal{L}_q$ and the existence of q–harmonic or q–superharmonic funcions. Observe that the above Proposition implies that if q is admissible; that is, $q=\mathfrak{q}(\omega,\lambda)$ with $\omega\in\Omega$ and $\lambda\ge 0$, then ω is q–superharmonic and q–harmonic only when $\lambda=0$. These properties are relative to q–harmonic and q–superharmonic weights, but below we prove that, basically, these are the only functions with this property. Previously to prove this result, we show the following key fact about positive semidefinite Schrödinger operators, known as the *strong Monotonicity Principle*, see [2, Proposition 4.10], that will be useful in many parts of this paper.

Proposition 2.2 (Monotonicity Principle)**.** *Let $q=\mathfrak{q}(\omega,\lambda)$, $\omega\in\Omega$ and $\lambda\ge 0$, be an admissible potential. Given $F\subset V$ a proper subset and $u\in\mathcal{C}(\bar F)$ satisfying that $\mathcal{L}_q(u)\ge 0$ on F and $u\ge 0$ on $\delta(F)$, then $u\ge 0$ on $\bar F$. Moreover, if $H\subset F$ is a connected component of F, then either $u=0$ on $\bar H$ or $u>0$ on H. In addition, if $\lambda>0$ and $u\in\mathcal{C}(V)$ satisfies that $\mathcal{L}_q(u)\ge 0$ on V then either $u=0$ or $u>0$ on V.*

The use the above result together the Doob Transform, in particular Identity (3), to study the relation between admissibility and the existence of q-harmonic or q–superharmonic functions.

Theorem 2.3. *Given $q\in\mathcal{C}(V)$, then a weight ω is q-harmonic iff $q=q_\omega$ and it is q-superharmonic iff $q\ge q_\omega$ on V. Moreover, if q is admissible and $q=\mathfrak{q}(\ \omega,\lambda)$, $\omega\in\Omega$, $\lambda\ge 0$, the following properties are satisfied:*

(i) *When $\lambda=0$, all q–superharmonic functions are q-harmonic and hence multiple of ω. In particular, ω is the unique unitary q-harmonic weight.*
(ii) *When $\lambda>0$, the only q–harmonic function is the null function, ω is a q–superharmonic weight and any q–superharmonic function, but not q-harmonic, is a weight.*

Proof. Observe that if ω is a weight, then $\mathcal{L}_q(\omega)=\mathcal{L}(\omega)+q\omega=\omega(q-q_\omega)$. Therefore, ω is q–harmonic iff $q=q_\omega$ and it is q–superharmonic iff $q\ge q_\omega$ and $q\ne q_\omega$.

(i) From Identity (3), any q–harmonic function is multiple of ω and hence, ω is the unique unitary q–harmonic weight. Finally, if $u\in\mathcal{C}(V)$ satisfies that $\mathcal{L}_{q_\omega}(u)\ge 0$, then

$$0=\langle\mathcal{L}_q(\omega),u\rangle=\langle\omega,\mathcal{L}_q(u)\rangle$$

and hence u is q–harmonic, since $\omega>0$.

(ii) From Proposition 2.1, $\mathcal{L}_q(\omega)=\lambda\omega>0$, which implies that ω is a q–superharmonic weight.

If u is a q–harmonic function, then $-u$ is also q–harmonic, and hence the last claim in the Monotonicity Principle implies that $u=0$.

Finally, if u is a q–superharmonic function, newly the last claim of the Monotonicity Principle implies that either $u=0$ on V or u is a weight. Since $u=0$ would imply that u is q–harmonic, necessarily u is a weight. □

3. Green Functions

Although on Γ many different self–adjoint boundary value problems can be defined, see for instance [3] where all of them were considered under the generic name of *mixed Dirichlet–Robin problems*, here we restrict ourselves to study two of them, namely Dirichlet problems and the Poisson equation. Therefore, the main reference for the terminology and main results continue being [2]. Actually, we treat both kind of problems in a common framework.

Given the potential $q \in \mathcal{C}(V)$ and its corresponding Schrödinger operator $\mathcal{L}_q$, for a non empty set $F \subset V$ we consider the boundary value problem consisting in

$$\text{given } f \in \mathcal{C}(F) \text{ find } u \in \mathcal{C}(F) \text{ such that } \mathcal{L}_q(u) = f \text{ on } F. \tag{4}$$

This is a self–adjoint problem in the sense that for any $u, v \in \mathcal{C}(F)$ it is satisfied that

$$\langle u, \mathcal{L}_q(v)\rangle = \frac{1}{2}\sum_{x,y\in \bar{F}} c(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big) + \sum_{x\in F} q(x)u(x)v(x) = \langle v, \mathcal{L}_q(u)\rangle.$$

When $F = V$, Problem (4) is known as the *Poisson Equation* on V, whereas when F is a proper subset it is known as the *Dirichlet Problem* on F. Notice that since $\mathcal{C}(F) = \big\{u \in \mathcal{C}(\bar{F}) : u = 0 \text{ on } \delta(F)\big\}$, when F is a proper subset, the Dirichelt problem can be rewritten, in its most common form, as

$$\text{given } f \in \mathcal{C}(F) \text{ find } u \in \mathcal{C}(\bar{F}) \text{ such that } \mathcal{L}_q(u) = f \text{ on } F \text{ and } u = 0 \text{ on } \delta(F).$$

The conditions on q under which the above problem (4) has solution, can be establish in the variational way known as the *Dirichlet Principle*, see [2, Proposition 3.5]. To obtain the above mentioned conditions, fixed a potential $q \in \mathcal{C}(V)$ for any $f \in \mathcal{C}(V)$ we define the quadratic funtional $\mathfrak{J}_f : \mathcal{C}(V) \longrightarrow \mathbb{R}$ as

$$\mathfrak{J}_f(u) = 2\langle f, u\rangle - \langle \mathcal{L}_q(u), u\rangle, \quad u \in \mathcal{C}(V). \tag{5}$$

Notice that for any $u \in \mathcal{C}(F)$ we have that

$$\mathfrak{J}_f(u) = 2\sum_{x\in F} f(x)u(x) - \frac{1}{2}\sum_{x,y\in\bar{F}} c(x,y)\big(u(x)-u(y)\big)^2 - \sum_{x\in F} q(x)u(x)^2.$$

Proposition 3.1 (Dirichlet Principle)**.** *Let $q = \mathsf{q}(\omega,\lambda)$, $\omega \in \Omega$ and $\lambda \geq 0$, be an admissible potential. Given $F \subset V$ a non empty subset and $f \in \mathcal{C}(F)$, then $u \in \mathcal{C}(F)$ is a solution of the problem (4) iff u maximizes $\mathfrak{J}_f$ on $\mathcal{C}(F)$ and then $\mathfrak{J}_f(u) = \langle f, u\rangle = \langle \mathcal{L}_q(u), u\rangle$. Moreover, the following properties hold:*

(i) *When F is a proper subset or $\lambda > 0$, then $\mathfrak{J}_f$ has a unique maximum.*
(ii) *When $F = V$ and $\lambda = 0$ then $\mathfrak{J}_f$ has a maximum iff $f \in \omega^\perp$. In this case, there exists a unique maximum $u \in \mathcal{C}(V)$ such that $u \in \omega^\perp$ and $\{u + a\omega : a \in \mathbb{R}\}$ describes the set of all maxima of $\mathfrak{J}_f$.*

As a by–product of the Dirichlet Principle we have that if the potential $q \in \mathcal{C}(V)$ is admissible, then for any proper subset $F \subset V$, $\mathcal{L}_q$ establishes an automorphism on $\mathcal{C}(F)$, and hence for any $f \in \mathcal{C}(F)$ the Problem (4) has a unique solution. The same properties are true for $F = V$ when $\lambda > 0$ and hence then $\mathcal{L}_q$ is invertible. In all these cases, the inverse of $\mathcal{L}_q$ on $\mathcal{C}(F)$ is called *Green operator for* F and denoted by $\mathcal{G}_q^F$. The function $G_q^F : F \times F \longrightarrow \mathbb{R}$ defined for any $y \in F$ as $G_q^F(\cdot, y) = \mathcal{G}_q^F(\varepsilon_y)$, the unique solution of Problem (4) corresponding to $f = \varepsilon_y$ is called the *Green function for* F. Next, we show the main properties of the Green function for a given subset of V.

Proposition 3.2. *Let $q = \mathsf{q}(\omega,\lambda)$, $\omega \in \Omega$ and $\lambda \geq 0$, be an admissible potential. Consider $F \subset V$ a non empty set subset except when $\lambda = 0$ in which case F must to be a proper subset. Then G_q is symmetric,*

$$0 \leq G_q^F(x,y)\omega(y) < G_q^F(y,y)\omega(x), \quad x, y \in F, \ x \neq y$$

and the first inequality is an equality iff $x \notin F_y$, where F_y is the connected component of F containing y. In particular, when $\lambda > 0$, then

$$0 < G_q^V(x,y)\omega(y) < G_q^V(y,y)\omega(x), \quad x, y \in V, \ x \neq y.$$

Moreover, given $f \in \mathcal{C}(F)$, the function $u \in \mathcal{C}(V)$ defined as

$$u(x) = \mathcal{G}_q^F(f)(x) = \sum_{y\in F} G_q^F(x,y)f(y), \quad x \in V,$$

is the unique solution of Problem (4).

Proof. If $u = \mathcal{G}_q^F(\varepsilon_y)$, then $\mathcal{L}_q(u) = \varepsilon_y \geq 0$ on F and $u = 0$ on $\delta(F)$. Applying the Monotonicity Principle, we get that $u > 0$ on F_y and $u = 0$ on $F \setminus F_y$.

On the other hand, if $a = \omega(y)^{-1} G_q^F(y,y)$ and $v = a\omega - u$, then $a > 0$ and $\mathcal{L}_q(v) = a\lambda\omega - \varepsilon_y \geq 0$ on $F \setminus \{y\}$ and $v = a\omega \geq 0$ on $\delta(F \setminus \{y\}) \subset \{y\} \cup \delta(F)$. Applying the Monotonicity Principle, we get that $u > 0$ on F_y and $u = 0$ on $F \setminus F_y$. □

Another consequence of the Dirichlet Principle is that when the potential $q \in \mathcal{C}(V)$ is admissible, then $\mathcal{L}_q$ establishes an automorphism on $\omega^\perp$, where $\omega \in \Omega$ is such that $q = \mathfrak{q}(\omega, \lambda)$ with $\lambda \geq 0$. Notice that under this hypothesis $\langle \omega, \mathcal{L}_q(u) \rangle = \lambda \langle \omega, u \rangle$, since $\mathcal{L}_q(\omega) = \lambda\omega$. The inverse of $\mathcal{L}_q$ on $\omega^\perp$ is called *Green operator for* Γ or simply *Green operator*, and denoted by $\mathcal{G}_q$. We can extend $\mathcal{G}_q$ to a self-adjoint and positive semidefinite endomorphism of $\mathcal{C}(V)$ by defining $\mathcal{G}_q(f) = \mathcal{G}_q(f - \langle f, \omega \rangle \omega)$ for any $f \in \mathcal{C}(V)$. The function $G_q \colon V \times V \longrightarrow \mathbb{R}$ defined for any $y \in V$ as $G_q(\cdot, y) = \mathcal{G}_q(\varepsilon_y) = \mathcal{G}_q(\varepsilon_y - \omega(y)\omega)$, the unique solution of Problem (4) corresponding to $f = \varepsilon_y - \omega(y)\omega$ is called the *Green function for* Γ or simply *Green function*. Next, we show the main properties of the Green function for Γ.

