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Decay rates of Saint-Venant type for functionally graded heat-conducting materials

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Abstract

This paper investigates decay rates for the spatial behaviour of solutions for functionally graded heat-conducting materials. From a mathematical point of view, we obtain a new inequality of Poincaré type. This new result allows us to give new decay rates for functionally graded materials when the inhomogeneity depends on the radial variable and the axial variable of the cylinder. The case when the cross-section is increasing is also considered. Besides, we propose to obtain estimates for the case of mixtures. Some pictures illustrate our estimates.

keywords: Functionally graded materials, heat conduction, spatial decay estimates

1 Introduction

Saint-Venant's principle has been investigated from a mathematical and a thermomechanical points of view in recents years [1–7]. For these studies the authors try to evaluate how the perturbations produced on a part of the boundary of a solid are damped far away where they were applied. This is relevant to clarify where the influence of the perturbations can be neglected.

A considerable interest has been developed in investigating the influence of material inhomogeneity on the decay rate of Saint-Venant end effects in mechanical theories of elasticity, mixtures and piezoelectric materials [4, 8–15]. Also lower bounds for the decay has been obtained by means of the logarithmic convexity argument [16]. It is suitable to say that these studies are motivated by the big interest on functionally graded materials (FGMs). These materials are characterized by the continuous varying properties tailored to satisfy particular applications in engineering.

In the previously cited papers the main idea is to consider anti-plane shear deformations of elastic or mixtures of elastic materials. Therefore the problem becomes a problem on a semi-infinite strip. We have changed completely the aim of our interest. We are going to consider an isotropic, but now inhomogeneous heat conducting material which occupies a circular semi-infinite cylinder (or cone). In this context we propose two situations. The first is when the inhomogeneity of the material depends on the radial coordinate and the second when the inhomogeneity depends on the radial coordinate and the axial coordinate. The case when the cross-section is increasing is also considered.

In the first case we are driven to a singular Sturm-Liouville problem and the question is to determinate a lower bound for the first eigenvalue. To this end, we use a couple of approaches. One of them uses a result obtained in [12] which is useful in our case and the second one is more innovating because we propose a new Poincaré's inequality type for suitable weight functions. This new inequality

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requires that the weight satisfies certain conditions and we will give a large quantity of examples where the conditions hold. Then we sketch how the use of the usual arguments for spatial decay estimates [1, 17, 18] bring us to use again the new proposed Poincaré's inequality. New spatial decay estimates are obtained. In Section 6 we also consider the case when the geometry of the body is not a cylinder. Last section is devoted to point out how to extend our results in the case of mixtures.

2 Basic Equations

In this section we recall the problem we want to work about in a mathematical language. As our analysis involves the study of spatial estimates we will define our problem in a semi-infinite cylinder $R = D \times [0, \infty)$, where D is a two-dimensional circle of radius a.

Our aim is to study the rate of decay for the solutions of the problem determined by the equation

$$\left(K(r, x_3)u_{,i}\right)_{,i} = 0 \quad \text{in} \quad D \times [0, \infty) \quad (K \geq 0) \tag{2.1}$$

with the boundary conditions

$$u(\mathbf{x}) = 0 \quad \text{on} \quad \partial D \times [0, \infty) \tag{2.2}$$

$$u(x_1, x_2, 0) = f(x_1, x_2) \quad \text{on} \quad D \times \{0\} \tag{2.3}$$

and the asymptotic condition

$$u \longrightarrow 0 \quad \text{as} \quad x_3 \longrightarrow \infty \text{ (uniformly).} \tag{2.4}$$

In this paper we assume an inhomogeneity in the sense that the thermal conductivity depends on the variable

$$r = \left(x_1^2 + x_2^2\right)^{1/2} \tag{2.5}$$

and with respect to the axial variable x_3.

As our problem is defined in terms of the polar coordinates, we will recall our equation (2.1) in these coordinates. That is, we assume that

$$u = u(r, \theta, x_3) \tag{2.6}$$

where r is defined at (2.5) and

$$\theta = \arctan\left(\frac{x_2}{x_1}\right). \tag{2.7}$$

We have

$$u_{,1} = u_{,r}\frac{x_1}{r} - u_{,\theta}\frac{x_2}{r^2}, \quad u_{,2} = u_{,r}\frac{x_2}{r} + u_{,\theta}\frac{x_1}{r^2}$$

and

$$\begin{aligned}(K(r, x_3)u_{,1})_{,1} =& \frac{\partial K(r, x_3)}{\partial r}\frac{x_1}{r}\left(u_{,r}\frac{x_1}{r} - u_{,\theta}\frac{x_2}{r^2}\right) \\ &+ K(r, x_3)\left[u_{,rr}\frac{x_1^2}{r^2} - 2u_{,r\theta}\frac{x_1x_2}{r^3} + u_{,\theta\theta}\frac{x_2^2}{r^4} + u_{,r}\frac{r^2 - x_1^2}{r^3} + u_{,\theta}\frac{2x_1x_2}{r^4}\right],\end{aligned}$$

$$\begin{aligned}(K(r, x_3)u_{,2})_{,2} =& \frac{\partial K(r, x_3)}{\partial r}\frac{x_2}{r}\left(u_{,r}\frac{x_2}{r} + u_{,\theta}\frac{x_1}{r^2}\right) \\ &+ K(r, x_3)\left[u_{,rr}\frac{x_2^2}{r^2} + 2u_{,r\theta}\frac{x_1x_2}{r^3} + u_{,\theta\theta}\frac{x_1^2}{r^4} + u_{,r}\frac{r^2 - x_2^2}{r^3} - u_{,\theta}\frac{2x_1x_2}{r^4}\right].\end{aligned}$$

In view of (2.5) and after adding the last two equalities, equation (2.1) can be written as

$$\frac{\partial K(r,x_3)}{\partial r}u_{,r}+K(r,x_3)\left[u_{,rr}+\frac{u_{,\theta\theta}}{r^2}+\frac{u_{,r}}{r}\right]+\left[K(r,x_3)u_{,3}\right]_{,3}=0. \tag{2.8}$$

The solutions $u(r,\theta,z)$ of (2.8) are combinations of the functions

$$\psi_n(r,x_3)\left\{\begin{array}{c}\sin n\theta\\ \cos n\theta\end{array}\right\}, \tag{2.9}$$

where $\psi_n(r,x_3)$ satisfies

$$\frac{\partial K(r,x_3)}{\partial r}\psi_{n,r}+K(r,x_3)\left[\psi_{n,rr}+\frac{1}{r}\psi_{n,r}\right]+\left[K(r,x_3)\psi_{n,3}\right]_{,3}=\frac{n^2}{r^2}K(r,x_3)\psi_n. \tag{2.10}$$

It is worth noting that the right hand side term of (2.10) comes from the evaluation of $K(r,x_3)u_{,\theta\theta}/r^2$. In this case the equation we want to study can be written as

$$\frac{\partial K(r,x_3)}{\partial r}u_{,r}+K(r,x_3)\left[u_{,rr}+\frac{u_{,r}}{r}\right]+\left[K(r,x_3)u_{,3}\right]_{,3}=\frac{n^2}{r^2}K(r,x_3)u, \tag{2.11}$$

which can be rewritten as

$$\big(rK(r,x_3)u_{,r}\big)_{,r}+\big(rK(r,x_3)u_{,3}\big)_{,3}=\frac{n^2}{r}K(r,x_3)u. \tag{2.12}$$

It is worth noting that whenever we assume that $K(r,x_3)\geq 0$, the slowest solution will be obtained when $n=0$. Therefore in this paper we will study the equation

$$\big(rK(r,x_3)u_{,r}\big)_{,r}+\big(rK(r,x_3)u_{,3}\big)_{,3}=0. \tag{2.13}$$

The boundary conditions (2.2), (2.3) become

$$u(a,x_3)=0,\quad \text{on}\ \ \{a\}\times[0,\infty), \tag{2.14}$$

$$u(r,0)=f(r)\quad \text{on}\ \ [0,a]\times\{0\}. \tag{2.15}$$

The asymptotic condition (2.4) is also imposed.

