# T-subgroups y T-indistinguishability operators

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**Abstract**

In this paper the fuzzy subgroup set of a group is studied. We use the close relation between indistinguishability operators and fuzzy subgroups to characterize the morphisms that transform fuzzy subgroups and indistinguishability operators into fuzzy subgroups and indistinguishability operators respectively. We show necessary and sufficient conditions over two t-norms to guarantee that the fuzzy subgroup set induced by the first one is into the fuzzy subgroup set induced by the second one.

*Keywords:* fuzzy subgroup, indistinguishability operator.

## 1. Introduction

$t$-norms generalize the crisp intersection to the fuzzy framework. They are very useful in many fields of fuzzy logic and they are a key tool in the development of fuzzy subgroup theory and fuzzy relation theory.

Indistinguishability operators are a particular case of fuzzy relations. They extend crisp equivalence relations to the fuzzy framework and were introduced on [1] by Zadeh. They are also named fuzzy equivalences, fuzzy equalities or similarity relations in different contexts too. These operators have been widely studied and have been useful in many investigation fields such as fuzzy control, cluster analysis and approximate reasoning. An overview of these operators is shown on [2].

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The first definition of fuzzy subgroup was given by Rosenfeld in [3], fuzzying conditions for subgroup crisp. He defined a fuzzy subgroup $\mu$ of a group $G$ as a fuzzy set of $G$ such that $\mu(x) \geq \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. Many authors have extended this notion using an arbitrary t-norm $T$ instead of the minimum t-norm $T_M$. Das studied widely the fuzzy subgroup with the minimum t-norm and he characterized them (see [4]). By definition, it is easy to check that $\mu(x) = \mu(x^{-1})$ for all $x \in G$ and, for the minimum t-norm $T_M$, $\mu(e) \geq \mu(x)$ for all $x \in G$ where $e$ is the neutral element of $G$. For this reason, many authors impose $\mu(e) = 1$, this is equivalent to the normalized condition (there exist an element which attains the maximum possible membership degree). In this work, $G_T$ will denote a set of all fuzzy subgroup of a group $G$ regarding a t-norm $T$.

Fuzzy subgroups and indistinguishability operators can be related as it is shown on [5]: To every fuzzy subgroup $\mu$ of a group $G$ regarding to a t-norm $T$, it is possible to associate two $T$-indistinguishability operators: $E_\mu$ and ${}_\mu E$. These operators are right and left invariant under translations respectively. Conversely, for every right (left) invariant under translation $T$-indistinguishability operators $E$, a fuzzy subgroup regarding to the t-norm $T$ is associated.

To mention only one field where invariance under translations is usually assumed and needed, we can consider Fuzzy Mathematical Morphology [6]. In the study of objects in the plane $\mathbb{R}^2$, isotropy is usually assumed and hence the used relations must be invariant under translations. For one of these relations $R$, two of the basic operators, dilation $D$ and erosion $E$, can be defined by $D(\mu)(\vec{x}) = \sup_{\vec{y}\in\mathbb{R}^2} T(R(\vec{y}-\vec{x}), \mu(\vec{y}))$ and $E(\mu)(\vec{x}) = \inf_{\vec{y}\in\mathbb{R}^2} \overrightarrow{T}(\mu(\vec{y})|R(\vec{y}-\vec{x}))$. If $R$ is an indistinguishability operator, then the structural element is the fuzzy subgroup $R((0,0),\cdot)$ of $\mathbb{R}^2$.

In Section 2, the basic definitions and needed properties are shown. Section 3 focus our work using the minimum t-norm, denoted by $T_M$, and t-norms with an additive generator $t$. Given a group $G$ and a t-norm $T$ of one of the types mentioned before, we study the morphisms on fuzzy subgroup set $G_T$. Let $\mu$ be a fuzzy subgroup of $G$ regarding to $T$ and $f : [0,1] \longrightarrow [0,1]$ a function, we characterise when $f \circ \mu$ is a $T$-subgroup. On [7], a similar study was done for $T$-indistinguishability operator, we support our investigation with these known results. Due to the close relation between $T$-indistinguishability operators and $T$-subgroups, it is not difficult to think that the conditions on $f$, will be very similar. Section 4 studies the relationship between two fuzzy subgroup sets. Given a group $G$ and two t-norms $T$ and $T'$, the purpose is to compare the fuzzy subgroup sets $G_T$ and $G_{T'}$. As main result, we show conditions on the t-norms to characterize that $G_T \subset G_{T'}$.

## 2. Preliminares

We introduce the notions which will be used in this work.

**Definition 2.1** ([8]). *A function $t : [0, 1] \longrightarrow [0, \infty]$ is called additive generator of a t-norm $T$ if $t$ is right-continuous in 0, strictly decreasing function with $t(1) = 0$ and for all $(x, y) \in [0, 1]^2$ we have*

$$t(x) + t(y) \in \mathrm{Im}\ t \cup [t(0), \infty]$$

$$T(x, y) = t^{[-1]}(t(x) + t(y))$$

*where $t^{[-1]} : [0, \infty] \longrightarrow [0, 1]$ is the pseudo-inverse of $t$ defined by*

$$t^{[-1]}(x) = \begin{cases} t^{-1}(x) & \text{if} \quad x \leq t(0) \\ 0 & \text{if} \quad x > t(0) \end{cases}$$

Please note that $(t^{[-1]} \circ t)(x) = x$ and

$$(t \circ t^{[-1]})(x) = \begin{cases} x & \text{if} \quad x \leq t(0) \\ t(0) & \text{if} \quad x > t(0) \end{cases}$$

**Definition 2.2** ([7]). *Given a t-norm $T$, a $T$-indistinguishability operator $E$ on a set $X$ is a fuzzy relation on $X$ that satisfies the following conditions:*

$(E1)$ $\quad E(x, x) = 1$ *for all* $x \in X$.

$(E2)$ $\quad E(x, y) = E(y, x)$ *for all* $x, y \in X$.

$(E3)$ $\quad E(x,z) \geq T(E(x,y), E(y,z))$ *for all* $x, y, z \in X$.

**Definition 2.3** ([5]). *Let* $G$ *be a group and* $E$ *a fuzzy relation on* $G$. $E$ *is invariant under translations if*

$$E(x,y) = E(zx, zy) \qquad \textit{(left invariant)}$$

*and*

$$E(x,y) = E(xz, yz) \qquad \textit{(right invariant)}$$

*for all* $x, y, z \in G$.

**Definition 2.4.** *Given a group* $G$ *and a t-norm* $T$, *we say* $\mu$ *is a fuzzy subgroup of* $G$ *regarding the t-norm* $T$, *to shorten, a* $T$*-subgroup of* $G$ *if it satisfies:*

$(G1)$ $\quad \mu(e) = 1$, *where* $e$ *is the neutral element of* $G$.

$(G2)$ $\quad \mu(x) = \mu(x^{-1})$ *for all* $x \in G$.

$(G3)$ $\quad \mu(xy) \geq T(\mu(x), \mu(y))$ *for all* $x, y \in G$.