Proposition 3.3. *Let $q = \mathfrak{q}(\omega, \lambda)$, $\omega \in \Omega$ and $\lambda \geq 0$, be an admissible potential. Then G_q is symmetric and given $f \in \omega^\perp$, the function $u \in \mathcal{C}(V)$ defined as*

$$u(x) = \mathcal{G}_q(f)(x) = \sum_{y \in F} G_q(x,y) f(y), \quad x \in V,$$

is the unique solution of Problem (4) *belonging to $\omega^\perp$. In particular, $\mathcal{G}_q(f) = 0$ iff $f = a\omega$, $a \in \mathbb{R}$. Moreover, $G_q(y,y) > 0$ for any $y \in V$ and*

$$G_q(x,y)\omega(y) < G_q(y,y)\omega(x), \quad x, y \in V, \ x \neq y.$$

In addition, when $\lambda > 0$, then

$$G_q^V(x,y) = G_q(x,y) + \lambda^{-1}\omega(x)\omega(y), \quad x, y \in V$$

and hence,

$$-\lambda^{-1}\omega(x)\omega(y) < G_q(x,y), \quad x, y \in V.$$

We remark that the Green operator and the Green function for the vertex set V only exist for those admissible potentials such that the corresponding Schrödinger operator is invertible, whereas the Green operator and the Green function for the network always exists for admissible potentials. The last part of the above Proposition determines the relation between the Green function for Γ and the Green function for V, when the last one exists.

4. Bottleneck functions

In this section we always assume that the potential $q \in \mathcal{C}(V)$ is admissible; that is, $q = \mathfrak{q}(\omega, \lambda)$ where $\omega \in \Omega$ and $\lambda \geq 0$. We denote by $\mathcal{L}_q$ the corresponding Schrödinger operator and by $\mathcal{G}_q$ and G_q the Green operator and the Green function for the network Γ.

For any $z \in V$ we consider the boundary value problem consisting in

(6) *given $f \in \mathcal{C}(V \setminus \{z\})$ find $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) - \lambda \langle u, \omega \rangle \, \omega = f$ on $V \setminus \{z\}$ and $u(z) = 0$.*

Theorem 4.1. *For any $f \in \mathcal{C}(V \setminus \{z\})$ the Problem* (6) *has a unique solution that is given by*

$$u = \mathcal{G}_q(f) - \omega(z)^{-1}\big(\langle f, \omega \rangle \, \mathcal{G}_q(\varepsilon_z) + \mathcal{G}_q(f)(z)\omega\big) + \omega(z)^{-2} G_q(z,z) \langle f, \omega \rangle \, \omega.$$

Proof. If $v \in \mathcal{C}(V)$ satisfies that $\mathcal{L}_q(v) - \lambda \langle v, \omega \rangle \, \omega = f$ on $V \setminus \{z\}$, then

$$\begin{aligned}\langle f, \omega \rangle &= \langle \mathcal{L}_q(v), \omega \rangle - \mathcal{L}_q(v)(z)\omega(z) - \lambda \langle v, \omega \rangle + \lambda \langle v, \omega \rangle \, \omega(z)^2 \\ &= \langle v, \mathcal{L}_q(\omega) \rangle - \lambda \langle v, \omega \rangle - \big(\mathcal{L}_q(v)(z) - \lambda \langle v, \omega \rangle\big)\omega(z) = -\big(\mathcal{L}_q(v)(z) - \lambda \langle u, \omega \rangle \, \omega(z)\big)\omega(z),\end{aligned}$$

and hence, $\mathcal{L}_q(v) - \lambda \langle v, \omega \rangle \, \omega = f - \omega(z)^{-1} \langle f, \omega \rangle \, \varepsilon_z$. In particular, when $f = 0$, then $\mathcal{L}_q(v) = \lambda \langle v, \omega \rangle \, \omega$ and hence $v = a\omega$, $a \in \mathbb{R}$.

Conversely, since $f-\omega(z)^{-1}\langle f,\omega\rangle\,\varepsilon_z\in\omega^\perp$, then $v=\mathcal{G}_q(f-\omega(z)^{-1}\langle f,\omega\rangle\,\varepsilon_z)\in\omega^\perp$ and moreover satisfies that $\mathcal{L}_q(v)=f-\omega(z)^{-1}\langle f,\omega\rangle\,\varepsilon_z$. In particular,

$$\mathcal{L}_q(v)-\lambda\langle v,\omega\rangle=\mathcal{L}_q(v)=f-\omega(z)^{-1}\langle f,\omega\rangle\,\varepsilon_z=f\ \text{ on } V\setminus\{z\}$$

In consequence, the set $\{v+a\omega : a\in\mathbb{R}\}$ describes all solutions of the problem $\mathcal{L}_q(v)-\lambda\langle v,\omega\rangle\,\omega=f$ on $V\setminus\{z\}$ and hence, $u=v-\omega(z)^{-1}v(z)\omega$ is the unique solution vanishing at z. □

The above Proposition establishes that for any $z\in V$, $\mathcal{L}_q-\lambda\langle\cdot,\omega\rangle\,\omega$ determines an automorphism on $\mathcal{C}(V\setminus\{z\})$, whose inverse is called *the Bottleneck operator at* z and denoted by $\mathcal{G}_q^z$. Since ε_z is null on $\mathcal{C}(V\setminus\{z\})$ we have that $\mathcal{G}_q^z(\varepsilon_z)=0$.

The function $G_q^z\colon V\times V\longrightarrow\mathbb{R}$ defined for any $y\in V$ as $G_q^z(\cdot,y)=\mathcal{G}_q^z(\varepsilon_y)$, the unique solution of Problem (6) corresponding to $f=\varepsilon_y$, is called the *Bottleneck function at* z. Next, we show some of the main properties of the bottleneck functions.

Proposition 4.2. [4, Theorem 3.7 (ii)] *For any $z\in V$ it is satisfied that G_q^z is symmetric, $G_q^z(x,z)=G_q^z(z,x)=0$ for any $x\in V$, and given $f\in\mathcal{C}(V\setminus\{z\})$, the function $u\in\mathcal{C}(V)$ defined as*

$$u(x)=\sum_{\substack{y\in V\\ y\neq z}}G_q^z(x,y)f(y),\quad x\in V,$$

is the unique solution of the Problem (6). Moreover,

$$0\le G_q^z(x,y)\omega(y)\le G_q^z(y,y)\omega(x),\quad \textit{for any } x,y\neq z,\ x\neq y$$

and the first inequality is an equality iff $\lambda=0$ and z separates x and y, whereas the second inequality is an equality iff $\lambda=0$ and y separates x and z.

We remark that that above Proposition and Theorem have been established under the hypothesis $\lambda\ge 0$. However, in the specific case when $\lambda=0$, we could have proved directly the existence and uniqueness of solutions for Problem (6), since then it is nothing but the Dirichlet Problem for $V\setminus\{z\}$ and hence $\mathcal{G}_q^z$ and G_q^z are the Green operator and the Green function for the the set $F=V\setminus\{z\}$. This corresponds to the standar case treated in the literature, see for instance [15, 16, 17] and also [21].

On the other hand, Theorem 4.1 shows that there exists a closed relation between the Green function and the bottleneck functions. Next we made explicit these relations.

Corollary 4.3. *Given $x,y,z,\hat{z}\in V$ the following identities are satisfied:*

(i) $G_q^z(x,y)=G_q(x,y)-\omega(z)^{-1}\Big[\omega(x)G_q(y,z)+\omega(y)G_q(x,z)\Big]+\omega(z)^{-2}\,G_q(z,z)\omega(x)\omega(y)$.

(ii) $G_q^z(x,y)=\omega(x)\omega(y)\left[\dfrac{G_q(x,y)}{\omega(x)\omega(y)}-\dfrac{G_q(y,z)}{\omega(y)\omega(z)}-\dfrac{G_q(x,z)}{\omega(x)\omega(z)}+\dfrac{G_q(z,z)}{\omega(z)^2}\right]$.

(iii) $G_q(x,y)=G_q^z(x,y)-\omega(x)\mathcal{G}_q^z(\omega)(y)-\mathcal{G}_q^z(\omega)(x)\omega(y)+\langle\mathcal{G}_q^z(\omega),\omega\rangle\,\omega(x)\omega(y)$.

(iv) $G_q^{\hat{z}}(x,y)=\omega(x)\omega(y)\left[\dfrac{G_q^z(x,y)}{\omega(x)\omega(y)}-\dfrac{G_q^z(\hat{z},y)}{\omega(y)\omega(\hat{z})}-\dfrac{G_q^z(x,\hat{z})}{\omega(x)\omega(\hat{z})}+\dfrac{G_q^z(\hat{z},\hat{z})}{\omega(\hat{z})^2}\right]$.

Proof. Part (i) is obtained simply by taking $f=\varepsilon_y$ in Theorem 4.1. In addition (ii) is a straightforward consequence of (i). To prove part (iii), from (i) and taking into account that $\mathcal{G}_q(\omega)=0$, we have that

$$\mathcal{G}_q^z(\omega)=-\omega(z)^{-1}\mathcal{G}_q(\varepsilon_z)+\omega(z)^{-2}G_q(z,z)\,\omega$$

and hence, $\langle\mathcal{G}_q^z(\omega),\omega\rangle=\omega(z)^{-2}G_q(z,z)$ and moreover $\omega(z)^{-1}G_q(x,z)=\langle\mathcal{G}_q^z(\omega),\omega\rangle\,\omega(x)-\mathcal{G}_q^z(\omega)(x)$, for any $x\in V$. Therefore (iii) follows. Finally, the combination of (i) with (iii) gives (iv). □

Notice that the results in the above Corollary can be also expressed as

$$\begin{aligned}G_q^z&=G_q-\omega(z)^{-1}\Big[\omega\otimes\mathcal{G}_q(\varepsilon_z)+\mathcal{G}_q(\varepsilon_z)\otimes\omega\Big]+\omega(z)^{-2}G_q(z,z)\omega\otimes\omega,\\ G_q&=G_q^z-\omega\otimes\mathcal{G}_q^z(\omega)-\mathcal{G}_q^z(\omega)\otimes\omega+\langle\mathcal{G}_q^z(\omega),\omega\rangle\omega\otimes\omega\end{aligned}$$

The matrix version of the second identity is widely used, see [15, 21], since leads to obtain the group inverse of a singular matrix in terms of the genuine inverse, the bottleneck matrix, of an invertible matrix.

