An equivalent formulation for the Shapley value

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Abstract. An equivalent explicit formula for the Shapley value is provided, its equivalence with the classical one is proven by double induction. The importance of this new formula, in contrast to the classical one, is its capability of being extended to more general classes of games, in particular to j-cooperative games or multichoice games, in which players choose among different levels of participation in the game.

Keywords: Cooperative games · Marginal contributions · Shapley value · Alternative formulations · Potential generalizations

1 Introduction

The Shapley value, see [Shapley(1953)] and [Shapley(1962)], admits a clear formulation in terms of marginal contributions. As Shapley described, his value is based on the following model: (1) starting with a single member, the coalition adds one player at a time until everyone has been admitted; (2) the order in which the players are to join is determined by chance, with all arrangements equally probable; (3) each player, on her admission, demands and is promised the amount which her adherence contributes to the value of the coalition as determined by the characteristic function, i.e., the marginal contribution. Such model was criticized by several authors as highly artificial (see e.g. [Luce and Raiffa(1957)] and [Brams(1975)] among others).

Instead, step (3) could be replaced by the following ones:

(3.1) In her turn, each player decides whether to cooperate or not in forming a proposed coalition.
(3.2) If in her turn the player has decided to cooperate, then she receives the marginal contribution of the coalition formed by those players preceding her that decided to cooperate.
(3.3) If in her turn the player has decided not to cooperate, then she receives the marginal contribution of the coalition formed by those players preceding her that decided to cooperate and all those subsequent players in the queue.

That is, the player in her turn has the choice to cooperate or not to do it. The first choice is rewarded to her by her gain capacity in the game, while in the second she is rewarded by her blocking capacity.

Thus, not only the $n!$ orderings of a queue need to be considered, also the two choices for players need to be considered. This brings us to a more general model of $n! \cdot 2^n$ equally likely queues versus binary choices for the n players. The probabilistic model associated to this procedure is the discrete uniform distribution for all the $n! \cdot 2^n$ roll-calls, i.e., pairs formed by a permutation and a vector for players which determines if each player decides to cooperate or not.

In [Bernardi and Freixas(2018)] we already proposed this new formulation and did a detailed analysis for it. The main proof of the coincidence of our formula with the well-known formula by [Shapley(1953)] was proven by using power series and generating functions. This allowed us to give a succinct but perhaps not very intuitive prove. The main purpose of this work is to prove it with a more intuitive procedure based on a simpler technique: induction.

The new explicit formula for the Shapley value is of great interest. The main reason is its capability to be extended to more general contexts than cooperative or simple games. [Felsenthal and Machover(1997)] naturally extended simple games to ternary games, i.e., simple games in which voters are allowed to abstain as an intermediate option to vote favorably or against the issue at hand. They defined for this class of games a new value, with the same flavor of the Shapley value, based on what they call the ternary space of roll-calls. In terms of [Freixas and Zwicker(2003),Freixas and Zwicker(2009)] ternary games are a particular case of $(3,2)$-simple games, which extend to $(j,2)$-simple games, i.e., games in which players can choose among several ordered levels of approval and the output is binary. The natural extension of $(j,2)$-simple games to the TU cooperative context is that of j-cooperative games, considered in [Freixas(2018)] in which players choose among different levels of activity and the characteristic function is defined on partitions capturing the choices of the players. The class of j-cooperative games is a bit more general than that of multichoice cooperative games, see [Hsiao and Raghavan(1993)].

The work in [Freixas(2018)] extends the formula we propose in this paper to j-cooperative games, and more particularly to: multichoice cooperative games, $(j,2)$-simple games, and ternary games. Of course, when $j = 2$ the formula reduces to the formula (3) we propose in this paper.

But, as far as we know, it does not exist any explicit formula in these further contexts by using the original Shapley probabilistic scheme based only on queues and assuming that in her turn each player agrees to cooperate with her predecessors.

The setup of the rest of the paper is as follows. Section 2 includes some necessary preliminaries on the Shapley value. Section 3 is devoted to demonstrate the main result. The conclusions end the work in Section 4.