Given a group $G$ and a t-norm $T$, we will denote the set of all $T$-subgroups of $G$ by $G_T$, this means, $G_T = \{\mu \in [0,1]^G \mid \mu \text{ is a } T\text{-subgroup of } G\}$. On [5], $T$-subgroup notion and $T$-indistinguishability operator notion are related as follows:

**Definition 2.5** ([5]). *Let* $\mu$ *be a fuzzy set of a group* $G$. *The fuzzy relations* $E_\mu$ *and* ${}_\mu E$ *on* $G$ *defined by*

$$E_\mu(x,y) = \mu(xy^{-1}) \text{ for all } x, y \in G$$

*and*

$${}_\mu E(x,y) = \mu(y^{-1}x) \text{ for all } x, y \in G$$

*are the right and left fuzzy relations associated to* $\mu$ *respectively.*

**Proposition 2.6** ([5]). *Given a t-norm* $T$, *let* $\mu$ *be a* $T$*-subgroup of a group* $G$. *Then* $E_\mu$ *and* ${}_\mu E$ *are right and left invariant* $T$*-indistinghishability operators on* $G$ *respectively.*

**Proposition 2.7** ([5])**.** *Given a t-norm $T$, let $E$ be a right invariant $T$-indistinguishability operator on a group $G$ with identify element $e$. Then the fuzzy set $\mu_E$ of $G$ defined by $\mu_E(x) = E(e, x)$ for all $x \in G$ is a $T$-subgroup of $G$ and $E = E_{\mu_E}$.*

**Proposition 2.8** ([5])**.** *Given a t-norm $T$, let $E$ be a left invariant $T$-indistinguishability operator on a group $G$ with identify element $e$. Then the fuzzy set $\mu_E$ of $G$ defined by $\mu_E(x) = E(e, x)$ for all $x \in G$ is a $T$-subgroup of $G$ and $E = {}_{\mu_E}E$.*

As a consequence of these propositions, we obtain the following result.

**Proposition 2.9.** *Let $\mu$ a fuzzy set on a group $G$ and $T$ a t-norm. Then, $\mu$ is a $T$-subgroup if and only if $E_\mu$ (${}_\mu E$) is a right (left) invariant $T$-indisguishability operator.*

**Proposition 2.10** ([5])**.** *Let $G$ be a group, $f : [0,1]^m \longrightarrow [0,1]$ a function, $\mu_1, ..., \mu_m$ fuzzy set and ${}_{\mu_1}E, ..., {}_{\mu_m}E$ and $E_{\mu_1}, ..., E_{\mu_m}$ their right and left invariant fuzzy relations. Then*

$$f({}_{\mu_1}E, ..., {}_{\mu_m}E) = {}_{f(\mu_1,..,\mu_m)}E$$

*and*

$$f(E_{\mu_1}, ..., E_{\mu_m}) = E_{f(\mu_1,...,\mu_m)}$$

## 3. Relating Fuzzy Subgroups with respect to different t-norms

Given a group $G$ and two t-norms $T$ and $T'$, we study conditions on a function $f : [0,1] \longrightarrow [0,1]$ to assure that $f \circ \mu$ is a $T'$-subgroup, where $\mu$ is a $T$-subgroup of $G$. We begin considering the minimum t-norm case, $T_M$. The following result described in [7] will help us:

**Proposition 3.1** ([7])**.** *Let $E$ be a $T_M$-indistinguishability operator on a set $X$ and $f : [0,1] \longrightarrow [0,1]$ a map. $f \circ E$ is a $T_M$-indistinguishability operator on $X$ if and only if $f(1) = 1$ and $f$, restricted to* Im *$E$, is a non-decreasing function.*

Using Proposition 2.9 and Proposition 2.10, we obtain the following result.

**Corollary 3.2.** *Let $\mu$ be a $T_M$-subgroup of a group $G$ and $f : [0,1] \longrightarrow [0,1]$ a map. $f \circ \mu$ is a $T_M$-subgroup of $G$ if and only if $f(1) = 1$ and $f$, restricted to* Im $\mu$*, is a non-decreasing function.*

*Proof.* Consider $\mu$ a $T_M$-subgroup, $E_\mu$ is a right invariant $T_M$-indistinghishability operator. By Proposition 3.1, $f \circ E_\mu$ is a $T_M$-indistinguishability operator on $G$ if and only if $f(1) = 1$ and $f$ restricted to Im $E_\mu$ is a non-decreasing function. By Proposition 2.10, $f \circ E_\mu = E_{f\circ\mu}$. Hence, by Proposition 2.9, $f \circ \mu$ is a $T_M$-subgroup. Note that Im $E_\mu =$ Im $\mu$ by construcction of $E_\mu$. □

Below, let us consider general t-norms. Firstly, we provide a characterization of $T$-indistinguishability operators (Proposition 3.5). Secondly, given a $T$-indistinguishability operator, we analyse when $f \circ E$ is a $T'$-indistinguishability operator. Please note that $T$ and $T'$ can coincide. Later on, some examples with particular t-norms are provided.

**Definition 3.3** ([10])**.** *We say that a triplet $(a, b, c) \in [0, \infty]^3$ is a triangular triplet if*

$$a \leq b + c\ , \quad b \leq a + c \quad \textit{and} \quad c \leq a + b$$

**Definition 3.4** ([10])**.** *Given a t-norm $T$, we say that a triplet $(a, b, c) \in [0, \infty]^3$ is a $T$-triangular triplet if*

$$a \geq T(b, c)\ , \quad b \geq T(a, c) \quad \textit{and} \quad c \geq T(a, b)$$

**Proposition 3.5.** *Let $X$ be a set, $T$ a t-norm and $E$ a fuzzy relation on $X$ such that $E(x, x) = 1$ for all $x \in X$ and $E(x, y) = E(y, x)$ for all $x, y \in X$. The following assertions are equivalent:*

(1) *$E$ is a $T$-indistinguishability operator.*

(2) *$\{E(x, y), E(y, z), E(x, z)\}$ is a $T$-triangular triplet for all $x, y, z \in X$*

*Moreover, if $T$ has an additive generator $t$, above assertions are equivalent to:*

(3) $\{t \circ E(x,y), t \circ E(y,z), t \circ E(x,z)\}$ *is a triangular triplet for all* $x, y, z \in X$

*Proof.*

(2) $\Rightarrow$ (1). Straightforward: Given $x, y, z \in X$, $E(x,z) \geq T(E(x,y), E(y,z))$ because $\{E(x,y), E(y,z), E(x,z)\}$ is a $T$-triangular triplet, hence $E$ is a $T$-indistinguishability operator.

(1) $\Rightarrow$ (2). Given $x, y, z \in X$, since $E$ is a $T$-indistinguishability operator, we have

$$E(x,z) \geq T(E(x,y), E(y,z))$$

$$E(x,y) \geq T(E(x,z), E(z,y)) = T(E(x,z), E(x,y))$$

$$E(y,z) \geq T(E(y,x), E(x,z)) = T(E(x,y), E(x,z))$$

Hence $\{E(x,y), E(y,z), E(x,z)\}$ is a $T$-triangular triplet.

Now, suppose that $T$ has an additive generator $t$.