We finish this section by taking advantage of the second identity in Corollary 4.3. If we define $\tilde{d}_q\colon V\times V\longrightarrow\mathbb{R}$ as

$$\tilde{d}_q(x,y)=\frac{G_q(y,y)}{\omega(y)^2}-\frac{G_q(x,y)}{\omega(x)\omega(y)},\quad x,y\in V,$$

then, from Proposition 3.3, $\tilde{d}_q(x,y)\geq 0$ and the equality holds iff $x=y$. Moreover, from part (ii) of Corollary 4.3 and taking into account the symmetry of G_q, we have that

$$\begin{aligned}0\leq G_q^z(x,y)&=\omega(x)\omega(y)\Big[\frac{G_q(x,y)}{\omega(x)\omega(y)}-\frac{G_q(y,y)}{\omega(y)^2}+\frac{G_q(y,y)}{\omega(y)^2}-\frac{G_q(z,y)}{\omega(y)\omega(z)}-\frac{G_q(x,z)}{\omega(x)\omega(z)}+\frac{G_q(z,z)}{\omega(z)^2}\Big],\\&=\omega(x)\omega(y)\Big[\tilde{d}_q(x,z)+\tilde{d}_q(z,y)-\tilde{d}_q(x,y)\Big]\quad x,y,z\in V,\end{aligned}$$

which implies that $\tilde{d}_q$ satisfies the triangular inequality. However, $\tilde{d}_q$ is not a distance on V since it is not symmetric. To solve this trouble, define $d_q\colon V\times V\longrightarrow\mathbb{R}$ as

$$d_q(x,y)=\tilde{d}_q(x,y)+\tilde{d}_q(y,x)=\frac{G_q(x,x)}{\omega(x)^2}+\frac{G_q(y,y)}{\omega(y)^2}-\frac{2G_q(x,y)}{\omega(x)\omega(y)},\quad x,y\in V,\tag{7}$$

where we have applied the symmetry of G_q. Clearly, d_q is symmetric, nonnegative and $d_q(x,y)=0$ iff $x=y$. Besides, for any $x,y,z\in V$ we have

$$G_q^z(x,y)=\omega(x)\omega(y)\Big[\tilde{d}_q(x,z)+\tilde{d}_q(z,y)-\tilde{d}_q(x,y)\Big],$$

$$G_q^z(y,x)=\omega(y)\omega(x)\Big[\tilde{d}_q(y,z)+\tilde{d}_q(z,x)-\tilde{d}_q(y,x)\Big],$$

Applying now the symmetry of G_q^z

$$G_q^z(x,y)=\frac{1}{2}\,\omega(x)\omega(y)\Big[d_q(x,z)+d_q(x,z)-d_q(x,y)\Big],\tag{8}$$

and, in particular

$$d_q(x,z)=\frac{G_q^z(x,x)}{\omega(x)^2},\quad\text{for any } x,z\in V.\tag{9}$$

Therefore, each admissible potential $q\in\mathcal{C}(V)$ determines the distance d_q on the network, that we call the *distance determined by* q. Moreover, if $q=\mathfrak{q}(\omega,\lambda)$, $\omega\in\Omega$ and $\lambda\geq 0$, when $\lambda=0$, the distance determined by q is cutpoint additive, whereas when $\lambda>0$ the distance determined by q satisfies that the triangular inequality is always strict. In this case, the relation between the Green function and the Green function for V implies that

$$d_q(x,y)=\frac{G_q^V(x,x)}{\omega(x)^2}+\frac{G_q^V(y,y)}{\omega(y)^2}-\frac{2G_q^V(x,y)}{\omega(x)\omega(y)},\quad x,y\in V,\tag{10}$$

Finally, from the relation between the Green function and the bottleneck function, part (iii) of Corollary 4.3 , for any $z\in V$ we also have that

$$d_q(x,y)=\frac{G_q^z(x,x)}{\omega(x)^2}+\frac{G_q^z(y,y)}{\omega(y)^2}-\frac{2G_q^z(x,y)}{\omega(x)\omega(y)},\quad x,y\in V.\tag{11}$$

5. Effective Resistances

In the standard setting, the effective resistance of the network Γ between vertices x and y is defined throughout the solution of the Poisson equation $\mathcal{L}(u)=f$ when the data is the dipole with poles at x and y; that is, $f=\varepsilon_x-\varepsilon_y$. Important properties of electrical networks can be deduced from the knowledge of the effective resistance, see for instance [13, 15, 19].

In [2] some of the present authors introduced the concept of effective resistance with respect to a weight that in turns was generalized in [4] by considering, in addition, a parameter $\lambda\geq 0$. The analysis of the generalized effective resistance from a Potential Theory point of view, see [?], leads to good bounds of the effective resistance and even to its explicit computation in some structured networks. Moreover, a general Foster's formula relating effective resistance and the iteration of the probability kernel was proved, see [?, Theorem 4.2] and also [18] for the standard setting.

Our aim here is to introduce new effective resistances in a network, specifically, for each admissible potential $q\in\mathcal{C}(V)$, we consider a family of the effective resistances associated, firstly to the weights and secondly to the family of q–superharmonic weights. As we will see, this family include the effective resistances defined in [?, 2, 4]. We follow here the same route than in these works to introduce the effective resistances.

In this section we newly assume that the potential $q\in\mathcal{C}(V)$ is admissible and hence that $q=\mathsf{q}(\omega,\lambda)$, where $\omega\in\Omega$ and $\lambda\geq 0$. As before, we denote by $\mathcal{L}_q$ the corresponding Schrödinger operator and by $\mathcal{G}_q$ and G_q the Green operator and the Green function for the network Γ. In addition for any $z\in V$, $\mathcal{G}_q^z$ and G_q^z are the bottleneck operator and the bottleneck function at z, respectively.

Fixed σ a weight, for any $x,y\in V$ the *σ-dipole between x and y* is the function $f^\sigma_{xy}=\dfrac{\varepsilon_x}{\sigma(x)}-\dfrac{\varepsilon_y}{\sigma(y)}$. Notice that $f^\sigma_{x,y}\in\sigma^\perp$ for any $x,y\in V$. In addition, $f^\sigma_{x,y}\in\omega^\perp$ for any $x,y\in V$ iff $\dfrac{\omega(x)}{\sigma(x)}=\dfrac{\omega(y)}{\sigma(y)}$ for any $x,y\in V$ and this property is equivalent to be $\sigma=a\omega$, $a>0$.

For any $x,y\in V$ we also consider the quadratic functional $\mathfrak{J}^\sigma_{x,y}\colon\mathcal{C}(V)\longrightarrow\mathbb{R}$ determined for any $u\in\mathcal{C}(V)$ by the expression

$$\mathfrak{J}^\sigma_{x,y}(u)=2\left[\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right]-\langle\mathcal{L}_q(u),u\rangle=2\langle f^\sigma_{x,y},u\rangle-\langle\mathcal{L}_q(u),u\rangle. \tag{12}$$

When $\lambda=0$, we know that $\mathfrak{J}^\sigma_{x,y}$ attains a maximum value iff $f^\sigma_{xy}\in\omega^\perp$ and hence, to apply the Dirichlet Principle for any $x,y\in V$, we must to demand that σ be a multiple of ω. So in the sequel, we assume that $\sigma=a\omega$, $a>0$ when $\lambda=0$. Under this constraint, applying the Dirichlet Principle, we conclude that $\mathfrak{J}^\sigma_{x,y}$ attains a maximum value. So, given $x,y\in V$, the *effective resistance between x and y, with respect to q and σ* is the value

$$R_{q,\sigma}(x,y)=\max_{u\in\mathcal{C}(V)}\{\mathfrak{J}^\sigma_{x,y}(u)\}. \tag{13}$$

Moreover, the *Kirchhoff Index of Γ, with respect to q and σ,* is the value

$$\mathsf{K}(q,\sigma)=\frac{1}{2}\sum_{x,y\in V}R_{q,\sigma}(x,y)\sigma^2(x)\,\sigma^2(y). \tag{14}$$

The standard case corresponds to take $q=0$; that is, ω constant and $\lambda=0$, and $\sigma=1$.

The Dirichlet Principle also establishes that u maximizes $\mathfrak{J}^\sigma_{x,y}$ iff

$$\mathcal{L}_q(u)=f^\sigma_{x,y}=\frac{\varepsilon_x}{\sigma(x)}-\frac{\varepsilon_y}{\sigma(y)} \tag{15}$$

and then

$$R_{q,\sigma}(x,y)=\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}=\langle\mathcal{L}_q(u),u\rangle. \tag{16}$$

Therefore, if we consider the Doob transform with respect to ω and also with respect to σ, we obtain that

$$\begin{aligned} R_{q,\sigma}(x,y) &= \frac{1}{2}\sum_{x,y\in V} c(x,y)\omega(x)\omega(y)\Big(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\Big)^2 + \lambda\sum_{x\in V} u(x)^2 \\ &= \frac{1}{2}\sum_{x,y\in V} c(x,y)\sigma(x)\sigma(y)\Big(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)}\Big)^2 + \sum_{x\in V}\big(q(x)-q_\sigma(x)\big)u(x)^2, \end{aligned}$$

where $u \in \mathcal{C}(V)$ maximizes $\mathfrak{J}^\sigma_{x,y}$.

Since $\mathfrak{J}^\sigma_{x,y}(0) = 0$ we have that $R_{q,\sigma}(x,y) \geq 0$ and the equality holds iff $u = 0$ maximizes $\mathfrak{J}^\sigma_{x,y}$, which in turns implies that $\mathcal{L}_q(u) = 0$ and hence $\dfrac{\varepsilon_x}{\sigma(x)} = \dfrac{\varepsilon_y}{\sigma(y)}$, what happens iff $x = y$. Therefore, $R_{q,\sigma}(x,y) \geq 0$ for any $x, y \in V$ and the equality only holds iff $x = y$.

On the other hand, since $\mathfrak{J}^\sigma_{x,y}(u) = \mathfrak{J}^\sigma_{y,x}(-u)$ for any $x, y \in V$ and any $u \in \mathcal{C}(V)$, we conclude that $R_{q,\sigma}(x,y) = R_{q,\sigma}(y,x)$ for any $x, y \in V$. Therefore, the effective resistance with respect to a weight is symmetric, nonnegative and null iff its two arguments coincide. Our main objective is to investigate when it determines a distance on the network Γ. Before to do this, we describe some properties of the effective resistance, that do not depend on it being a distance.

Given $a > 0$, if we consider the weight $\hat\sigma = a\sigma$ then we have that $R_{q,\hat\sigma}(x,y) = a^{-2}R_{q,\sigma}(x,y)$, for any $x, y \in V$ and hence that $\mathsf{K}(q,\hat\sigma) = a^2\mathsf{K}(q,\sigma)$. Therefore, we could always restrict ourselves to consider unitary weights to compute these parameters. However, in this paper we prefer to consider arbitrary; that is non normalized, weights.