2 Preliminaries

Let N be a finite fixed set of cardinality n. The elements of N are called *players*, while a subset of N is called *coalition.* A *TU-cooperative game* is a function

$v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The cardinality of a coalition S is denoted by s. Let $\mathcal{CG}_N$ be the set of all TU-cooperative games on N.

The *Shapley value* is a function $\phi : \mathcal{CG}_N \rightarrow \mathbb{R}^n$, that assigns to each player a real number $\phi_a(v)$. [Shapley(1953)] and [Shapley(1962)] defined this function following a deductive procedure, by showing that it is uniquely characterized by the axioms of: efficiency, null player, symmetry and additivity.

The Shapley value has an explicit expression, in terms of the marginal contributions of the characteristic function, which is widely used to compute it:

$$\phi_a(v) = \sum_{S \subseteq (N \setminus \{a\})} \rho^n(s)[v(S \cup \{a\}) - v(S)], \tag{1}$$

where $s = |S|$ and

$$\rho^n(s) = \frac{s!(n-s-1)!}{n!}. \tag{2}$$

The coefficient $\rho^n(s)$, for each coalition $S \subseteq (N \setminus \{a\})$, is the proportion of permutations in which player a is occupying the $(s+1)$-position in the queue, where the preceding players are those in coalition S, no matter in which ordering, and the remaining players are occupying positions after $s+1$ in the queue, no matter in which ordering. Figures 1 and 2 schematically represents it.

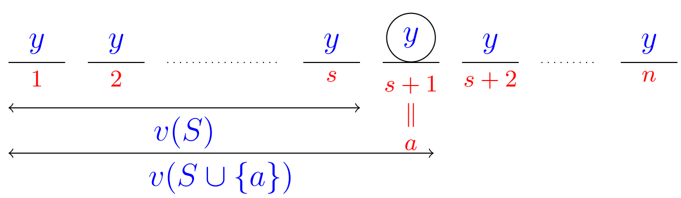

Fig. 1. Standard scheme for the classical Shapley value: "y" means that players choose forming part of the coalition.

The marginal contribution of a is weighted by a coefficient that counts all possible orderings of players before and after a as schematically described in Figure 2.

3 The alternative explicit formula for the Shapley value

The alternative explicit formula following the more detailed model of roll-calls takes into account that in her turn, player a can decide either to cooperate or not.

$$\Phi_a(v) = \sum_{S \subseteq (N \setminus \{a\})} \Gamma^n(s)[v(S \cup \{a\}) - v(S)], \tag{3}$$

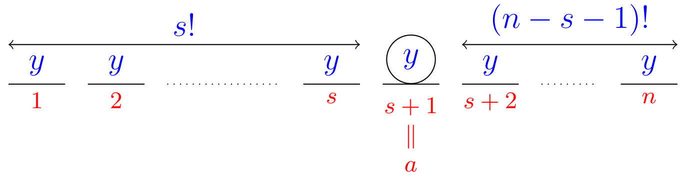

Fig. 2. Standard scheme for the classical Shapley value: counting all orderings.

where $s = |S|$ and for any $s = 0, \ldots, n-1$:

$$\Gamma^n(s) = \frac{s!}{2^n n!} \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k + \frac{(n-s-1)!}{2^n n!} \sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-1-k)!} 2^k. \quad (4)$$

As we will see later Φ and ϕ coincide. The new formula is based on the following assumptions (see Theorem 4 in [Bernardi and Freixas(2018)]):

1. Players act in a randomly chosen order and all $n!$ orderings are equally likely.
2. In her turn, each player decides whether to cooperate or not in forming a coalition by either gaining collective value of blocking collective gain.
3. If in her turn the player has decided to cooperate, then she receives the marginal contribution of the coalition formed by those players preceding her that decided to cooperate.
4. If in her turn the player has decided not to cooperate, then she receives the marginal contribution of the coalition formed by those players preceding her that decided to cooperate and all those subsequent players to a in the queue.

According to the third item, player a receives the *direct gain* of joining to the coalition (say R) of members who decided to cooperate and preceded her in the queue. According to the fourth item, player a receives the *indirect gain*, i.e. the gain due to her blocking capacity, of joining to the coalition (say T) of members who decided to cooperate and preceded her in the queue (those in R) and all those players who follow her in the queue. The reason for the addition of the subsequent players to a in the queue is because we are assuming that in their turn they will decide to cooperate, which is the worst scenario for the capacity of player a to block gain.

