Aggregating Fuzzy Subgroups and T-vague Groups

D. Boixader, G. Mayor and J. Recasens

Abstract Fuzzy subgroups and T-vague groups are interesting fuzzy algebraic structures that have been widely studied. While fuzzy subgroups fuzzify the concept of crisp subgroup, T-vague groups can be identified with quotient groups of a group by a normal fuzzy subgroup and there is a close relation between both structures and T-indistinguishability operators (fuzzy equivalence relations).
In this paper the functions that aggregate fuzzy subgroups and T-vague groups will be studied. The functions aggregating T-indistinguishability operators have been characterized ([9]) and the main result of this paper is that the functions aggregating T-indistinguishability operators coincide with the ones that aggregate fuzzy subgroups and T-vague groups. In particular, quasi-arithmetic means and some OWA operators aggregate them if the t-norm is continuous Archimedean.

1 Introduction

T-indistinguishability operators, also called fuzzy equivalence relations or fuzzy equalities, fuzzify the concepts of crisp equivalence relation and crisp equality. They appear naturally when studying fuzzy systems and have an extensive amount of literature since its first definition by Zadeh in [18]. One important example is the study of fuzzy algebras (see [10] for example) and the works by Demirci ([2]). In many situations more than one T-indistinguishability operator is defined on a system and it is necessary to aggregate them into a unique of such fuzzy relations. There are many works dealing with this problem [12] [13] [6]. The last one is [9]

D. Boixader and J. Recasens
Universitat Politècnica de Catalunya
Sant Cugat del Vallès, Barcelona, Spain. e-mail: {dionis.boixader, j.recasens}@upc.edu

G. Mayor
Universitat de les Illes Balears
Palma (Mallorca), Spain. e-mail: gmayor@uib.es

where the aggregation operators that aggregate T-indistinguishability operators have been completely characterized regardless of the nature of the t-norm T (continuity, archimedianity,...). This result will be recalled and used in this work.

Fuzzy subgroups of a group $(G,\circ)$ were introduced by Rosenfeld [16] as a natural generalization of subgroup of G and have been widely studied [11]. They are defined as fuzzy subsets of $(G,\circ)$ satisfying some properties fuzzifying the definition of crisp subgroup.

In the crisp case, if $(G,\circ)$ is a set with an operation $\circ : G\times G \to G$ and $\sim$ is an equivalence relation on G, then $\circ$ is compatible with $\sim$ if and only if

$$a\sim a' \text{ and } b\sim b' \text{ implies } a\circ b\sim a'\circ b'.$$

In this case, an operation $\tilde{\circ}$ can be defined on $\overline{G} = G/\sim$ by

$$\overline{a}\tilde{\circ}\overline{b} = \overline{a\circ b}$$

where $\overline{a}$ and $\overline{b}$ are the equivalence classes of a and b with respect to $\sim$.

If the equivalence relation is replaced by a T-indistinguishability operator we obtain fuzzy algebras ([10] [5] [4]). Demirci generalized this idea by introducing the concept of vague algebra, which basically consists of a fuzzy operation on a set G (i.e.: a mapping $\tilde{\circ} : G\times G\times G \to [0,1]$ where $\tilde{\circ}(a,b,c)$ is interpreted as the degree in which $a\circ b$ is equivalent or indistinguishable from c) compatible with a given indistinguishability operator [2].

To every fuzzy subgroup μ of a group $(G,\circ)$ two T-indistinguishability operators E_μ and ${}_\mu E$ can be associated, that are left and a right invariant under translations. If the fuzzy subgroup μ is normal, then the left and right T-indistinguishability operators E_μ and ${}_\mu E$ coincide. The operation of the group is compatible with the T-indistinguishability operator E_μ if and only if μ is a normal fuzzy subgroup of G.

A group $(G,\circ)$ with the vague operation $\tilde{\circ}(a,b,c) = E(a\circ b,c)$ is a T-vague group and, reciprocally, for every T-vague group of G there exists a T-indistinguishability operator E that is invariant under translations such that $\tilde{\circ}(a,b,c) = E(a\circ b,c)$. Moreover, there is a bijection between the fuzzy normal subgroups of G and the T-vague groups of G. In particular, T-vague groups can be thought of as the fuzzy counterparts of crisp quotient groups (if μ is the normal fuzzy subgroup associated to the vague group $(G,\tilde{\circ})$, then $(G,\tilde{\circ})$ can be identified with G/μ).

