# Uniform clutters and dominating sets of graphs[1]

Jaume Martí-Farré, Mercè Mora, José Luis Ruiz

*Departament de Matemàtiques, Universitat Politècnica de Catalunya*

**Abstract**

A (simple) clutter is a family $\mathcal{H}$ of pairwise incomparable subsets of a finite set $\Omega$. We say that a clutter $\mathcal{H}$ is a *domination clutter* if there is at least a graph $G$ such that the collection of the inclusion-minimal dominating sets of vertices of $G$ is equal to $\mathcal{H}$. Given a clutter $\mathcal{H}$, we are interested in determining if it is a domination clutter and, if this is not the case, we want to find domination clutters in some sense close to it: the *domination completions* of $\mathcal{H}$. Here we will focus on the family of clutters containing all the subsets with the same cardinality; the *uniform clutters* of maximum size. Specifically, we characterize those clutters $\mathcal{H}$ in this family that are domination clutters and, in any other case, we prove that the domination completions exist. Moreover, we then demonstrate that the clutter $\mathcal{H}$ is uniquely determined by some of its domination completions, in the sense that $\mathcal{H}$ can be recovered from some of these domination completions by using a suitable operation between clutters.

Keywords: *clutters, uniform clutters, dominating sets of graphs*

## 1. Introduction

A *vertex dominating set* of a graph $G$ is a set of vertices $D$ such that every vertex of $G$ is either in $D$ or adjacent to some vertex of $D$ (see [6]). Domination in graphs is a widely researched branch of graph theory, both from a theoretical and algorithmic point of view. In part, it is due to its applications to several fields where graphs are used to model the relationships between a finite number of objects. In this way, for instance, some concepts from domination in graphs appear in problems involving finding sets of representatives, as well as in facility location problems or in problems in monitoring communication, in electrical networks or in network routing.

The starting point of this work is a question concerning the design of networks on a finite set of nodes $\Omega$ whose dominating sets satisfy specific properties. Concretely, in this paper we focus our attention on the collection $\mathcal{D}(G)$ of all

[1]Partially supported by projects MINECO MTM2014-60127-P, MTM2015-63791-R (MINECO/FEDER), Gen.Cat. DGR2014SGR1147, and Gen.Cat. DGR2014SGR46. This project has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

the *inclusionwise minimal vertex dominating sets* of a graph $G$. Specifically, we are looking for graphs $G$ whose collection of vertex dominating sets $\mathcal{D}(G)$ is equal or close to a given collection $\{A_1, \dots, A_r\}$ of subsets of nodes $A_i \subseteq \Omega$.

*Clutters* become the natural framework of this problem. A (simple) *clutter* $\mathcal{H}$ on a finite set $\Omega$ is a collection of subsets of $\Omega$ none of which is a proper subset of another (see [1]). The *domination clutter of a graph* $G$ is the collection $\mathcal{D}(G)$ of all the inclusion-minimal vertex dominating sets of the graph $G$. A clutter $\mathcal{H}$ is said to be a *domination clutter* if $\mathcal{H}$ is the domination clutter of a graph; that is, if $\mathcal{H} = \mathcal{D}(G)$ for some graph $G$.

Since in general a clutter $\mathcal{H}$ is not the domination clutter of a graph, a natural question that arises at this point is to determine domination clutters close to $\mathcal{H}$, the *domination completions* of $\mathcal{H}$. This paper deals with this issue. Specifically, we focus our attention on this problem for the uniform clutters $\mathcal{H} = \mathcal{U}_{r,\Omega}$ containing all the subsets with cardinality $r$ of a finite set $\Omega$. The goal is to prove that the domination completions of the uniform clutters $\mathcal{U}_{r,\Omega}$ exist. Moreover, by taking into account a suitable partial order $\leqslant$, we will prove that the set of domination completions of $\mathcal{U}_{r,\Omega}$ is a partially ordered set, and that the uniform clutter $\mathcal{U}_{r,\Omega}$ is univocally determined by the minimal elements of this poset, the *minimal domination completions*. Namely we will prove that there is a clutter operation $\sqcap$ that allows us to express the uniform clutter $\mathcal{U}_{r,\Omega}$ as a combination of its optimal completions $\mathcal{D}(G_1), \dots, \mathcal{D}(G_s)$; that is, $\mathcal{U}_{r,\Omega} = \mathcal{D}(G_1) \sqcap \cdots \sqcap \mathcal{D}(G_s)$. We thus speak of a *decomposition* of the clutter $\mathcal{U}_{r,\Omega}$ into domination clutters. In addition, we study the number of completions appearing in the decomposition of $\mathcal{U}_{r,\Omega}$. Summarizing, in this paper we present results concerning the completions and decompositions of clutters into domination clutters (a previous version of this work was presented at the European Conference on Combinatorics, Graph Theory and Applications — EUROCOMB 2015 [11]).

Closest in spirit to our work are the papers [9] and [10], in which the authors present some results on the completion and decompositions of clutters into *matroidal clutters*. It is worth mentioning that even though our results are formally analogous to those in [9, 10], they can not be proved using the same techniques. The difference between our results on domination clutters and those presented in [9, 10] concerning matroidal clutters are pointed out in this paper.

The present paper is structured as follows. First, in Section 2 we recall the properties of vertex dominating sets of graphs that we will use throughout the paper (Subsection 2.1); we present the basic definitions on clutters and domination clutters (Subsection 2.2); and we characterize the uniform clutters of maximum size that are domination clutters (Subsection 2.3). The main theoretical results of this paper are gathered in Section 3. In this section we introduce the poset of domination completions (Subsection 3.1), and we present our results on domination completion and decomposition of the uniform clutters $\mathcal{U}_{r,\Omega}$ (Theorem 9, Theorem 10 and Proposition 11). Finally, in Section 4, we describe the set of the minimal domination completions of some uniform clutters $\mathcal{U}_{r,\Omega}$; we present their graph realization; and, we discuss some issues on the corresponding domination decomposition. Concretely, we analyze these questions for the uniform clutters $\mathcal{U}_{r,\Omega}$ when $r = 2$ (Subsection 4.1), when $r = |\Omega| - 1$ (Subsection 4.2), and when $r$ arbitrary and $|\Omega| \le 5$ (Subsection 4.3).

## 2. Dominating sets of graphs. Domination clutters

As mentioned in the introduction, the aim of this section is to present those general results on dominating sets that we will use throughout the paper.

### *2.1. Vertex dominating sets of graphs*

A *graph* $G$ is an ordered pair $(V(G), E(G))$ comprising a finite set $V(G)$ of *vertices* together with a (possibly empty) set $E(G)$ of *edges* which are two-element subsets of $V(G)$ (for general references on graph theory see [4, 12]). If $e = \{x, y\} \in E(G)$ is an edge of $G$, then $x$ and $y$ are said to be *adjacent vertices*. An *isolated vertex* is a vertex of the graph that is not adjacent to any other vertices; that is, a vertex that does not belong to any edge of the graph. Let us denote by $V_0(G)$ the set of all the isolated vertices of $G$.

A *dominating set* for a graph $G = (V(G), E(G))$ is a subset $D$ of $V(G)$ such that every vertex not in $D$ is adjacent to at least one member of $D$. Since any superset of a dominating set of $G$ is also a dominating set of $G$, the collection $D(G)$ of the dominating sets of a graph $G$ is a monotone increasing family of subsets of $V(G)$. Therefore, $D(G)$ is uniquely determined by the family $\min\big(D(G)\big)$ of its inclusion-minimal elements. Let us denote by $\mathcal{D}(G)$ the family of the *inclusion-minimal dominating sets* of the graph $G$.

Dominating sets of a graph are closely related to independent sets. An *independent set* of a graph $G$ is a set of vertices such that no two of them are adjacent. It is clear that an independent set is also a dominating set if and only if it is an inclusion-maximal independent set (see [4]). Therefore, any inclusion-maximal independent set of a graph is necessarily also an inclusion-minimal dominating set. The next lemma follows from this fact and from the definitions.

**Lemma 1.** *If $G$ is a graph, then $V(G) = \bigcup_{D \in \mathcal{D}(G)} D$ and $V_0(G) = \bigcap_{D \in \mathcal{D}(G)} D$.*

Next, in Lemma 2, we recall the well-known relation between dominating sets and *star systems* (see [2, 3, 8]).