When $\sigma = \omega$, since $R_{q,\omega}(x,y) = \dfrac{u(x)}{\omega(x)} - \dfrac{u(y)}{\omega(y)}$, where $\mathcal{L}_q(u) = \dfrac{\varepsilon_x}{\omega(x)} - \dfrac{\varepsilon_y}{\omega(y)} \in \omega^\perp$, we can take $u = \mathcal{G}_q(f^\omega_{x,y}) = \omega(x)^{-1}\mathcal{G}_q(\varepsilon_x) - \omega(y)^{-1}\mathcal{G}_q(\varepsilon_y)$ and hence, for any $x, y \in V$ we have

$$R_{q,\omega}(x,y) = \frac{G_q(x,x)}{\omega(x)^2} - \frac{G_q(x,y)}{\omega(y)\omega(x)} - \frac{G_q(y,x)}{\omega(x)\omega(y)} + \frac{G_q(y,y)}{\omega(y)^2} = d_q(x,y).$$

So, we have proved that the effective resistance with respect the weight ω such that $q = \mathsf{q}(\omega,\lambda)$ is a distance on Γ that in fact, coincides with the distance determined by q, defined in the previous section.

Definitely, the study of the conditions under which $R_{q,\sigma}$ determines a distance on Γ can be limited to $\lambda > 0$ since for $\lambda = 0$ necesarily $\sigma = a\omega$, $a > 0$ and the answer is positive. In fact, the complete analysis for $\sigma = \omega$ and $\lambda \geq 0$ was carried out in previous works by the authors, see [**?**, 2, 4].

Assume that $\lambda > 0$ and consider $\mathcal{G}^V_q$ and G^V_q the Green operator and the Green function for V, respectively, that are defined under this hypothesis. Applying newly the identities (15) and (16) we obtain that

$$R_{q,\sigma}(x,y) = \frac{G^V_q(x,x)}{\sigma(x)^2} + \frac{G^V_q(y,y)}{\sigma(y)^2} - \frac{2G^V_q(x,y)}{\sigma(y)\sigma(x)}, \quad x, y \in V \tag{17}$$

that implies that

$$\mathsf{K}(q,\sigma) = ||\sigma||_2^2\,\mathsf{tr}(G^V_q) - \langle \mathcal{G}^V_q(\sigma), \sigma\rangle \tag{18}$$

Taking into account the relation between the Green function for V and the Green function for Γ, the above identities can be rewritten as

$$R_{q,\sigma}(x,y) = \frac{G_q(x,x)}{\sigma(x)^2} + \frac{G_q(y,y)}{\sigma(y)^2} - \frac{2G_q(x,y)}{\sigma(y)\sigma(x)} + \lambda^{-1}\Big(\frac{\omega(x)}{\sigma(x)} - \frac{\omega(y)}{\sigma(y)}\Big)^2, \quad x, y \in V, \tag{19}$$

$$\mathsf{K}(q,\sigma) = ||\sigma||_2^2\,\mathsf{tr}(G_q) - \langle \mathcal{G}_q(\sigma), \sigma\rangle + \lambda^{-1}\big(||\sigma||_2^2 - \langle \sigma, \omega\rangle^2\big). \tag{20}$$

Finally, bearing in mind the relation between the Green function for Γ and the bottleneck functions, part (iii) of Corollary 4.3, for any $x,y,z\in V$ we have that

$$\begin{aligned}R_{q,\sigma}(x,y)=&\frac{G_q^z(x,x)}{\sigma(x)^2}+\frac{G_q^z(y,y)}{\sigma(y)^2}-\frac{2G_q^z(x,y)}{\sigma(y)\sigma(x)}+2\Big(\frac{\omega(x)}{\sigma(x)}-\frac{\omega(y)}{\sigma(y)}\Big)\Big(\frac{\mathcal{G}_q^z(\omega)(y)}{\sigma(y)}-\frac{\mathcal{G}_q^z(\omega)(x)}{\sigma(x)}\Big)\\&+\big(\lambda^{-1}+\langle\mathcal{G}_q^z(\omega),\omega\rangle\big)\Big(\frac{\omega(x)}{\sigma(x)}-\frac{\omega(y)}{\sigma(y)}\Big)^2,\end{aligned}\tag{21}$$

for any $x,y\in V$. Therefore,

$$\mathsf{K}(q,\sigma)=||\sigma||_2^2\mathsf{tr}(G_q^z)-\langle\mathcal{G}_q^z(\sigma),\sigma\rangle+2\langle\sigma,\omega\rangle\langle\mathcal{G}_q^z(\omega),\sigma\rangle-\langle\mathcal{G}_q^z(\omega),\omega\rangle\big(||\sigma||_2^2+\langle\sigma,\omega\rangle^2\big)+\lambda^{-1}\big(||\sigma||_2^2-\langle\sigma,\omega\rangle^2\big)$$

In particular, taking $\sigma=\omega$ we recover the well known identities, see [4, Corollary 4.4] and also the identities (10) and (11),

$$R_{q,\omega}(x,y)=\frac{G_q(x,x)}{\omega(x)^2}+\frac{G_q(y,y)}{\omega(y)^2}-\frac{2G_q^z(x,y)}{\omega(y)\omega(x)}=\frac{G_q^z(x,x)}{\omega(x)^2}+\frac{G_q^z(y,y)}{\omega(y)^2}-\frac{2G_q^z(x,y)}{\omega(y)\omega(x)}\tag{22}$$

$$\mathsf{K}(q,\omega)=\mathsf{tr}(G_q^V)-\lambda^{-1}=\mathsf{tr}(G_q)=\mathsf{tr}(G_q^z)-\langle\mathcal{G}_q^z(\omega),\omega\rangle.\tag{23}$$

6. Extending Networks

In the previous section, given an admissible potential $q=\mathfrak{q}(\omega,\lambda)$, we have defined the effective resistance between two vertices with respect to the potential q and an unitary weight $\sigma\in\Omega$ and have proved some of its main properties. However, except for the case in which σ is a positive multiple of ω, we have not yet analyzed when the effective resistance determines a metric on Γ, that we declared as our main objective. In fact, we only have to analyze when the triangular inequality holds, since we have proved the other properties.

In this section we show as the q–harmonicity of the chosen weight σ is essential to prove that the corresponding effective resistance is a distance on the network. This property was assured in the case $\lambda=0$, since then σ should be a multiple of ω and hence it is q–harmonic. Therefore, in this section we assume that the admissible potential q satisfies that $q=\mathfrak{q}(\omega,\lambda)$ where $\omega\in\Omega$ and $\lambda>0$ and consider $\mathcal{L}_q$, $\mathcal{G}_q^V$, G_q^V its corresponding Schrödinder operator, Green operator for V and Green function for V, respectively.

We also denote by $\mathcal{H}_q$ the set of q–superharmonic weights. Since $\lambda>0$, G_q^V is positive, and moreover for any $y\in V$ we can define the weight $\sigma_q^y=G_q^V(\cdot,y)$. Then, we have the following characterization of the set $\mathcal{H}_q$.

Proposition 6.1. *For any $y\in V$ we have that $\sigma_q^y\in\mathcal{H}_q$. Moreover,*

$$\mathcal{H}_q=\Big\{\mathcal{G}_q^V(f)=\sum_{y\in V}f(y)\sigma_q^y : f\ge 0 \text{ and } f\neq 0\Big\}.$$

In particular, $\omega=\lambda^{-1}\sum\limits_{y\in V}\omega(y)\sigma_q^y\in\mathcal{H}_q$ and moreover $1\in\mathcal{H}_q$ iff $q\ge 0$ (and $q\neq 0$).

Given $\sigma\in\mathcal{H}_q$, we also consider $R_{q,\sigma}$ the effective resistance with respect to q and σ. To prove that $R_{q,\sigma}$ determines a distance on Γ we use a well known technique originally implemented by M. Fiedler, see [12] for nonnegative potentials and also [4] for admissible potentials and when $\sigma=\omega$. The main idea is to embed the given network Γ in other one in such a way the Green function for V appears as the Green function of a subset in the host network. The most simple way to do this, is consider a new vertex, *the grounded vertex,* $\hat{x}$ and extend the conductance to form a host network $\widehat{\Gamma}=(\hat{V},\hat{c})$, where $\widehat{V}=V\cup\{\hat{x}\}$. In addition, given σ, a weight on V, we consider $\hat{\sigma}>0$ on $\widehat{V}$ such that $\hat{\sigma}(x)=\sigma(x)$ for any $x\in V$. So, the extension $\hat{\sigma}$ of the weight σ is determined by its value at the grounded vertex, $\sigma(\hat{x})$. For any $a>0$, we call the a–extension of σ, the weight $\hat{\sigma}$ on $\widehat{V}$ defined as $\hat{\sigma}=\sigma$ on V and as $\hat{\sigma}(\hat{x})=a$.

To define the value of the conductance $\hat{c}$ at the pairs $(x,\hat{x})$, $x\in V$, consider $u\in\mathcal{C}(V)$ and the Doob transform with respect to σ. Then, for any $x\in V$ we have

$$\mathcal{L}_q(u)(x)=\mathcal{L}_{q_\sigma}(u)(x)+(q-q_\sigma)(x)u(x)=\frac{1}{\sigma(x)}\sum_{y\in V}c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)+(q-q_\sigma)(x)u(x).$$

If we impose $u(\hat{x})=0$, that electrically means that the new vertex $\hat{x}$ is grounded, then for any $x\in V$ we have

$$(q-q_\sigma)(x)u(x)=\frac{1}{\sigma(x)}\,(q-q_\sigma)(x)\sigma(x)^2\frac{u(x)}{\sigma(x)}=\frac{1}{\sigma(x)}\left[(q-q_\sigma)(x)\sigma(x)\sigma(\hat{x})^{-1}\right]\sigma(x)\sigma(\hat{x})\left(\frac{u(x)}{\sigma(x)}-\frac{u(\hat{x})}{\sigma(\hat{x})}\right).$$

Since $\sigma\in\mathcal{H}_q$, we know that $q-q_\sigma\geq 0$ on V and that $q\neq q_\sigma$. Therefore, if we define the conductance $\hat{c}(x,\hat{x})=(q-q_\sigma)(x)\sigma(x)\sigma(\hat{x})^{-1}=\sigma(\hat{x})^{-1}\mathcal{L}_q(\sigma)(x)$ for any $x\in V$, then

$$\mathcal{L}_q(u)(x)=\frac{1}{\sigma(x)}\sum_{y\in\hat{V}}\hat{c}(x,y)\hat{\sigma}(x)\hat{\sigma}(y)\left(\frac{u(x)}{\hat{\sigma}(x)}-\frac{u(y)}{\hat{\sigma}(y)}\right),\quad x\in V.$$

Fixed $\sigma\in\mathcal{H}_q$ and $a>0$ we consider the a–extension of σ and call the *Fiedler extension of the network* $\Gamma=(V,c)$ *with respect to* σ *and* a, the network $\widehat{\Gamma}=(V\cup\{\hat{x}\},\hat{c})$ where $\hat{c}=c$ on $V\times V$ and

$$\hat{c}(x,\hat{x})=\hat{c}(\hat{x},x)=a^{-1}\left(q(x)-q_\sigma(x)\right)\sigma(x)=a^{-1}\,\mathcal{L}_q(\sigma)(x),\quad x\in V.$$

Since $\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)\neq\emptyset$, the Fiedler extension consists in joining each vertex $x\in\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$ with the grounded vertex $\hat{x}$ through an edge whose conductance depends on the value of the weight at both extremes and on the *potential excess* at x, $q(x)-q_\sigma(x)$. In particular, the host network $\widehat{\Gamma}$ is always connected. Next, we prove that any connected host network of Γ with exactly one more vertex, is in fact a Fiedler extension of Γ.