Due to the importance of fuzzy groups and T-vague groups, it seems interesting to study how can they be fused or aggregated. More concrete, finding aggregation functions that aggregate them. This will be done in the present work, where the functions aggregating them will be characterized. Due to the close relation between fuzzy subgroups, T-vague groups and T-indistinguishability operator, it is not surprising that the aggregation functions aggregating them coincide. This is the main result of the work. As a consequence, quasi arithmetic means and some OWA operators aggregate fuzzy subgroups and T-vague groups if T is a continuous Archimedean t-norm.

2 Preliminaries

In this section, the basic definitions and properties of T-indistinguishability operators, fuzzy subgroups and T-vague groups are recalled.

T-indistinguishability operators fuzzify the concepts of crisp equivalence and crisp equality. An overview of these operators can be found in [14].

Definition 2.1 *Let T be a t-norm. A fuzzy relation E on a set X is a T-indistinguishability operator if and only if for all $x,y,z \in X$*

1. *$E(x,x) = 1$ (Reflexivity)*
2. *$E(x,y) = E(y,x)$ (Symmetry)*
3. *$T(E(x,y),E(y,z)) \leq E(x,z)$ (T-transitivity).*

E separates points if and only if

c) *$E(x,y) = 1$ implies $x = y$.*

Fuzzy subgroups were introduced by Rosenfeld [16] by fuzzifying the definition of crisp subgroup. The use of the minimum to model the conjunction has been generalized to any t-norm later on. The reader interested in the study of fuzzy subgroups is referred to [11].

Definition 2.2 *Let $(G,\circ)$ be a group, e its identity element and μ a fuzzy subset of G. μ is a fuzzy subgroup of G if and only if for all $x,y \in G$*

- $\mu(e) = 1$
- $T(\mu(x),\mu(y)) \leq \mu(x \circ y^{-1})\ \forall x,y \in X.$

T-vague algebras were introduced by Demirci considering fuzzy operations compatible with given T-indistinguishability operators and an extensive study of vague operations and T-vague groups can be found in [2].

Definition 2.3 *A fuzzy binary operation on a set G is a map $\tilde{\circ} : G \times G \times G \to [0,1]$.*

$\tilde{\circ}(x,y,z)$ is interpreted as the degree in which z is $x \circ y$.

Definition 2.4 *Let E be a T-indistinguishability operator on G. A vague binary operation on G is a fuzzy binary operation $\tilde{\circ}$ satisfying for all $x,x',y,y' \in G$*

a) *$T(\tilde{\circ}(x,y,z),E(x,x'),E(y,y'),E(z,z')) \leq \tilde{\circ}(x',y',z')$.*
b) *$T(\tilde{\circ}(x,y,z),\tilde{\circ}(x,y,z')) \leq E(z,z')$.*
c) *For all $x,y \in G$ there exists a unique $z \in G$ such that $\tilde{\circ}(x,y,z) = 1$.*

N.B. The usual definition of vague binary operation does not require uniqueness in the third property and the vague binary operations with this property are called perfect.

Definition 2.5 *Let $\tilde{\circ}$ be a T-vague binary operation on G with respect to a T-indistinguishability operator E on G. Then $(G,\tilde{\circ})$ is a T-vague group if and only if it satisfies the following properties.*

1. *Associativity.* $\forall x,y,z,t,m,q,w, \in G$

$$T(\tilde{\circ}(y,z,t),\tilde{\circ}(x,t,m),\tilde{\circ}(x,y,q),\tilde{\circ}(q,z,w)) \leq E(m,w)).$$

2. *Identity. There exists a (two sided) identity element* $e \in G$ *such that*

$$T(\tilde{\circ}(e,x,x),\tilde{\circ}(x,e,x)) = 1$$

   for each $a \in G$.
3. *Inverse. For each* $x \in G$ *there exists a (two-sided) inverse element* $x^{-1} \in G$ *such that*

$$T(\tilde{\circ}(x^{-1},x,e),\tilde{\circ}(x,x^{-1},e)) = 1.$$

A T*-vague group is Abelian or commutative if and only if*

$$\forall x,y,m,w \in G, T((\tilde{\circ}(x,y,m),\tilde{\circ}(y,x,w))) \leq E(m,w)).$$