Note that the expression in (3) is referenced to an arbitrary coalition S that does not contain player a as well as for the coefficients in (4). Thus, we need to consider $R = S$ when computing the direct gain, while for the indirect gain S is then the union of those who decided to cooperate and preceded player a in the queue and all those players who follow her in the queue. See Figures 3 and 4.

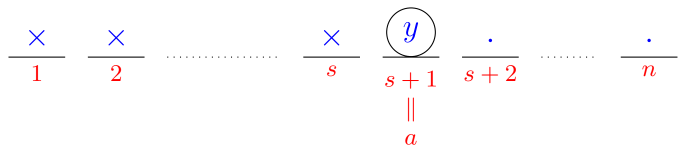

Fig. 3. Scheme for direct gain: "×" means that players preceding a already decided to cooperate (y) or not (n). No matter if players after a are going to cooperate or not, which is represented by "·". Thus, coalition S is formed by those with $\times = y$.

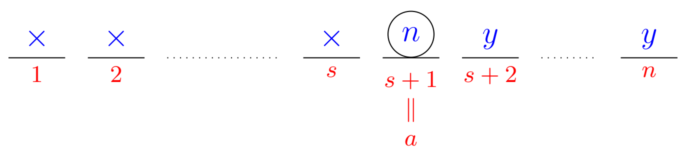

Fig. 4. Scheme for indirect gain: "×" means that players preceding a already decided to cooperate (y) or not (n). For measuring a's blocking gain capacity it is needed to assume that all players after a will choose to cooperate "y". Thus, coalition S is formed by those with $\times = y$ union all those after a in the queue.

Theorem 1. *The values Φ and ϕ for TU-cooperative games coincide.*

Proof. To prove Theorem 1 it is enough to deduce the equality of the coefficients $\rho^n(s)$ and $\Gamma^n(s)$ for all positive integers n and $0 \leq s \leq n-1$. Thus we need to prove that for any n and any $s = 0, \dots, n-1$, it holds

$$\frac{s!(n-s-1)!}{n!} = \frac{s!}{2^n n!} \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k + \frac{(n-s-1)!}{2^n n!} \sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-1-k)!} 2^k .$$

By simplifying the identity, the previous equality is equivalent to the following equation

$$s!(n-s-1)!2^n = s! \sum_{k=0}^{s} 2^k \frac{(n-k-1)!}{(s-k)!} + (n-s-1)! \sum_{k=0}^{n-s-1} 2^k \frac{(n-k-1)!}{(n-s-k-1)!} \quad (5)$$

for all n and any $s = 0, \dots, n-1$.

Observe that if $n = 1$, then $s = 0$, then this equality reduces to $2 = 1 + 1$, that is trivially true. Since we are dealing with voting games, we assume that there is not only one player and so $n \geq 2$.
We proceed in proving (5) using induction on n.

If $n = 2$ and $s = 0$ we have $2^2 = 1 + 1 + 2$. If $n = 2$ and $s = 1$ we have $2^2 = 1 + 2 + 1$. So the thesis is true for $n = 2$.

Now, we assume that (5) is true for n and all $0 \leq s \leq n-1$ and we prove it for $n+1$ and $0 \leq s \leq n$.

We first consider the extreme cases $s = 0$ and $s = n$ and prove them directly. Secondly, we prove the statement for each s with $0 < s < n$, using the induction hypothesis for n with s and $s - 1$.

First step:
For $n + 1$ and $s = 0$ (or $s = n$), equality (5) becomes

$$2^{n+1}n! = n! + n!\sum_{k=0}^{n} 2^k.$$

By the induction hypothesis (for n and $s = 0$) we have

$$2^n(n-1)! = (n-1)! + (n-1)!\sum_{k=0}^{n-1} 2^k.$$

Then we can write the right side of our claim as

$$\begin{aligned} n! + n!\sum_{k=0}^{n} 2^k &= n! + 2^n n! + n!\sum_{k=0}^{n-1} 2^k \\ &= n! + 2^n n! + n[2^n(n-1)! - (n-1)!] \\ &= n! + 2^n n! + 2^n n! - n! = 2^{n+1}n! \end{aligned}$$

and this proves the first part.