Lemma 6.2. *Let $\hat{x}\notin V$ and $\widehat{\Gamma}=(V\cup\{\hat{x}\},\hat{c})$ a connected host network of Γ. Then, for any $a>0$ there exists $\sigma\in\mathcal{H}_q$ such that $\widehat{\Gamma}$ is the Fiedler extension of Γ, with respect to σ and a.*

Proof. If we define $f\in\mathcal{C}(V)$ as $f(x)=a\hat{c}(x,\hat{x})$, then $f\geq 0$ and moreover $f\neq 0$. Therefore, from Proposition 6.1, if $\sigma=\mathcal{G}_q(f)$, then $\sigma\in\mathcal{H}_q$ and moreover $\mathcal{L}_q(\sigma)=f$. This last identity implies that $\hat{c}(x,\hat{x})=a^{-1}\mathcal{L}_q(\sigma)$ and hence, $\widehat{\Gamma}$ is the Fiedler extension of Γ, with respect to σ and a. □

The combinatorial Laplacian corresponding to the Fiedler extension is denoted by $\widehat{\mathcal{L}}$. The next result establishes the relationship between the original Schrödinger operator $\mathcal{L}_q$ and a new singular and positive semi-definite Schrödinger operator on $\widehat{\Gamma}$.

Proposition 6.3. *If we consider $\hat{q}=\mathsf{q}(\hat{\sigma},0)$, then*

$$\hat{q}=q-a^{-1}\mathcal{L}_q(\sigma)\quad\text{on } V\qquad\text{and}\qquad \hat{q}(\hat{x})=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle u(\hat{x})-a\langle\mathcal{L}_q(\sigma),u\rangle\big).$$

Moreover, for any $u\in\mathcal{C}(\widehat{V})$ we get that

$$\widehat{\mathcal{L}}_{\hat{q}}(u)=\mathcal{L}_q(u)-a^{-1}\mathcal{L}_q(\sigma)\,u(\hat{x})\quad\text{on } V\qquad\text{and}\qquad \widehat{\mathcal{L}}_{\hat{q}}(u)(\hat{x})=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle u(\hat{x})-a\langle\mathcal{L}_q(\sigma),u\rangle\big).$$

Proof. Given $u\in\mathcal{C}(\hat{V})$, then for any $x\in V$ we get that

$$\widehat{\mathcal{L}}(u)(x)=\mathcal{L}(u_{|_V})(x)+a^{-1}\mathcal{L}_q(\sigma)(x)u(x)-a^{-1}\mathcal{L}_q(\sigma)(x)u(\hat{x}),\quad x\in V,$$
$$\widehat{\mathcal{L}}(u)(\hat{x})=a^{-1}\big(u(\hat{x})\langle q,\sigma\rangle-\langle\mathcal{L}_q(\sigma),u\rangle\big),$$

since $\langle\mathcal{L}_q(\sigma),1\rangle=\langle q,\sigma\rangle$. In particular, taking $u=\hat{\sigma}$ we obtain

$$\hat{q}=-\hat{\sigma}^{-1}\widehat{\mathcal{L}}(\hat{\sigma})=q_\sigma-a^{-1}\mathcal{L}_q(\sigma)+\sigma^{-1}\mathcal{L}_q(\sigma)=q-a^{-1}\mathcal{L}_q(\sigma),\quad\text{on } V,$$
$$\hat{q}(\hat{x})=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle-a\langle q,\sigma\rangle\big)$$

and hence

$$\widehat{\mathcal{L}}_{\hat q}(u) = \mathcal{L}_q(u) - a^{-1}\mathcal{L}_q(\sigma)\, u(\hat x), \qquad \text{on } V,$$
$$\widehat{\mathcal{L}}_{\hat q}(u)(\hat x) = a^{-2}\big(\langle \mathcal{L}_q(\sigma),\sigma\rangle u(\hat x) - a\langle \mathcal{L}_q(\sigma),u\rangle\big).$$

□

Corollary 6.4. *Given $f\in\mathcal{C}(V)$ consider $u\in\mathcal{C}(V)$, the unique solution of the Poisson equation $\mathcal{L}_q(u)=f$ on V. Then, $u=v_{|_V}$ where v is the unique solution of the Dirichlet Problem on $\widehat\Gamma$*

$$\widehat{\mathcal{L}}_{\hat q}(v) = f \quad \text{on } V \quad \text{and} \quad v(\hat x)=0.$$

In particular, G_q^V, the Green function for V is the bottleneck function at $\hat x$ for $\widehat\Gamma$.

Proof. We know that both, the Poisson equation on Γ and the Dirichlet problem have a unique solution, say $u,v\in\mathcal{C}(V)$, respectively. From the above Proposition we have that

$$f=\widehat{\mathcal{L}}_{\hat q}(v) = \mathcal{L}_q(v_{|_V}) - a^{-1}\mathcal{L}_q(\sigma)\, v(\hat x) = \mathcal{L}_q(v_{|_V}) \ \text{ on } V$$

which implies that $u=v_{|_V}$. □

Now we have all ingredients to prove the result we are looking for.

Theorem 6.5. *Given $\sigma\in\mathcal{H}_q$, then $R_{q,\sigma}$, the effective resistance with respect to q and σ, determines a distance on Γ. Moreover given $x,y,z\in V$, $R_{q,\sigma}(x,z)+R_{q,\sigma}(z,y)=R_{q,\sigma}(x,y)$ iff z separates x,y in Γ and moreover either $x\notin \mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$ or $y\notin\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$.*

Proof. Consider the potential $\hat q=\mathsf{q}(\hat\sigma,0)$ in the host network $\widehat\Gamma$ and $\widehat R_{\hat q,\hat\sigma}$ its associated effective resistance. Then $\widehat R_{\hat q,\hat\sigma}=d_{\hat q}$ and hence determines a distance on $\widehat\Gamma$. Therefore, its restriction ot $V\times V$ also determines a distance on Γ.

On the other hand, applying the Identity (22), or equivalently the Identity (11), and taking into account that G_q^V is the bottleneck function for $\widehat\Gamma$ at $\hat x$, we have that

$$d_{\hat q}(x,y)=R_{\hat q,\hat\sigma}(x,y)=\frac{G_q^V(x,x)}{\sigma(x)^2}+\frac{G_q^V(y,y)}{\sigma(y)^2}-\frac{2G_q^V(x,y)}{\sigma(y)\sigma(x)}=R_{q,\sigma}(x,y),\quad x,y\in V.$$

□

We remark that, applying the identity (9), in the host network we also have the following identity

$$\widehat R_{\hat q,\hat\sigma}(x,\hat x)=\frac{G_q^V(x,x)}{\sigma(x)^2},\quad \text{for any } x\in V.$$

As a by–product, we have the following relation between the Kirchhoff indexes

$$\mathsf{K}(\hat q,\hat\sigma)=\mathsf{K}(q,\sigma)+a^2\mathsf{tr}(\mathcal{G}_q^V)=\big(a^2+||\sigma||_2^2\big)\mathsf{tr}(G_q^V)-\langle\mathcal{G}_q^V(\sigma),\sigma\rangle \tag{24}$$

In addition, we can newly apply the part (iii) of Corollary 4.3 to obtain the Green function for $\widehat\Gamma$

$$\widehat G_{\hat q}(x,y)=\begin{cases} \sigma(x)\sigma(y)\Big[\dfrac{G_q^V(x,y)}{\sigma(x)\sigma(y)}-\dfrac{\mathcal{G}_q^V(\sigma)(y)}{(a^2+||\sigma||_2^2)\sigma(y)}-\dfrac{\mathcal{G}_q^V(\sigma)(x)}{(a^2+||\sigma||_2^2)\sigma(x)}+\dfrac{\langle\mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2}\Big], & x,y\in V,\\[2mm] a\sigma(x)\Big[\dfrac{\langle\mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2}-\dfrac{\mathcal{G}_q^V(\sigma)(x)}{(a^2+||\sigma||_2^2)\sigma(x)}\Big] & x\in V,\ y=\hat x,\\[2mm] \dfrac{a^2\langle\mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2} & x=y=\hat x.\end{cases}$$

Moreover applying the part (iv) of the Corollary 4.3, we have that the bottleneck function for Γ and any vertex $z\in V$ is given by

$$\widehat{G}_{\hat{q}}^{z}(x,y)=\begin{cases}\sigma(x)\sigma(y)\Big[\dfrac{G_q^V(x,y)}{\sigma(x)\sigma(y)}-\dfrac{G_q^V(y,z)}{\sigma(y)\sigma(z)}-\dfrac{G_q^V(x,z)}{\sigma(x)\sigma(z)}+\dfrac{G_q^V(z,z)}{\sigma(z)^2}\Big], & x,y\in V,\\[2mm] \sigma(x)a\Big[\dfrac{G_q^V(z,z)}{\sigma(z)^2}-\dfrac{G_q^V(x,z)}{\sigma(x)\sigma(z)}\Big], & x\in V,\ y=\hat{x},\\[2mm] \dfrac{a^2G_q^V(z,z)}{\sigma(z)^2}, & x=y=\hat{x}.\end{cases}$$

If given $z\in V$ we consider the weight σ_q^z, we have the following result.

Corollary 6.6. *For any $z\in V$, the function*

$$d_z(x,y)=\frac{G_q^V(x,x)}{G_q^V(x,z)^2}+\frac{G_q^V(y,y)}{G_q^V(y,z)^2}-\frac{2G_q^V(x,y)}{G_q^V(x,z)G_q^V(y,z)},\quad x,y\in V,$$

determines a cutpoint additive distance on Γ.

Notice that

$$d_z(z,y)=\frac{G_q^V(z,z)}{G_q^V(z,z)^2}+\frac{G_q^V(y,y)}{G_q^V(y,z)^2}-\frac{2G_q^V(z,y)}{G_q^V(z,z)G_q^V(y,z)}=\frac{G_q^V(y,y)}{G_q^V(y,z)^2}-\frac{1}{G_q^V(z,z)}$$

and hence the positiveness of $d_z(z,y)$ is equivalent to the Cauchy-Schwarz inequality

$$G_q^V(z,y)^2=\langle\mathcal{G}_q^V(\varepsilon_z),\varepsilon_y\rangle^2\le\langle\mathcal{G}_q^V(\varepsilon_z),\varepsilon_z\rangle\langle\mathcal{G}_q^V(\varepsilon_y),\varepsilon_y\rangle=G_q^V(z,z)G_q^V(y,y)$$

with equality iff $z=y$, since $\mathcal{G}_q$ is positive definite.

We end this section by observing that the effective resistance on $\widehat{\Gamma}$ with respect to $\hat{q}=\mathfrak{q}(\hat{\sigma},0)$ in fact does not depend of a the value of the extended weight $\hat{\sigma}$ at the grounded vertex $\hat{x}$.