2.1 Case $K=K(r)$

An important problem is determined in the case where the function K is independent of x_3; that is $K=K(r)$. Whenever $K(r)$ satisfy suitable conditions, we have that the problem is well posed. In fact, the general solutions of the equation (2.8) are combinations of the functions

$$u(r,\theta,x_3)=\phi_m(r)\left\{\begin{array}{c}\sin n\theta\\ \cos n\theta\end{array}\right\}\exp\left(-\lambda_{nm}x_3\right), \tag{2.16}$$

where $\phi_m(r)$ are the solutions of the Sturm-Liouville problem

$$\big(rK(r)\phi'(r)\big)'+\left(\lambda^2 r-\frac{n^2}{r}\right)K(r)\phi(r)=0 \tag{2.17}$$

with the boundary conditions

$$\phi(a)=0,\ |\phi(0)|<\infty. \tag{2.18}$$

The problem determined by (2.17)-(2.18) is a singular Sturm-Liouville problem and we will obtain the smallest value for λ when $n=0$.

As we want to obtain the smallest rate of decay, we will restrict our attention to the problem determined by the equation

$$\big(rK(r)\phi'(r)\big)'+\lambda^2 rK(r)\phi(r)=0 \tag{2.19}$$

with the corresponding boundary conditions (2.18).

3 K independent of x_3: First Approach

In this section we will try to obtain a lower bound for the first eigenvalue λ_1^2 determined by the problem (2.18)-(2.19).

3.1 Theoretical Analysis

We know that the first eigenvalue of the problem can be characterized by means of the minimum value of the Rayleigh quotient (see [19], p.184)

$$\lambda_1^2 = \min \frac{\int_0^a rK(r)\,(\phi'(r))^2\,dr}{\int_0^a rK(r)\,(\phi(r))^2\,dr} \tag{3.1}$$

This characterization shows that

$$\int_0^a rK(r)\,(\phi(r))^2\,dr \le \lambda_1^{-2} \int_0^a rK(r)\,\left(\phi'(r)\right)^2 dr \tag{3.2}$$

for every continuous function $\phi(r)$ satisfying condition (2.18).

Our first approach to this eigenvalue can be obtained with the help of the result ([12], p.119).

Lemma 3.1 *Let $F(r) \in \mathcal{C}^2(0,a)$ such that $F^{1/2}(r) > 0$ on $(0,a)$, $F(0) = 0$ and*

$$\left(F^{1/2}\right)'' \ge -c_1 \left(F^{1/2}\right)' - c_2 \left(F^{1/2}\right) \quad on\ (0,a), \tag{3.3}$$

where c_1, c_2 are constants such that

$$c_2 < \pi^2/a^2. \tag{3.4}$$

Then, there exists a positive constant $k_1 = k_1(|c_1|, c_2, a)$ such that

$$\int_0^a F(r)\phi^2(r)dr \le k_1 \int_0^a F(r)\left(\phi'(r)\right)^2 dr \tag{3.5}$$

for all continuous functions $\phi(r)$ such that $|\phi(0)|$ is bounded and $\phi(a) = 0$.

It is worth noting that (see [12], p.121)

$$k_1 = \frac{a^2(1 + |c_1|/(2\varepsilon))}{\pi^2\left[1 - (c_2 + |c_1|\varepsilon/2)\,a^2/\pi^2\right]}, \tag{3.6}$$

where

$$\varepsilon = -B + \sqrt{B^2 + (1 - CA)/A} \tag{3.7}$$

and

$$A = \frac{a^2}{\pi^2},\ B = \frac{|c_1|}{2},\ C = c_2. \tag{3.8}$$

3.2 Discussion

We now give several illustrative examples where the previous result can be applied.

Example 1. Suppose that

$$K^{1/2}(r) = K_0^{1/2}\left(\frac{r}{a^2}\right)^{1/2}\exp\left(\frac{mr}{a}\right), \tag{3.9}$$

where m is a dimensionless constant and K_0 is a positive constant.

If we take

$$F(r) = rK(r)$$

we see that

$$F^{1/2}(r) = K_0^{1/2}\frac{r}{a}\exp\left(\frac{mr}{a}\right).$$

It is not difficult to see ([12], p.123) that $F^{1/2}$ satisfies

$$\left(F^{1/2}\right)'' = -c_1\left(F^{1/2}\right)' - c_2F^{1/2}, \tag{3.10}$$

where

$$c_1 = -\frac{2m}{a},\ c_2 = \frac{m^2}{a^2}.$$

Inequality (3.4) holds, whenever

$$m^2 < \pi^2.$$

We can calculate (see [12], p.123) that

$$k_1 = \frac{a^2}{\pi^2}\left(1 - \frac{|m|}{\pi}\right)^{-2}\left(1 + \frac{2|m|}{\pi}\right)^{-1}.$$

Therefore a lower bound for the rate of decay is

$$k \equiv \frac{\pi}{a}\left(1 - \frac{|m|}{\pi}\right)\left(1 + \frac{2|m|}{\pi}\right)^{1/2}. \tag{3.11}$$

Example 2. Let us consider the case

$$K^{1/2}(r) = \left(\frac{K_0}{r}\right)^{1/2}\exp\left(\frac{mr}{a}\right)\sin\frac{br}{a}, \tag{3.12}$$

where m is a dimensionless parameter and K_0 is a positive constant. We take

$$F^{1/2}(r) = K_0^{1/2}\exp\left(\frac{mr}{a}\right)\sin\frac{br}{a}.$$

The function $F^{1/2}(r)$ is non-negative whenever

$$b < \pi.$$

Relation (3.10) holds, where

$$c_1 = -\frac{2m}{a},\ c_2 = \frac{m^2 + b^2}{a^2}.$$

Therefore condition (3.4) holds whenever $m^2 + b^2 < \pi^2$.

If we consider

$$\varepsilon = \frac{\pi}{a}\left[\sqrt{1-\frac{b^2}{\pi^2}}-\frac{|m|}{b}\right]$$

we can obtain (see [12], p.125)

$$k_1 = \frac{a^2}{\pi^2}\left(1+\frac{|m|}{\varepsilon a}\right)\left[1-\left(\frac{a}{\pi}\right)^2\left(\frac{m^2+b^2}{a^2}+\frac{|m|\varepsilon}{a}\right)\right]^{-1}$$

and the lower bound for the rate of decay will be

$$k \equiv \frac{\pi}{a}\left(1+\frac{|m|}{\varepsilon a}\right)^{-1/2}\left[1-\left(\frac{a}{\pi}\right)^2\left(\frac{m^2+b^2}{a^2}+\frac{|m|\varepsilon}{a}\right)\right]^{1/2}. \tag{3.13}$$

Example 3. Let us consider the case

$$K^{1/2}(r) = \left(\frac{K_0}{r}\right)^{1/2}\frac{1}{m}\left(1-\exp\left(-\frac{mr}{a}\right)\right), \tag{3.14}$$

where $m > 0$ is a dimensionless constant and K_0 is also a positive parameter. We have that

$$F^{1/2}(r) = \frac{K_0^{-1/2}}{m}\left(1-\exp\left(-\frac{mr}{a}\right)\right).$$

We know that

$$\left(F^{1/2}\right)'' = -\frac{m}{a}\left(F^{1/2}\right)'.$$

Therefore we can take

$$c_1 = \frac{m}{a},\ c_2 = 0.$$

And so

$$k_1 = \left(\frac{a}{\pi}\right)^2\frac{\left(1+d+\sqrt{1+d}\right)^2}{d(1+d)},$$

where $d = \left(\frac{2\pi}{m}\right)^2$. A lower bound for the rate of decay becomes

$$k \equiv \frac{\pi}{a}\left[\sqrt{1+\frac{m^2}{4\pi^2}}-\frac{m}{2\pi}\right]. \tag{3.15}$$

4 K independent of x_3: Second Approach

The estimates obtained in the previous section give some information on the lower bounds for the rate of decay for the solutions. However, it is worth noting that the theoretical result proposed in Section 3 could not be applied to the easier case where $K(r) = 1$. For this reason we have proposed to state another Poincaré's type inequality in such a way that the homogeneous case could be also applied. This new inequality will be used in a large quantity of examples.