Second step:
We now want to prove the thesis for $n + 1$, thus, we have to show that the following is true

$$2^{n+1}s!(n-s)! \stackrel{?}{=} s!\sum_{k=0}^{s} \frac{(n-k)!}{(s-k)!}2^k + (n-s)!\sum_{k=0}^{n-1} \frac{(n-k)!}{(n-s-k)!}2^k. \qquad (6)$$

By induction hypothesis, if we take n and s we have

$$s!(n-s-1)!2^n = s!\sum_{k=0}^{s} 2^k \frac{(n-k-1)!}{(s-k)!} + (n-s-1)!\sum_{k=0}^{n-s-1} 2^k \frac{(n-k-1)!}{(n-s-k-1)!} \qquad (7)$$

and if we take $s - 1$

$$(s-1)!(n-s)!2^n = (s-1)!\sum_{k=0}^{s-1} 2^k \frac{(n-k-1)!}{(s-1-k)!} + (n-s)!\sum_{k=0}^{n-s} 2^k \frac{(n-k-1)!}{(n-s-k)!}. \qquad (8)$$

We work on the right-hand side of equation (6) and rewrite each of the two addends in the following way

$$\begin{aligned} s!\sum_{k=0}^{s}\frac{(n-k)!}{(s-k)!}2^k &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1}\frac{(n-k)!}{(s-k)!}2^k \\ &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k\frac{n-k}{s-k}, \end{aligned}$$

writing $\frac{n-k}{s-k}$ as $\frac{n-s}{s-k}+1$,

$$\begin{aligned} &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k\left(\frac{n-s}{s-k}+1\right) \\ &= s!(n-s)!2^s + s!(n-s)\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k)!}2^k + s!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k, \end{aligned}$$

the first term can be moved inside the sum, to get

$$= s!(n-s)\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k + s!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k.$$

Analogously the second term in (6) can be written as

$$(n-s)!\sum_{k=0}^{n-s}\frac{(n-k)!}{(n-s-k)!}2^k = s(n-s)!\sum_{k=0}^{n-s}\frac{(n-k-1)!}{(n-s-k)!}2^k + (n-s)!\sum_{k=0}^{n-s-1}\frac{(n-k-1)!}{(n-s-k-1)!}2^k.$$

If we now sum these expressions the right-hand side of (6) becomes

$$s\left[(s-1)!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k + (n-s)!\sum_{k=0}^{n-s}\frac{(n-k-1)!}{(n-s-k)!}2^k\right] +$$
$$(n-s)\left[s!\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k + (n-s-1)!\sum_{k=0}^{n-s-1}\frac{(n-k-1)!}{(n-s-k-1)!}2^k\right].$$

Using the induction hypothesis and in particular (7) and (8) and replacing everything in the right-hand side of (6), we finally get

$$\begin{aligned} 2^{n+1}s!(n-s)! &= s[2^n(s-1)!(n-s)!] + (n-s)[2^n s!(n-s-1)!] \\ &= 2^n s!(n-s)! + 2^n s!(n-s)! \\ &= 2^{n+1}s!(n-s)! \end{aligned}$$

4 Conclusion

A new probabilistic approach to the Shapley value for TU-cooperative games has been proposed. Instead of just considering permutations as in the classical approach, the model also takes into account if the player wants to cooperate or not in her turn. From this new model we obtain a formula depending on the marginal contributions. This formula is proven by induction, which is a bit more transparent than using power series and generating functions as we recently did in [Bernardi and Freixas(2018)]. This proof pretends to bring a little more light on different formulations of the Shapley value for TU-cooperative games.

Its main importance is revealed when trying extensions of the value to further contexts. Indeed, a natural extension of the formula (3) to the so-called j-cooperative games, games where players can choose among different (say j) ordered levels of activity, has been obtained in [Freixas(2018)]. Nevertheless, as far as we know, it does not exist any formula of the value on j-cooperative games inspired in the original probabilistic model by Shapley based on marginal contributions on permutations.

Thus, the application potential of the model we present is enormous. All the theory and applications of cooperative games based on the Shapley value can be studied for the more general model of j-cooperative games, which contains multi-choice games.

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