The *star system of a graph* $G = (V(G), E(G))$ is the multiset $N[G]$ of *closed neighborhoods* of all the vertices of the graph; that is, the multiset $N[G] = \{N[x] : x \in V(G)\}$ where $N[x] = \{x\} \cup \{y \in V(G) : \{x, y\} \in E(G)\}$. Let us denote by $\mathcal{N}[G]$ the inclusion-minimal elements of the star system; that is, $\mathcal{N}[G] = \min\big(N[G]\big)$ is the family of the *inclusion-minimal closed neighborhoods of the graph* $G$. The relation between $\mathcal{D}(G)$ and $\mathcal{N}[G]$ involves the *transversal* or *blocker* of a family of subsets. Let $\mathcal{A}$ be a collection of subsets none of which is a proper subset of another. The *transversal* $\mathrm{tr}(\mathcal{A})$ of the family $\mathcal{A}$ consists of those inclusion-minimal subsets that have non-empty intersection with every member of $\mathcal{A}$; that is, $\mathrm{tr}(\mathcal{A}) = \min\{X : X \cap A \neq \emptyset \text{ for all } A \in \mathcal{A}\}$.

**Lemma 2.** *Let $G$ be a graph. Then $\mathcal{D}(G) = \mathrm{tr}\big(\mathcal{N}[G]\big)$ and $\mathcal{N}[G] = \mathrm{tr}\big(\mathcal{D}(G)\big)$.*

*Proof.* From the definitions it is clear that a subset $D$ of vertices is a dominating set of the graph $G$ if and only if $D \cap N[x] \neq \emptyset$ for every vertex $x \in V(G)$. Hence it follows that $\mathcal{D}(G) = \operatorname{tr}\big(\mathcal{N}[G]\big)$. The transversal map is involutive, that is, $\operatorname{tr}(\operatorname{tr}(\mathcal{A})) = \mathcal{A}$ (see [1]). Therefore we get that $\mathcal{N}[G]$ is, at once, the transversal of the family $\mathcal{D}(G)$. $\square$

To conclude this subsection we recall two graph operations that we will use: the *disjoint union* and the *join* of graphs.

Let $G_1, \ldots, G_r$ be $r \geq 2$ graphs with pairwise disjoint vertex sets $V(G_1), \ldots, V(G_r)$. The *disjoint union* $G_1 + \cdots + G_r$ of $G_1, \ldots, G_r$ is the graph with $V(G_1) \cup \cdots \cup V(G_r)$ as set of vertices and $E(G_1) \cup \cdots \cup E(G_r)$ as set of edges; while the *join* $G_1 \vee \cdots \vee G_r$ of $G_1, \ldots, G_r$ is the graph with set of vertices $V(G_1) \cup \cdots \cup V(G_r)$ and set of edges $E(G_1) \cup \cdots \cup E(G_r) \cup \{\{x_1, x_2\} : x_1 \in V(G_{i_1}), x_2 \in V(G_{i_2}) \text{ and } i_1 \neq i_2\}$. The following lemma deals with the minimal dominating sets of these graphs. Its proof is a straightforward consequence of the definitions.

**Lemma 3.** *Let $G_1, \ldots, G_r$ be $r \geq 2$ graphs with pairwise disjoint set of vertices. Then:*

1. $\mathcal{D}(G_1 + \cdots + G_r) = \{D_1 \cup \cdots \cup D_r : D_i \in \mathcal{D}(G_i)\}$.
2. $\mathcal{D}(G_1 \vee \cdots \vee G_r) = \mathcal{D}(G_1) \cup \cdots \cup \mathcal{D}(G_r)$
   $\cup \{\{x_1, x_2\} : x_k \in V(G_{i_k}),\ N[x_k] \neq V(G_{i_k}),\ i_1 \neq i_2\}$.

*2.2. Clutters. Domination clutters*

Let $\Omega$ be a non-empty finite set. A *(simple) clutter* on $\Omega$ is a non-empty collection $\mathcal{H}$ of non-empty different subsets of $\Omega$, none of which is a proper subset of another; that is, if $A, A' \in \mathcal{H}$ and $A \subseteq A'$ then $A = A'$. Clutters are also known as *antichains*, *Sperner systems* or *hypergraphs* (for general references on clutter theory see [1, 5]). In general, if $\mathcal{H}$ is a clutter on $\Omega$ then $\bigcup_{A \in \mathcal{H}} A \subseteq \Omega$. We say that $\mathcal{H}$ is a *clutter with ground set* $\Omega$ whenever the equality $\Omega = \bigcup_{A \in \mathcal{H}} A$ holds.

There are several clutters that can be associated to a graph. In this paper we are interested in those clutters defined by the dominating sets of the graph. Namely, if $G$ is a graph with vertex set $V(G)$, we consider the collection $\mathcal{D}(G)$ of the inclusion-minimal dominating sets of the graph. It is clear that $\mathcal{D}(G)$ is a clutter on the finite set $V(G)$. Moreover, by Lemma 1, $\mathcal{D}(G)$ is a clutter with ground set $V(G)$.

The *domination clutters* are those clutters that can be *realized* by the dominating sets of a graph; that is, we will say that a clutter $\mathcal{H}$ on $\Omega$ is a *domination clutter* if there exists a graph $G$ such that $\mathcal{H} = \mathcal{D}(G)$ (notice that then the set of vertices of $G$ is $V(G) = \bigcup_{A \in \mathcal{H}} A \subseteq \Omega$). If $\mathcal{H} = \mathcal{D}(G)$, we say that the graph $G$ is a *realization* of the domination clutter $\mathcal{H}$.

**Remark 1.** Observe that there exist domination clutters $\mathcal{H}$ with more than one graph realization. For example, let us consider the clutter $\mathcal{H} = \{\{1, 3\}, \{1, 4\},$

$\{2,3\}, \{2,4\}\}$ on the finite set $\Omega = \{1,2,3,4\}$. Then $\mathcal{H} = \mathcal{D}(G) = \mathcal{D}(G') = \mathcal{D}(G'')$ where $G$, $G'$ and $G''$ are the graphs with vertex sets $V(G) = V(G') = V(G'') = \Omega$ and edge sets $E(G) = \{\{1,2\}, \{3,4\}\}$, $E(G') = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and $E(G'') = \{\{1,2\}, \{1,4\}, \{3,4\}\}$.

The following lemma provides a necessary condition for a clutter to be a domination clutter.

**Lemma 4.** *Let $\mathcal{H}$ be a clutter with ground set $\Omega$. Assume that $\mathcal{H}$ is a domination clutter. Then $|\operatorname{tr}(\mathcal{H})| \leq |\Omega|$.*

*Proof.* Let $G$ be a graph with vertex set $V(G) = \Omega$ and such that $\mathcal{H} = \mathcal{D}(G)$. Then, by applying Lemma 2 we get that $\operatorname{tr}(\mathcal{H}) = \operatorname{tr}(\mathcal{D}(G)) = \mathcal{N}[G]$. So, $|\operatorname{tr}(\mathcal{H})| = |\mathcal{N}[G]| \leq |V(G)| = |\Omega|$. □

From the above, we get that not all clutters are domination clutters. Indeed, let $\mathcal{A} = \{A_1, \ldots, A_r\}$ be a family of $r \geq |\Omega| + 1$ non-empty different subsets of $\Omega$ with $A_i \not\subseteq A_j$ if $i \neq j$ (for instance, the family $\mathcal{A} = \{A \subseteq \Omega : |A| = 2\}$ where $|\Omega| \geq 4$). Since $\operatorname{tr}(\operatorname{tr}(\mathcal{A})) = \mathcal{A}$, from Lemma 4 it follows that the clutter $\mathcal{H} = \operatorname{tr}(\mathcal{A})$ is not a domination clutter.

Therefore, a natural question that arises at this point is to characterize whenever a clutter $\mathcal{H}$ is a domination clutter. The following subsection deals with this issue for a special family of clutters.

### *2.3. Uniform clutters. Domination clutters of the form $\mathcal{U}_{r,\Omega}$*

Let $\Omega$ be a finite set of size $|\Omega| = n$ and let $1 \leq r \leq n$. We say that a clutter $\mathcal{H}$ on $\Omega$ is *$r$-uniform* if $|A| = r$ for all $A \in \mathcal{H}$. Let us denote by $\mathcal{U}_{r,\Omega}$ the $r$-uniform clutter on $\Omega$ whose elements are all the subsets of $\Omega$ of size $r$; that is, $\mathcal{U}_{r,\Omega} = \{A \subseteq \Omega : |A| = r\}$.