4.1 A new theorem

The aim of this subsection is to state and prove the following theorem and corollaries.

Theorem 4.1 *Let $K^{1/2}(r) \in \mathcal{C}^2(0,a)$ such that $K^{1/2}(r) > 0$ on $(0,a)$ and such that $K^{1/2}(r) \geq 0$. Let us also suppose that there exist three constants c_1, c_2 and c_3 such that*

$$\left(K^{1/2}\right)'' \geq -c_1 \left(K^{1/2}\right)' - c_2 K^{1/2} \quad \text{on } (0,a), \tag{4.1}$$

and

$$\frac{c_1}{2} K^{1/2} + \left(K^{1/2}\right)' \geq -c_3 r K^{1/2} \quad \text{on } (0,a), \tag{4.2}$$

where

$$c_2 + c_3 < (z_0/a)^2 \tag{4.3}$$

and z_0 is the first zero of the Bessel function $J_0(r)$[1]*. Then, there exists a positive constant $k_1 = k_1(|c_1|, c_2, c_3)$ such that*

$$\int_0^a rK(r)\phi^2(r)dr \leq k_1 \int_0^a rK(r)\left(\phi'\right)^2(r)dr, \tag{4.4}$$

for all continuous function $\phi(r)$ such that $|\phi(0)| < \infty$ and $\phi(a) = 0$.

Proof: Let us consider $G(r) = K^{1/2}(r)\phi(r)$. We have that (see [19], p.300)

$$\int_0^a rG^2(r)dr \leq \left(\frac{a}{z_0}\right)^2 \int_0^a r\left(G'(r)\right)^2 dr. \tag{4.5}$$

In view of the definition of the function $G(r)$, we have after an integration by parts

$$\begin{aligned}\int_0^a r\left(G'(r)\right)^2 dr = \int_0^a rK(r)\left(\phi'\right)^2(r)\,dr - \int_0^a rK^{1/2}(r)\left(K^{1/2}\right)''(r)\phi^2(r)\,dr \\ - \int_0^a K^{1/2}(r)\left(K^{1/2}\right)'(r)\phi^2(r)dr.\end{aligned} \tag{4.6}$$

Therefore, we see that

$$\begin{aligned}\int_0^a rK(r)\phi^2(r)\,dr \leq \left(\frac{a}{z_0}\right)^2 \Bigg[\int_0^a rK(r)\left(\phi'\right)^2(r)\,dr - \int_0^a rK^{1/2}(r)\left(K^{1/2}\right)''(r)\phi^2(r)\,dr \\ - \int_0^a K^{1/2}(r)\left(K^{1/2}\right)'(r)\phi^2(r)\,dr\Bigg].\end{aligned} \tag{4.7}$$

But

$$-\int_0^a rK^{1/2}\left(K^{1/2}\right)''\phi^2(r)\,dr \leq c_1\int_0^a rK^{1/2}\left(K^{1/2}\right)'\phi^2(r)\,dr + c_2\int_0^a rK(r)\phi^2(r)\,dr, \tag{4.8}$$

and

$$\frac{1}{2}\int_0^a r\left(K(r)\right)'\phi^2(r)\,dr = -\int_0^a rK(r)\phi(r)\phi'(r)\,dr - \frac{1}{2}\int_0^a K(r)\phi^2(r)\,dr. \tag{4.9}$$

[1]It is known that $z_0 = 2.4048255577...$

From (4.7), (4.8) and (4.9) we obtain

$$\begin{aligned}\int_0^a rK(r)\phi^2(r)\,dr &\le \left(\frac{a}{z_0}\right)^2 \left[\int_0^a rK(r)\left(\phi'\right)^2(r)\,dr - c_1\int_0^a rK(r)\phi(r)\phi'(r)\,dr\right.\\ &\left.+c_2\int_0^a rK(r)\phi^2(r)\,dr\right] - \left(\frac{a}{z_0}\right)^2 \int_0^a \left(\frac{c_1}{2}K^{1/2}(r) + \left(K^{1/2}\right)'(r)\right)K^{1/2}(r)\phi^2(r)\,dr.\end{aligned} \tag{4.10}$$

In view of the condition (4.2), the term containing the last integral on the right hand side of (4.10) can be bounded by

$$\left(\frac{a}{z_0}\right)^2 c_3 \int_0^a rK(r)\phi^2(r)\,dr. \tag{4.11}$$

Therefore,

$$\begin{aligned}\int_0^a rK(r)\phi^2(r)\,dr \le \left(\frac{a}{z_0}\right)^2 &\left[\int_0^a rK(r)\left(\phi'\right)^2(r)\,dr - c_1\int_0^a rK(r)\phi(r)\phi'(r)\,dr\right.\\ &\left.+(c_2+c_3)\int_0^a rK(r)\phi^2(r)\,dr\right].\end{aligned} \tag{4.12}$$

So, if we apply the arithmetic-geometric mean inequality, we obtain that

$$\left[1-\left(c_2+c_3+\frac{|c_1|\varepsilon}{2}\right)\left(\frac{a}{z_0}\right)^2\right]\int_0^a rK(r)\phi^2(r)\,dr \le \left(\frac{a}{z_0}\right)^2\left(1+\frac{|c_1|}{2\varepsilon}\right)\int_0^a rK(r)\left(\phi'\right)^2(r)\,dr. \tag{4.13}$$

Here ε is an arbitrary positive constant. We may choose ε so small that

$$1-\left(c_2+c_3+\frac{|c_1|\varepsilon}{2}\right)\left(\frac{a}{z_0}\right)^2 > 0 \tag{4.14}$$

and so it follows that the following inequality

$$\int_0^a rK(r)\phi^2(r)\,dr \le Q(\varepsilon)\int_0^a rK(r)\left(\phi'\right)^2(r)\,dr \tag{4.15}$$

holds, where

$$Q(\varepsilon) = \frac{a^2\left(1+|c_1|/(2\varepsilon)\right)}{z_0^2\left[1-\left(c_2+c_3+|c_1|\varepsilon/2\right)a^2/z_0^2\right]}. \tag{4.16}$$

We now choose ε to minimize $Q(\varepsilon)$ and satisfying (4.14). Thus, we get

$$\varepsilon = -B+\sqrt{B^2+(1-CA)/A}, \tag{4.17}$$

where

$$A=\left(\frac{a}{z_0}\right)^2,\quad B=\frac{|c_1|}{2},\quad C=c_2+c_3. \tag{4.18}$$

For this choice of ε, the value for $Q(\varepsilon)$ is

$$Q(\varepsilon) = \frac{A\left(1+B/\varepsilon\right)}{1-\left(C+B\varepsilon\right)A}. \tag{4.19}$$

The desired inequality is obtained by taking

$$k_1 = Q(\varepsilon). \quad \square \tag{4.20}$$

In the case where $c_3 = 0$ in the previous theorem, we obtain directly the following result.