Proposition 2.6 *Let* $(G,\tilde{\circ})$ *be a* T*-vague group. Then* $(G,\circ)$ *with* $\circ$ *defined for all* $x,y \in G$ *by* $x \circ y = z$*, where* z *is the unique element of* G *with* $\tilde{\circ}(x,y,z) = 1$*, is a group.*

3 Relationship between Indistinguishability Operators, Fuzzy Subgroups and Vague groups

3.1 Fuzzy Subgroups

To every fuzzy subset μ of a group $(G,\circ)$ a pair of fuzzy relations can be associated that are indistinguishability operators if and only if μ is a fuzzy subgroup of G. This two indistinguishability operators coincide when μ is a fuzzy normal subgroup and there is a compatibility between it and the operation $\circ$ of the group. These properties and their relation with the invariance under translations will be analyzed in this section.

In the crisp case, given a subgroup H of a group $(G,\circ)$, the relations $\sim_r$ and $\sim_l$ on G defined by $x \sim_r y$ if and only if $x \circ y^{-1} \in H$ and $x \sim_l y$ if and only if $y^{-1} \circ x \in H$ respectively are equivalence relations. The operation $\circ$ of G is compatible with $\sim_r$ and $\sim_l$ if and only if H is a normal subgroup of G.

These results can be generalized to fuzzy subgroups and T-indistinguishability operators.

Definition 3.1 *Let* $\circ$ *be a binary operation on* G *and* E *a fuzzy relation on* G. E *is invariant under translations with respect to* $\circ$ *if and only if*

a)

$$E(x,y) = E(z \circ x, z \circ y) \;\; (left\ invariant)$$

and

b)

$$E(x,y) = E(x \circ z, y \circ z) \ \ (\textit{right invariant}),$$

$\forall x,y,z \in G$.

To every fuzzy subset μ of a group $(G,\circ)$ two fuzzy relations E_μ and ${}_\mu E$ can be assigned that are right and left invariant T-indistinguishability operators respectively if and only if μ is a fuzzy subgroup of G.

Definition 3.2 *Let μ be a fuzzy subset of $(G,\circ)$. The fuzzy relations E_μ and ${}_\mu E$ on G defined by*

$$E_\mu(x,y) = \mu(x \circ y^{-1}) \ \forall x,y \in G$$

and

$${}_\mu E(x,y) = \mu(y^{-1} \circ x) \ \forall x,y \in G$$

are the right and left fuzzy relations associated to μ respectively.

Proposition 3.3 *Let μ be a fuzzy subgroup of a group $(G,\circ)$. Then E_μ and ${}_\mu E$ are right and left invariant T-indistinguishability operators on G respectively.*

Lemma 3.4 *If μ is a fuzzy subgroup of $(G,\circ)$ and e is the identity element of G, then $E_\mu(x,y) = E_\mu(e, x \circ y^{-1})$ and ${}_\mu E(x,y) = {}_\mu E(e, y \circ x^{-1})$ $\forall x,y \in G$.*

Proof. Trivial.

Reciprocally, to every right (left) T-indistinguishability operator on $(G,\circ)$ a fuzzy subgroup of G can be assigned.

Proposition 3.5 *Let E be a T-indistinguishability operator on a group $(G,\circ)$ with identity element e such that E is right invariant. Then the column μ_e of E (i.e., the fuzzy subset μ_e of G defined by $\mu_e(x) = E(e,x)$ $\forall x \in G$) is a fuzzy subgroup of G and $E = E_{\mu_e}$.*

Similarly,

Proposition 3.6 *Let E be a T-indistinguishability operator on a group $(G,\circ)$ with identity element e such that E is left invariant. Then the column μ_e of E is a fuzzy subgroup of G and $E = {}_{\mu_e} E$.*

Corollary 3.7 *Let $(G,\circ)$ be a group. There exist bijections between the set* **FSG** *of fuzzy subgroups of G, the set* **RIG** *of right invariant indistinguishability operators on G and the set* **LIG** *of left invariant indistinguishability operators on G mapping every fuzzy subgroup μ of G into its associated T-indistinguishability operators E_μ and ${}_\mu E$.*

The following definition fuzzifies the concept of normal subgroup.

