Corrigendum to “Fuzzy implication functions based on powers of continuous t-norms” [Int. J. Approx. Reason. 83 (2017) 265-279]

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Abstract

In (Int. J. Approx. Reason. 83 (2017) 265-279) a new property of fuzzy implication functions, called the invariance property with respect to powers of a continuous t-norm, was introduced and its application in approximate reasoning was studied. In the same paper, the novel family of power based implications was presented as a family of fuzzy implication functions that satisfy such invariance property. Unfortunately, this fact is not entirely true since as we will prove in this paper, there exist some power based implications generated from specific ordinal sum t-norms which do not fulfil the invariance property. Thus, a characterization of which continuous t-norms can be used to generate power based implications satisfying the power based invariance property is presented. Additionally, as an alternative solution, we introduce in this paper a slight modification of this property in such a way that all power based implications satisfy it.

Keywords: Fuzzy implication function, power based implications, invariance with respect to powers of a continuous t-norm.

1. Introduction

Power based implications were introduced in [1] as a new class of fuzzy implication functions based on the use of powers of a continuous t-norm T (see [1] for further details on powers with respect to continuous t-norms).

Definition 1 ([1, Definition 4]). A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a *T-power based implication* if there exists a continuous t-norm T such that

$$I(x,y) = \sup\{r \in [0,1] \mid y_T^{(r)} \geq x\} \qquad \text{for all } \ x,y \in [0,1]$$

where $y_T^{(r)}$ denotes the r-th power of y with respect to T. If I is a T-power based implication, then it will be denoted by I^T.

This kind of implications, jointly with their properties, were extensively studied in [1] where their general expression was given depending on the t-norm T. In particular, T-power based implications constitute a new family of fuzzy implication functions which does not intersect the most well-known families. Indeed, while they satisfy **(OP)** and **(IP)**, they do not satisfy **(NP)**, a property fulfilled by the most well-known families. Moreover, in [1], the conditions under which they satisfy other additional properties as the contrapositive symmetry or the T-transitivity, were also determined. Nevertheless, the importance of the power based implications relied on their applications in approximate reasoning. Consider, for instance, the classical example of tomatoes in [2]:

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If the tomato is red, then it is ripe.
If the tomato is very red, then it is very ripe.
If the tomato is little red, then it is little ripe.

where the linguistic modifiers "very" and "little" are modelled using powers of continuous t-norms. It is reasonable to expect that these three fuzzy conditionals have the same truth value. This led to the introduction of the so-called invariance property with respect to powers of a continuous t-norm.

Definition 2 ([1, Definition 5]). Let T be a continuous t-norm and $I : [0,1]^2 \to [0,1]$ be a binary function . It is said that I is *invariant with respect to* T-powers, or simply that it is T*-power invariant* when

$$I(x,y) = I\left(x_T^{(r)}, y_T^{(r)}\right). \qquad (\mathbf{PI_T})$$

holds for all real number $r > 0$ and for all $x, y \in [0,1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$.

In [1], it was proved that given a continuous t-norm T, all T-power based implications are T-power invariant.

Proposition 1 ([1, Proposition 7]). *Let T be a continuous t-norm and I^T its power based implication. Then I^T is T-power invariant.*

Unfortunately, this result is wrong (there is a mistake in the proof). Indeed, there is a special kind of ordinal sum t-norms T for which I^T are not T-power invariant. Let us give an example of this fact.

Example 1. Consider the ordinal sum t-norm given by $T = (\langle 1/2, 1, T_{\mathbf{L}} \rangle)$ and let us take $y = 1/2 < 3/4 = x$. A simple calculation shows that $x_T^{(2)} = y_T^{(2)} = 1/2$ and consequently $I^T(x_T^{(2)}, y_T^{(2)}) = I^T(1/2, 1/2) = 1$. On the other hand, $I^T(x,y) = I^T(3/4, 1/2) = 1/2$ and so I^T is not T-power invariant.

Therefore, in this corrigendum, we will provide the correct result by characterizing which continuous t-norms T generate T-power based implications satisfying the invariance property with respect to T-powers.

2. Correction

The mistake made in the proof of Proposition 7 in [1] was to assume that for any continuous t-norm T, $r > 0$ and $x, y \in [0,1]$ such that $x > y$ and $y_T^{(r)} \neq 0$, it holds that $x_T^{(r)} > y_T^{(r)}$. This is of course not true in general as Example 1 shows, in which $y = 1/2 < 3/4 = x$ but $x_T^{(2)} = y_T^{(2)} = 1/2$.

Taking into account the previous example, it is clear that the problem lies in the ordinal summands $([a_j, b_j], T_j)$ with T_j a nilpotent t-norm and $a_j \neq 0$. To have this particular case under control, we will adopt the following notation.

Definition 3. A continuous t-norm T is said to be a *strict ordinal sum* t-norm (SOS t-norm for short) whenever T is an ordinal sum t-norm of the form $T = (\langle a_j, b_j, T_j \rangle)_{j \in J}$ being T_j a strict t-norm for all summands $[a_j, b_j]$ such that $a_j \neq 0$.