Corollary 4.1 *Let $K^{1/2}(r) \in \mathcal{C}^2(0,a)$ such that $K^{1/2}(r) > 0$ on $(0,a)$ and such that $K^{1/2}(r) \geq 0$. Let us also suppose that there exist two constants c_1 and c_2 such that*

$$\left(K^{1/2}\right)'' \geq -c_1 \left(K^{1/2}\right)' - c_2 K^{1/2} \quad \text{on } (0,a), \tag{4.21}$$

and

$$\frac{c_1}{2} K^{1/2} + \left(K^{1/2}\right)' \geq 0 \quad \text{on } (0,a), \tag{4.22}$$

where

$$c_2 < \left(z_0/a\right)^2. \tag{4.23}$$

Then, there exists a positive constant $k_1 = k_1(|c_1|, c_2)$ such that

$$\int_0^a rK(r)\phi^2(r)dr \leq k_1 \int_0^a rK(r)\left(\phi'\right)^2(r)dr, \tag{4.24}$$

for all continuous function $\phi(r)$ such that $|\phi(0)| < \infty$ and $\phi(a) = 0$.

In the case $c_1 = c_2 = 0$ we obtain the following result.

Corollary 4.2 *Let $K^{1/2}(r) \in \mathcal{C}^2(0,a)$ such that $K^{1/2}(r) > 0$ on $(0,a)$ and such that $K^{1/2}(r) \geq 0$. Let us also assume that*

$$\left(K^{1/2}\right)' \geq 0, \quad \text{and} \quad \left(K^{1/2}\right)'' \geq 0. \tag{4.25}$$

Then,

$$\int_0^a rK(r)\phi^2(r)dr \leq \left(\frac{a}{z_0}\right)^2 \int_0^a rK(r)\left(\phi'(r)\right)^2 dr, \tag{4.26}$$

for all continuous function $\phi(r)$ such that $|\phi(0)| < \infty$ and $\phi(a) = 0$.

Proof: In this case we can take $c_1 = c_2 = c_3 = 0$. The conditions of the theorem 4.1 are clearly satisfied. The estimate (4.1) holds. □

Remark. It is worth noting that if we have

$$0 \leq K_m^{1/2} \bar{K}^{1/2}(r) \leq K^{1/2}(r) \leq K_M^{1/2} \bar{K}^{1/2}(r), \tag{4.27}$$

where $\bar{K}^{1/2}(r)$ satisfies that the inequality

$$\int_0^a r\bar{K}(r)\phi^2(r)dr \leq k^* \int_0^a r\bar{K}(r)\left(\phi'\right)^2(r)dr \tag{4.28}$$

holds, we can obtain a similar inequality for $K(r)$. In fact, we have

$$\int_0^a rK(r)\phi^2(r)dr \leq K_M \int_0^a r\bar{K}(r)\phi^2(r)dr \leq \frac{K_M}{K_m} k^* \int_0^a rK(r)\left(\phi'(r)\right)^2 dr. \tag{4.29}$$

4.2 Discussion

Example 4. We consider the case

$$K^{1/2}(r) = K_{00}^{1/2} \exp\left(\frac{mr}{a}\right) + K_{01}^{1/2} r \exp\left(\frac{mr}{a}\right), \tag{4.30}$$

where $K_{00}^{1/2}$, $K_{01}^{1/2}$ are non-negative constants and m is a dimensionless parameter. We have that

$$\left(K^{1/2}\right)' = \frac{m}{a}\left[K_{00}^{1/2} + K_{01}^{1/2}\left(r + \frac{a}{m}\right)\right] \exp\left(\frac{mr}{a}\right), \tag{4.31}$$

$$\left(K^{1/2}\right)'' = \left(\frac{m}{a}\right)^2 \left[K_{00}^{1/2} + K_{01}^{1/2}\left(r + 2\left(\frac{a}{m}\right)^2\right)\right] \exp\left(\frac{mr}{a}\right) \tag{4.32}$$

So,

$$\left(K^{1/2}\right)'' = \frac{2m}{a}\left(K^{1/2}\right)' - \left(\frac{m}{a}\right)^2 K^{1/2} \tag{4.33}$$

and condition (4.21) is satisfied with

$$c_1 = -\frac{2m}{a},\ c_2 = \left(\frac{m}{a}\right)^2. \tag{4.34}$$

At the same time

$$-\frac{m}{a}K^{1/2} + \left(K^{1/2}\right)' = K_{01}^{1/2} \exp\left(\frac{mr}{a}\right) \geq 0 \tag{4.35}$$

and then condition (4.22) is also satisfied. However, to guarantee condition (4.23), we need to impose that

$$m^2 < z_0^2. \tag{4.36}$$

In this case we have

$$A = \left(\frac{a}{z_0}\right)^2,\ B = \left|\frac{m}{a}\right|,\ C = \left(\frac{m}{a}\right)^2. \tag{4.37}$$

So that

$$\varepsilon = \frac{z_0}{a}\left(1 - \frac{|m|}{z_0}\right) \tag{4.38}$$

and we obtain

$$k_1 = \left(\frac{a}{z_0}\right)^2 \left(1 - \frac{|m|}{z_0}\right)^{-2} \left(1 + \frac{2|m|}{z_0}\right)^{-1}. \tag{4.39}$$

Therefore a lower bound for the rate of decay is

$$k \equiv \frac{z_0}{a}\left(1 - \frac{|m|}{z_0}\right)\left(1 + \frac{2|m|}{z_0}\right)^{1/2}. \tag{4.40}$$

Several sub-families could be considered. When $K_{01} = 0$, we have an estimate for the function

$$K^{1/2}(r) = K_0^{1/2} \exp\left(\frac{mr}{a}\right). \tag{4.41}$$

However, this family is restricted to the condition (4.36). Later, we will use another approach that improves the results obtained in the case of the family (4.41) when $m > 0$.

A second family could be considered if we assume that $m \to 0$ in (4.30). We have

$$K^{1/2}(r) = K_{00}^{1/2} + rK_{01}^{1/2}. \tag{4.42}$$

In this case the estimate (4.40) gives the lower bound

$$k \equiv \frac{z_0}{a} \tag{4.43}$$

which is exact in the particular case of homogeneous materials. In fact, this estimate could be also obtained directly from corollary 4.2.

Example 5. We now consider the family of functions

$$K^{1/2}(r) = K_0^{1/2} \exp\left(\frac{mr}{a}\right) \sin\left(\frac{br}{a} + \varphi_0\right), \tag{4.44}$$

where $\varphi_0 \geq 0$, $b > 0$ and m are dimensionless constants. If we assume that $\varphi_0 < \pi$ and $b < \pi - \varphi_0$, the function (4.44) is strictly positive in the interior of $(0, a)$. We have that

$$\left(K^{1/2}\right)' = \frac{K_0^{1/2}}{a} \exp\left(\frac{mr}{a}\right) \left[m \sin\left(\frac{br}{a} + \varphi_0\right) + b \cos\left(\frac{br}{a} + \varphi_0\right)\right] \tag{4.45}$$

$$\left(K^{1/2}\right)'' = \frac{K_0^{1/2}}{a} \exp\left(\frac{mr}{a}\right) \left[(m^2 - b^2) \sin\left(\frac{br}{a} + \varphi_0\right) + 2mb \cos\left(\frac{br}{a} + \varphi_0\right)\right] \tag{4.46}$$

So,

$$\left(K^{1/2}\right)'' = \frac{2m}{a} \left(K^{1/2}\right)' - \frac{m^2 + b^2}{a^2} K^{1/2}. \tag{4.47}$$

Our inequality is satisfied, where

$$c_1 = -\frac{2m}{a}, \ c_2 = \frac{m^2 + b^2}{a^2}. \tag{4.48}$$

To guarantee the condition (4.23), we must impose

$$m^2 + b^2 < z_0^2. \tag{4.49}$$

On the other hand, we have

$$-\frac{m}{a} K^{1/2} + \left(K^{1/2}\right)' = \frac{bK_0^{1/2}}{a} \exp\left(\frac{mr}{a}\right) \cos\left(\frac{br}{a} + \varphi_0\right). \tag{4.50}$$

To guarantee that the right hand side of (4.50) is positive we need to impose that

$$b + \varphi_0 < \pi/2, \tag{4.51}$$

which is a stronger condition to the one proposed before on b and φ_0.