Definition 3.8 *A fuzzy subgroup μ of a group $(G,\circ)$ is called a normal fuzzy subgroup if and only if $\mu(x \circ y) = \mu(y \circ x)$ $\forall x,y \in G$.*

Proposition 3.9 *Let $(G, \circ)$ be a group and μ a normal fuzzy subgroup of G. The associated T-indistinguishability operators E_μ and ${}_\mu E$ to μ coincide and are invariant under translations.*

Reciprocally,

Proposition 3.10 *Let $(G, \circ)$ be a group, μ a fuzzy subgroup of G and E_μ and ${}_\mu E$ its associated T-indistinguishability operators. If E_μ and ${}_\mu E$ are invariant under translations, then they coincide and μ is a normal fuzzy subgroup of G.*

Corollary 3.11 *Let $(G, \circ)$ be a group. There is a bijection between the set of normal fuzzy subgroups of G and the set of T-indistinguishability operators on G invariant under translations with respect to $\circ$.*

The following proposition links normality of a fuzzy subgroup μ with compatibility with respect to its associated T-indistinguishability operator E_μ.

Proposition 3.12 *Let $(G, \circ)$ be a group, μ a fuzzy normal subgroup of G and E_μ its associated T-indistinguishability operator. Then $\circ$ is extensional with respect to E_μ (i.e., $T(E_\mu(x,x'), E_\mu(y,y')) \leq E_\mu(x \circ y, x' \circ y')$).*

3.2 Vague Groups

Vague groups were introduced in [2] as structures compatible with given indistinguishability operators. They are also closely related to fuzzy normal subgroups [3].

The next two propositions relate normal fuzzy subgroups and indistinguishability operators invariant under translations with vague groups.

Proposition 3.13 *[3] Let $(G, \circ)$ be a group, μ a normal fuzzy subgroup of $(G, \circ)$ and E_μ its associated T-indistinguishability operator on G. If $\tilde{\circ} : G \times G \times G \rightarrow [0,1]$ is defined for all $x,y,z \in G$ by $\tilde{\circ}(x,y,z) = \mu(x \circ y \circ z^{-1}) = E_\mu(x \circ y, z)$, then $(G, \tilde{\circ})$ is a T-vague group.*

Proposition 3.14 *[3] Let $(G, \tilde{\circ})$ be a T-vague group with respect to the T-indistinguishability operator E. Then,*

a) $\tilde{\circ}(x,y,z) = E(x \circ y, z)$ $\forall x,y,z \in G$.
b) $\circ$ *is extensional with respect to* E.
c) E *is invariant under translations with respect to* $\circ$.

Proposition 3.15 *Let $(G, \circ)$ be a group. There exist bijective maps between its T-vague groups, its fuzzy normal subgroups and its T-indistinguishability operators invariant under translations.*

Proof. The bijections are given by

$$\tilde{\circ}(x,y,z) = E(x \circ y, z) = \mu(x \circ y \circ z^{-1}).$$

4 Aggregating Fuzzy Subgroups and Vague Groups

In [9] the functions preserving indistinguishability operators have been characterized by means of the so called T-triangular triplets. In this section we first recall these results and we will use them in the next two subsections to obtain the functions preserving fuzzy subgroups and vague groups thanks to the results of the previous Section 3.

Definition 4.1 *We say that a triplet $(a,b,c) \in [0,\infty]^3$ is triangular if and only if*

$$a \leq b+c, \quad b \leq a+c, \quad c \leq a+b.$$

Being $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in [0,\infty]^m$, $m \geq 1$, we say that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is a (m-dimensional) triangular triplet if (a_i,b_i,c_i) is triangular for all $i = 1,\ldots,m$, where $\boldsymbol{a} = (a_1,\ldots,a_m), \boldsymbol{b} = (b_1,\ldots,b_m), \boldsymbol{c} = (c_1,\ldots,c_m)$.