(2) $\Leftrightarrow$ (3). Take $x, y, z \in X$.

$$T(E(x,y), E(y,z)) \leq E(x,z) \Rightarrow$$

$$t^{[-1]}(t \circ E(x,y) + t \circ E(y,z)) \leq E(x,z) \Rightarrow$$

$$t \circ t^{[-1]}(t \circ E(x,y) + t \circ E(y,z)) \geq t \circ E(x,z) \Rightarrow$$

$$t \circ E(x,y) + t \circ E(y,z) \geq t \circ E(x,z)$$

Conversely,

$$t \circ E(x,y) + t \circ E(y,z) \geq t \circ E(x,z) \Rightarrow$$

$$t^{[-1]}(t \circ E(x,y) + t \circ E(y,z)) \leq t^{[-1]} \circ t \circ E(x,z) = E(x,z) \Rightarrow$$

$$T(E(x,y), E(y,z)) \leq E(x,z)$$

Similarly,

$$T(E(x,y), E(x,z)) = T(E(y,x), E(x,z)) \leq E(y,z) \Leftrightarrow$$

$$t \circ E(x,y) + t \circ E(x,z) \geq t \circ E(y,z)$$

and

$$T(E(x,z), \circ E(y,z)) = T(E(x,z), E(z,y)) \leq E(x,y) \Leftrightarrow$$

$$t \circ E(x,z) + t \circ E(y,z) \geq t \circ E(x,y)$$

□

**Corollary 3.6.** *Let $T$ and $T'$ be two t-norms and $t$ an additive generator of $T'$. If $E$ is a $T$-indistinguishability operator on a set $X$ and $f : [0,1] \longrightarrow [0,1]$ a function, then, $f \circ E$ is a $T'$-indistinguishability operator on $X$ if and only if:*

*$f(1) = 1$ and*

*$f = t^{[-1]} \circ \varphi \circ t : \mathrm{Im}\ E \longrightarrow [0,1]$, where $\varphi : \mathrm{Im}\ (t \circ E) \longrightarrow [0, t(0)]$ and*

$$\{\varphi \circ t(E(x,y)), \varphi \circ t(E(y,z)), \varphi \circ t(E(x,z))\}$$

*is a triangular triplet for all $x, y, z \in X$.*

*Proof.* Taking $\varphi = t \circ f \circ t^{[-1]}$ and using Proposition 3.5, the proof is complete.

□

A similar results for $T$-subgroups defined on a group $G$ can be proved.

**Proposition 3.7.** *Let $G$ be a group, $T$ a t-norm and $\mu$ a fuzzy set of $G$ such that $\mu(e) = 1$ and $\mu(x) = \mu(x^{-1})$. The following assertions are equivalent:*

(1) *$\mu$ is a $T$-subgroup*

(2) *$\{\mu(x), \mu(y), \mu(xy)\}$ is a $T$-triangular triplet for all $x, y \in G$*

*Moreover, if $T$ has an additive generator $t$, above assertions are equivalent to:*

(3) *$\{t\mu(x), t\mu(y), t\mu(xy)\}$ is a triangular triplet for all $x, y \in G$*

**Proposition 3.8.** *Let $T$ and $T'$ be two t-norms and $t$ an additive generator of $T'$. If $\mu$ is a $T$-subgroup of a group $G$ and $f : [0,1] \longrightarrow [0,1]$ a function, then $f \circ \mu$ is a $T'$-subgroup of $G$ if and only if:*

*$f(1) = 1$ and*

*$f = t^{[-1]} \circ \varphi \circ t : \mathrm{Im}\ \mu \longrightarrow [0,1]$, where $\varphi : \mathrm{Im}\ (t \circ \mu) \longrightarrow [0, t(0)]$ and*

$$\{\varphi \circ t(\mu(x)), \varphi \circ t(\mu(y)), \varphi \circ t(\mu(xy))\}$$

*is a triangular triplet for all $x, y \in G$.*

Below, different ways to obtain $T'$-subgroups from other $T$-subgroups with $T$ and $T'$ not necessarily equal will be provided. Interesting is the case where the two t-norms are isomorphic. Also when $T = T'$, it will be shown that the powers of given $T$-subgroup are similar $T$-subgroups accordingly to the definition in this section. We provide several examples to illustrate our results.

**Definition 3.9** ([9])**.** *Given a t-norm $T$ and $x \in [0,1]$, we define $x_T^{(n)}$ recursively:*

$$x_T^{(1)} = x \qquad \textit{and} \qquad x_T^{(n)} = T(x_T^{(n-1)}, x)$$

In some contexts beyond this work, if the t-norm is clear $x_T^{(n)}$ is denoted by $x^{(n)}$, but we always denote it by the first.

**Definition 3.10** ([9])**.** *Given a continuous t-norm $T$ and $x \in [0,1]$, the $n$-th root $x_T^{(\frac{1}{n})}$ of $x$ with respect to $T$ is defined by*

$$x_T^{(\frac{1}{n})} = \sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

*and for $m, n \in \mathbb{N}$, $x_T^{(\frac{m}{n})} = (x_T^{(\frac{1}{n})})^{(m)}$.*

**Lemma 3.11.** *If $k, m, n \in \mathbb{N},\ k, n \neq 0$ then $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.*

Assuming continuity for the t-norm $T$, the powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

**Definition 3.12** ([9])**.** *If $r \in \mathbb{R}^+$ is a positive real number, let $(a_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers with $\lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is*

$$x^{(r)} = \lim_{n\to\infty} x^{(a_n)}.$$

Usual continuity assures the existence of this limit and independence of the sequence $(a_n)_{n\in\mathbb{N}}$.

**Proposition 3.13** ([9])**.** *Let $T$ be a continuous Archimedean t-norm with additive generator $t$, $x \in [0,1]$ and $r \in \mathbb{R}^+$. Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Given a t-norm $T$ and a $T$-subgroup $\mu$ of a group $G$, $\mu_T^{(n)}$ is defined by

$$\mu_T^{(n)}(x) = \mu(x)_T^{(n)}.$$

**Proposition 3.14.** *If $\mu$ is a $T$-subgroup of a group $G$, $T$ a continuous t-norm, and $r \in \mathbb{R}^+$, then $\mu_T^{(r)}$ is a $T$-subgroup of $G$.*

*Proof.*

$(G1)$ $\mu_T^{(r)}(e) = 1$ because $1_T^{(r)} = 1$.

$(G2)$ $\mu_T^{(r)}(x) = \mu(x)_T^{(r)} = \mu(x^{-1})_T^{(r)} = \mu_T^{(r)}(x^{-1})$.

$(G3)$ Consider first the case $r = \frac{1}{n}$ with $n \in \mathbb{N}$. If $\mu_T^{(\frac{1}{n})} = \nu$, then $\nu_T^{(n)} = \mu$.

$$T(\nu(x), \nu(y))_T^{(n)} = T(\nu_T^{(n)}(x), \nu_T^{(n)}(y)) \leq \nu_T^{(n)}(xy)$$

because $\mu$ is a $T$-subgroup of $G$. Hence

$$(T(\nu(x), \nu(y))_T^{(n)})_T^{(\frac{1}{n})} \leq (\nu_T^{(n)}(xy))_T^{(\frac{1}{n})}$$

and thanks to Lemma 3.11,

$$T(\nu(x), \nu(y)) \leq \nu(xy)$$

this is,

$$T(\mu_T^{(\frac{1}{n})}(x), \mu_T^{(\frac{1}{n})}(y)) \leq \mu_T^{(\frac{1}{n})}(xy).$$

Now, take $r = \frac{m}{n}$ with $m, n \in \mathbb{N}$. Remark that $\mu_T^{(\frac{m}{n})} = (\mu_T^{\frac{1}{n}})^{(m)}$, since $\mu_T^{(\frac{1}{n})}$ is $T$-subgroup,

$$T(\mu_T^{(\frac{1}{n})}(x), \mu_T^{(\frac{1}{n})}(y)) \leq \mu_T^{(\frac{1}{n})}(xy).$$

for all $x, y \in [0, 1]$. This implies that

$$(T(\mu_T^{(\frac{1}{n})}(x), \mu_T^{(\frac{1}{n})}(y)))^{(m)} \leq \mu_T^{(\frac{1}{n})}(xy)^{(m)}$$

this is,

$$T(\mu_T^{(\frac{1}{n})}(x)^{(m)}, \mu_T^{(\frac{1}{n})}(y)^{(m)}) \leq \mu_T^{(\frac{1}{n})}(xy)^{(m)}$$

and by Definition 3.10, we conclude that

$$T(\mu_T^{(\frac{m}{n})}(x), \mu_T^{(\frac{m}{n})}(y)) \leq \mu_T^{(\frac{m}{n})}(xy)$$

From continuity for all $r > 0$ the result is also valid.