In this situation we can take

$$A = \left(\frac{a}{z_0}\right)^2, \ B = \left|\frac{m}{a}\right|, \ C = \frac{m^2 + b^2}{a^2}. \tag{4.52}$$

So,

$$\varepsilon = \frac{z_0}{a} \left[\sqrt{1 - \frac{b^2}{z_0^2}} - \frac{|m|}{z_0}\right], \tag{4.53}$$

which gives

$$Q = \left(\frac{a}{z_0}\right)^2 \left(1 + \left|\frac{m}{a\varepsilon}\right|\right) \left[1 - \left(\frac{a}{z_0}\right)^2 \left(\frac{m^2 + b^2}{a^2} + \frac{|m|\varepsilon}{a}\right)\right]^{-1}, \tag{4.54}$$

which gives a lower bound for the decay

$$k \equiv \frac{z_0}{a}\left(1+\frac{|m|}{a\varepsilon}\right)^{-1/2}\left[1-\left(\frac{a}{z_0}\right)^2\left(\frac{m^2+b^2}{a^2}+\frac{|m|\varepsilon}{a}\right)\right]^{1/2}. \tag{4.55}$$

Notice the symmetry with respect to the sign of m. Taylor's development of k centered at zero for $m \geq 0$ is[2]

$$\begin{aligned}&\frac{z_0}{a}\left[1-\left(\frac{b}{z_0}\right)^2\right]^{1/2}-\frac{1}{a}m+\frac{1.66868...\times 10^{-17}}{a(1-0.172915...b^2)^{1/2}}m^2+\frac{6.03144...\times 10^{-17}}{a(-5.78319...+b^2)}m^3\\&+\frac{2.60732...\times 10^{-18}}{a(1-0.172915...b^2)^{3/2}}m^4-\frac{2.08585...\times 10^{-18}}{a(1-0.172915...)^2}m^5+O[m^6].\end{aligned} \tag{4.56}$$

So we can approximate our lower bound (4.55) as

$$k \approx \frac{z_0}{a}\left[1-\left(\frac{b}{z_0}\right)^2\right]^{1/2}-\frac{|m|}{a}. \tag{4.57}$$

In case that $m \to 0$ we obtain the function

$$K^{1/2}(r)=K_0^{1/2}\sin\left(\frac{br}{a}+\varphi_0\right). \tag{4.58}$$

The lower bound for the decay rate for the family (4.58) is

$$k \equiv \frac{z_0}{a}\left[1-\left(\frac{b}{z_0}\right)^2\right]^{1/2}. \tag{4.59}$$

Example 6. Another interesting example is given when

$$K^{1/2}(r)=K_{01}^{1/2}\exp\left(\frac{m_1 r}{a}\right)+K_{02}^{1/2}\exp\left(\frac{m_2 r}{a}\right), \tag{4.60}$$

where $K_{01}^{1/2}$ and $K_{02}^{1/2}$ are non-negative and m_1, m_2 are dimensionless constants.

We know that

$$\left(K^{1/2}\right)''=\frac{m_1+m_2}{a}\left(K^{1/2}\right)'-\frac{m_1 m_2}{a^2}K^{1/2}. \tag{4.61}$$

We can take

$$c_1=-\frac{m_1+m_2}{a},\ c_2=\frac{m_1 m_2}{a^2}. \tag{4.62}$$

Then condition (4.23) is satisfied whenever

$$m_1 m_2 < z_0^2. \tag{4.63}$$

On the other hand,

$$\frac{1}{2}c_1 K^{1/2}+\left(K^{1/2}\right)'=\frac{1}{2a}(m_2-m_1)\left[K_{02}^{1/2}\exp\left(\frac{m_2 r}{a}\right)-K_{01}^{1/2}\exp\left(\frac{m_1 r}{a}\right)\right] \tag{4.64}$$

[2] We note that this approximation has been obtained with the help of Wolfram Mathematica.

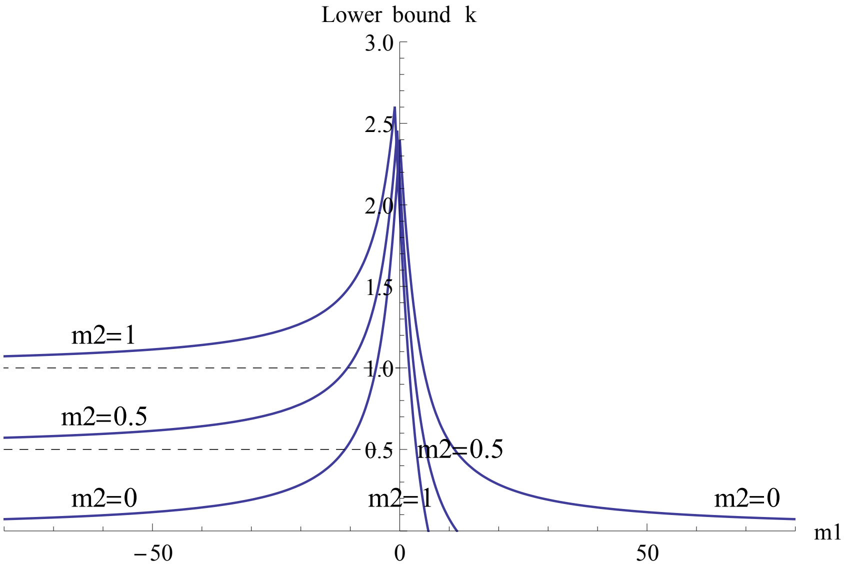

Figure 1: *Lower bound for the decay for Example 6 with $a = 1$ and some small $m_2 \geq 0$.*

which is positive if we assume that

$$(m_2 - m_1)\left(K_{02}^{1/2} - K_{01}^{1/2}\right) \geq 0. \tag{4.65}$$

In this case, we can take

$$A = \left(\frac{a}{z_0}\right)^2, \; B = \frac{|m_1 + m_2|}{2a}, \; C = \frac{m_1 m_2}{a^2}. \tag{4.66}$$

We then obtain that

$$\varepsilon = \frac{-|m_1 + m_2| + \sqrt{|m_1 + m_2|^2 + 4z_0^2 - 4m_1 m_2}}{2a} \tag{4.67}$$

and

$$Q = \frac{\left(\frac{a}{z_0}\right)^2 \left(1 + \frac{|m_1+m_2|}{2a\varepsilon}\right)}{1 - \left(\frac{m_1 m_2}{a^2} + \frac{|m_1+m_2|\varepsilon}{2a}\right)\left(\frac{a}{z_0}\right)^2}. \tag{4.68}$$

The lower bound for the rate of decay becomes

$$k \equiv \frac{\left[\left(\frac{z_0}{a}\right)^2 - \left(\frac{m_1 m_2}{a^2} + \frac{|m_1+m_2|\varepsilon}{2a}\right)\right]^{1/2}}{\left(1 + \frac{|m_1+m_2|}{2a\varepsilon}\right)^{1/2}} \tag{4.69}$$

In Figures 1 and 2 we have represented the dependence of the lower bound for the decay with respect to the parameters (4.69). We have fixed the radius of the cylinder $a = 1$. We note that the domain of m_1 varies according to m_2 in order to satisfy condition (4.63). Figure 1 corresponds to some small values of $0 \leq m_2 \leq 1$. The graph for the small negative values of m_2 is symmetrical with respect to the ordinate axis of the graph of Figure 1.