Definition 4.2 *Let T be a t-norm. We say that $(a,b,c) \in [0,1]^3$ is T-triangular if and only if*

$$a \geq T(b,c), \quad b \geq T(a,c), \quad c \geq T(a,b).$$

Being $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in [0,1]^m$, $m \geq 1$, we say that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is a (m-dimensional) T-triangular triplet if (a_i,b_i,c_i) is T-triangular for all $i = 1,\ldots,m$, where $\boldsymbol{a} = (a_1,\ldots,a_m), \boldsymbol{b} = (b_1,\ldots,b_m), \boldsymbol{c} = (c_1,\ldots,c_m)$.

Proposition 4.3 *Let T be a left continuous t-norm and $\overleftrightarrow{T}$ its bi-residuation. A triplet $(a,b,c) \in [0,1]^3$ is T-triangular if and only if $T(a,b) \leq c \leq \overleftrightarrow{T}(a,b)$.*

Example 4.4

- *A triplet is T-triangular with respect to the minimum t-norm if and only if there exists a reordering (a,b,c) such that $a = b$ and $c \geq a$.*
- *A triplet is T-triangular with respect to the Łukasiewicz t-norm if and only if there exists a reordering (a,b,c) such that $\max(a+b-1,0) \leq c \leq 1- \mid a-b \mid$.*
- *A triplet is T-triangular with respect to the product t-norm if and only if it is $(0,0,0)$ or there exists a reordering (a,b,c) with $a,b,c > 0$, such that $ab \leq c \leq \min(\frac{a}{b}, \frac{b}{a})$.*

Definition 4.5 *A function $F : [0,1]^m \to [0,1], m \geq 1$, aggregates T-indistinguishability operators if for any set X and any collection of T-indistinguishability operators on X, $(E_1,\ldots,E_m)$, then $F(E_1,\ldots,E_m)$ is also a T-indistinguishability operators on X, where $F(E_1,\ldots,E_m)$ is the fuzzy binary relation $F(E_1,\ldots,E_m)(x,y) = F(E_1(x,y),\ldots,E_m(x,y))$.*

The next result characterizes the functions that aggregate T-indistinguishability operators.

Proposition 4.6 *[9] A function $F : [0,1]^m \to [0,1], m \geq 1$, aggregates T-indistinguishability operators if and only if the following conditions hold:*

(i) $F(\overbrace{1,\dots,1}^{m}) = 1$.

(ii)F transforms m-dimensional T-triangular triplets into 1*-dimensional T-triangular triplets.*

Proposition 4.7 *A function $F : [0,1]^m \to [0,1]$, aggregates* min*-indistinguishability operators if and only if it is increasing in each variable and $F(1,\dots,1) = 1$.*

When T is a continuous Archimedean t-norm, a characterization of those functions that aggregate T-equivalence relations can be formulated in terms of an additive generator of T as follows.

Proposition 4.8 *If T is a continuous Archimedean t-norm with additive generator g, then $F : [0,1]^m \longrightarrow [0,1]$ aggregates T-indistinguishability operators if and only if the function $G = gF(g^{(-1)} \times ... \times g^{(-1)})$ transforms (ordinary) triangular triplets of $[0,\infty]^m$ (with elements in $[0,g(0)]^m$) into (ordinary) triangle triplets of $[0,\infty]$ (with elements in $[0,g(0)]$) and $G(0,\dots,0) = 0$.*

Example 4.9 *A function $F : [0,1]^m \to [0,1], m \geq 1$, aggregates T-indistinguishability operators with T the Łukasiewicz t-norm if and only if $G(a_1,\dots,a_m) = 1 - F(max(1-a_1,0),\dots,max(1-a_m,0))$ transforms triangular triplets of $[0,\infty]^m$ (with elements in $[0,1]^m$) into triangle triplets of $[0,\infty]$ (with elements in $[0,1]$) and $G(0,\dots,0) = 0$.*

Under increasingness, subadditivity ($G(\mathbf{a}+\mathbf{b}) \leq G(\mathbf{a}) + G(\mathbf{b})$) is equivalent to the property of transforming triangular triplets into triangle triplets.

Proposition 4.10 *Consider $G : [0,\infty]^m \longrightarrow [0,\infty]$. Then:*

(i) If G transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$ then it is subadditive.
(ii)If G is increasing and subadditive then it transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$.