$\square$

**Proposition 3.15.** *Let $T$ and $T'$ be continuous Archimedean t-norms with additive generators $t$ and $s$ respectively. If $\mu$ is a $T$-subgroup of a group $G$, then $\mu' = s^{[-1]} \circ t \circ \mu$ is a $T'$-subgroup of $G$.*

*Proof.* It is trivial to prove that $\mu'(e) = 1$ and that $\mu'(x) = \mu'(x^{-1})$.

Let us prove $T'(\mu'(x), \mu'(y)) \leq \mu'(xy)$. It is equivalent to prove that $s(\mu'(x)) + s(\mu'(y)) \geq s(\mu(xy))$.

- If $t(\mu(x)) \leq s(0)$ and $t(\mu(y)) \leq s(0)$, then

$$\begin{aligned} s(\mu'(x)) + s(\mu'(y)) &= s(s^{[-1]}(t(\mu(x)))) + s(s^{[-1]}(t(\mu(y)))) \\ &= s(s^{-1}(t(\mu(x)))) + s(s^{-1}(t(\mu(y)))) \\ &= t(\mu(x)) + t(\mu(y)) \\ &\geq t(\mu(xy)) \\ &\geq s(s^{[-1]}(t(\mu(xy))) \end{aligned}$$

- If $t(\mu(x)) \geq s(0)$, then $\mu'(x) = 0$ and $T'(\mu'(x), \mu'(y)) \leq \mu'(xy)$ is satisfied.

- If $t(\mu(y)) \geq s(0)$ is similar to the previous case.

$\square$

**Example 3.16.** *Let $T$ be the Łukasiewicz t-norm and $\mu$ a $T$-subgroup of a group $G$. Then $e^{\mu-1}$ is a $T'$-subgroup of $G$ where $T'$ is the Product t-norm and $e^{\mu-1}$ is defined by $e^{\mu-1}(x) = e^{\mu(x)-1}$.*

The way $\mu'$ is obtained is not canonical, but depends on the additive generators of $T$ and $T'$.

**Proposition 3.17.** *Let $t$ and $t'$ be additive generators of a continuous t-norm $T$ such that $t' = \alpha t$ with $\alpha > 0$ and $\mu$ a $T$-subgroup of a group $G$. Then $t'^{[-1]} \circ t \circ \mu = \mu_T^{(\frac{1}{\alpha})}$.*

*Proof.* Using Proposition 3.13, we obtain for each $x \in G$,

$$t'^{[-1]} \circ t \circ \mu(x) = t^{[-1]}(\frac{1}{\alpha} t(\mu(x))) = \mu(x)_T^{(\frac{1}{\alpha})} = \mu_T^{(\frac{1}{\alpha})}(x)$$

□

Please note that this is an alternative proof of Proposition 3.14.

**Definition 3.18** ([8])**.** *Two continuous t-norms $T, T'$ are isomorphic if and only if there exists a bijective map $f : [0,1] \to [0,1]$ such that $f \circ T = T' \circ (f \times f)$.*

- Isomorphisms $f$ are continuous and increasing maps.
- It is well known ([8]) that all strict continuous Archimedean t-norms are isomorphic. In particular, they are isomorphic to the Product t-norm.
- Also, all non-strict continuous Archimedean t-norms are isomorphic ([8]). In particular, they are isomorphic to the Łukasiewicz t-norm.

The next proposition relates the isomorphisms of continuous Archimedean t-norms with their additive generators.

**Proposition 3.19.** *Let $f$ be a bijective map $f : [0,1] \to [0,1]$, $T$, $T'$ two continuous Archimedean t-norms and $t$, $t'$ additive generators of $T$ and $T'$ respectively. $f$ is an isomorphism between $T$ and $T'$ if and only if there exists $\alpha \in (0,1]$ such that $f = t'^{[-1]}(\alpha t)$.*

*Proof.* $\forall x, y \in [0,1]$, $f(T(x,y)) = T'(f(x), f(y))$

is equivalent to

$$\begin{aligned} f \circ t^{[-1]}(t(x) + t(y)) &= t'^{[-1]}((t' \circ f)(x) + (t' \circ f)(y)) \\ t^{[-1]}(t(x) + t(y)) &= (f^{-1} \circ t'^{[-1]})((t' \circ f)(x) + (t' \circ f)(y) \end{aligned}$$

which means that $f$ is an isomorphism between $T$ and $T'$ if and only if $t' \circ f$ is an additive generator of $T$. Since two additive generators of a t-norm differ only by a multiplicative positive constant, $t = k(t' \circ f)$ with $k > 0$ and putting $\alpha = 1/k$, we have $t = k(t' \circ f)$ is equivalent to $\alpha t = t' \circ f$. Using that $t'^{[-1]} \circ t'(z) = z$ for all $z \in [0,1]$, we conclude that

$$f = t'^{[-1]} \alpha t.$$

□

**Example 3.20.**

- *The only automorphism of the Łukasiewicz t-norm is the identity map.*

  *Indeed, taking* $t(x) = 1 - x$, *then* $f(x) = 1 - \alpha + \alpha\, x$ *and the only bijective linear map in [0,1] is the identity.*

- *The automorphisms of the Product t-norm are* $f(x) = x^\alpha$ *with* $\alpha > 0$.

*More general,*

- *The only automorphism of a non-strict Archimedean t-norm is the identity map*

- *For strict t-norms, every* $\alpha > 0$ *produces an isomorphism* $f_\alpha$ *with* $f_\alpha \neq f_\beta$ *if* $\alpha \neq \beta$.

The next proposition shows that t-norm isomorphisms preserve indistinguishability operators.

**Proposition 3.21** ([7]). *If $E$ is a $T$-indistinguishability operator on a set $X$ for a given t-norm $T$ and $f$ is a continuous, increasing and bijective map $f : [0,1] \longrightarrow [0,1]$, then $f \circ E$ is a $T'$-indistinguishability operator with respect to the t-norm $T' = f \circ T \circ (f^{-1} \times f^{-1})$.*

Note that if $f$ is an increasing and bijective map, then $f$ is a continuous map.

**Corollary 3.22.** *Let $G$ be a group, $\mu$ a $T$-subgroup of $G$ for a given t-norm $T$ and $f$ an increasing and bijective map $f:[0,1]\to[0,1]$. Then $f\circ\mu$ is a $T'$-subgroup of $G$ where $T'=f\circ T\circ(f^{-1}\times f^{-1})$.*

*Proof.* Since $\mu$ is a $T$-subgroup, we have $E_\mu$ is a right invariant $T$-indistinghishability. By Proposition 3.21, $f\circ E_\mu$ is a $T'$-indistinguishability operator on $G$ and by Proposition 2.10, $f\circ E_\mu=E_{f\circ\mu}$. Hence, by Proposition 2.9, $f\circ\mu$ is a $T'$-subgroup. □

Below is shown several interesting examples.

