Group inverse matrix of the normalized laplacian on subdivision graphs[Ⓢ]

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Abstract. In this paper, given a network we consider a subdivision network of it and we show how the group inverse matrix of the normalized laplacian on the subdivision network is related to the group inverse matrix of the normalized laplacian of the initial given network. Our approach establishes a relationship between solutions of related Poisson problems on both structures and takes advantage on the properties of the group inverse matrix.

Keywords. Subdivision network, normalized laplacian, group inverse, Poisson problem

1 Preliminaries

In this paper $\Gamma = (V, E, c)$ denotes a simple *network;* that is, a finite, with no loops, nor multiple edges, connected graph, with n vertices in V and m edges in E. We call *conductance* to the symmetric function $c : V \times V \to [0, +\infty)$ satisfying $c(x, y) > 0$ iff $x \sim y$, which means that $\{x, y\} \in E$. For every vertex in V, let $k(x) = \sum_{y \in V} c(x, y)$ be the *degree* of vertex x, then $\mathrm{vol}(\Gamma) = \sum_{x \in V} k(x)$.

Let $\mathcal{C}(V)$ be the set of real functions on V, the *normalized laplacian* of Γ, introduced by Chung and Langlands in [2], is the linear operator $\mathcal{L} : \mathcal{C}(V) \to \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ and at each $x \in V$

$$\mathcal{L}(u)(x) = \frac{1}{\sqrt{k(x)}} \sum_{y \in V} c(x, y) \left(\frac{u(x)}{\sqrt{k(x)}} - \frac{u(y)}{\sqrt{k(y)}} \right).$$

Easily $\ker(\mathcal{L}) = span(\sqrt{k})$ and given $f \in \mathcal{C}(V)$ the Poisson problem, *i.e.* the linear system $\mathcal{L}(u)(x) = f(x)$, is compatible iff $f \perp \sqrt{k}$. In this case, two different solutions differ up to a multiple of $\sqrt{k}$, so there exists a unique solution orthogonal to $\sqrt{k}$ to every compatible linear system $\mathcal{L}(u) = f$ (Fredholm's alternative).

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In particular, $\mathcal{L}u = \varepsilon_x - \frac{\sqrt{k(x)}}{\mathrm{vol}(\Gamma)}\sqrt{k}$ has a unique solution which is orthogonal to $\sqrt{k}$ that we denote by $\mathcal{L}^{\#}(\cdot, x)$. The $n \times n$ matrix $\mathcal{L}^{\#}$, whose columns are $\mathcal{L}^{\#}(\cdot, x)$ for any $x \in V$, is the *group inverse* of $\mathcal{L}$.

A *subdivision network* $\Gamma^S = (V^S, E^S, c^s)$ of Γ is obtained from it by inserting a new vertex in every edge, so that each $\{x, y\} \in E$ is replaced by two new edges, say $\{x, v_{xy}\}$ and $\{v_{xy}, y\}$ where v_{xy} is the new inserted vertex. We denote by V' the set of new generated vertices assuming that $v_{xy} = v_{yx}$. Thus $V^S = V \cup V'$ and the order of the subdivision network is $n + m$ while its size is $2m$. Moreover, we define the conductances on the new edges as $c^S(x, v_{xy}) = c^S(y, v_{xy}) = 2c(x, y)$. Then the degree function on Γ^S, $k^S \in \mathcal{C}(V^S)$, satisfies $k^S(x) = 2k(x)$ when $x \in V$, and $k^S(v_{xy}) = 4c(x, y)$ for those vertices in V'. Moreover, it holds that $\mathrm{vol}(\Gamma^S) = 4\mathrm{vol}(\Gamma)$.

The subdivision procedure has attracted the attention of some researchers, see for instance [3, 4] in the context of graphs, and [1] for the case of networks. In these works different parameters associated with the combinatorial Laplacian have been studied as eigenvalues, resistances, kirchhoff index.