The following proposition provides a characterization of the domination clutters of the form $\mathcal{U}_{r,\Omega}$, as well as the description of their graph realizations. This proposition was partially stated in [7].

Before stating the proposition, let us introduce some notation. The *complete graph* with $n$ vertices is denoted by $K_n$, whereas the *complete graph* with vertex set $\Omega$ will be denoted by $K_\Omega$, and the *empty graph* with vertex set $\Omega$ will be denoted by $\overline{K_\Omega}$. Observe that if $|\Omega| = 2m$, then the graph $G$ obtained from the complete graph $K_\Omega$ by deleting the edges of a perfect matching is the join graph of $m$ empty graphs on sets of size two; that is, $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}$ where $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $|\Omega_i| = 2$ for $1 \leq i \leq m$ (namely the vertices of the sets $\Omega_i$ are the endpoints of each one of the edges of the perfect matching).

**Proposition 5.** *Let $\Omega$ be a finite set of size $|\Omega| = n$. Let $1 \leq r \leq n$. Then the clutter $\mathcal{U}_{r,\Omega}$ is a domination clutter if and only if $r = 1$, or $r = n$, or $r = 2$ and $n$ is even. Moreover, the following statements hold:*

1. *The complete graph $K_\Omega$ is the unique graph $G$ such that $\mathcal{U}_{1,\Omega} = \mathcal{D}(G)$.*
2. *The empty graph $\overline{K_\Omega}$ is the unique graph $G$ such that $\mathcal{U}_{n,\Omega} = \mathcal{D}(G)$.*

3. *If $n = 2m$, then there are $(2m)!/(2^m m!)$ graphs $G$ such that $\mathcal{U}_{2,\Omega} = \mathcal{D}(G)$. Namely, $G$ is any graph of the form $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}$ where $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $|\Omega_i| = 2$ for $1 \le i \le m$.*

*Proof.* Statements (1) and (2) are a straightforward consequence of the definitions. Let us prove the third statement.

Assume that $n = 2m$ is even. From the description of the minimal domination sets of the join graph (Lemma 3) it follows that $\mathcal{D}(\overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}) = \mathcal{U}_{2,\Omega}$ if $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $|\Omega_i| = 2$ for $1 \le i \le m$. So the uniform clutter $\mathcal{U}_{2,\Omega}$ is a domination clutter and the graphs of the form $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}$ are domination realizations of $\mathcal{U}_{2,\Omega}$. Conversely, let us prove that if $G$ is a graph such that $\mathcal{D}(G) = \mathcal{U}_{2,\Omega}$, then $G$ is obtained from the complete graph $K_\Omega$ by deleting the edges of a perfect matching. So, assume that $\mathcal{D}(G) = \mathcal{U}_{2,\Omega}$. Then from Lemma 2 it follows that $\mathcal{N}[G] = \mathrm{tr}(\mathcal{U}_{2,\Omega})$. Since $\mathrm{tr}(\mathcal{U}_{2,\Omega}) = \mathcal{U}_{2m-1,\Omega}$, hence $\mathcal{N}[G] = \mathcal{U}_{2m-1,\Omega}$, and therefore all the vertices of $G$ have degree $2m-2$. Consequently, the graph $G$ is obtained from the complete graph $K_\Omega$ by deleting the edges of a perfect matching, as we wanted to prove.

From the above we conclude that if $n = 2m$ is even, then $\mathcal{U}_{2,\Omega} = \mathcal{D}(G)$ if and only if $G$ is a graph obtained from $K_\Omega$ by deleting the edges of a perfect matching. It is well known that the number of perfect matchings in a complete graph $K_{2m}$ is given by the double factorial $(2m-1)!!$, that is, $(2m)!/(2^m m!)$. Hence, if $n = 2m$ is even, then there are $(2m)!/(2^m m!)$ graphs $G$ such that $\mathcal{D}(G) = \mathcal{U}_{2,\Omega}$. This completes the proof of the third statement.

To finish the proof of the proposition we must demonstrate that if $3 \le r \le n-1$, or if $r = 2$ and $n$ is odd, then $\mathcal{U}_{r,\Omega}$ is not a domination clutter. Otherwise, assume that there exists a graph $G$ with vertex set $V(G) = \Omega$ and such that $\mathcal{D}(G) = \mathcal{U}_{r,\Omega}$. Since $\mathrm{tr}(\mathcal{U}_{r,\Omega}) = \mathcal{U}_{n-r+1,\Omega}$, from Lemma 2 we get that $\mathcal{N}[G] = \mathcal{U}_{n-r+1,\Omega}$. On one hand, the size of $\mathcal{N}[G]$ is at most $n$ because $V(G) = \Omega$. On the other hand, $\mathcal{U}_{n-r+1,\Omega}$ has size $\binom{n}{n-r+1}$. Therefore $\binom{n}{n-r+1} \le n$, and thus $r = 2$. At this point we have that $G$ is a graph of order $n$ with $\mathcal{N}[G] = \mathcal{U}_{n-r+1,\Omega} = \mathcal{U}_{n-1,\Omega}$. So, $G$ is a $(n-2)$-regular graph of order $n$, which is not possible if $n$ is odd. This completes the proof of the proposition. $\square$

## 3. Domination completions and decompositions of the unifom clutters $\mathcal{U}_{r,\Omega}$

Not all clutters are domination clutters. Therefore, given a clutter $\mathcal{H}$ a natural question is to study domination clutters "close" to $\mathcal{H}$. Here, we consider this problem whenever $\mathcal{H}$ is the *uniform* clutter $\mathcal{U}_{r,\Omega}$. Our goal is to introduce the poset of domination completions of $\mathcal{U}_{r,\Omega}$ (Subsection 3.1), and to prove that the minimal elements of this poset provide a decomposition of $\mathcal{U}_{r,\Omega}$ into domination clutters (Subsection 3.2).

### *3.1. Poset of domination completions of $\mathcal{U}_{r,\Omega}$*

Let $\Omega$ be a finite set and let us consider the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$. Observe that if $\mathcal{H}$ is a clutter on $\Omega$, the elements of $\mathcal{H}$ are pairwise non-comparable sets, and hence it follows that $\mathcal{U}_{r,\Omega} \subseteq \mathcal{H}$ if and only if $\mathcal{U}_{r,\Omega} = \mathcal{H}$. Therefore, if $\mathcal{U}_{r,\Omega}$ is not a domination clutter, then there does not exist a graph $G$ with vertex set $V(G) = \Omega$ such that $\mathcal{U}_{r,\Omega} \subseteq \mathcal{D}(G)$. Thus, a question that arises at this point is to determine domination clutters $\mathcal{D}(G)$ *close* to the clutter $\mathcal{U}_{r,\Omega}$, the *domination completions* of $\mathcal{U}_{r,\Omega}$.

A crucial point when looking for the domination completions $\mathcal{D}(G)$ of $\mathcal{U}_{r,\Omega}$ is to take into account all the dominating sets of the graph $G$ instead of considering only the inclusion-minimal dominating sets of $G$; that is, taking into account the family $D(G)$ instead of the family $\mathcal{D}(G)$. In order words, now we are looking for graphs $G$ such that the elements of $\mathcal{U}_{r,\Omega}$ are dominating sets of vertices of $G$ (not necessarily minimal dominating sets); that is, we look for graphs $G$ such that $\mathcal{U}_{r,\Omega} \subseteq D(G)$.

In order to seek the domination clutters close to $\mathcal{U}_{r,\Omega}$ we introduce a suitable partial order $\leqslant$ on the set of clutters that involves the *monotone increasing family of subsets* $\mathcal{H}^+$ associated to a clutter $\mathcal{H}$.

Let $\Omega$ be a finite set. Let $\mathcal{H}$ be a clutter on $\Omega$. Then we define $\mathcal{H}^+$ as the family whose elements are the subsets $A \subseteq \Omega$ such that there exists $A_0 \in \mathcal{H}$ with $A_0 \subseteq A$. Observe that $\mathcal{H}^+$ is a monotone increasing family of subsets of $\Omega$ whose inclusion-minimal elements are the subsets of $\mathcal{H}$; that is, $\mathcal{H} = \min\left(\mathcal{H}^+\right)$. Therefore, the clutter $\mathcal{H}$ is uniquely determined by the monotone increasing family $\mathcal{H}^+$. For instance, if $G$ is a graph then $\mathcal{D}(G)$ is a clutter on $V(G)$ whose associated monotone increasing family of subsets is $\mathcal{D}(G)^+ = D(G)$, and so $D(G)$ is uniquely determined by $\mathcal{D}(G)$.