**Example 3.23.** *Let $\mu$ be a $T$-subgroup of a group $G$ with respect to a t-norm $T$ and $f_\alpha(x)=x^\alpha$ for some $\alpha>0$. Then $f_\alpha\circ\mu$ is a $T_\alpha$-subgroup of $G$ with*

$$T_\alpha(x,y)=\left(T\left(x^{\frac{1}{\alpha}},y^{\frac{1}{\alpha}}\right)\right)^\alpha.$$

*If $T$ is the Łukasiewicz t-norm, then $(T_\alpha)_{\alpha>0}$ is the Schweizer-Sklar family of t-norms*

$$T_\alpha(x,y)=\left(\max\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}}-1,0\right)\right)^\alpha$$

*Therefore, given a member $T_\alpha$ of this family and a $T_\alpha$-subgroup $\mu_\alpha$, it is easy to find a similar $T_\beta$-subgroup $\mu_\beta$ of $G$.*

**Example 3.24.** *Let $\mu$ be a $T$-subgroup of a group $G$ with respect to a non-strict continuous Archimedean t-norm $T$ with normalized additive generator $t$, this means $t(0)=1$. Taking $f(x)=1-(t(x))^\alpha$, $\alpha>0$, then*

$$f\circ\mu(x)=1-(t(\mu(x)))^\alpha$$

*is a $T_\alpha$-subgroup of $G$ where $T_\alpha$ is the non-strict Archimedean t-norm*

$$T_\alpha(x,y)=1-\min((1-x^{\frac{1}{\alpha}})+(1-y)^{\frac{1}{\alpha}},1)^\alpha.$$

*Let us observe that, in this case, the family $\{T_\alpha\}_{\alpha>0}$, known as Yager family, is independent from the original t-norm $T$, more precisely, from the generator $t$.*

*In particular, for $\alpha=1$, $T_\alpha$ is the Łukasiewicz t-norm, and the fuzzy set $\mu_1=1-t(\mu)$ is a $T_1$-subgroup.*

Please also note that thanks to Proposition 3.17 the powers of a $T$-subgroup are similar.

The following results show some ways to obtain min-subgroups from a given one.

**Corollary 3.25.** *Let $\mu$ be a* min-*subgroup of a group $(G, \circ)$, $\varphi$ a strong negation and $t$ an additive generator of a continuous Archimedean t-norm. Then $t^{[-1]} \circ \varphi \circ \mu$ is a* min-*subgroup of $G$.*

*Proof.* $t^{[-1]}$ and $\varphi$ are strict decreasing bijections; so their composition is a non increasing function $f$ with $f(1) = 1$. Using Corollary 3.2, we conclude the proof. $\square$

For example, if $\mu$ is a min-subgroup of $G$, then taking the additive generator $t(x) = -\ln x$ of the Product t-norm and $\varphi(x) = 1 - x$, then $e^{\mu - 1}$ is a min-subgroup of $G$.

Taking $t(x) = \alpha(1 - x)$, $\alpha > 0$ and $\varphi(x) = 1 - x$, we get that $\max(0, \frac{\alpha - 1 + \mu}{\alpha})$ is a min-subgroup of $G$.

**Corollary 3.26.** *Let $\mu$ be a* min-*subgroup of $G$ and $\varphi$, $\psi$ two strong negations. The fuzzy subset $\nu$ of $G$ defined by $\nu = \varphi \circ \psi \circ \mu$ is a* min-*subgroup of $G$.*

*Proof.* $\varphi$ and $\psi$ are strict decreasing bijections; so their composition is a non increasing function $f$ with $f(1) = 1$. Using Corollary 3.2, we conclude the proof. $\square$

## 4. On the relationship between the fuzzy subgroup sets of two t-norms

Given two t-norms $T$ and $T'$ and a group $G$, the aim of this section is to find a relationship between $G_T$ and $G_{T'}$. More concretely, we characterize the fact $G_T \subset G_{T'}$ under certain conditions on $T$ and $T'$. Remember that $G_T = \{\mu \in [0,1]^G \mid \mu$ is a $T$-subgroup of $G\}$ and $G_{T'} = \{\mu \in [0,1]^G \mid \mu$ is a $T'$-subgroup of $G\}$

**Lemma 4.1.** *Given a t-norm $T$,a $T$-subgroup $\mu$ of a group $G$ and $k \in \mathbb{N}$, we have*

$$\mu(x^k) \geq x_T^{(k)}$$

*Proof.* If $k = 1$, $\mu(x) = \mu(x)_T^{(1)}$. Suppose it is true for $k-1$, this is, $\mu(x^{k-1}) \geq \mu(x)_T^{(k-1)}$. For $k$, using the hypothesis and $\mu$ is a $T$-subgroup, we have:

$$\mu(x^k) \geq T(\mu(x^{k-1}), \mu(x)) \geq T(\mu(x)_T^{(k-1)}, \mu(x)) = \mu(x)_T^{(k)}$$

□

**Lemma 4.2.** *Given a t-norm $T$ and $m \in \mathbb{N}$ greater than or equal to 2, we have*

$$a_T^{(m)} \geq T(a_T^{(n_1)}, a_T^{(n_2)})$$

*with $n_1 + n_2 = m$ for all $a \in [0,1]$.*

*Proof.* Let $a$ be in $[0,1]$. For $m = 2$, we have that $n_1 = n_2 = 1$, $a_T^{(2)} = T(a,a) = T(a_T^{(1)}, a_T^{(1)})$. For induction, suppose it is true for $m-1$, this is, $a_T^{(m-1)} \geq T(a_T^{(m_1)}, a_T^{(m_2)})$ for all $m_1, m_2$ such that $m_1 + m_2 = m-1$. Consider $n_1, n_2$ arbitrary numbers such that $n_1 + n_2 = m$, then

$$a_T^{(m)} = T(a_T^{(m-1)}, a) \geq T(T(a_T^{(n_1)}, a_T^{(n_2)}), a) =$$

$$= T(a_T^{(n_1)}, T(a_T^{(n_2-1)}, a)) = T(a_T^{(n_1)}, a_T^{(n_2)})$$

□

**Lemma 4.3.** *Given a t-norm $T$ and $n \in \mathbb{N}$ greater than or equal to 2, we have*

$$a_T^{(n)} \geq T(a_T^{(n_1)}, a_T^{(n_2)})$$

*with $n_1 + n_2 \geq n$ for all $a \in [0,1]$.*

*Proof.* If $m = n_1 + n_2$, by Lemma 4.2, $a_T^{(m)} \geq T(a_T^{(n_1)}, a_T^{(n_2)})$ for all $a \in [0,1]$. Since $a_T^{(n)} \geq a_T^{(m)}$ because $n \leq m$, we conclude

$$a_T^{(m)} \geq T(a_T^{(n_1)}, a_T^{(n_2)})$$

□

**Lemma 4.4.** *Let $T$ and $T'$ two t-norms and $G$ a group, if $T'(x,y) \leq T(x,y)$ for all $x,y \in [0,1]$, then every $T$-subgroup is a $T'$-subgroup.*

*Proof.* Straightforward. $\square$

As $T_M \geq T \geq T_D$ for any t-norm $T$ where $T_D$ denotes a drastic t-norm, we conclude that

$$G_{T_M} \subset G_T \subset G_{T_D}$$

for any t-norm $T$.