2 The group inverse of the normalized laplacian for a subdivision network

The first main result we present in this work sets the precise relation between the solution of a compatible Poisson problem for the normalized laplacian on a subdivision network Γ^S and the solution of a conveniently well posed Poisson problem for the normalized Laplacian on the base network Γ.

With the aim of usefulness we consider some definitions: firstly, a coefficient $\alpha(x, y) = \sqrt{\dfrac{c^S(x, v_{xy})}{k^S(x)}} = \sqrt{\dfrac{c(x, y)}{k(x)}}$ for every pair of adjacent vertices $\{x, y\} \in E$, and secondly two function operators: (a) let $h \in \mathcal{C}(V^S)$ its *contraction to* V is

$$\underline{h}(x) = h(x) + \frac{1}{\sqrt{2}} \sum_{y \sim x} \alpha(x, y)\, h(v_{xy})$$

and (b) for any $u \in \mathcal{C}(V)$ (and related to an $h \in \mathcal{C}(V^S)$) the *extension* to V^S as

$$u^h(v_{xy}) = h(v_{xy}) + \frac{1}{\sqrt{2}}\alpha(x, y)\, u(x) + \frac{1}{\sqrt{2}}\alpha(y, x)\, u(y)$$

for $v_{xy} \in V'$, while $u^h(x) = u(x)$ for those vertices in V.

The following result links the solution of a Poisson problem in the subdivision network with an appropriate Poisson problem on the base network. From now on $\mathcal{L}_S$ denotes the normalized laplacian of the subdivision network Γ^S.

Theorem 1. *Given $h \in \mathcal{C}(V^S)$ such that $\langle h, \sqrt{k^S}\rangle_{V^S} = 0$, then $\langle \underline{h}, \sqrt{k}\rangle_V = 0$. Moreover, $\overline{u} \in \mathcal{C}(V^S)$ is a solution of the Poisson equation $\mathcal{L}_S(\overline{u}) = h$ in V^S iff $u = \overline{u}_{|V}$ is a solution of the Poisson equation $\mathcal{L}(u) = 2\underline{h}$ in V. In this case, the identity $\overline{u} = u^h$ holds.*

Corollary 1. *Given $h \in \mathcal{C}(V^S)$, such that $\langle h, \sqrt{k^S}\rangle_{V^S} = 0$, let $\underline{h} \in \mathcal{C}(V)$ be its contraction to V; let $u \in \mathcal{C}(V)$ be the unique solution of $\mathcal{L}(u) = 2\underline{h}$ that satisfies $\langle u, \sqrt{k}\rangle_V = 0$; and the constant*

$$\lambda = -\frac{1}{\mathrm{vol}(\Gamma^S)} \sum_{v_{xy}\in V'} u^h(v_{xy})\sqrt{k^S(v_{xy})}.$$

Then, $u^\perp = u^h + \lambda\sqrt{k}$ is the unique solution of $\mathcal{L}_S(u^\perp) = h$ that satisfies $\langle u^\perp, \sqrt{k}\rangle_{V^S} = 0$.

Taking into account the relation between Poisson problems for the normalized laplacian on Γ^S and Γ, we obtain the expression of the group inverse for the normalized laplacian of a subdivision network, $\mathcal{L}_S^\#$, in terms of the group inverse of the base network $\mathcal{L}^\#$.

From now on we consider $\nu = \mathrm{vol}(\Gamma^S)$, and $\pi^S \in \mathcal{C}(V)$, defined as $\pi^S(x) = \dfrac{1}{\sqrt{2}} \sum\limits_{y\sim x} \alpha(x,y)\sqrt{k^S(v_{xy})}$.

So we definitely obtain the expression of the group inverse matrix of $\mathcal{L}^S$ in terms of the group inverse matrix of $\mathcal{L}$.