To compare two clutters $\mathcal{H}_1, \mathcal{H}_2$ on $\Omega$, we use their associated monotone increasing families of subsets $\mathcal{H}_1^+, \mathcal{H}_2^+$. It is clear that if $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then $\mathcal{H}_1^+ \subseteq \mathcal{H}_2^+$. However, the converse is not true; that is, there exist clutters with $\mathcal{H}_1 \not\subseteq \mathcal{H}_2$ and $\mathcal{H}_1^+ \subseteq \mathcal{H}_2^+$ (for instance the clutters $\mathcal{H}_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $\mathcal{H}_2 = \{\{1,2\},\{3\}\}$). This fact leads us to consider the binary relation $\leqslant$, also used in [9, 10], which is defined as follows: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two clutters on the finite set $\Omega$, then we say that:

$$\mathcal{H}_1 \leqslant \mathcal{H}_2 \text{ if and only if } \mathcal{H}_1^+ \subseteq \mathcal{H}_2^+.$$

Next lemma will be used several times throughout this paper. The proofs of the three statements of the lemma are a straightforward consequence of the definition of the family $\mathcal{H}^+$ and of the fact that $\mathcal{H} = \min(\mathcal{H}^+)$.

**Lemma 6.** *Let $\Omega$ be a finite set. The following statements hold:*

1. *If $\mathcal{H}_1, \mathcal{H}_2$ are two clutters on $\Omega$ then, $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if and only if $\mathcal{H}_1 \subseteq \mathcal{H}_2^+$.*

2. *If $\mathcal{H}_1, \mathcal{H}_2$ are two clutters on $\Omega$ then, $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if and only if for all $A_1 \in \mathcal{H}_1$ there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq A_1$.*

3. *The binary relation $\leqslant$ is a partial order on the set of clutters on $\Omega$.*

Now, by using the partial order $\leqslant$, we define the *domination completions* of the clutter $\mathcal{U}_{r,\Omega}$ as any domination clutter $\mathcal{H}$ with ground set $\Omega$ such that

$\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}$. We denote by $\mathrm{Dom}(r,\Omega)$ the set whose elements are the domination completions of the clutter $\mathcal{U}_{r,\Omega}$; that is:

$$\mathrm{Dom}(r,\Omega) = \big\{\mathcal{H} : \mathcal{H} \text{ is a domination clutter with ground set } \Omega \text{ and } \mathcal{U}_{r,\Omega} \leqslant \mathcal{H}\big\}.$$

**Proposition 7.** *Let $\Omega$ be a finite set. Then the set $\mathrm{Dom}(r,\Omega)$ is non-empty.*

*Proof.* The uniform clutter $\mathcal{U}_{1,\Omega}$ is a domination clutter, and it is clear that $\mathcal{U}_{r,\Omega} \leqslant \mathcal{U}_{1,\Omega}$. So $\mathcal{U}_{1,\Omega} \in \mathrm{Dom}(r,\Omega)$. □

The partial order $\leqslant$ induces a poset structure in the set $\mathrm{Dom}(r,\Omega)$ of the domination completions of the uniform clutter $\mathcal{U}_{r,\Omega}$. Therefore, the minimal elements of the poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$ are the *optimal* domination completions of $\mathcal{U}_{r,\Omega}$. This subsection is completed by showing that if $\mathcal{U}_{r,\Omega}$ is not a domination clutter, then $\mathcal{U}_{r,\Omega}$ has at least two different optimal domination completions (Theorem 9). The following technical lemma is a key point in order to prove this result.

**Lemma 8.** *Let $\Omega$ be a finite set. Let $\mathcal{H}$ be a clutter on $\Omega$ such that $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}$ and $\mathcal{U}_{r,\Omega} \neq \mathcal{H}$. Then there exists a domination clutter $\mathcal{H}_0 \in \mathrm{Dom}(r,\Omega)$ such that $\mathcal{H} \nleqslant \mathcal{H}_0$.*

*Proof.* First notice that for $r = 1$ the hypotheses of the lemma do not hold because there is no clutter $\mathcal{H}$ on $\Omega$ different from $\mathcal{U}_{1,\Omega}$ such that $\mathcal{U}_{1,\Omega} \leqslant \mathcal{H}$. Moreover, observe that if $r = n$, then $\mathcal{U}_{r,\Omega}$ is a domination clutter, and so the clutter $\mathcal{H}_0 = \mathcal{U}_{n,\Omega}$ fulfills the required conditions. Therefore, from now on we may assume that $2 \leq r \leq n-1$.

Let $2 \leq r \leq n-1$. By assumption, $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}$ and $\mathcal{U}_{r,\Omega} \neq \mathcal{H}$. Thus, since $\leqslant$ is a partial order, it follows that $\mathcal{H} \nleqslant \mathcal{U}_{r,\Omega}$. Therefore, there exists $A_0 \in \mathcal{H}$ such that $A \nsubseteq A_0$ for all $A \subseteq \Omega$ with $|A| = r$; that is, there exists $A_0 \in \mathcal{H}$ with $|A_0| = t < r$. Without loss of generality we may assume that $\Omega = \{w_1, \ldots, w_t, w_{t+1}, \ldots, w_r, \ldots, w_n\}$ and that $A_0 = \{w_1, \ldots, w_t\}$. Set $\Omega_1 = \{w_1, \ldots, w_t, w_{t+1}, \ldots, w_r\}$ and set $\Omega_2 = \Omega \setminus \Omega_1$.

At this point let us consider the domination clutter $\mathcal{H}_0 = \mathcal{D}(G_0)$ where $G_0$ is the join graph $G_0 = \overline{K_{\Omega_1}} \vee K_{\Omega_2}$. So, from Lemma 3 we get that $\mathcal{H}_0 = \{\Omega_1\} \cup \{\{w\} : w \in \Omega_2\}$. The proof will be completed by showing that $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}_0$ and that $\mathcal{H} \nleqslant \mathcal{H}_0$.

In order to prove the inequality $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}_0$ we must demonstrate that for all $A \in \mathcal{U}_{r,\Omega}$ there exists $A' \in \mathcal{H}_0$ such that $A' \subseteq A$. So let $A \subseteq \Omega$ with $|A| = r$. If $A = \Omega_1$, then set $A' = \Omega_1 \in \mathcal{H}_0$; whereas if $A \neq \Omega_1$, then there exists $w \in \Omega_2$ such that $w \in A$ and, so, we can consider $A' = \{w\} \in \mathcal{H}_0$.

To finish we must demonstrate that $\mathcal{H} \nleqslant \mathcal{H}_0$. On the contrary, let us assume that $\mathcal{H} \leqslant \mathcal{H}_0$. Then, since $A_0 \in \mathcal{H}$, there exists $A \in \mathcal{H}_0$ such that $A \subseteq A_0$. So either $\Omega_1 \subseteq A_0$ or there exists $w \in \Omega_2$ such that $\{w\} \subseteq A_0$. In any case a contradiction is obtained because, by construction, $A_0 \subsetneq \Omega_1$ and $A_0 \cap \Omega_2 = \emptyset$. This completes the proof of the lemma. □

**Theorem 9.** *Let $\Omega$ be a finite set. Then the non-empty poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$ of the domination completions of the clutter $\mathcal{U}_{r,\Omega}$ has a unique minimal element if and only if the clutter $\mathcal{U}_{r,\Omega}$ is a domination clutter.*

*Proof.* It is clear that if $\mathcal{U}_{r,\Omega}$ is a domination clutter, then the non-empty poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$ has a unique minimal element, namely $\min\big(\mathrm{Dom}(r,\Omega), \leqslant\big) = \{\mathcal{U}_{r,\Omega}\}$. Therefore, we must only prove that if the poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$ has a unique minimal element, then $\mathcal{U}_{r,\Omega}$ is a domination clutter. So let us assume that $\min\big(\mathrm{Dom}(r,\Omega), \leqslant\big) = \{\mathcal{H}\}$. On one hand, $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}$ because $\mathcal{H} \in \mathrm{Dom}(r,\Omega)$. On the other hand, since $\mathcal{H}$ is the unique minimal element of the poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$, we get that $\mathcal{H} \leqslant \mathcal{H}_0$ for all domination clutters $\mathcal{H}_0 \in \mathrm{Dom}(r,\Omega)$. Therefore, from Lemma 8 we conclude that $\mathcal{U}_{r,\Omega} = \mathcal{H}$. In particular, $\mathcal{U}_{r,\Omega}$ is a domination clutter, as we wanted to prove. $\square$

In the following example we present the description of the domination completions of the 2-uniform clutter $\mathcal{U}_{2,\Omega}$ where $\Omega = \{1,2,3\}$ (in Subsection 4.1 we study the general case $\mathcal{U}_{2,\Omega}$ where $\Omega$ is a finite set of odd size).