Thus, from the two previous propositions, we can enunciate the following

Proposition 4.11 *Let T be a continuous Archimedean t-norm with additive generator g. An increasing function $F : [0,1]^m \to [0,1]$, with $F(1,\dots,1) = 1$, aggregates T-indistinguishability operators if and only if the function $G = gFg^{(-1)}$ is subadditive.*

Consequences of the previous propositions are two known results concerning the role of weighted arithmetic means and ordered weighted arithmetic means (OWA operators) in this approach.

Proposition 4.12 *Let T be a continuous Archimedean t-norm with additive generator g. Any weighted quasi-arithmetic mean $F(a_1,\dots,a_m) = g^{-1}(\Sigma w_i g(a_i))$ where $(w_1,\dots,w_m)$ are non-negative real numbers satisfying $\Sigma w_i = 1$ aggregates T-indistinguishability operators.*

Proposition 4.13 *Let T be a continuous Archimedean t-norm with additive generator g. An ordered weighted quasi-arithmetic mean $F(a_1,\ldots,a_m)=g^{-1}(\Sigma w_i g(a_{(m-i)}))$ where $a_{(k)}$ denotes the k-largest input in the list $(a_1,\ldots,a_m)$ aggregates T-indistinguishability operators if $w_i \geq w_j$ for $i \leq j$.*

4.1 Aggregating Fuzzy Subgroups

The relationship between fuzzy subgroups and indistinguishability operators will be used in this subsection to characterize the functions aggregating fuzzy subgroups.

Definition 4.14 *A function $S:[0,1]^m \to [0,1], m \geq 1$, aggregates T-fuzzy subgroups if for any group $(G,\circ)$ and any collection of T-fuzzy subgroups of G, $\mu_1,\ldots,\mu_m$, then $S(\mu_1,\ldots,\mu_m)$ is also a T-fuzzy subgroup of G, where $S(\mu_1,\ldots,\mu_m)$ is the fuzzy subset $S(\mu_1,\ldots,\mu_m)(x)=S(\mu_1(x),\ldots,\mu_m(x))$.*

Proposition 4.15 *Let $(G,\circ)$ be a group, $E_1,E_2,...,E_m$ left (right) invariant T-indistinguishability operators on G and $F:[0,1]^m \to [0,1]$ a function aggregating T-indistinguishability operators. Then $F(E_1,E_2,...,E_m)$ is a left (right) invariant T-indistinguishability operator.*

Proof.

$$\begin{aligned} F(E_1,E_2,...,E_m)(z\circ x,z\circ y) &= F(E_1(z\circ x,z\circ y),E_2(z\circ x,z\circ y),...,E_m(z\circ x,z\circ y)) \\ &= F(E_1(x,y),E_2(x,y),...,E_m(x,y)) \\ &= F(E_1,E_2,...,E_m)(x,y). \end{aligned}$$

Proposition 4.16 *Let $(G,\circ)$ be a group, $F:[0,1]^m \to [0,1]$ a function, $\mu_1,...\mu_m$ fuzzy subsets of G and ${}_{\mu_1}E,...,{}_{\mu_m}E$ ($E_{\mu_1},....,E_{\mu_m}$) their respective left (right) invariant fuzzy relations. Then*

a) $F({}_{\mu_1}E,...,{}_{\mu_m}E) = {}_{F(\mu_1,...,\mu_m)}E$
b) $F(E_{\mu_1},....,E_{\mu_m}) = E_{F(\mu_1,...,\mu_m)}$.

Proof. We will prove a):

$$\begin{aligned} F({}_{\mu_1}E,...,{}_{\mu_m}E)(x,y) &= F({}_{\mu_1}E(x,y),...,{}_{\mu_m}E(x,y)) \\ &= F(\mu_1(y^{-1}\circ x),...,\mu_m(y^{-1}\circ x)) \\ &= F(\mu_1,...,\mu_m)(y^{-1}\circ x) \\ &= {}_{F(\mu_1,...,\mu_m)}E(x,y). \end{aligned}$$

As a corollary we obtain:

Corollary 4.17 *With the same notations as in the preceding Proposition 4.16,*

- *if F aggregates T-indistinguishability operators and $\mu_1,...,\mu_m$ are fuzzy subgroups of G, then $F(\mu_1,...,\mu_m)$ is a fuzzy subgroup of G.*