7. Contracting networks: The Neighborhood Transformation

In this section we introduce a transformation on the network Γ by deleting a given vertex $x\in V$ but maintaining the connectedness. In what follows, we fix $x_0\in V$ and $F=V\setminus\{x_0\}$. Therefore, $\mathcal{C}(F)$ is the set of real functions on V vanishing at x_0.

Let us consider a potential $q\in\mathcal{C}(F)$ and hence such that $q(x_0)=0$, and the Poisson equation $\mathcal{L}_q(u)=f$, where $f\in\mathcal{C}(V)$. Then, Identity (1) implies that

$$\kappa(x_0)u(x_0)=f(x_0)+\sum_{y\in V}c(x_0,y)\,u(y)$$

and hence, we get

$$u(x_0)=\frac{1}{\kappa(x_0)}\Big[f(x_0)+\sum_{z\in F}c(x_0,z)\,u(z)\Big].\tag{25}$$

Lemma 7.1. *If $q\in\mathcal{C}(F)$, then for any $f\in\mathcal{C}(V)$, $u\in\mathcal{C}(V)$ is a solution of the Poisson equation $\mathcal{L}_q(u)=f$ iff for any $x\in F$ we have*

$$f(x)+\frac{c(x_0,x)}{\kappa(x_0)}\,f(x_0)=\sum_{y\in F}\Big[\frac{c(x,x_0)c(x_0,y)}{\kappa(x_0)}+c(x,y)\Big]\big(u(x)-u(y)\big)++q(x)\,u(x),$$

and, in addition, $u(x_0)=\dfrac{1}{\kappa(x_0)}\Big[f(x_0)+\displaystyle\sum_{y\in F}c(x_0,y)\,u(y)\Big].$

Proof. Given $x \in F$, then Identity (25) implies that

$$u(x) - u(x_0) = \frac{1}{\kappa(x_0)} \sum_{y \in F} c(x_0, y)\big(u(x) - u(y)\big) - \frac{f(x_0)}{\kappa(x_0)}$$

and hence

$$\begin{aligned} f(x) &= c(x, x_0)\big(u(x) - u(x_0)\big) + \sum_{y \in F} c(x, y)\big(u(x) - u(y)\big) + q(x)u(x) \\ &= -\frac{c(x_0, x)}{\kappa(x_0)} f(x_0) + \sum_{y \in F} \left[\frac{c(x, x_0)c(x_0, y)}{\kappa(x_0)} + c(x, y)\right] \big(u(x) - u(y)\big) + q(x)u(x). \end{aligned}$$

□

The identity given in the above Lemma motivates the definition of the function $c^{x_0}\colon F \times F \longrightarrow [0, +\infty)$, given by $c^{x_0}(x, x) = 0$ for any $x \in F$ and by

(26) $$c^{x_0}(x, x) = 0, \quad x \in F \quad \text{and} \quad c^{x_0}(x, y) = c(x, y) + \frac{c(x, x_0)c(x_0, y)}{\kappa(x_0)}, \quad x, y \in F, \ x \neq y.$$

If we define in $F \times F$ the adjacency relation $x \overset{x_0}{\sim} y$ iff $c^{x_0}(x, y) > 0$ and denote the new graph as Γ^{x_0}, clearly c^{x_0} is a conductance on Γ^{x_0} and then (Γ^{x_0}, c^{x_0}) is a new network that, in the sequel, we denote simply by Γ^{x_0}. Moreover, its corresponding combinatorial Laplacian is denoted by $\mathcal{L}^{x_0}$.

We say that the network Γ^{x_0} has been obtained from Γ after the *Neighborhood Transformation at vertex* x_0. Observe that any pair of vertices that are adjacent to x_0 in Γ are adjacent in the network Γ^{x_0}, because if $x, y \sim x_0$ in Γ, then $c(x, x_0)c(x_0, y) > 0$. In other words, the subnetwork in Γ^{x_0} consisting in the neighborhood of x_0 in Γ is complete. For this reason, the Neighborhood Transformation is also named *Star–Mesh transformation.* Notice that , when $x \not\sim x_0$ in Γ, then $c^{x_0}(x, y) = c(x, y)$ for all $y \in F$.

Lemma 7.2. *The network* Γ^{x_0} *is connected.*

Proof. Let $x, y \in V^x$ and consider $x = z_0 \sim z_1 \sim \cdots \sim z_n \sim z_{n+1} = y$ a path joining x and y in Γ. If $z_j \neq x_0$ for any $j = 1, \ldots, n$, then $c^{x_0}(z_i, z_{i+1}) \geq c(z_i, z_{i+1}) > 0$ and hence $xz_1 \cdots z_n y$ is a path in Γ^{x_0}.

On the other hand, if $z_i = x$ for some $i = 1, \ldots, n$, then $z_{i-1}, z_{i+1} \sim x_0$ and hence $z_{i-1} \sim z_{i+1}$ in Γ^{x_0}. So, we can delete vertex z_i in the above path and continue having a path on Γ^{x_0}. □

Given $u \in \mathcal{C}(F)$, *the harmonic extension of* u *at* x_0 is $u_h \in \mathcal{C}(V)$ defined as

(27) $$u_h(y) = u(y) \quad \text{for any} \quad y \in F \quad \text{and} \quad u_h(x_0) = \frac{1}{\kappa(x_0)} \sum_{y \in F} c(x_0, y)\, u(y).$$

Observe that if $u \geq 0$ on the neighborhood of x_0, then $u_h(x_0) \geq 0$ and the equality holds iff $u(x) = 0$ for any $x \sim x_0$.

Lemma 7.3. *Given* $q \in \mathcal{C}(F)$, *if* $u \in \mathcal{C}(F)$ *then* $\mathcal{L}_q(u_h)(x_0) = 0$ *and conversely if* $u \in \mathcal{C}(V)$ *satisfied that* $\mathcal{L}_q(u)(x_0) = 0$, *then* $u = v_h$ *where* $v = u_{|_F}$.

After the above definitions, (26) and (27), the result of Lemma 7.1 can straightforwardly be re-written as follows.

Proposition 7.4. *Given* $q, f \in \mathcal{C}(F)$, *then* $u \in \mathcal{C}(F)$ *is a solution of the Poisson equation* $\mathcal{L}^{x_0}_q(u) = f$ *on* F *iff* $u_h \in C(V)$ *is a solution of the Poisson equation* $\mathcal{L}_q(u_h) = f$ *on* V. *In particular, if* $\omega \in \mathcal{C}(F)$ *satisfies that* $\omega > 0$ *on* F, *then* ω_h *is a weight,* $\mathcal{L}(\omega_h) \in \mathcal{C}(F)$ *and moreover* $\mathcal{L}^{x_0}(\omega) = \mathcal{L}(\omega_h)$ *on* F.

In the sequel we consider the set $\Omega(F) = \{\omega \in \mathcal{C}(F) : \omega > 0 \text{ on } F\}$ and denote by Ω_{x_0}, the set of weights on Γ that are harmonic at x_0. It is clear that $q_\omega \in \mathcal{C}(F)$ for all $\omega \in \Omega_{x_0}$. Moreover it is clearly satisfied that

(28) $$\Omega(F) = \big\{\omega_{|_F} : \omega \in \Omega_{x_0}\big\} \quad \text{and} \quad \Omega_{x_0} = \big\{\omega_h : \omega \in \Omega(F)\big\}$$

Lemma 7.5. *Given $\omega \in \Omega(F)$, then $\omega^{-1}\mathcal{L}^{x_0}(\omega) = \omega_h^{-1}\mathcal{L}(\omega_h)$ on F. Conversely, given $\omega \in \Omega_{x_0}(V)$, then $\omega^{-1}\mathcal{L}^{x_0}(\omega) = \omega^{-1}\mathcal{L}(\omega)$ on F.*

Given $\omega \in \Omega(F)$, the above Lemma permits to identify the potential on Γ^{x_0} associated with ω with the potential on Γ associated with ω_h; that is with $\omega_h^{-1}\mathcal{L}(\omega_h)$. In the sequel we systematically use this identification and then both will be denoted by q_ω.

Corollary 7.6. *Given $\omega \in \Omega(F)$ and $f \in \mathcal{C}(F)$ such that $f \in \omega^\perp$, then $u \in \mathcal{C}(F)$ is a solution of the Poisson equation $\mathcal{L}^{x_0}_{q_\omega}(u) = f$ on F iff $u_h \in C(V)$ is a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u_h) = f$ on V.*

Consider now fixed $\omega \in \Omega(F)$ and $G^{x_0}_{q_\omega}$ the Green function for Γ^{x_0} and G_{q_ω} the Green function for Γ. In addition, $R^{x_0}_{q_\omega}$ and R_{q_ω} denote their corresponding effective resistances, respectively.

Theorem 7.7. *For any $x, y \in F$, we get*

$$G^{x_0}_{q_\omega}(x,y) = G_{q_\omega}(x,y) + \omega_h(x_0)\Big[\omega(y)G_{q_\omega}(x,x_0) + \omega(x)G_{q_\omega}(x_0,y)\Big] + \omega(x)\omega(y)\omega_h^2(x_0)G_{q_\omega}(x_0,x_0),$$

where $\omega_h(x_0) = \dfrac{1}{\kappa(x_0)}\sum\limits_{y\in F} c(x_0,y)\,\omega(y)$.

Proof. Given $y \in F$, consider $f = \varepsilon_y - \omega(y)\omega$ and u the unique solution of the Poisson equation $\mathcal{L}^{x_0}_{q_\omega}(u) = f$ such that $u \in \omega^\perp$. According with Corollary 7.6,

$$u = G_{q_\omega}(\cdot,y) - \omega(y)\mathcal{G}_{q_\omega}(\omega_h - \omega_h(x_0)\varepsilon_{x_0}) + \alpha\omega = G_{q_\omega}(\cdot,y) + \omega(y)\omega_h(x_0)G_{q_\omega}(\cdot,x_0) + \alpha\omega$$

where

$$\begin{aligned} 0 = \langle u,\omega\rangle &= \sum_{x\in F}\omega(x)G_{q_\omega}(x,y) + \omega(y)\omega_h(x_0)\sum_{x\in F}\omega(x)G_{q_\omega}(x,x_0) + \alpha \\ &= -\omega_h(x_0)G_{q_\omega}(x_0,y) - \omega(y)\omega_h^2(x_0)G_{q_\omega}(x_0,x_0) + \alpha \end{aligned}$$

that is, $\alpha = \omega_h(x_0)\Big[G(x_0,y) + \omega(y)\omega_h(x_0)G(x_0,x_0)\Big]$. □

Corollary 7.8. *$R^{x_0}_{q_\omega}$ is the restriction of R_{q_ω} to $F\times F$.*

Proof. Applying the Identity (19), for any $x, y \in F$,

$$\begin{aligned} R^{x_0}_{\omega}(x,y) &= \frac{G^{x_0}_{q_\omega}(x,x)}{\omega^2(x)} + \frac{G^{x_0}_{q_\omega}(y,y)}{\omega^2(y)} - \frac{2G^{x_0}_{q_\omega}(x,y)}{\omega(x)\omega(y)} \\ &= \frac{G_{q_\omega}(x,x)}{\omega^2(x)} + 2\omega_h(x_0)\frac{G_{q_\omega}(x,x_0)}{\omega(x)} + \omega_h^2(x_0)G_{q_\omega}(x_0,x_0) \\ &+ \frac{G_{q_\omega}(y,y)}{\omega^2(y)} + 2\omega_h(x_0)\frac{G_{q_\omega}(y,x_0)}{\omega(y)} + \omega_h^2(x_0)G_{q_\omega}(x_0,x_0) \\ &- \frac{2G_{q_\omega}(x,y)}{\omega(x)\omega(y)} - 2\omega_h(x_0)\Big[\frac{G_{q_\omega}(x,x_0)}{\omega(x)} + \frac{G_{q_\omega}(x_0,y)}{\omega(y)}\Big] - 2\omega_h^2(x_0)G_{q_\omega}(x_0,x_0) = R_{q_\omega}(x,y). \end{aligned}$$

□

We end this section considering newly the connected network $\Gamma = (V,c)$ and the admissible potential $q = \mathfrak{q}(\omega,\lambda)$, where $\omega \in \Omega$. Therefore, $\mathcal{L}_q$ is the corresponding Schrödinger operator, $\mathcal{G}_q$ and G_q the Green operator for the network, and R_q the associated effective resistance. In addition we also consider fixed a new vertex $\hat{x} \notin V$.