**Example 1.** Let $\Omega = \{1,2,3\}$ and let us consider the 2-uniform clutter $\mathcal{U}_{2,\Omega}$; that is, $\mathcal{U}_{2,\Omega} = \{\{1,2\},\{1,3\},\{2,3\}\}$. From Proposition 5 we know that $\mathcal{U}_{2,\Omega}$ is not a domination clutter. Therefore, by applying Theorem 9 we conclude that the non-empty poset $\big(\mathrm{Dom}(2,\Omega), \leqslant\big)$ has at least two minimal elements. Let us compute these minimal elements. Let $G$ be a graph with vertex set $V(G) = \{1,2,3\}$. It is clear that if $E(G) = \emptyset$, then $\mathcal{D}(G) = \big\{\{1,2,3\}\big\}$; whereas if $\{a,b,c\} = \{1,2,3\}$ and $E(G) = \big\{\{a,b\}\big\}$, then $\mathcal{D}(G) = \big\{\{a,c\},\{b,c\}\big\}$; while if $E(G) = \big\{\{a,b\},\{a,c\}\big\}$, then $\mathcal{D}(G) = \big\{\{a\},\{b,c\}\big\}$; but if $|E(G)| = 3$ then $\mathcal{D}(G) = \big\{\{1\},\{2\},\{3\}\big\}$. Therefore we conclude that $\mathcal{U}_{2,\Omega} \leqslant \mathcal{D}(G)$ if and only if either $|E(G)| = 2$ or $|E(G)| = 3$. Thus, the clutter $\mathcal{U}_{2,\Omega}$ has four domination completions, namely, the three domination clutters defined by the graphs of size 2 and the domination clutter defined by the graph of size 3; that is, $\mathrm{Dom}(2,\Omega) = \big\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{U}_{1,\Omega}\big\}$ where $\mathcal{H}_i = \{\{i\},\{j,k\}\}$ being $\{i,j,k\} = \{1,2,3\}$. Observe that $\mathcal{H}_i \leqslant \mathcal{U}_{1,\Omega}$ and that $\mathcal{H}_i \not\leqslant \mathcal{H}_j$ if $i \neq j$. Therefore, the poset $\big(\mathrm{Dom}(2,\Omega), \leqslant\big)$ has three minimal elements $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. So $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are the optimal domination completions of $\mathcal{U}_{2,\Omega}$.

### *3.2. Domination decompositions of $\mathcal{U}_{r,\Omega}$*

Let $\Omega$ be a finite set. We say that a domination clutter $\mathcal{H}$ is a *minimal domination completion* of the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ if $\mathcal{H}$ is a minimal element of the poset $\big(\mathrm{Dom}(r,\Omega), \leqslant\big)$. Let us denote by $\mathcal{D}om(r,\Omega)$ the set whose elements are the minimal domination completions $\mathcal{H}$ of $\mathcal{U}_{r,\Omega}$; that is:

$$\mathcal{D}om(r,\Omega) = \min\big(\mathrm{Dom}(r,\Omega), \leqslant\big).$$

We have seen in Theorem 9 that $\mathcal{D}om(r,\Omega)$ has cardinality one if and only if $\mathcal{U}_{r,\Omega}$ is a domination clutter. The following theorem deals with the case of cardinality greater than one, and shows that we can recover uniquely the clutter from the elements of $\mathcal{D}om(r,\Omega)$. Before stating our result we introduce the clutter operation $\sqcap$, also used in the matroidal framework in [9, 10].

Let $\Omega$ be a finite set and let $\mathcal{H}_1,\dots,\mathcal{H}_\ell$ be $\ell$ clutters on $\Omega$. Then we define the clutter $\mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_\ell$ as:

$$\mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_\ell = \min\left(\mathcal{H}_1^+ \cap \cdots \cap \mathcal{H}_\ell^+\right).$$

Observe that from the definition of $\mathcal{H}^+$ it is not hard to prove that:

$$\mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_\ell = \min\left\{A_1 \cup \cdots \cup A_\ell \,:\, A_i \in \mathcal{H}_i \text{ for } 1 \le i \le \ell\right\}.$$

**Theorem 10.** *Let $\Omega$ be a finite set. Let $\mathcal{D}om(r,\Omega) = \{\mathcal{H}_1,\dots,\mathcal{H}_s\}$ be the set of the minimal domination completions of $\mathcal{U}_{r,\Omega}$. Then:*

$$\mathcal{U}_{r,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_s.$$

*Proof.* Let us denote $\mathcal{H}_0 = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_s = \min\left\{ A_1 \cup \cdots \cup A_s \,:\, A_i \in \mathcal{H}_i \text{ for } 1 \le i \le s\right\}$.

First let us show that $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}_0$; that is, we must prove that if $A \in \mathcal{U}_{r,\Omega}$, then there exists $A_0 \in \mathcal{H}_0$ such that $A_0 \subseteq A$. So, let $A \in \mathcal{U}_{r,\Omega}$. Let $1 \le i \le s$. Since $\mathcal{U}_{r,\Omega} \leqslant \mathcal{H}_i$, for all $A \in \mathcal{U}_{r,\Omega}$ there exists $A_i \in \mathcal{H}_i$ with $A_i \subseteq A$. Therefore we get that $A_1 \cup \cdots \cup A_s \subseteq A$. So there exists $A_0 \in \mathcal{H}_0$ such that $A_0 \subseteq A$.

Next we are going to prove that $\mathcal{U}_{r,\Omega} = \mathcal{H}_0$. Observe that if $\mathcal{U}_{r,\Omega} \neq \mathcal{H}_0$, then, by applying Lemma 8, we get that there exists a domination clutter $\mathcal{H}_0' \in \mathrm{Dom}(r,\Omega)$ such that $\mathcal{H}_0 \not\leqslant \mathcal{H}_0'$. The proof will be completed by showing that this leads us to a contradiction. On one hand, since $\mathcal{H}_0' \in \mathrm{Dom}(r,\Omega)$ and $\mathcal{D}om(r,\Omega) = \{\mathcal{H}_1,\dots,\mathcal{H}_s\}$, we conclude that there exists $i_0 \in \{1,\dots,s\}$ such that $\mathcal{H}_{i_0} \leqslant \mathcal{H}_0'$. On the other hand, from the definition of $\mathcal{H}_0$ and by applying Lemma 6 it is easy to check that the inequality $\mathcal{H}_0 \leqslant \mathcal{H}_{i_0}$ holds. Therefore we conclude that $\mathcal{H}_0 \leqslant \mathcal{H}_0'$ because the binary relation $\leqslant$ is a partial order. Hence a contradiction is achieved, as we wanted to prove. $\square$

**Remark 2.** We observe that Theorem 9 is a corollary of Theorem 10. Indeed, if the clutter $\mathcal{U}_{r,\Omega}$ has a unique minimal element, then $\mathcal{D}om(r,\Omega) = \{\mathcal{H}_1,\dots,\mathcal{H}_s\}$ with $s = 1$ and hence $\mathcal{U}_{r,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_s = \mathcal{H}_1$, which is a domination clutter.

**Remark 3.** Theorems 9 and 10 are related to [10, Theorem 14], but this latter result cannot be applied in the domination framework. Indeed, if we want to apply [10, Theorem 14], then, using the notation of this theorem, we have to take $\Sigma = \{\mathcal{H} : \mathcal{H} \text{ is a domination clutter with ground set } \Omega\}$. But this family does not contain the clutters of the form $\mathcal{U}_{1,X}$ for $X \subset \Omega$, which is just the hypothesis needed in [10, Theorem 14]. ($\mathcal{U}_{1,X}$ is represented as $\Lambda_X$ in [10, Theorem 14].)