Figure 2 illustrates the dependence of the lower bound with respect to the parameters for some negative values $m_2 < -1$. The graph for positive values of m_2 is symmetrical with respect to the ordinate axis of the previous figure.

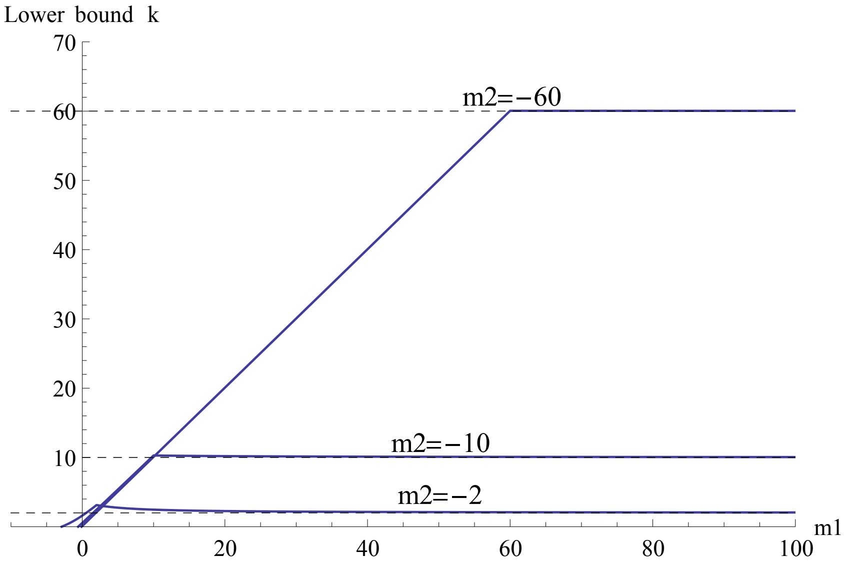

Figure 2: *Lower bound for the decay for Example 6 with $a = 1$ and some $m_2 \leq 0$.*

It is worth noting that, for $a = 1$ and fixed m_2, the lower bound $k = k(m_1)$ given by (4.69) tends to $|m_2|$ as $m_1 \to -\infty$ or $m_1 \to +\infty$ according to $m_2 \geq 0$ or $m_2 \leq 0$, respectively.

A particular family of examples of (4.60) is when

$$m_1 = m \;\text{ and }\; m_2 = -m, \; m > 0. \tag{4.70}$$

We then have

$$c_1 = 0, \; c_2 = -\frac{m^2}{a^2}. \tag{4.71}$$

To satisfy (4.65), we need to suppose that $K_{01}^{1/2} \geq K_{02}^{1/2}$. We have that

$$A = \left(\frac{a}{z_0}\right)^2, \; B = 0, \; C = -\frac{m^2}{a^2} \tag{4.72}$$

and

$$\varepsilon = \frac{\sqrt{z_0^2 + m^2}}{a}. \tag{4.73}$$

We arrive that a lower decay rate is

$$k \equiv \frac{\sqrt{z_0^2 + m^2}}{a}. \tag{4.74}$$

For instance, if we assume

$$K^{1/2}(r) = K_{01}^{1/2} \cosh\left(\frac{mr}{a}\right) + K_{02}^{1/2} \sinh\left(\frac{mr}{a}\right) \tag{4.75}$$

we can write

$$K^{1/2}(r) = \frac{1}{2}\left(K_{01}^{1/2} + K_{02}^{1/2}\right) \exp\left(\frac{mr}{a}\right) + \frac{1}{2}\left(K_{01}^{1/2} - K_{02}^{1/2}\right) \exp\left(-\frac{mr}{a}\right). \tag{4.76}$$

Therefore, if we assume that $K_{01}^{1/2} \geq \left|K_{02}^{1/2}\right|$, we can apply the previous arguments.

Another interesting case corresponds to the function

$$K^{1/2}(r) = K_0^{1/2} \exp\left(\frac{mr}{a}\right). \tag{4.77}$$

When we assume that $m > 0$ and $K_0^{1/2} > 0$, we can recover the previous case and we obtain the lower bound

$$k \equiv \frac{\sqrt{z_0^2 + m^2}}{a}. \tag{4.78}$$

This estimate improves that the one obtained in the example 4 for the case of the exponential. However, here we need to impose that $m > 0$.

Example 7. Now, we give an example satisfying the assumptions of the corollary 4.2. If we consider

$$K^{1/2}(r) = K_0^{1/2} \left(1 + \frac{mr}{a}\right)^{\beta}, \; \beta \geq 1, \tag{4.79}$$

where m is a dimensionless positive constant, we have that z_0/a is a lower bound for the decay.

Example 8. If we take

$$K^{1/2}(r) = K_0^{1/2} \left(1 + \frac{mr}{a}\right)^{\beta}, \; 0 < \beta \leq 1 \tag{4.80}$$

we can not apply directly the theorem neither the corollaries. However, we have that

$$\left(1 + \frac{mr}{a}\right)^{\beta} = \left(1 + \frac{mr}{a}\right)^{\beta-1} \left(1 + \frac{mr}{a}\right). \tag{4.81}$$

We note that

$$(1+m)^{\beta-1} \leq \left(1 + \frac{mr}{a}\right)^{\beta-1} \leq 1. \tag{4.82}$$

In view of the remark after the corollary 4.2 and the comments in the Example 7, we can see that a lower bound for the decay in this case can be

$$\frac{z_0}{a(1+m)^{1-\beta}}. \tag{4.83}$$

We can extend the analysis to the case when $\beta < 0$ by means a recurrence.

We could also consider lower bounds for the cases when we combine $\left(1 + \frac{mr}{a}\right)^{\beta}$, $\beta < 1$, with the examples 4 to 7 considered previously. But we left to obtain lower bounds for the decay as an exercise for the reader.

So far we have seen several examples to obtain lower bounds by means of the corollaries. We now consider an example to apply Theorem 4.1.

Example 9. We take

$$K^{1/2}(r) = K_0^{1/2} \exp\left(\frac{mr^2}{a^2}\right), \tag{4.84}$$

where m is a dimensionless real constant. We have that

$$\left(K^{1/2}\right)' = K_0^{1/2} \frac{2mr}{a^2} \exp\left(\frac{mr^2}{a^2}\right) \tag{4.85}$$

$$\left(K^{1/2}\right)'' = K_0^{1/2} \left(\frac{2m}{a^2} + \frac{4r^2m^2}{a^4}\right) \exp\left(\frac{mr^2}{a^2}\right). \tag{4.86}$$

Thus, we have that

$$\left(K^{1/2}\right)'' \geq \frac{2m}{a^2} K_0^{1/2} \exp\left(\frac{mr^2}{a^2}\right) \tag{4.87}$$

and condition (4.1) is satisfied with

$$c_1 = 0,\ c_2 = -\frac{2m}{a^2}. \tag{4.88}$$

Since

$$\left(K^{1/2}\right)' = K_0^{1/2} \frac{2m}{a^2} r \exp\left(\frac{mr^2}{a^2}\right), \tag{4.89}$$

condition (4.2) holds with

$$c_3 = -\frac{2m}{a^2}. \tag{4.90}$$

Moreover, if $m \geq 0$, condition (4.3) is satisfied

$$c_2 + c_3 = -\frac{4m}{a^2} \leq 0 < (z_0/a)^2. \tag{4.91}$$

But, if $m < 0$, condition (4.3) is satisfied whenever

$$-\frac{4m}{a^2} < \frac{z_0^2}{a^2}. \tag{4.92}$$

That is, $-z_0^2/4 < m < 0$. Now, for $m > -z_0^2/4$, we consider

$$A = \left(\frac{a}{z_0}\right)^2,\ \ B = 0,\ \ C = -\frac{4m}{a^2} \tag{4.93}$$

and

$$Q(\varepsilon) = \frac{a^2}{z_0^2 + 4m}. \tag{4.94}$$

Therefore, a lower bound for the rate of decay is

$$k \equiv \frac{\sqrt{z_0^2 + 4m}}{a},\ \ m > -z_0^2/4. \tag{4.95}$$

Clearly, for $m > 0$ the lower bound is greater than for $m < 0$. So, the decay is faster when $m > 0$.