Fixed $\sigma \in \mathcal{H}_q$ and $a > 0$, we consider the a–extension of σ and $\widehat{\Gamma} = (V \cup \{\hat{x}\}, \hat{c})$, the Fiedler extension of the network Γ with respect to σ and a, where $\hat{c} = c$ on $V \times V$ and $\hat{c}(x,\hat{x}) = a^{-1}\mathcal{L}_q(\sigma)(x)$, for any $x \in V$.

Observe that $\hat{c}(x,\hat{x})>0$ only when $x\in \mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$. Then $\widehat{\kappa}(\hat{x})=a^{-1}\langle q,\sigma\rangle$ and hence, if σ_h is the harmonic extension of σ to $\widehat{\Gamma}$, then

$$\sigma_h(\hat{x})=\frac{1}{\widehat{\kappa}(\hat{x})}\sum_{y\in F}\hat{c}(\hat{x},y)\,\sigma(y)=\langle q,\sigma\rangle^{-1}\langle \mathcal{L}_q(\sigma),\sigma\rangle.$$

Therefore, $\hat{\sigma}=\sigma_h$ iff $a=\langle q,\sigma\rangle^{-1}\langle \mathcal{L}_q(\sigma),\sigma\rangle$. In particular, when $\sigma=\omega$, then $\hat{c}(x,\hat{x})=a^{-1}\lambda\,\omega(x)$, for any $x\in V$ and $\hat{\omega}=\omega_h$ iff $a=||\omega||_1^{-1}$.

Now, consider the Neighborhood Transformation at vertex $\hat{x}$. Then for any $x,y\in V$, $x\neq y$, we have that

$$c^{\hat{x}}(x,y)=\hat{c}(x,y)+\frac{\hat{c}(x,\hat{x})\hat{c}(\hat{x},y)}{\hat{\kappa}(\hat{x})}=c(x,y)+a^{-1}\langle q,\sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y).$$

As a consequence of the Corollary 7.8, we have the following relation between the effective resistance of the Fiedler extension and the effective resistance after a Neighborhood Transformation.

Theorem 7.9. *Given $\sigma\in\mathcal{H}_q$, consider the conductance $c^\sigma\colon V\times V\longrightarrow\mathbb{R}$ defined as*

$$c^\sigma(x,y)=c(x,y)+\langle \mathcal{L}_q(\sigma),\sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y),\quad \textit{for any } x,y\in V,\ x\neq y.$$

Then $R_{q,\sigma}=R^\sigma$, where R^σ is the effective resistance of the network $\Gamma^\sigma=(V,c^\sigma)$ with respect to q_σ.

Observe that $\Gamma^\sigma=(V,c^\sigma)$ appears as the perturbation of the initial network Γ that consists in to take the perturbation $\varepsilon(x,y)=\langle \mathcal{L}_q(\sigma),\sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y)$, $x\neq y$. In particular, for any $z\in V$, we have that $c^{\sigma_z}=c$ and hence

$$R_{q_\sigma}=R_{q,\sigma}.$$

8. Stars and recoverable complete networks

Given $x_0\in V$ we say that $V(x_0)$ the subset of vertices adjacent to x_0, is a *star in* Γ if $c(z,y)=0$ for all $z,y\in V(x_0)$. In this case, x is called the *center of the star*. The degree of the star is $k(x)$ the degree of its center.

If S is the star centered at x_0, then the Neighborhood Transformation at x transforms Γ into a connected network Γ^{x_0} with $n-1$ vertices and $\frac{1}{2}\,k(x_0)\big(k(x_0)-3\big)$ more edges. Notice that the this value is also true when $k(x_0)=2$, since then Γ^{x_0} has one less edge than Γ. In fact, under this transformation the subnetwork induced by $V(x_0)$ in Γ is transformed into a complete subnetwork in Γ^{x_0}. For this reason the Neighborhood Transformation at x is, is this case, also called *Star-Complete Transformation*, see [20, 22].

Conversely, suppose that an specific subnetwork with vertex set K of a given network $(\hat{\Gamma},\hat{c})$ is complete, that is $\hat{c}(z,y)>0$ for any $z,y\in K$. The main question here is when the complete subnetwork is obtained from a star in a network Γ, that is, when $\hat{\Gamma}=\Gamma^{x_0}$ for a network Γ and a vertex $x_0\in V$, where V is the set or vertices of Γ and $V(x_0)$ is a star. In this case we say that the complete subnetwork is *recoverable from a star*, or simply that the complete subnetwork is *recoverable*. In addition, we are also interested in when a complete network is uniquely recoverable. We remark that in [22] this notion is referred as *response-equivalent*.

Our objective is to characterize what complete subnetworks are recoverable. In all cases we denote by x_0 a new vertex, that is not belonging to the vertex set of $\hat{\Gamma}$. We use this new vertex as the center of the star that recovers K.

Proposition 8.1. *Consider K a complete subnetwork in $(\hat{\Gamma},\hat{c})$. Then, K is the transformation of a star after a neighborhood transformation iff there exists $\alpha\in\mathcal{C}(V)$ satisfying the following properties:*

(i) $\alpha\geq 0$.
(ii) $\mathsf{supp}(\alpha)=K$.
(iii) $\hat{c}(x,y)=\alpha(x)\alpha(y)$ *for all* $x,y\in K$ *with* $x\neq y$.

Moreover, if we add a new vertex, x_0, to the vertex set of $\hat{\Gamma}$ and define $c(x,x_0) = \alpha(x)||\alpha||_1$ for any $x \in K$, $c(x,y) = 0$ for any $x,y \in K$ and $c(x,y) = \hat{c}(x,y)$ otherwise, then $K = V(x_0)$ is a star in the network (Γ, c) and the complete network is obtained from the star after a Neighborhood Transformation at x_0. In addition K is uniquely recoverable iff there exists a unique function $\alpha \in \mathcal{C}(V)$ satisfying the above properties.

Proof. Assume that $V(x_0)$ is a star in Γ and consider the function $\alpha \in \mathcal{C}(V)$ defined as $\alpha(x) = \dfrac{c(x,x_0)}{\sqrt{\kappa(x_0)}}$. Then $\mathsf{supp}(\alpha) = V(x_0)$, $\alpha(x)\alpha(y) = c^{x_0}(x,y)$ for any $x,y \in V(x_0)$ with $x \neq y$ and moreover $||\alpha||_1 = \sqrt{\kappa(x_0)}$. Therefore $c(x,x_0) = \alpha(x)||\alpha||_1$, for any $x \in V(x_0)$. □

Corollary 8.2. *Any recoverable complete network on more than two vertices is uniquely recoverable. On the other hand if K is a digon, that is, K is the complete network on two vertices, then K is infinitely recoverable. Moreover if $K = \{x,y\}$ and $\hat{c}(x,y)$ is the conductance, then for any $a > 0$, K is recoverable from the star with conductances $c(x,x_0) = a^2 + \hat{c}(x,y)$ and $c(y,x_0) = \hat{c}(x,y)\big[1 + a^{-2}\hat{c}(x,y)\big]$.*

Proof. Let $(K,\hat{c})$ a recoverable complete network and suppose that there exist $\alpha, \beta \in \mathcal{C}(V)$ with $\alpha, \beta \geq 0$, $\mathsf{supp}(\alpha) = \mathsf{supp}(\beta) = K$ and such that $\alpha(x)\alpha(y) = \hat{c}(x,y) = \beta(x)\beta(y)$ for any $x,y \in K$ with $x \neq y$. Fixed $y \in K$ and defining $a = \beta^{-1}(y)\alpha(y)$, the above identity implies that $\beta(x) = a\alpha(x)$ for $x \neq y$ and moreover $\beta(y) = a^{-1}\alpha(y)$. any $x \in K$.

If $|K| \geq 3$, and $x,y,z \in K$ are different from each other, then $\alpha(x)\alpha(z) = \beta(x)\beta(z) = a^2\alpha(x)\alpha(z)$, and hence $a = 1$, which implies that $\beta = \alpha$.

If $|K| = 2$, $K = \{x,y\}$ and we consider $\beta(y) = 1$ and $\beta(x) = \hat{c}(x,y)$, then $\alpha \in \mathcal{C}(V)$ with $\alpha \geq 0$ and $\mathsf{supp}(\alpha) = K$ satisfies that $\alpha(x)\alpha(y) = \hat{c}(x,y)$ iff there exists $a > 0$ such that $\alpha(x) = a$ and $\alpha(y) = a^{-1}\hat{c}(x,y)$. Then $||\alpha||_1 = a + a^{-1}\hat{c}(x,y)$ and hence $c(x,x_0) = a\big(a + a^{-1}\hat{c}(x,y)\big)$ and $c(y,x_0) = a^{-1}\hat{c}(x,y)\big(a + a^{-1}\hat{c}(x,y)\big)$. □

Now we obtain a geometrical characterization of the recoverable complete networks on, at least, three vertices. The result is a reformulation of that obtained in [22].

Proposition 8.3 (Triangle Condition)**.** *Given $(K,\hat{c})$ a complete network such that $|K| \geq 3$, then $(K,\hat{c})$ is recoverable iff for any three different vertices $x,y,z \in K$ the value $\dfrac{\hat{c}(x,y)\hat{c}(x,z)}{\hat{c}(y,z)}$ depends only on x. Moreover if the above condition holds, then for any $x,y \in K$ such that $x \neq y$, we have that $\hat{c}(x,y) = \alpha(x)\alpha(y)$, where $\alpha(x) = \sqrt{\dfrac{\hat{c}(x,y)\hat{c}(x,z)}{\hat{c}(y,z)}}$ and z is any vertex different from x and y.*

Proof. If the network is recoverable, then there exists $\alpha \in \mathcal{C}(V)$ such that $\alpha \geq 0$, $\mathsf{supp}(\alpha) = K$ and moreover $\hat{c}(x,y) = \alpha(x)\alpha(y)$ for any $x,y \in K$ with $x \neq y$. Clearly the above identity implies the triangle conditions and moreover the equality $\alpha(x) = \sqrt{\dfrac{\hat{c}(x,y)\hat{c}(x,z)}{\hat{c}(y,z)}}$ for any $y,z \in K \setminus \{x\}$ with $y \neq z$.