The previous theorem leads us to the following definition. Let $\Omega$ be a finite set. We say that a family $\{\mathcal{H}_1,\dots,\mathcal{H}_t\}$ of $t \ge 1$ distinct domination clutters with ground set $\Omega$ is a *$t$-decomposition* of the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ if $\mathcal{U}_{r,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_t$. Let us denote $\mathfrak{D}(r,\Omega) = \min\{t \,:\, \text{there exists a } t\text{-decomposition of } \mathcal{U}_{r,\Omega}\}$. It is clear that $\mathfrak{D}(r,\Omega)=1$ if and only if the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ is a domination clutter.

From Theorem 10 we get that the domination clutters in $\mathcal{D}om(r,\Omega)$ provide a decomposition of $\mathcal{U}_{r,\Omega}$, and therefore if the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ has $s$ minimal domination completions, then $\mathfrak{D}(r,\Omega)\leq s$. The next proposition states that, in fact, to compute $\mathfrak{D}(r,\Omega)$ it is enough to consider only those decompositions consisting of minimal domination completions of $\mathcal{U}_{r,\Omega}$.

**Proposition 11.** *Let $\Omega$ be a finite set. Let $\mathfrak{D}(r,\Omega)=\delta$. Then there exist $\delta$ minimal domination completions $\mathcal{H}_1,\dots,\mathcal{H}_\delta\in\mathcal{D}om(r,\Omega)$ of the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ such that $\{\mathcal{H}_1,\dots,\mathcal{H}_\delta\}$ is a $\delta$-decomposition of $\mathcal{U}_{r,\Omega}$.*

*Proof.* To prove the proposition it is enough to show that any decomposition of $\mathcal{U}_{r,\Omega}$ can be transformed into a decomposition consisting of minimal domination completions of $\mathcal{U}_{r,\Omega}$; that is, we must demonstrate that if $\{\mathcal{H}'_1,\dots,\mathcal{H}'_t\}$ is a $t$-decomposition of $\mathcal{U}_{r,\Omega}$, then there exist $\ell$ distinct clutters $\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\ell}\in\mathcal{D}om(r,\Omega)$ (with $\ell\leq t$) such that $\{\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\ell}\}$ is an $\ell$-decomposition of $\mathcal{U}_{r,\Omega}$.

So, assume that $\{\mathcal{H}'_1,\dots,\mathcal{H}'_t\}$ is a $t$-decomposition of the clutter $\mathcal{U}_{r,\Omega}$, and let $\mathcal{D}om(r,\Omega)=\{\mathcal{H}_1,\dots,\mathcal{H}_s\}$.

It is clear that $\mathcal{H}'_1\sqcap\cdots\sqcap\mathcal{H}'_t\leqslant\mathcal{H}'_k$ for all $k\in\{1,\dots,t\}$. Therefore, $\mathcal{U}_{r,\Omega}\leqslant\mathcal{H}'_k$, and so $\mathcal{H}'_k\in\mathrm{Dom}(r,\Omega)$. Since $\mathcal{H}_1,\dots,\mathcal{H}_s$ are the minimal elements of the poset $(\mathrm{Dom}(r,\Omega),\leqslant)$, for all $k\in\{1,\dots,t\}$ there exists $\alpha_k\in\{1,\dots,s\}$ such that $\mathcal{H}_{\alpha_k}\leqslant\mathcal{H}'_k$. Let $\{\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\ell}\}=\{\mathcal{H}_{\alpha_1},\dots,\mathcal{H}_{\alpha_t}\}$, where $\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\ell}$ are different (observe that $\ell\leq t$). On one hand we have that $\mathcal{H}_{i,1}\sqcap\cdots\sqcap\mathcal{H}_{i_\ell}=\mathcal{H}_{\alpha_1}\sqcap\cdots\sqcap\mathcal{H}_{\alpha_t}\leqslant\mathcal{H}'_1\sqcap\cdots\sqcap\mathcal{H}'_t=\mathcal{U}_{r,\Omega}$. On the other hand, $\mathcal{U}_{r,\Omega}\leqslant\mathcal{H}_{i_1}\sqcap\cdots\sqcap\mathcal{H}_{i_\ell}$ because $\mathcal{U}_{r,\Omega}\leqslant\mathcal{H}_{i_k}$ for all $k$. Since $\leqslant$ is a partial order, we conclude that the equality $\mathcal{U}_{r,\Omega}=\mathcal{H}_{i_1}\sqcap\cdots\sqcap\mathcal{H}_{i_\ell}$ holds; that is, $\{\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\ell}\}$ is an $\ell$-decomposition of $\mathcal{U}_{r,\Omega}$. □

To conclude this subsection let us show an example of decomposition, namely we compute the domination related parameter $\mathfrak{D}(2,\Omega)$ where $\Omega=\{1,2,3\}$ (in Subsection 4.1 we study the general case $\mathfrak{D}(2,\Omega)$ where $\Omega$ is a finite set of odd size).

**Example 2.** Let us consider the 2-uniform clutter $\mathcal{U}_{2,\Omega}$ with ground set $\Omega=\{1,2,3\}$. From Example 1 we get that $\mathcal{U}_{2,\Omega}$ has three minimal domination completions, namely, $\mathcal{D}om(2,\Omega)=\{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3\}$, where $\mathcal{H}_i=\{\{i\},\{j,k\}\}$ being $\{i,j,k\}=\{1,2,3\}$. Therefore, by applying Theorem 10 we get that $\mathcal{U}_{2,\Omega}=\mathcal{H}_1\sqcap\mathcal{H}_2\sqcap\mathcal{H}_3$. So the minimal domination completions of the non-domination clutter $\mathcal{U}_{2,\Omega}$ provides the domination decomposition $\mathcal{U}_{2,\Omega}=\mathcal{D}(G_1)\sqcap\mathcal{D}(G_2)\sqcap\mathcal{D}(G_3)$ where $G_i$ is the graph with vertex set $V(G_i)=\{1,2,3\}$ and edge set $E(G_i)=\{\{i,j\},\{i,k\}\}$ being $\{i,j,k\}=\{1,2,3\}$. Thus we have that $2\leq\mathfrak{D}(2,\Omega)\leq 3$. However observe that in this case, if $i_1\neq i_2$, then $\mathcal{H}_{i_1}\sqcap\mathcal{H}_{i_2}=\big\{\{i_1\},\{i_2,i_3\}\big\}\sqcap\big\{\{i_2\},\{i_1,i_3\}\big\}=\min\big\{\{i_1,i_2\},\{i_1,i_3\},\{i_2,i_3\},\{i_1,i_2,i_3\}\big\}=\big\{\{i_1,i_2\},\{i_1,i_3\},\{i_2,i_3\}\big\}=\mathcal{U}_{2,\Omega}$. Therefore we conclude that $\mathfrak{D}(2,\Omega)=2$.

## 4. Determining the minimal domination completions of some uniform clutters $\mathcal{U}_{r,\Omega}$

Let $\Omega$ be a finite set of size $n$. From Proposition 5, the study of the uniform clutters of maximum size will be completed with the computation of the minimal domination completions of $\mathcal{U}_{r,\Omega}$ either when $r = 2$ and $n$ is odd, or when $3 \leq r \leq n-1$. The description of the set $\mathcal{D}om(r,\Omega)$ of all the minimal domination completions of $\mathcal{U}_{r,\Omega}$ is a problem which is far from being solved. However, here we present the description in three cases. Namely, the case $r = 2$ and $n$ odd (Subsection 4.1), the case $r = n-1$ (Subsection 4.2), and the case $r$ arbitrary and $n \leq 5$ (Subsection 4.3).

### *4.1. Minimal domination completions of $\mathcal{U}_{2,\Omega}$*

From Proposition 5 and Theorem 9 we get that $\mathcal{D}om(2,\Omega) = \{\mathcal{U}_{2,\Omega}\}$ if and only if $\Omega$ has even size. In this subsection we determine the set $\mathcal{D}om(2,\Omega)$ of the minimal domination completions of the uniform clutter $\mathcal{U}_{2,\Omega}$ whenever the finite set $\Omega$ has odd size.