Notice that, for $m < 0$, it is not possible to apply any of the above corollaries (4.1), (4.2) because $\left(K^{1/2}\right)' < 0$ and neither (4.22) nor (4.25) are satisfied.

5 Inhomogeneity also in the axial direction

The aim of this section is to study the problem determined by the equation (2.12) together with the boundary conditions (2.14)-(2.15).

We want to obtain lower bounds for the rate of decay. To do that, we cannot continue with the proposed arguments in the previous section. We will use the usual energy arguments joined with the new Poincaré type inequality we proposed before.

5.1 Theoretical Aspects

We now consider the problem determined by the equation (2.12) in the strip $[0,a]\times[0,\infty)$. We assume the boundary conditions

$$u(a,x_3)=0, \quad \text{on } \ \{a\}\times[0,\infty), \tag{5.1}$$

$$|u(0,x_3)| \ \text{ bounded on } \ \{0\}\times[0,\infty), \tag{5.2}$$

$$u(r,0)=f(r) \quad \text{on } \ [0,a]\times\{0\}. \tag{5.3}$$

The asymptotic condition (2.4) is also imposed.

In this section we suppose that $K^{1/2}(r,x_3)$ satisfies:

(I) $\dfrac{\partial^2\left(K^{1/2}\right)}{\partial r^2} \geq -c_1(x_3)\dfrac{\partial\left(K^{1/2}\right)}{\partial r} - c_2(x_3)K^{1/2}$.

(II) $\dfrac{c_1(x_3)}{2}K^{1/2} + \dfrac{\partial\left(K^{1/2}\right)}{\partial r} \geq -rc_3(x_3)K^{1/2}$.

Here, $c_1(x_3)$, $c_2(x_3)$ and $c_3(x_3)$ are three functions such that

$$c_2(x_3)+c_3(x_3) < \left(\frac{z_0}{a}\right)^2, \quad \text{for all } \ x_3\geq 0. \tag{5.4}$$

We note that in this situation we have

$$\int_0^a rK(r,x_3)u^2dr \leq k_1(|c_1(x_3)|,c_2(x_3),c_3(x_3),a)\int_0^a rK(r,x_3)u_{,r}^2dr \tag{5.5}$$

for every function $u(r,x_3)$ such that satisfies conditions (5.1)–(5.3) and (2.4).

In fact, we can calculate

$$k_1(x_3)=\frac{A\left(1+B(x_3)/\varepsilon(x_3)\right)}{1-\left(C(x_3)+B(x_3)\varepsilon(x_3)\right)A}, \tag{5.6}$$

where

$$A=\frac{a^2}{z_0^2}, \quad B(x_3)=\frac{|c_1(x_3)|}{2}, \quad C(x_3)=c_2(x_3)+c_3(x_3) \tag{5.7}$$

and

$$\varepsilon(x_3)=-B+\sqrt{B^2+(1-AC)/A}. \tag{5.8}$$

Now, if we define

$$H(x_3)=-\int_0^a rK(r,x_3)uu_{,3}\,dr \tag{5.9}$$

we see that

$$H'(x_3)=\int_0^a rK(r,x_3)(u_{,r}^2+u_{,3}^2)dr. \tag{5.10}$$

The key point is to evaluate $H(x_3)$ in terms of the derivative. We see

$$|H(x_3)| \leq \left(\int_0^a rK(r,x_3)u^2dr\right)^{1/2}\left(\int_0^a rK(r,x_3)u_{,3}^2dr\right)^{1/2}. \tag{5.11}$$

If we use the inequality (5.5), we see that

$$|H(x_3)| \leq \frac{1}{2}k_1^{1/2}(x_3)H'(x_3) \tag{5.12}$$

From where we obtain that the function

$$E(x_3) = \int_{x_3}^{\infty} \int_0^a rK(r, x_3) \left(u_{,r}^2 + u_{,3}^2\right) dr dx_3 \tag{5.13}$$

satisfies the inequality (see [1, 17, 18] for the details in the analysis)

$$E(x_3) \le E(0) \exp\left[-2\int_0^{x_3} k_1^{-1/2}(\xi)\, d\xi\right]. \tag{5.14}$$

If we denote by

$$\bar{k}(x_3) = \int_0^{x_3} k_1^{-1/2}(\xi)\, d\xi, \tag{5.15}$$

we obtain a lower bound for the decay which is

$$\exp\left(\bar{k}(x_3)\right). \tag{5.16}$$

We also note that with the help of the arguments recalled in Section 3, we could also obtain an alternative approach to this problem. However, we believe that this is not a difficult task after the arguments proposed in this section and the ideas developed at [12].

5.2 Discussion

Example 10. An easy example corresponds to the case that

$$K^{1/2}(r, x_3) = K_{00}^{1/2} + rK_{01}^{1/2}(x_3), \tag{5.17}$$

where $K_{00}^{1/2}$ and $K_{01}^{1/2}$ are non-negative. This case corresponds to

$$\frac{\partial^2 \left(K^{1/2}\right)}{\partial r^2} = 0, \quad \frac{\partial \left(K^{1/2}\right)}{\partial r} = K_0^{1/2}(x_3). \tag{5.18}$$

We can take $c_1(x_3) = c_2(x_3) = c_3(x_3) = 0$ and (II) is also satisfied. We have that $k_1 = a/z_0$ and then,

$$\exp\left(\bar{k}(x_3)\right) = \exp\left(\frac{z_0}{a}x_3\right). \tag{5.19}$$

Example 11. Another illustrative example corresponds to the case

$$K^{1/2}(r, x_3) = K_{01}^{1/2}(x_3)\cosh\left(\frac{mr}{a}\right) + K_{02}^{1/2}(x_3)\sinh\left(\frac{mr}{a}\right), \tag{5.20}$$

where $m > 0$ and $K_{01}^{1/2}(x_3)$, $K_{02}^{1/2}(x_3)$ are two non-negative functions. Following the ideas proposed in Example 6, we can see that

$$\exp\left(\bar{k}(x_3)\right) = \exp\left(\frac{\sqrt{z_0^2 + m^2}}{a}x_3\right). \tag{5.21}$$

Example 12. We now consider the case

$$K^{1/2}(r, x_3) = K_{01}^{1/2}\cosh\left[m\left(\frac{x_3}{a}\right)^{1/2}\frac{r}{a}\right] + K_{02}^{1/2}\sinh\left[m\left(\frac{x_3}{a}\right)^{1/2}\frac{r}{a}\right], \tag{5.22}$$

where $m > 0$ is a dimensionless parameter and $K_{01}^{1/2}$, $K_{02}^{1/2}$ are non-negative. We see that

$$\frac{\partial \left(K^{1/2}\right)}{\partial r} \ge 0 \tag{5.23}$$

and

$$\frac{\partial^2 \left(K^{1/2}\right)}{\partial r^2} = \left(\frac{m}{a}\right)^2 \left(\frac{x_3}{a}\right) K^{1/2}. \tag{5.24}$$

We have that $c_1(x_3) = 0$ and $c_2(x_3) = -\left(\frac{m}{a}\right)^2 \left(\frac{x_3}{a}\right)$. In this case,

$$\bar{k}(x_3) = \frac{z_0}{a} \int_0^{x_3} \left(1 + \frac{m^2 \xi}{z_0^2 a}\right)^{1/2} d\xi = \frac{2z_0^3}{3m^2} \left[\left(1 + \frac{m^2 x_3}{z_0^2 a}\right)^{3/2} - 1\right]. \tag{5.25}$$

So,

$$\bar{k}(x_3) \sim \frac{2m}{3} \left(\frac{x_3}{a}\right)^{3/2}, \tag{5.26}$$

which is faster than the usual linear exponential decay obtained in Examples 9 and 10.