**Lemma 4.5.** *Let $G$ the cyclic group of order 2 or 3. Then a fuzzy set $\mu$ of $G$ is a $T$-subgroup for any t-norm $T$ if and only if $\mu$ is built as follows:*

$$\mu(t) = \begin{cases} 1 \quad if \quad t = e \\ z \qquad\qquad \textit{otherwise} \end{cases}$$

*for some element $z \in [0,1]$.*

*Proof.* If $\mu$ is a $T$-subgroup for each t-norm $T$, $\mu(e) = 1$. If $G = \mathbb{Z}_3$, by $(G2)$, $\mu(a) = \mu(a^2) = z$ for some $z \in [0,1]$. Reciprocally, if $\mu$ is defined as

$$\mu(t) = \begin{cases} 1 \quad if \quad t = e \\ z \qquad\qquad \text{otherwise} \end{cases}$$

then $\mu$ satisfies $(G1)$, $(G2)$ and $(G3)$. $\square$

**Lemma 4.6.** *Let $T$ and $T'$ two t-norms and $G$ the cyclic group of $p$ elements, where $p$ is a prime greater than or equal to 7. If there is $x \in [0,1]$ such that $T'(x,x) > T(x,x)$, then there exists a $T$-subgroup $\mu$ of $G$ such that $\mu$ is not a $T'$-subgroup.*

*Proof.* We define the next $T$-subgroup:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ x & if \quad t \in \{a, a^{p-1}\} \\ T(x,x) & \quad \text{otherwise} \end{cases}$$

$(G1)$ and $(G2)$ are satisfied by $\mu$. We check $(G3)$ for the t-norm $T$, this is:

$$\mu(rs) \geq T(\mu(r), \mu(s)) \text{ for all } r, s \in G$$

If $r = e$ or $s = e$, it is straightforward. In other case, we have that

$$T(\mu(r), \mu(s)) \in \{T(x,x), T(x, T(x,x), T(T(x,x), T(x,x))\}$$

Since $\mu(rs) \in \{x, T(x,x)\}$, we conclude that $\mu(rs) \geq T(\mu(r), \mu(s))$ $\square$

The next theorem determines when $G_T \subset G_{T'}$ for any group $G$ and t-norms $T$ and $T'$.

**Theorem 4.7.** *Let $T$ and $T'$ t-norms and $G$ a group.*

(1) *If $G$ is a cyclic group of two or three elements, then $G_T \subset G_{T'}$.*

(2) *If $G$ is a cyclic group of four or five elements, then $G_T \subset G_{T'}$ if and only if $T'(x,x) \leq T(x,x)$ for all $x \in [0,1]$.*

(3) *If $G$ is a cyclic group of $p$ elements, where $p$ is a prime greater than $5$, then $G_T \subset G_{T'}$ if and only if for all $x, y \in [0,1]$ such that $T(x,y) < T'(x,y)$ with $x \geq y$, the inequality $y < x_T^{(\frac{p-1}{2})}$ is satisfied.*

(4) *If $G$ is a cyclic group which is not isomorphic to $\mathbb{Z}_p$ for some prime number $p$, $G \neq \mathbb{Z}_4$ or if $G$ is not a cyclic group, then $G_T \subset G_{T'}$ if and only if for all $x, y \in [0,1]$, $T'(x,y) \leq T(x,y)$.*

*Proof.* (1) By Lemma 4.5, we have that every $T$-subgroup is $T'$-subgroup. More exactly, $G_T = G_{T'}$.

(2) $| \Longrightarrow |$ Equivalently, we show that if there is $x \in [0,1]$ with $T(x,x) < T'(x,x)$ then there exists a $T$-subgroup $\mu$ such that $\mu$ is not a $T'$-subgroup. Take $x \in [0,1]$ with $T'(x,x) > T(x,x)$. If $G = \mathbb{Z}_4$, we consider the fuzzy set $\mu$ defined by:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ x & if \quad t \in \{a, a^3\} \\ T(x,x) & if \quad t = a^2 \end{cases}$$

and if $G = \mathbb{Z}_5$, we consider the fuzzy set $\mu$ defined by:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ x & if \quad t \in \{a, a^4\} \\ T(x,x) & if \quad t \in \{a^2, a^3\} \end{cases}$$

It is easy to prove that $\mu$ is $T$-subgroup of $G$, but $\mu$ is not a $T'$-subgroup because

$$\mu(a^2) = T(x,x) < T'(x,x) = T'(\mu(a), \mu(a))$$

$| \Longleftarrow |$ Conversely, suppose that for all $x \in [0,1]$, $T'(x,x) \leq T(x,x)$. By reductio ad absurdum, suppose that there exists a $T$-subgroup $\mu$ such that $\mu$ is not $T'$-subgroup. Since $\mu$ satisfies $(G1)$ and $(G2)$, for $G = \mathbb{Z}_4$ we have $\mu$ is the following fuzzy set:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ z_1 & if \quad t \in \{a, a^3\} \\ z_2 & if \quad t = a^2 \end{cases}$$

for some $z_1, z_2 \in [0,1]$. And for $G = \mathbb{Z}_5$, we have $\mu$ is the following fuzzy set:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ z_1 & if \quad t \in \{a, a^4\} \\ z_2 & if \quad t \in \{a^2, a^3\} \end{cases}$$

for some $z_1, z_2 \in [0,1]$. Now, we show that $\mu$ is $(G3)$ for $T'$: $\mu(rs) \geq T(\mu(r), \mu(s))$ for all $r, s \in G$. Take $r, s \in G$, if $r = e$ or $s = e$, then $\mu(rs) \geq T(\mu(r), \mu(s))$ is true. Suppose $r \neq e$ and $s \neq e$. If $\mu(r) = \mu(s)$ by hypothesis we have $T'(\mu(r), \mu(s)) \leq T(\mu(r), \mu(s)) \leq \mu(rs)$. If $\mu(r) \neq \mu(s)$, using Im $\mu \in \{z_1, z_2, 1\}$ we have $T'(\mu(r), \mu(s)) = T'(z_1, z_2) \leq \min\{z_1, z_2\} \leq \mu(rs)$. Hence $\mu$ is a $T'$-subgroup of $G$. This is a contradiction.

(3) Equivalently, we show that there are $x, y \in [0,1]$ with $T(x,y) < T'(x,y)$ with $x \geq y$ and $y \geq x_T^{(k)}$ for some $k \in \{1, ..., \frac{p-1}{2}\}$ if and only if there exists a $T$-subgroup $\mu$ such that $\mu$ is not a $T'$-subgroup.

Since $G$ is the cyclic group $(\mathbb{Z}_p, +)$ for $p \geq 7$ with $p$ a prime number, we use the additive notation in the proof and the neutral element will be the number 0. $| \Longrightarrow |$ Suppose that $\mu$ is a $T$-subgroup of $G$ but it is not a $T'$-subgroup.

Then, there are $a, b \in G$ such that $\mu(a+b) < T'(\mu(a), \mu(b))$. Since $\mu(0) = 1$, because $\mu$ is a $T$-subgroup, we have that $a \neq 0$ and $b \neq 0$. Without loss of generality, suppose $\mu(a) \geq \mu(b)$ and consider $x = \mu(a)$ and $y = \mu(b)$. We have that $x, y \in [0,1]$ with $x \geq y$. Moreover,

$$T(x,y) = T(\mu(a), \mu(b)) \leq \mu(a+b) < T'(\mu(a), \mu(b)) = T'(x,y).$$

Since $a \neq 0$ and $G$ is cyclic, $G = \langle a \rangle$. We have $b = na$ with $n \in \{1, ..., p\}$. Using that $\mu$ satisfies $(G2)$, we obtain that $\mu(b) = \mu(na) = \mu((p-n)a)$. We distinguish two cases: (1.) $n \leq p-n$ or (2.) $p-n \leq n$. If (1.), then $n \in \{1, ..., \frac{p-1}{2}\}$. If (2.), $p-n \leq n \Rightarrow p-n \leq \frac{p}{2} \Rightarrow p-n \in \{1, ..., \frac{p-1}{2}\}$.

Take $k = n$ if (1) or $k = p-n$ if (2), using Lemma 4.1, we conclude

$$y = \mu(b) = \mu(ka) \geq \mu(a)_T^{(k)} = x_T^{(k)}$$

with $k \in \{1, ..., \frac{p-1}{2}\}$.