- *if F aggregates T-fuzzy subgroups and $E_1,\ldots,E_m$ are left (right) invariant T-indistinguishability operators, then $F(E_1,\ldots,E_m)$ is a left (right) invariant T-indistinguishability operator.*

Corollary 4.18 *If $F([0,1]^m \to [0,1]$ aggregates T-indistinguishability operators, then F aggregates T-fuzzy subgroups.*

Proposition 4.19 *Let $F : [0,1] \to [0,1]$ be a function that aggregates fuzzy subgroups. Then F transforms m-dimensional T-triangular triplets into 1-dimensional triplets.*

Proof. Consider the group $\mathbb{Z}/(2)\times\mathbb{Z}/(2)$ and put $e=(\overline{0},\overline{0})$, $a=(\overline{0},\overline{1})$, $b=(\overline{1},\overline{0})$, $c=(\overline{1},\overline{1})$.

Let $\mathbf{a}=(a_1,\ldots,a_m),\mathbf{b}=(b_1,\ldots,b_m),\mathbf{c}=(c_1,\ldots,c_m)$ be m-dimensional T-triplets. For every $i=1.,,m$ consider the fuzzy subgroup μ_i defined by

$$\mu_i(e)=1,\ \ \mu_i(a)=a_i,\ \ \mu_i(b)=b_i,\ \ \mu_i(c)=c_i.$$

Corollary 4.20 *Let $F : [0,1] \to [0,1]$ be a function that aggregates left (right) invariant T-indistinguishability operators. Then F transforms m-dimensional T-triangular triplets into 1-dimensional triplets.*

From Propositions 4.6 and 4.19 we obtain the following important result.

Proposition 4.21 *A function $F : [0,1]^m \to [0,1], m \geq 1$, aggregates fuzzy subgroups if and only if it aggregates T-indistinguishability operators.*

Proposition 4.22 *A function $F : [0,1]^m \to [0,1], m \geq 1$, aggregates fuzzy subgroups if and only if the following conditions hold:*

(i) $F(\overbrace{1,\ldots,1}^{m})=1$.

(ii)F transforms m-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

4.2 Aggregating Vague Groups

In this subsection we will obtain results on the aggregation of T-vague groups similar to the ones in the previous subsection.

Definition 4.23 *A function $V : [0,1]^m \to [0,1], m \geq 1$, aggregates T-vague groups if for any group $(G,\circ)$ and any collection of T-vague groups of G, $(G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m)$, then $V((G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m))$ is also a T-vague group G, where $V((G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m))$ is the T-vague group of G with the vague operation defined for all $x,y,z \in G$ by $V(\tilde{\circ}_1,\ldots,\tilde{\circ}_m)(x,y,z)=V(\tilde{\circ}_1(x,y,z),\ldots,\tilde{\circ}_m(x,y,z))$.*

Proposition 4.24 *Let $(G,\circ)$ be a group, $(G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m)$ T-vague groups of G, $E_1,\ldots,E_m$ their respective associated T-indistinguishability operators (i.e., $\tilde{\circ}_i(x,y,z) = E_i(x\circ y,z)$ for $i=1,\ldots,m$) and $F:[0,1]^m \to [0,1]$ a function. Then*

$$F(E_1,\ldots,E_m) = E_{F((G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m))}.$$

Corollary 4.25 *With the same notations as in the preceding Proposition 4.24,*

- *if F aggregates T-indistinguishability operators and $(G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m)$ are T-vague groups of G, then $F((G,\tilde{\circ}_1),\ldots,(G,\tilde{\circ}_m))$ is a T-vague group of G.*
- *if F aggregates T-vague groups $E_1,\ldots,E_m$ are invariant T-indistinguishability operators, then $F(E_1,\ldots,E_m)$ is an invariant T-indistinguishability operator.*

Similarly to fuzzy subgroups we obtain the following important result.