We will denote by $\mathcal{T}_{SOS}$ the set of all strict ordinal sum t-norms. Note that this set includes in particular the Minimum t-norm (taking J the empty set), all strict and nilpotent t-norms (taking $J = \{1\}$ and $[a_1, b_1] = [0,1]$) and also all proper ordinal sums with the required condition. Now, with this notation we can easily correct the Proposition 7 in [1] as follows.

Proposition 2. *Let T be a continuous t-norm and I^T its power based implication. Then I^T is T-power invariant if and only if T is a strict ordinal sum t-norm (i.e., if and only if $T \in \mathcal{T}_{SOS}$).*

PROOF. Let us prove first that if $T \in \mathcal{T}_{SOS}$ then I^T is T-power invariant. The key fact in this proof is that for any t-norm in $\mathcal{T}_{SOS}$, the condition $x > y$ with $y_T^{(r)} \neq 0$ implies $x_T^{(r)} > y_T^{(r)}$. Indeed, let $T = (\langle a_j, b_j, T_j \rangle)_{j \in J}$ be an ordinal sum t-norm such that T_j is a strict t-norm for all summands $[a_j, b_j]$ such that $a_j \neq 0$. In this case, by using Definition 22 in [3], we have that

$$x_T^{(r)} = \begin{cases} a_j + (b_j - a_j) \cdot t_j^{-1}\left(\left(t_j\left(\frac{x-a_j}{b_j-a_j}\right)\right)^r\right) & \text{if } a_j \le x \le b_j \text{ for some } j \in J \text{ and } a_j \neq 0, \\ f(x) & \text{if } a_j \le x \le b_j \text{ for some } j \in J \text{ and } a_j = 0, \\ x & \text{otherwise,} \end{cases}$$

where $t_j : [0,1] \to [0,+\infty]$ is an additive generator of the strict t-norm T_j for all $j \in J$ and two cases arise for the expression of $f(x)$:

Case 1: if the summand T_j such that $a_j = 0$ is a strict t-norm then

$$f(x) = b_j \cdot t_j^{-1}\left(\left(t_j\left(\frac{x}{b_j}\right)\right)^r\right)$$

where $t_j : [0,1] \to [0,+\infty]$ is an additive generator of the strict t-norm T_j,

Case 2: if the summand T_j such that $a_j = 0$ is a nilpotent t-norm then

$$f(x) = b_j \cdot t_j^{-1}\left(\max\left\{r \cdot t_j\left(\frac{x}{b_j}\right) - r + 1, 0\right\}\right)$$

where $t_j : [0,1] \to [0,+\infty)$ is an additive generator of the nilpotent t-norm T_j.

Now, if $x > y$ with $y_T^{(r)} \neq 0$, then we will prove that $x_T^{(r)} > y_T^{(r)}$. On the one hand, the proof of Case 1 is straightforward due to the strictly increasingness of the powers. On the other hand, in Case 2, the only problematic case would be when $a_j \le y < x \le b_j$ for some $j \in J$ and $a_j = 0$, that is, when the nilpotent summand is applied. However, since we know that $y_T^{(r)} \neq 0$, this implies that

$$y_T^{(r)} = b_j \cdot t_j^{-1}\left(rt_j\left(\frac{y}{b_j}\right) - r + 1\right) < b_j \cdot t_j^{-1}\left(rt_j\left(\frac{x}{b_j}\right) - r + 1\right) = x_T^{(r)}$$

and the result is proved. Due to this fact, an analogous proof to the one of Proposition 7 in [1] works now. For the sake of completeness, we include here the complete proof.

If $x \le y$, then $x_T^{(r)} \le y_T^{(r)}$ and in this case

$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(x, y) = 1.$$

Thus, we only need to prove the $(\mathbf{PI_T})$ property for values x, y such that $x > y$. We will do it depending on the t-norm $T \in \mathcal{T}_{SOS}$.

- If $T = T_{\mathbf{M}}$, x is T-idempotent for all $x \in [0,1]$ and the result is trivial because $x_T^{(r)} = x$ for all $r > 0$.
- If T is Archimedean with additive generator t, take $x > y$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$. In this case, T-power of x are given by $x_T^{(r)} = t^{-1}(rt(x))$ and so $x_T^{(r)} > y_T^{(r)}$ and

$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(t^{-1}(rt(x)), t^{-1}(rt(y))) = \frac{rt(x)}{rt(y)} = \frac{t(x)}{t(y)} = I^T(x, y).$$