$| \Longleftarrow |$ Our hypothesis is that there exist $x, y \in [0,1]$ satisfying $T(x,y) < T'(x,y)$, $x \geq y$ and $y \geq x_T^{(k)}$ for some $k \in \{1, ..., \frac{p-1}{2}\}$. If $x = y$, using Lemma 4.6, the proof is completed. For this observation, in the rest of the proof we suppose $x > y$. Consider $m = \min\{s \in \{1, ..., \frac{p-1}{2}\} \mid x_T^{(s)} \leq y\}$, note that $\{s \in \{1, ..., \frac{p-1}{2}\} \mid x_T^{(s)} \leq y\}$ is a non-empty finite set, so we can ensure the existence of $m$. Moreover, $x_T^{(s)} > y$ if and only if $s < m$. Now, we are going to show that $m \geq 2$: If $m = 1$, $x_T^{(1)} \leq y \Rightarrow x \leq y$, but $x > y$. Hence, $m \geq 2$. We define the fuzzy set $\mu$ as follows:

$\mu(0) = 1$

$\mu(n) = \mu(p-n) = x_T^{(n)}$ when $n \in \{1, ..., m-1\}$

$\mu(m) = \mu(p-m) = y$

$\mu(n) = \mu(p-n) = T(x,y)$ when $n \in \{m+1, ..., \frac{p-1}{2}\}$

We would like to remark that $T(x,y) \geq x_T^{(k)}$ for all $k \geq m+1$, this fact will be used in some parts of the proof. By construction, $\mu$ fulfills $(G1)$ and $(G2)$. Moreover,

$$T(x,y) < y < x < 1$$

because $T(x,y) < T'(x,y) \leq y$ and if $x = 1$, then $y = T(1,y) = T(x,y) < T'(x,y) = T'(1,y) = y$. For $(G3)$, we need to prove $\mu(a+b) \geq T(\mu(a),\mu(b))$ for all $a, b \in G$. It is easy to check that if $a = 0$, $b = 0$ or $a+b = 0$ the condition is straightforward. We suppose $a \neq 0$, $b \neq 0$ and $a+b \neq 0$ and consider five cases:

(1.) $a+b = n$ with $n \in \{1, ..., m-1\}$

(2.) $a+b = n$ with $n \in \{p-(m-1), ..., p-1\}$

(3.) $a+b = n$ with $n \in \{m+1, ..., p-(m+1)\}$

(4.) $a+b = m$

(5.) $a+b = p-m$

(1.) If $a+b = n$ with $n \in \{1, ..., m-1\}$, consider $n_1, n_2 \in \{1, ..., p-1\}$ which are representative elements of the class of $a$ and the class of $b$ respectively, then $n_1 + n_2 = n$ $(mod\ p)$ and $\mu(n) = x_T^{(n)}$. We have two possibilities: (1.1) $n_1 + n_2 = n$ or (1.2) $n_1 + n_2 = n + p$. If (1.1) happens, $n_1 < n$ and $n_2 < n$, hence $\mu(n_1) = x_T^{(n_1)}$ and $\mu(n_2) = x_T^{(n_2)}$. We conclude using Lemma 4.3 that

$$\mu(n) = x_T^{(n)} \geq T(x_T^{(n_1)}, x_T^{(n_2)}) = T(\mu(n_1), \mu(n_2))$$

If (1.2) happens, $n_1 \geq \frac{p-1}{2}$ or $n_2 \geq \frac{p-1}{2}$. Without loss of generality, $n_2 \geq \frac{p-1}{2}$, then $\mu(n_2) = x_T^{(p-n_2)}$. If $n_1 \in \{1, ..., \frac{p-1}{2}\}$, since

$$n_1 + n_2 = n + p \Leftrightarrow p - n_2 = n_1 - n$$

and $p - n_2 > 0$, $n_1 - n > 0$. We have that

$$\mu(n) = x_T^{(n)} \geq x_T^{(n_1)} \geq T(x_T^{(n_1)}, x_T^{(p-n_2)}) = T(\mu(n_1), \mu(n_2))$$

If $n_1 \in \{\frac{p+1}{2}, ..., p-1\}$, we have that $\mu(n_1) = x_T^{(p-n_1)}$ and since

$$n_1 + n_2 = n + p \Leftrightarrow -n = p - n_1 - n_2 \Leftrightarrow p - n = p - n_1 + p - n_2$$

we obtain that $n \leq p - n = p - n_1 + p - n_2$. Using again Lemma 4.3, we conclude that

$$\mu(n) = x_T^{(n)} \geq T(x_T^{(p-n_1)}, x_T^{(p-n_2)}) = T(\mu(n_1), \mu(n_2)).$$

(2.) If $a+b = n$ with $n \in \{p-(m-1), ..., p-1\}$, the procedure is the same described in (1) due to the observation $\mu(n) = \mu(p-n)$.

(3.) If $a+b=n$ with $n \in \{m+1, ..., p-(m+1)\}$, consider $n_1, n_2 \in \{1, ..., p-1\}$ which are representative elements of the class of $a$ and the class of $b$ respectively, then $n_1 + n_2 = n \ (mod\ p)$ and $\mu(n) = T(x, y)$. We have two possibilities: (3.1) $n_1 + n_2 = n$ or (3.2) $n_1 + n_2 = n + p$. Suppose (3.1), then $0 \neq n_1 < n$ and $0 \neq n_2 < n$. If $n_1 > m$ or $n_2 > m$, $T(\mu(n_1), \mu(n_2)) = T(T(x, y), T(x, y)) \leq T(x, y) = \mu(n)$. If $n_1 = m$, since $\mu(n_2) \leq x$ because $\mu(t) \leq x$ for all $t \neq 0$, we conclude $\mu(n) = T(x, y) \geq T(\mu(n_2), y) = T(\mu(n_2), \mu(n_1))$. If $n_2 = m$ the procedure is the same. Finally, if $n_1 < m$ and $n_2 < m$, using that $T(x, y) \geq x_T^{(n)}$ because $n \geq m+1$ and Lemma 4.3, we obtain that

$$\mu(n) = T(x, y) \geq x_T^{(n)} \geq T(x_T^{(n_1)}, x_T^{(n_2)}) = T(\mu(n_1), \mu(n_2))$$

Finally, if $n_1 > m$ or $n_2 > m$, since $n_1 < n$ and $n_2 < n$ we have that $\mu(n) \geq T(\mu(n_1), \mu(n_2))$. Suppose (3.2), since $n_1 + n_2 = p + n \Leftrightarrow n_1 - n = p - n_2 \Leftrightarrow n_2 - n = p - n_1$ and $p - n_i > 0$ we obtain that $n_1 > n$ and $n_2 > n$. This implies that for each $i \in \{1, 2\}$, $\mu(n_i) \in \{T(x, y), y, x_T^{(p-n_i)}\}$. If $\mu(n_2) = T(x, y)$ or $\mu(n_2) = y$ we have that

$$\mu(n) = T(x, y) \geq T(x, \mu(n_2)) \geq T(\mu(n_1), \mu(n_2))$$

Similarly, if $\mu(n_1) = T(x, y)$ or $\mu(n_1) = y$. Therefore, suppose that $\mu(n_1) = x_T^{(p-n_1)}$ and $\mu(n_2) = x_T^{(p-n_2)}$. Since $p - n = p - n_1 + p - n_2$, using Lemma 4.3 we conclude that

$$\mu(n) = T(x, y) \geq x_T^{(p-n)} \geq T(x_T^{(p-n_1)}, x_T^{(p-n_2)}) = T(\mu(n_1), \mu(n_2))$$