**Lemma 12.** *Let $\Omega'$ be a non-empty subset of a finite set $\Omega$. Let $\mathcal{H}$ be a clutter on $\Omega$ and let $\mathcal{H}[\Omega'] = \{A \in \mathcal{H} : A \subseteq \Omega'\}$. Let $G'$ be a graph with vertex set $V(G') \subseteq \Omega'$, and suppose that $\mathcal{H}[\Omega'] \neq \emptyset$. Then $\mathcal{H}[\Omega'] \leqslant \mathcal{D}(G')$ if and only if $\mathcal{H} \leqslant \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$.*

*Proof.* First assuming that $\mathcal{H}[\Omega'] \leqslant \mathcal{D}(G')$ we are going to prove that $\mathcal{H} \leqslant \mathcal{D}(G' \vee K_{\{\omega\}})$. Recall that by Lemma 3 we get that $\mathcal{D}(G' \vee K_{\Omega\setminus\Omega'}) = \mathcal{D}(G') \cup \{\{w\} : w \in \Omega \setminus \Omega'\}$. Therefore, by applying Lemma 6, we must demonstrate that if $A \in \mathcal{H}$, then there exists $D \in \mathcal{D}(G') \cup \{\{w\} : w \in \Omega \setminus \Omega'\}$ such that $D \subseteq A$. Let $A \in \mathcal{H}$. If $A \nsubseteq \Omega'$, then there is $\omega \in A \cap (\Omega \setminus \Omega')$ and so we can set $D = \{\omega\}$. Now assume that $A \subseteq \Omega'$. Then $A \in \mathcal{H}[\Omega'] \leqslant \mathcal{D}(G')$, and hence there exists $D' \in \mathcal{D}(G')$ such that $D' \subseteq A$. Thus, in such a case, we can consider $D = D'$.

Now suppose that $\mathcal{H} \leqslant \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$. We want to prove that $\mathcal{H}[\Omega'] \leqslant \mathcal{D}(G')$; that is, we must demonstrate that if $A' \in \mathcal{H}[\Omega']$, then there exists $D' \in \mathcal{D}(G')$ such that $D' \subseteq A'$. Let $A' \in \mathcal{H}[\Omega']$. Since $A' \in \mathcal{H}[\Omega'] \subseteq \mathcal{H}$ and $\mathcal{H} \leqslant \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$, there exists $D \in \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$ such that $D \subseteq A'$. But $A' \subseteq \Omega'$ and, by Lemma 3, $\mathcal{D}(G' \vee K_{\Omega\setminus\Omega'}) = \mathcal{D}(G') \cup \{\{w\} : w \in \Omega \setminus \Omega'\}$. Therefore $D \in \mathcal{D}(G')$. Now the proof is completed by setting $D' = D$. $\square$

**Theorem 13.** *Let $\Omega$ be a finite set of size $|\Omega| = n$. Assume that $n$ is odd. Then the following statements hold:*

1. *For all $w \in \Omega$, the clutter $\mathcal{H}_\omega = \big\{\{\omega\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}}$ is a domination clutter. Moreover, if $G$ is a graph with vertex set $\Omega$, then $\mathcal{D}(G) = \mathcal{H}_\omega$ if and only if $G = K_{\{\omega\}} \vee G'$ where $G'$ is a graph realization of the domination clutter $\mathcal{U}_{2,\Omega\setminus\{\omega\}}$.*
2. *The uniform clutter $\mathcal{U}_{2,\Omega}$ has $n$ minimal domination completions. Namely, $\mathcal{D}om(2,\Omega) = \{\mathcal{H}_\omega : \omega \in \Omega\}$.*

3. *If $w_{i_1}, w_{i_2}$ are distinct elements of $\Omega$, then $\{\mathcal{H}_{\omega_{i_1}}, \mathcal{H}_{\omega_{i_2}}\}$ is a 2-decomposition of $\mathcal{U}_{2,\Omega}$. In particular, $\mathfrak{D}(2,\Omega) = 2$.*

*Proof.* Let $w \in \Omega$. Since $\Omega \setminus \{\omega\}$ has even size, the clutter $\mathcal{U}_{2,\Omega\setminus\{\omega\}}$ is a domination clutter, and so there exists a graph $G_0'$ with vertex set $V(G_0') = \Omega\setminus\{\omega\}$ such that $\mathcal{U}_{2,\Omega\setminus\{\omega\}} = \mathcal{D}(G_0')$. Let $G_0$ be the join graph $G_0 = K_{\{\omega\}} \vee G_0'$. Then $G_0$ is a graph with vertex set $\Omega$ and minimal dominating sets $\mathcal{D}(G_0) = \mathcal{D}(K_{\{\omega\}} \vee G_0') = \big\{\{\omega\}\big\} \cup \mathcal{D}(G_0') = \big\{\{\omega\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}} = \mathcal{H}_\omega$. So $\mathcal{H}_\omega$ is a domination clutter.

To conclude the proof of the first statement we must demonstrate that if $G$ is a graph with $\mathcal{D}(G) = \big\{\{\omega\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}}$, then $G = K_{\{\omega\}} \vee G'$ for some graph $G'$ with vertex set $\Omega \setminus \{w\}$ and minimal dominating sets $\mathcal{D}(G') = \mathcal{U}_{2,\Omega\setminus\{\omega\}}$. Let $G' = G - \omega$ be the graph obtained by deleting the vertex $\omega$ from $G$. Since $\{\omega\} \in \mathcal{D}(G)$, the vertex $\omega$ is universal in $G$ and so $G = K_{\{\omega\}} \vee (G - \omega) = K_{\{\omega\}} \vee G'$. Moreover, from Lemma 3 we have $\mathcal{D}(G) = \big\{\{\omega\}\big\} \cup \mathcal{D}(G')$. Thus we conclude that $\mathcal{D}(G') = \mathcal{U}_{2,\Omega\setminus\{\omega\}}$ because $\mathcal{D}(G) = \big\{\{\omega\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}}$. This completes the proof of the first statement.

Next we are going to prove the second statement; that is, we must demonstrate that $\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}$ are the minimal domination completions of $\mathcal{U}_{2,\Omega}$.

Let $1 \leq i \leq n$. It is clear that $\mathcal{U}_{2,\Omega} \leqslant \big\{\{\omega_i\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega_i\}}$; that is, $\mathcal{U}_{2,\Omega} \leqslant \mathcal{H}_{\omega_i}$. Moreover, from statement (1) the clutter $\mathcal{H}_{\omega_i}$ is a domination clutter. So the domination clutters $\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}$ are domination completions of $\mathcal{U}_{2,\Omega}$; that is, $\{\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}\} \subseteq \mathrm{Dom}(2,\Omega)$.

Now let us prove that $\mathcal{D}om(2,\Omega) \subseteq \{\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}\}$. In order to do this it is enough to show that if $\mathcal{H}$ is a domination completion of $\mathcal{U}_{2,\Omega}$, then there exists $i_0$ such that $\mathcal{H}_{\omega_{i_0}} \leqslant \mathcal{H}$. Let $\mathcal{H}$ be a domination completion of $\mathcal{U}_{2,\Omega}$; that is, $\mathcal{H}$ is a dominatoin clutter such that $\mathcal{U}_{2,\Omega} \leqslant \mathcal{H}$. Recall that $\mathcal{U}_{2,\Omega}$ is not a domination clutter, so $\mathcal{U}_{2,\Omega} \neq \mathcal{H}$. Hence, since the clutter $\mathcal{U}_{2,\Omega}$ consists of all subsets $A \subseteq \Omega$ of size $|A| = 2$ and $\mathcal{H}$ is a clutter, there exists $\omega_{i_0} \in \Omega$ such that $\{\omega_{i_0}\} \in \mathcal{H}$. Therefore we have that $\mathcal{U}_{2,\Omega} \leqslant \mathcal{H}$ and that $\{\omega_{i_0}\} \in \mathcal{H}$, and so $\big\{\{\omega_{i_0}\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega_{i_0}\}} \leqslant \mathcal{H}$, that is, $\mathcal{H}_{\omega_{i_0}} \leqslant \mathcal{H}$.

From the above we have that $\mathcal{D}om(2,\Omega) = \min\{\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}\}$. Observe that if $i \neq j$, then $\mathcal{H}_{\omega_i} \not\leqslant \mathcal{H}_{\omega_j}$. So, $\mathcal{D}om(2,\Omega) = \{\mathcal{H}_{\omega_1}, \dots, \mathcal{H}_{\omega_n}\}$. This completes the proof of the second statement.