Proposition 4.26 *A function $F:[0,1]^m \to [0,1], m\geq 1$, aggregates T-indistinguishability operators if and only if it aggregates T-vague groups.*

Proposition 4.27 *A function $F:[0,1]^m \to [0,1], m\geq 1$, aggregates T-vague groups if and only if the following conditions hold:*

(i) $F(\overbrace{1,\ldots,1}^{m}) = 1.$

(ii)F transforms m-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

5 Concluding Remarks

In this work we have dealt with the problem of aggregating fuzzy subgroups and T-vague groups. Thanks to the close relation between these objects and T-indistinguishability operators, it results that the functions that aggregate them are the functions preserving T-triplets. In other words, the functions preserving T-indistinguishability operators, fuzzy subgroups and T-vague groups coincide. Interesting examples of these functions are

- The t-norm
- The minimum
- If the t-norm T is continuous Archimedean and g and additive of T, then
  - The weighted quasi-arithmetic means m_g generated by g
  - The ordered quasi-arithmetic means generated by g with decreasing weights.

We have studied the aggregation of fuzzy subgroups and T-vague groups from a functional point of view. Nevertheless, there are other ways to fusion or aggregate them that are not functional. Given a collection of T-indistinguishability operators, fuzzy subgroups or T-vague groups, a very natural way to aggregate them is calculating the transitive closure of their union, which is not a functional procedure. We have pointed out in this paper that there are a bijections between the sets of left and of right invariant Tindistinguishability operators and the set of fuzzy subgroups, and the set of T-vague groups, normal fuzzy subgroups and invariant T-indistinguishability operators. In fact these sets are isomorphic lattices [15] and the bijections preserve transitive closures (if $E_1,...,E_m$ are the left (right) T-indistinguishability operators associated to the fuzzy subgroups $\mu_1,...,\mu_m$ respectively and E is the transitive closure of $E_1 \cup ... \cup E_m$, then E is the left (right) T-indistinguishability operator associated to the transitive closure of $\mu_1 \cup ... \cup \mu_m$ and similarly with T-vague groups).

References

1. Inheung Chon On T-Fuzzy Groups Kangweon-Kyungki Math. Jour. 9 (2001) 149-156.
2. M. Demirci, Vague groups, J. Math. Anal. Appl. 230 (1999) 142-156.
3. M. Demirci, J. Recasens Fuzzy groups, fuzzy functions and fuzzy equivalence relations Fuzzy Sets and Systems 144 (2004) 441-458
4. S. Gottwald Fuzzy Sets and Fuzzy Logic: the Foundations of Applicationfrom a Mathematical Point of Wiew. Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Wiesbaden (1993)
5. P. Hájek P Metamathematics of fuzzy logic. Kluwer Academic Publishers, New York (1998).
6. J. Jacas, J. Recasens Aggregation of T-Transitive Relations International Journal of Intelligent Systems 18 (2003) 1193-1214.
7. E. P. Klement, R. Mesiar, E. Pap Triangular norms. Kluwer. Dordrecht (2000).
8. A.S. Mashour, M.H. Ghanim, F.I. Sidky Normal Fuzzy Subgroups Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat 20,2 (1990) 53-59.
9. G. Mayor, J. Recasens, Preserving T-transitivity CCIA2016 Barcelona (2016) 79–87.
10. R. Mesiar and V. Novak, Operations fitting triangular-norm-based biresiduation, Fuzzy Sets and Systems, 104 (1999) 77-84.
11. J. Mordeson, K. Bhutani, A. Rosenfeld Fuzzy Group Theory. Studies in Fuzziness and Soft Computing 182. Springer (2005).
12. A. Pradera, E Trillas, E Castiñeira On the aggregation of some classes of fuzzy relations.In:B. Bouchon-Meunier,J. Gutiérrez,L. Magdalena, R. Yager editors. Technologies for constructing intelligent systems. Heidelberg: Springer-Verlag (2002) 125-147.
13. A. Pradera, E. Trillas A note on pseudometrics aggregation. Int J Gen Syst. (2002) 41-51.
14. J. Recasens Indistinguishability operators. Modelling Fuzzy Equalities and Fuzzy Equivalence Relations. Studies in Fuzziness and Soft Computing 260. Springer (2010)
15. J. Recasens Permutable Indistinguishability Operators, Perfect Vague Groups and Fuzzy Subgroups. Information Sciences 196 (2012), 129-142.
16. A. Rosenfeld Fuzzy groups J. Math. Anal. Appl. 35 (1971) 512-517.
17. B. Schweizer, A. Sklar Probabilistic Metric Spaces. North-Holland. Amsterdam (1983).
18. L.A.Zadeh Similarity relations and fuzzy orderings Information Science 3 (1971) 177-200.