(4.) If $a+b=m$, consider $n_1, n_2 \in \{1, ..., p-1\}$ which are representative elements of the class of $a$ and the class of $b$ respectively, then $n_1 + n_2 = m \ (mod\ p)$ and $\mu(n) = y$. We have two possibilities: (4.1) $n_1 + n_2 = m$ or (4.2) $n_1 + n_2 = m + p$. If (4.1) happens, by construction of $m$, $y \geq x_T^{(m)}$ and if we apply Lemma 4.3, we obtain that

$$\mu(m) = y \geq x_T^{(m)} \geq T(x_T^{(n_1)}, x_T^{(n_2)}) = T(\mu(n_1), \mu(n_2)).$$

Suppose (4.2), since $n_1 + n_2 = p + m \Leftrightarrow n_1 - m = p - n_2 \Leftrightarrow n_2 - m = p - n_1$ and $p - n_i > 0$ we obtain that $n_1 > m$ and $n_2 > m$. This implies that for each

$i \in \{1, 2\}$, $\mu(n_i) \in \{T(x, y), y, x_T^{(p-n_i)}\}$. Using the similar procedure described in (3.2), if $\mu(n_2) = T(x, y)$ or $\mu(n_2) = y$ we have that

$$\mu(m) = y \geq T(x, y) \geq T(x, \mu(n_2)) \geq T(\mu(n_1), \mu(n_2))$$

and the same if $\mu(n_1) = T(x, y)$ or $\mu(n_1) = y$. Finally, suppose that $\mu(n_1) = x_T^{(p-n_1)}$ and $\mu(n_2) = x_T^{(p-n_2)}$. Since $p - m = p - n_1 + p - n_2$, using Lemma 4.3, we obtain that

$$\mu(m) = y \geq x_T^{(m)} \geq T(x_T^{(p-n_1)}, x_T^{(p-n_2)}) = T(\mu(n_1), \mu(n_2)).$$

If $\mu(n_2) = x_T^{(n_2)}$, then $n_2 < m$ by construction of $\mu$, but $n_2 > m$ because $n_1 + n_2 = p + m \Leftrightarrow n_2 - m = p - n_1 > 0$.

(5.), If $a + b = p - m$ the procedure is the same one described in (4) due to the observation $\mu(m) = \mu(p - m)$.

Therefore, $\mu$ satisfies $(G3)$, hence $\mu$ is a $T$-subgroup. Now, we will check $\mu$ is not a $T'$-subgroup:

$$\mu(1 + m) = T(x, y) < T'(x, y) = T(\mu(1), \mu(m)).$$

Hence $\mu$ is not a $T'$-subgroup.

(4) $| \Longleftarrow |$ Lemma 4.4

$| \Longrightarrow |$ Equivalently, we show that if there are $x, y \in [0, 1]$ with $T(x, y) < T'(x, y)$, then there exists a $T$-subgroup $\mu$ such that $\mu$ is not a $T'$-subgroup. Suppose without loss of generality that $x \leq y$, we divide the proof in two blocks:

(1.) $G$ has an element with infinite order.

(2.) $G$ has not elements with infinite order.

If (1.) happens, there is $a \in G$ with infinite order. Take $b = a^2$ and the fuzzy set $\mu$ defined as follows:

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ x & if \quad t \in \{a, a^{-1}\} \\ y & if \quad t \in \langle b \rangle - \{e\} \\ T(x, y) & \quad \text{otherwise} \end{cases}$$

Observe that $a, a^{-1} \notin \langle b \rangle - \{e\}$.

If (2.) happens, there is an element $b \in G$ with $o(b) = p$, where $p$ is some prime number. Since $G \neq \mathbb{Z}_q$ for all prime $q$, there exists $a \in G$ satisfying $a \notin \langle b \rangle$. Consider the following fuzzy set $\mu$

$$\mu(t) = \begin{cases} 1 & if \quad t = e \\ x & if \quad t \in \{a, a^{-1}\} \\ y & if \quad t \in \langle b \rangle - \{e\} \\ T(x,y) & \quad \text{otherwise} \end{cases}$$

In both cases, we have that $\mu(ab) = T(x, y)$ because $a \notin \langle b \rangle$ and $\mu$ satisfies $(G1)$ and $(G2)$ by construction. To prove $(G3)$ we need to check that $\mu(rs) \geq T(\mu(r), \mu(s))$ for all $r, s \in G$. If $r = e$ or $s = e$, it is straightforward. In other cases, we have that

$$T(\mu(r), \mu(s)) \in \{T(x, x), T(x, y), T(x, T(x, y)), T(y, y), T(y, T(x, y)), T(T(x, y), T(x, y))\}$$

Note that $\mu(rs)$ is greater than or equal to $T(x, y)$, so if

$$T(\mu(r), \mu(s)) \in \{T(x, x), T(x, y), T(x, T(x, y)), T(y, T(x, y)), T(T(x, y), T(x, y))\}$$

we conclude that $T(\mu(r), \mu(s)) \leq T(x, y) \leq \mu(rs)$. If $T(\mu(r), \mu(s)) = T(y, y)$ and

$$T(y, y) \in \{T(x, x), T(x, y), T(x, T(x, y)), T(y, T(x, y)), T(T(x, y), T(x, y))\}$$

$(G3)$ would be true. Therefore, suppose that $T(\mu(r), \mu(s)) = T(y, y)$ is different to each element of the previous set. This means that $r \in \langle b \rangle - \{e\}$ and $s \in \langle b \rangle - \{e\}$, hence $rs \in \langle b \rangle$. If $rs = e$, $\mu(rs) = 1$ and if $rs \in \langle b \rangle - \{e\}$, $\mu(rs) = y$. In both cases, using the monotony of $T$, $\mu(rs) \geq T(y, y) = T(\mu(r), \mu(s))$, hence $\mu$ satisfies $(G3)$ for the t-norm $T$. But $\mu$ is not a $T'$-subgroup because $\mu(ab) = T(x, y) < T'(x, y) = T'(\mu(a), \mu(b))$.

□

Due to the close relation between $T$-subgroup notion and right (right) invariant under translation $T$-indistinguishability operator, we obtain the following result.

**Corollary 4.8.** *Let $T$ and $T'$ t-norms be and $G$ a group.*

(1) *If $G$ is a cyclic group of two or three elements, then every right (left) invariant $T$-indistinguishability operator of $G$ is a right (left) invariant $T'$-indistinguishability operator.*

(2) *If $G$ is a cyclic group of four or five elements, then every right (left) invariant $T$-indistinguishability operator of $G$ is a right (left) invariant $T'$-indistinguishability operator if and only if $T'(x,x) \leq T(x,x)$ for all $x \in [0,1]$.*

(3) *If $G$ is a cyclic group of $p$ elements, where $p$ is a prime greater than or equal to $7$, then every right (left) invariant $T$-indistinguishability operator of $G$ is a right (left) invariant $T'$-indistinguishability operator if and only if for all $a, b \in [0,1]$ such that $T(a,b) < T'(a,b)$ with $a \geq b$ satisfy $b < a_T^{\frac{p-1}{2}}$.*

(4) *If $G$ is a cyclic group of $n$ elements where $n$ is not a prime integer and $n > 5$ or if $G$ is not a cyclic group, then every right (left) invariant $T$-indistinguishability operator of $G$ is a right (left) invariant $T'$-indistinguishability operator if and only if for all $x, y \in [0,1]$, $T'(x,y) \leq T(x,y)$.*

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