6 Increasing Cross-Section

Another interesting aspect is considered in the case when we assume that the radius of the cross-section is increasing with the variable x_3. That is, we will have again the problem determined by the equation (2.12) in the region determined by

$$\{(r, x_3) | x_3 \geq 0, 0 \leq r \leq h(x_3)\}, \tag{6.1}$$

where $h(x_3)$ is a positive function. Then we assume that

$$|u(0, x_3)| < \infty, \quad u(h(x_3), x_3) = 0. \tag{6.2}$$

As in Section 5, if we define

$$H(x_3) = -\int_0^{h(x_3)} rK(r, x_3) u u_{,3} dr, \tag{6.3}$$

we have that

$$H'(x_3) = \int_0^{h(x_3)} rK(r, x_3) \left(u_{,r}^2 + u_{,3}^2\right) dr. \tag{6.4}$$

And so

$$|H(x_3)| \leq \frac{1}{2} k_1^{1/2}(x_3) H'(x_3), \tag{6.5}$$

where

$$k_1 = k_1(|c_1(x_3)|, c_2(x_3), c_3(x_3), h(x_3)). \tag{6.6}$$

That is in this case we have

$$A(x_3) = \frac{h^2(x_3)}{z_0^2}, \quad B(x_3) = \frac{|c_1(x_3)|}{2}, \quad C(x_3) = c_2(x_3) + c_3(x_3). \tag{6.7}$$

If we assume a similar condition to (5.4), but with $h(x_3)$ depending on the variable x_3, we see that

$$c_2(x_3) + c_3(x_3) \leq \left(\frac{z_0}{h(x_3)}\right)^2. \tag{6.8}$$

In fact, when c_2 and c_3 are independent of x_3 and the length of the interval tends to infinite, we see that $c_2 + c_3 \leq 0$. If we consider Example 11, we see that

$$\bar{k}(x_3) = \int_0^{x_3} \frac{\sqrt{z_0^2 + m^2}}{h(\xi)} d\xi. \tag{6.9}$$

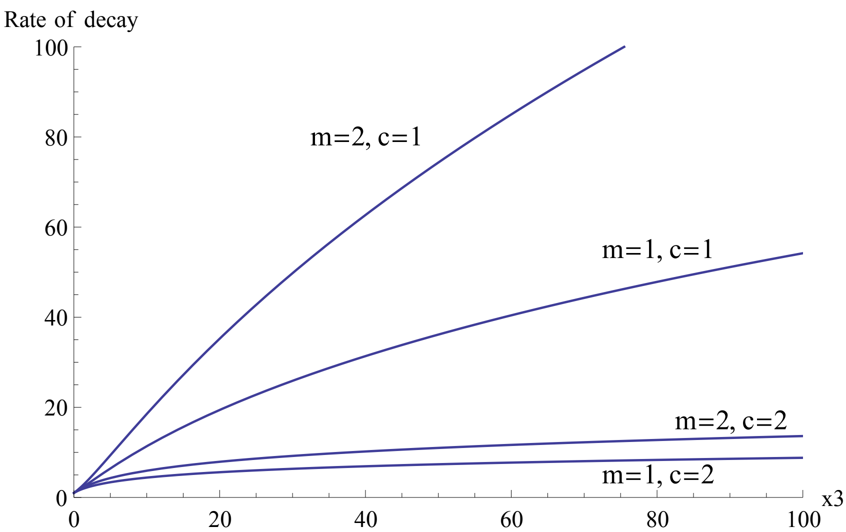

Figure 3: *Rate of decay* $\exp(\bar{k}(x_3))$ *for some* m *and* c.

In case we consider

$$h(\xi) = 1 + c\xi, \quad c > 0 \tag{6.10}$$

we see that

$$\bar{k}(x_3) = \frac{\sqrt{z_0^2 + m^2}}{c} \ln(1 + cx_3) \tag{6.11}$$

and therefore

$$\exp\left(\bar{k}(x_3)\right) = (1 + cx_3)^{\frac{\sqrt{z_0^2+m^2}}{c}}, \tag{6.12}$$

which is a decay rate of polynomial type.

If we assume that

$$h(\xi) = (e + c\xi)\ln(e + c\xi), \tag{6.13}$$

we have that

$$\bar{k}(x_3) = \frac{\sqrt{z_0^2 + m^2}}{c} \ln\left[\ln(e + cx_3)\right] \tag{6.14}$$

and

$$\exp\left(\bar{k}(x_3)\right) = \left[\ln(e + cx_3)\right]^{\frac{\sqrt{z_0^2+m^2}}{c}}, \tag{6.15}$$

which gives a rate of decay slower that the polynomial decay. We have represented the graph of the rate of decay (6.15) in Figure 3, for some values of m and c.

7 Case of a mixture

The arguments proposed in the previous sections can be adapted to study the case of a mixture of heat conducting rigid solid. In this case, we have to study the system (see [20–22])

$$\begin{cases} \left(a_{11}K(r, x_3)u_{,i}\right)_{,i} + \left(a_{12}K(r, x_3)w_{,i}\right)_{,i} - cK(r, x_3)(u - w) = 0 \\ \left(a_{12}K(r, x_3)u_{,i}\right)_{,i} + \left(a_{22}K(r, x_3)w_{,i}\right)_{,i} + cK(r, x_3)(u - w) = 0 \end{cases} \tag{7.1}$$

To this system we adjoin the boundary conditions the boundary conditions

$$u(\mathbf{x}) = w(\mathbf{x}) = 0 \quad \text{on} \quad \partial D \times [0, \infty), \tag{7.2}$$

$$\left. \begin{array}{l} u(x_1, x_2, 0) = f(x_1, x_2) \\ w(x_1, x_2, 0) = g(x_1, x_2) \end{array} \right\} \quad \text{on} \quad D \times \{0\}, \tag{7.3}$$

and the asymptotic conditions

$$u, w \longrightarrow 0 \quad \text{as} \quad x_3 \longrightarrow \infty \quad \text{(uniformly).} \tag{7.4}$$

We here assume that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \tag{7.5}$$

is positive definite and $c > 0$. Adding the two equations of (7.1), it is clear that the function

$$z_1 = (a_{11} + a_{12})u + (a_{12} + a_{22})w \tag{7.6}$$

satisfies the equation (2.1). On the other side, the function

$$z_2 = u - w \tag{7.7}$$

satisfies

$$(K(r, x_3)z_{2,i})_{,i} - \delta K(r, x_3)z_2 = 0, \tag{7.8}$$

where

$$\delta = c\, \frac{a_{11} + a_{22} + 2a_{12}}{a_{11}a_{22} - a_{12}^2} > 0. \tag{7.9}$$

In fact, if we take the first equation of (7.1) minus $\frac{a_{11}+a_{12}}{a_{22}+a_{12}}$ times the second one and we simplify the calculations, we obtain (7.8).

It is clear that the lower bounds for the decay rates for z_1 and z_2 can be obtained by means of the arguments proposed before.

On the other side, from (7.6) and (7.7) we get u and w in terms of z_1 and z_2:

$$u = \frac{z_1 + (a_{12} + a_{22})z_2}{a_{11} + a_{22} + 2a_{12}}, \quad w = \frac{z_1 - (a_{11} + a_{12})z_2}{a_{11} + a_{22} + 2a_{12}}. \tag{7.10}$$

Then, we can obtained the corresponding estimates for u and w.

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