- Consider T a proper strict ordinal sum t-norm $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}$, where each T_i has additive generator t_i for all $i \in I$. Take again $x > y$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$. In this case, since $T \in \mathcal{T}_{SOS}$, it holds that $x_T^{(r)} > y_T^{(r)}$. Now, some subcases arise:
  - If there is some T-idempotent element α such that $x > \alpha > y$ then $x_T^{(r)} > \alpha > y_T^{(r)}$ and
$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(x, y) = 0.$$
  - If there is some $i \in I$ such that $a_i \leq y < x \leq b_i$ then it is also $a_i \leq y_T^{(r)} < x_T^{(r)} \leq b_i$ and
$$\begin{aligned} I^T(x_T^{(r)}, y_T^{(r)}) &= I^T\left(a_i + (b_i - a_i)t_i^{-1}\left(rt_i\left(\frac{x-a_i}{b_i-a_i}\right)\right), a_i + (b_i - a_i)t_i^{-1}\left(rt_i\left(\frac{x-a_i}{b_i-a_i}\right)\right)\right) \\ &= \frac{rt_i\left(\dfrac{x-a_i}{b_i-a_i}\right)}{rt_i\left(\dfrac{y-a_i}{b_i-a_i}\right)} = \frac{t_i\left(\dfrac{x-a_i}{b_i-a_i}\right)}{t_i\left(\dfrac{y-a_i}{b_i-a_i}\right)} = I^T(x, y). \end{aligned}$$

On the contrary, suppose now that T is an ordinal sum of the form $T = (\langle a_j, b_j, T_j\rangle)_{j\in J}$ and there is some $j \in J$ such that $a_j \neq 0$ and T_j is a nilpotent t-norm. In this case there exists some $x > a_j$ such that $(a_j)_T^{(2)} = x_T^{(2)} = a_j$ and then it is clear that
$$I^T\left((a_j)_T^{(2)}, x_T^{(2)}\right) = I^T(a_j, a_j) = 1, \quad \text{whereas} \quad I^T(a_j, x) < 1,$$
which implies that I^T is not T-power invariant. □

From this correction, it is obviously not true that all T-power based implications are T-power invariant. However, we have been able to characterize which continuous t-norms T can be used to generate T-power based implications satisfying this property. Since there are still many suitable continuous t-norms to fulfil this property, power based implications remain an important family of fuzzy implication functions due to their applications in approximate reasoning.

In fact, an alternative solution to the one presented in this corrigendum would be possible. Note that the invariance property given by $(\mathbf{PI_T})$ is only required to hold for all real numbers $r > 0$ and for all $x, y \in [0, 1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$, the property could be redefined in the following way.

Definition 4. Let $T = (\langle a_j, b_j, T_j\rangle)_{j\in J}$ be a continuous t-norm and $I : [0, 1]^2 \to [0, 1]$ be a binary function. It is said that I is *invariant with respect to* T-powers, or simply that it is T-*power invariant* when
$$I(x, y) = I\left(x_T^{(r)}, y_T^{(r)}\right). \qquad (\mathbf{PI_T^*})$$
holds for all real number $r > 0$ and for all $x, y \in [0, 1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1, a_j$ for all $j \in J$.

Taking into account this definition of power invariance, it would remain true the statement that all T-power based implications are T-power invariant, as the following result proves.

Proposition 3. *Let T be a continuous t-norm and I^T its power based implication. Then I^T is T-power invariant, that is, I^T satisfies $(\mathbf{PI_T^*})$.*

PROOF. The proof is the same than the one of Proposition 7 in [1], which is now valid since for all $x, y \in [0, 1]$ such that $x > y$ and $r > 0$, if $x_T^{(r)} = y_T^{(r)}$, then either $x_T^{(r)}$ or $y_T^{(r)}$ are equal to $0, 1, a_j$ for some $j \in J$. Indeed, the only difference is that nilpotent summands with $a_j > 0$ are now allowed. Let us consider a nilpotent summand T_j with $a_j > 0$. It is clear from Definition 22 in [3] that in this case, for all $a_j \leq x \leq b_j$,
$$x_T^{(r)} = a_j + (b_j - a_j)\cdot t_j^{-1}\left(\max\left\{r \cdot t_j\left(\frac{x-a_j}{b_j-a_j}\right) - r + 1, 0\right\}\right)$$

where $t_j : [0,1] \to [0,+\infty)$ is an additive generator of the nilpotent t-norm T_j. Now, for all $a_j \leq y < x \leq b_j$, either $y_T^{(r)} < x_T^{(r)}$ or $y_T^{(r)} = x_T^{(r)} = a_j$. Since $(\mathbf{PI}^*_{\mathbf{T}})$ is only required to hold when $x_T^{(r)}, y_T^{(r)} \neq 0, 1, a_j$ for all $j \in J$, the result clearly follows. □

Although this alternative version of the invariance property solves in a more direct way the mistake carried out in the proof of Proposition 7 in [1], we prefer to keep Definition 5 in [1], i.e., the original definition of the invariance property with respect to powers of a continuous t-norm. The main reason is that the original definition is quite natural, whereas the corrected one is more artificial since it involves a different behaviour depending on the considered t-norm. However, it is clear that Definition 4 could be also studied in a similar way as it was done for Definition 2 in [1]. There is no doubt that such study would lead to similar properties but with some slight differences.

3. Conclusions

In this corrigendum, we have corrected Proposition 7 in [1] proving that not all T-power based implications generated from a continuous t-norm T are T-power invariant. Indeed, we have proved that the result holds for any continuous t-norm in $\mathcal{T}_{SOS}$, the set of all strict ordinal sum t-norms. Finally, an alternative solution has been also proposed by modifying the definition of the invariance property with respect to powers of a continuous t-norm.

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