Proposition 1. *Let Γ^S be the subdivision network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$, the entries of group inverse of Γ^S are given by*

$$\begin{aligned}\mathcal{L}_S^\#(x,z) = 2\mathcal{L}^\#(x,z) - \frac{2\sqrt{2}}{\nu}\sum_{\ell\in V}\Big[\sqrt{k(z)}\mathcal{L}^\#(x,\ell) + \sqrt{k(x)}\mathcal{L}^\#(z,\ell)\Big]\pi^S(\ell)\\ + 4\frac{\sqrt{k(x)}\sqrt{k(z)}}{\nu^2}\sum_{r,s}\mathcal{L}^\#(s,r)\pi^S(r)\pi^S(s) + 2\frac{\sqrt{k(x)}\sqrt{k(z)}}{\nu^2}\sum_{r\sim s}k^S(v_{rs}),\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_S^{\#}(v_{xy},z) = {} & \frac{\sqrt{2}\sqrt{k^S(v_{xy})}\sqrt{k(z)}}{\nu^2}\left(-\nu+\sqrt{2}\sum_{r,s}\mathcal{L}^{\#}(s,r)\pi^S(r)\pi^S(s)+\sum_{r\sim s}k^S(v_{rs})\right)\\
& +\sqrt{2}\Big(\alpha(x,y)\mathcal{L}^{\#}(x,z)+\alpha(y,x)\mathcal{L}^{\#}(y,z)\Big)\\
& -2\frac{\sqrt{k(z)}}{\nu}\sum_{\ell\in V}\Big(\alpha(x,y)\mathcal{L}^{\#}(x,\ell)+\alpha(y,x)\mathcal{L}^{\#}(y,\ell)\Big)\pi^S(\ell)\\
& -\frac{\sqrt{2}\sqrt{k^S(v_{xy})}}{\nu}\sum_{\ell\in V}\mathcal{L}^{\#}(z,\ell)\pi^S(\ell),
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_S^{\#}(v_{xy},v_{zt}) = {} & \varepsilon_{v_{zt}}(v_{xy})-2\frac{\sqrt{k^S(v_{xy})}\sqrt{k^S(v_{zt})}}{\nu}+\frac{\sqrt{k^S(v_{xy})}\sqrt{k^S(v_{zt})}}{\nu^2}\sum_{r\sim s}k^S(v_{rs})\\
& +2\frac{\sqrt{k^S(v_{xy})}\sqrt{k^S(v_{zt})}}{\nu^2}\sum_{r,s}\mathcal{L}^{\#}(s,r)\pi^S(r)\pi^S(s)\\
& +\alpha(x,y)\alpha(z,t)\mathcal{L}^{\#}(x,z)+\alpha(x,y)\alpha(t,z)\mathcal{L}^{\#}(x,t)\\
& +\alpha(y,x)\alpha(z,t)\mathcal{L}^{\#}(y,z)+\alpha(y,x)\alpha(t,z)\mathcal{L}^{\#}(y,t)\\
& -\frac{\sqrt{2}\sqrt{k^S(v_{zt})}}{\nu}\sum_{\ell\in V}\Big(\alpha(x,y)\mathcal{L}^{\#}(x,\ell)+\alpha(y,x)\mathcal{L}^{\#}(y,\ell)\Big)\pi^S(\ell)\\
& -\frac{\sqrt{2}\sqrt{k^S(v_{xy})}}{\nu}\sum_{\ell\in V}\Big(\alpha(z,t)\mathcal{L}^{\#}(z,\ell)+\alpha(t,z)\mathcal{L}^{\#}(t,\ell)\Big)\pi^S(\ell).
\end{aligned}$$

References

[1] Carmona, A., Mitjana, M., Monsó, E. Effective resistances and Kirchhoff index in subdivision networks, *Linear and Multilinear Algebra* (http://dx.doi.org/10.1080/03081087.2016.1256945)

[2] Chung, F., Langlands, R.P. A combinatorial Laplacian with vertex weights, *J. Combin. Theory (A)*, **75** (1996), 316–327.

[3] Xie, P., Zhang, Z., Comellas, F. The normalized Laplacian spectrum of subdivisions of a graph. *Appl. Math. and Comp.* **286** (2016, 250–256)

[4] Yang, Y., The Kirchhoff index of subdivisions of graphs. *Discrete Appl. Math.* **171** (2014, 153–157)