Conversely, if the Triangle Condition is in force and we define $\alpha(x) = \sqrt{\dfrac{\hat{c}(x,y)\hat{c}(x,z)}{\hat{c}(y,z)}}$ for any pair of vertices $y,z \in K \setminus \{x\}$ with $y \neq z$, then given $v \in K \setminus \{x\}$, if we consider $u \in K \setminus \{x,v\}$, we get

$$\alpha(x)\alpha(v) = \sqrt{\frac{\hat{c}(x,v)\hat{c}(x,u)}{\hat{c}(v,u)}}\sqrt{\frac{\hat{c}(v,x)\hat{c}(v,u)}{\hat{c}(x,u)}} = \hat{c}(x,v)$$

and hence the network $(K,\hat{c})$ is recoverable. □

Corollary 8.4. *Let $(K,\hat{c})$ be the triangle of vertices x,y,z. Then K is uniquely recoverable from the star with conductances*

$$c(x,x_0) = \frac{\beta}{\hat{c}(y,z)}, \quad c(y,x_0) = \frac{\beta}{\hat{c}(x,z)} \quad \textit{and} \quad c(z,x_0) = \frac{\beta}{\hat{c}(x,y)},$$

where $\beta=\hat{c}(x,y)\hat{c}(x,z)+\hat{c}(x,y)\hat{c}(y,z)+\hat{c}(x,z)\hat{c}(y,v)$.

Proof. Clearly $(K,\hat{c})$ satisfies the Triangle Condition. Therefore, if we define

$$\alpha(x)=\sqrt{\frac{\hat{c}(x,y)\hat{c}(x,z)}{\hat{c}(y,z)}},\qquad \alpha(y)=\sqrt{\frac{\hat{c}(y,x)\hat{c}(y,z)}{\hat{c}(x,z)}}\qquad\text{and}\qquad \alpha(z)=\sqrt{\frac{\hat{c}(z,x)\hat{c}(z,y)}{\hat{c}(x,y)}}$$

then

$$||\alpha||_1=\frac{\hat{c}(x,y)\hat{c}(x,z)+\hat{c}(y,x)\hat{c}(y,z)+\hat{c}(z,x)\hat{c}(z,y)}{\sqrt{\hat{c}(y,z)\hat{c}(x,z)\hat{c}(x,y)}}=\frac{\beta}{\sqrt{\hat{c}(y,z)\hat{c}(x,z)\hat{c}(x,y)}}$$

and the results appears as a consequence of Proposition 8.1. □

Observe that the cases $|K|=2$, the digon and $|K|=3$, the triangle, correspond to the well-known transformations on electrical networks, see for instance [6]:

If we have a star of degree 2, then we have two resistances $r(x,x_0)=c^{-1}(x,x_0)$ and $r(y,x_0)=c^{-1}(y,x_0)$ connected *in series*. Then, after the Neighborhood Transformation at x_0 the two resistances are glued forming a resistance whose value equals to

$$r^{x_0}(x,y)=\frac{1}{c^{x_0}(x,y)}=\frac{c(x,x_0)+c(y,x_0)}{c(x,x_0)c(y,x_0)}=\frac{1}{c(x,x_0)}+\frac{1}{c(y,x_0)}=r(x,x_0)+r(y,x_0).$$

If we have a star of degree 3, then we have three resistances $r(x,x_0)=c^{-1}(x,x_0)$, $r(y,x_0)=c^{-1}(y,x_0)$ and $r(z,x_0)=c^{-1}(z,x_0)$ forming a Y*–structure centered at* x_0. Then, after the Neighborhood Transformation at x_0, in this case called *Star-Triangle transformation*, the three resistances are transformed into a triangle of resistances, with vertices at x,y,z and whose values equal to

$$r^{x_0}(x,y)=\frac{1}{c^{x_0}(x,y)}=\frac{c(x,x_0)+c(x_0,y)+c(x_0,z)}{c(x,x_0)c(x_0,y)}=\frac{r(x_0,y)r(x_0,z)+r(x,x_0)r(x_0,z)+r(x,x_0)r(x_0,y)}{r(x_0,z)}$$

and we have analogous expressions for $r^{x_0}(x,z)$ and $r^{x_0}(y,z)$.

Now we obtain another geometrical characterization of the recoverable complete networks on more than four vertices. Newly, the result is a reformulation of that obtained in [22].

Proposition 8.5 (Square Condition)**.** *Given $(K,\hat{c})$ a complete network such that $|K|\ge 4$, then $(K,\hat{c})$ is recoverable iff for any four different vertices $x,y,u,v\in K$ it is satisfied that $\hat{c}(x,y)\hat{c}(u,v)=\hat{c}(x,u)\hat{c}(y,v)$.*

Proof. It suffices to prove that the square and the triangle conditions are equivalent.

If we assume that the triangle condition holds, then for any four different vertices $x,y,u,v\in K$, we get

$$\frac{\hat{c}(x,y)\hat{c}(x,v)}{\hat{c}(y,v)}=\frac{\hat{c}(x,u)\hat{c}(x,v)}{\hat{c}(u,v)}$$

that in turns, clearly implies $\hat{c}(x,y)\hat{c}(u,v)=\hat{c}(x,u)\hat{c}(y,v)$.

Conversely, if the square condition holds, then for any four different vertices $x,y,u,v\in K$, we get $\hat{c}(x,y)\hat{c}(u,v)=\hat{c}(x,u)\hat{c}(y,v)$. Multiplying both sides of the above identity by $\hat{c}(x,v)$ we obtain

$$\hat{c}(x,y)\hat{c}(u,v)\hat{c}(x,v)=\hat{c}(x,u)\hat{c}(y,v)\hat{c}(x,v)$$

and hence that $\dfrac{\hat{c}(x,y)\hat{c}(x,v)}{\hat{c}(y,v)}=\dfrac{\hat{c}(x,u)\hat{c}(x,v)}{\hat{c}(u,v)}$. □

Theorem 8.6. *Let $(K,\hat{c})$ a recoverable complete network and consider $\alpha\in\mathcal{C}(V)$ such that $\alpha\ge 0$, $\mathsf{supp}(\alpha)=K$ and moreover $\hat{c}(x,y)=\alpha(x)\alpha(y)$ for any $x,y\in K$, $x\ne y$. Then, for any $\omega\in\Omega(K)$, the Green function of $(K,\hat{c})$ with respect to ω is given by*

$$G(x,y)=\frac{\omega(x)\omega(y)}{\langle\alpha,\omega\rangle}\left[\sum_{v\in K}\frac{\omega^3(v)}{\alpha(v)}-\frac{\omega(x)}{\alpha(x)}-\frac{\omega(y)}{\alpha(y)}+\frac{\varepsilon_x(y)}{\omega(x)\alpha(x)}\right],\qquad x,y\in K.$$

Proof. From Proposition 8.1, $(K,\hat{c})$ is obtained, after a Neighborhood Transformation at x_0, from the star whose conductance is given by $c(x,x_0)=\alpha(x)||\alpha||_1$. Taking into account that $\omega_h(x_0)=\dfrac{\langle\alpha,\omega\rangle}{||\alpha||_1}$, the Green function of the star, with respect to the weight $\frac{1}{\sqrt{1+\omega_h^2(x_0)}}\,\omega_h=\dfrac{||\alpha||_1}{\sqrt{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}}\,\omega_h$ is given by

$$
\begin{aligned}
G^S(x_0,x_0)&=\frac{\langle\alpha,\omega\rangle||\alpha||_1^2Q}{\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)^2},\\
G^S(x,x_0)&=\frac{\omega(x)||\alpha||_1}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}\left[\frac{||\alpha||_1^2Q}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}-\frac{\omega(x)}{\alpha(x)}\right],\\
G^S(x,y)&=\frac{\omega(x)\omega(y)||\alpha||_1^2}{\langle\alpha,\omega\rangle\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)}\left[\frac{||\alpha||_1^2Q}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}-\frac{\omega(x)}{\alpha(x)}-\frac{\omega(y)}{\alpha(y)}+\frac{\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)\varepsilon_x(y)}{\omega(x)\alpha(x)||\alpha||_1^2}\right],
\end{aligned}
$$

for any $x,y\in K$, where $Q=\displaystyle\sum_{v\in K}\frac{\omega^3(v)}{\alpha(v)}$, see [8]. Applying now Theorem 7.7, for any $z,y\in K$ we obtain that

$$
\begin{aligned}
G(x,y)&=G^S(x,y)+\frac{\langle\alpha,\omega\rangle}{||\alpha||_1}\Big[\omega(y)G^S(x,x_0)+\omega(x)G^S(x_0,y)\Big]+\frac{\langle\alpha,\omega\rangle^2\omega(x)\omega(y)}{||\alpha||_1^2}\,G^S(x_0,x_0)\\
&=\frac{\omega(x)\omega(y)||\alpha||_1^2}{\langle\alpha,\omega\rangle\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)}\left[\frac{||\alpha||_1^2Q}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}-\frac{\omega(x)}{\alpha(x)}-\frac{\omega(y)}{\alpha(y)}+\frac{\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)\varepsilon_x(y)}{\omega(x)\alpha(x)||\alpha||_1^2}\right]\\
&+\frac{\langle\alpha,\omega\rangle\omega(x)\omega(y)}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}\left[\frac{2||\alpha||_1^2Q}{||\alpha||_1^2+\langle\alpha,\omega\rangle^2}-\frac{\omega(x)}{\alpha(x)}-\frac{\omega(y)}{\alpha(y)}\right]+\frac{\langle\alpha,\omega\rangle^3Q\omega(x)\omega(y)}{\big(||\alpha||_1^2+\langle\alpha,\omega\rangle^2\big)^2}\\
&=\frac{\omega(x)\omega(y)}{\langle\alpha,\omega\rangle}\left[Q-\frac{\omega(x)}{\alpha(x)}-\frac{\omega(y)}{\alpha(y)}+\frac{\varepsilon_x(y)}{\omega(x)\alpha(x)}\right]. \qquad \square
\end{aligned}
$$

Corollary 8.7. *Any complete network K on n vertices and with constant conductance, c, is recoverable from the star with n vertices and constant conductance c. Moreover, for any weight $\omega\in\Omega(V)$ its Green function is given by*

$$G(x,y)=\frac{\omega(z)\omega(y)}{c\displaystyle\sum_{v\in K}\omega(v)}\left[\sum_{v\in K}\omega^3(v)-\omega(x)-\omega(y)+\frac{\varepsilon_x(y)}{\omega(x)}\right],\qquad x,y\in K.$$

In particular, when the weight is also constant, the Green function is given by

$$G(z,y)=\frac{1}{n^2c}\big[n\varepsilon_x(y)-1\big],\qquad x,y\in K.$$

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