To complete the proof of the proposition we must prove that if $\omega_1 \neq \omega_2$, then $\{\mathcal{H}_{\omega_1}, \mathcal{H}_{\omega_2}\}$ is a 2-decomposition of $\mathcal{U}_{2,\Omega}$; that is, we must demonstrate that $\mathcal{U}_{2,\Omega} = \mathcal{H}_{\omega_1} \sqcap \mathcal{H}_{\omega_2}$. Since $\mathcal{H}_{\omega_i} = \big\{\{\omega_i\}\big\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega_i\}}$, the union $A_1 \cup A_2$ has at least size two whenever $A_1 \in \mathcal{H}_{\omega_1}$ and $A_2 \in \mathcal{H}_{\omega_2}$ if $w_1 \neq w_2$. Moreover, it is clear that every subset $\{\omega_k, \omega_\ell\}$ with $\omega_k \neq \omega_\ell$ can be obtained as $A_1 \cup A_2$, for some $A_1 \in \mathcal{H}_{\omega_1}$ and $A_2 \in \mathcal{H}_{\omega_2}$. Hence we conclude that $\mathcal{H}_{\omega_1} \sqcap \mathcal{H}_{\omega_2} = \mathcal{U}_{2,\Omega}$. □

### *4.2. Minimal domination completions of $\mathcal{U}_{n-1,\Omega}$*

From Proposition 5 we get that if $\Omega$ has size $n \geq 3$, then the clutter $\mathcal{U}_{n-1,\Omega}$ is not a domination clutter. The goal of this subsection is to provide a complete description of the set $\mathcal{D}om(n-1,\Omega)$ of the minimal domination completions of $\mathcal{U}_{n-1,\Omega}$ (Theorem 19), and to display their graph realizations (Proposition 20).

In addition, we present an upper bound for the decomposition parameter $\mathfrak{D}(n-1,\Omega)$ (Proposition 21). Up to now, the computation of the exact value of this parameter remains as an open problem.

In order to prove our results we will use the following five technical lemmas. Three of these lemmas are concerned with graphs that are disjoint union of stars; whereas the other two lemmas involve some properties of the partial order $\leqslant$.

A tree $T$ of order $n \geq 2$ is a *star* if it is isomorphic to the complete bipartite graph $K_{1,n-1}$. Observe that a tree $T$ of order $n \geq 2$ is a star if and only if $T$ has at most one vertex of degree at least 2, the *center* of the star. If a star $T$ has no vertices of degree at least 2, then $T$ is isomorphic to $K_2$ and both vertices can be considered as the center of the star. Stars can also be characterized as non-empty connected graphs such that all its edges are incident to a *leaf*; that is, a vertex of degree 1. It is clear that every graph without isolated vertices and such that all its edges have at least one endpoint of degree 1 is a disjoint union of stars. The following result is a consequence of this fact.

**Lemma 14.** *Let $G$ be a graph without isolated vertices. Then $G$ is a disjoint union of stars if and only if $\mathcal{N}[G] = E(G)$.*

*Proof.* Suppose first that $G$ is a disjoint union of stars. If $x$ is a leaf, then $N[x] = \{x, y\} \in E(G)$; whereas if $x$ is not a leaf, then $x$ a vertex of degree $r \geq 2$, and so $N[x] = \{x, y_1, \dots, y_r\}$ where $y_1, \dots, y_r$ are the leaves hanging from $x$. Therefore, we conclude that $\mathcal{N}[G] = \{N[x] : x \text{ is a leaf }\}$. So, $\mathcal{N}[G] \subseteq E(G)$. Now if $\{x, y\} \in E(G)$, then either $x$ or $y$ is a leaf; hence $\{x, y\}$ is either $N[x]$ or $N[y]$ and so $\{x, y\} \in \mathcal{N}[G]$. Thus $E(G) \subseteq \mathcal{N}[G]$. Therefore, the equality follows.

Now suppose that $\mathcal{N}[G] = E(G)$. Then, every edge has an endpoint of degree 1, because it is the closed neighborhood of some vertex. Therefore, $G$ is a disjoint union of stars. $\square$

**Lemma 15.** *Every graph $G$ without isolated vertices contains a spanning subgraph that is a disjoint union of stars.*

*Proof.* It is sufficient to prove that the statement holds for connected graphs $G$ of order $n \geq 2$. We proceed by induction on $n$. The result is trivial for $n = 2$. Now assume that $G$ is a connected graph of order $n \geq 3$. Consider a spanning tree $T$ of $G$. If $T$ is a star, then the result follows. So we may assume that $T$ is not a star. In such a case $T$ has at least two vertices of degree $\geq 2$. Consider an edge of the path joining these two vertices. By removing this edge, we obtain two trees $T_1$ and $T_2$ of order at least 2 and without isolated vertices. By inductive hypothesis, both trees contain a spanning subgraph that is a disjoint union of stars. To finish observe that the union of those subgraphs is a spanning subgraph of $G$ that is a disjoint union of stars. $\square$

**Lemma 16.** *Let $G$ be the disjoint union of the stars $S_1, \dots, S_r$. Then $G$ has exactly $2^r$ minimal dominating sets. Namely, the minimal dominating sets of $G$*

*are the sets of vertices of the form:*

$$\{c_j : j \in J\} \cup \left( \bigcup_{i \in \{1,\dots,r\} \setminus J} L_i \right)$$

*where $J \subseteq \{1, \dots, r\}$, and where $c_i$ and $L_i$ are respectively the center and the set of leaves of the star $S_i$ (whenever $S_i$ is isomorphic to $K_2$, choose one of the two vertices as the center and the other as the leaf).*

*Proof.* It is clear that a star $S$ has exactly two minimal dominating sets. Namely, if the star $S$ is not isomorphic $K_2$, then the minimal dominating sets of $S$ are the set of leaves and the set containing only the center; whereas if the star $S$ is isomorphic to $K_2$, then the minimal dominating sets of $S$ are the sets containing exactly one vertex. Now, the result follows by applying Lemma 3 because if $G$ is the disjoint union of the stars $S_1, \dots, S_r$, then $\mathcal{D}(G) = \{D_1 \cup \cdots \cup D_r : D_i \in \mathcal{D}(S_i)\}$. □

**Lemma 17.** *Let $G'$ be a spanning subgraph of $G$. Then $\mathcal{D}(G') \leqslant \mathcal{D}(G)$.*

*Proof.* From $V(G) = V(G')$ and $E(G') \subseteq E(G)$, we have that every dominating set of $G'$ is also a dominating set of $G$. In particular, if $D' \in \mathcal{D}(G')$, then $D'$ contains a minimal dominating set $D$ of $G$. Therefore, the inequality $\mathcal{D}(G') \leqslant \mathcal{D}(G)$ holds. □

**Lemma 18.** *If $\mathcal{H}$ and $\mathcal{H}'$ are clutters such that $\mathcal{H} \leqslant \mathcal{H}'$, then $\operatorname{tr}(\mathcal{H}') \leqslant \operatorname{tr}(\mathcal{H})$.*

*Proof.* Let $X' \in \operatorname{tr}(\mathcal{H}')$. We want to prove that there exists $X \in \operatorname{tr}(\mathcal{H})$ such that $X \subseteq X'$. To do this, it is enough to demonstrate that $X' \cap A \neq \emptyset$ for every $A \in \mathcal{H}$. Let $A \in \mathcal{H}$. Since $\mathcal{H} \leqslant \mathcal{H}'$, there exists $A' \in \mathcal{H}'$ such that $A' \subseteq A$. By assumption $X' \in \operatorname{tr}(\mathcal{H}')$. So $X' \cap A' \neq \emptyset$ and thus $X' \cap A \neq \emptyset$, as we wanted to prove. □

Now, by using these lemmas, we are going to prove the following theorem which provides a complete description of all the minimal domination completions of the uniform clutter $\mathcal{U}_{n-1,\Omega}$.

**Theorem 19.** *Let $\Omega$ be a finite set of size $n \geq 3$. Then the minimal domination completions of $\mathcal{U}_{n-1,\Omega}$ are the domination clutters $\mathcal{H}$ of the form $\mathcal{H} = \mathcal{D}(G)$ where $G$ is a disjoint union of stars; that is:*

$$\mathcal{D}om(n-1, \Omega) = \{\mathcal{D}(G) : G \text{ is a disjoint union of stars with vertex set } \Omega\}.$$

*Proof.* Let $\Sigma = \{G : G$ is a disjoint union of stars with vertex set $\Omega\}$. First we will prove that $\mathcal{U}_{n-1,\Omega} \leq \mathcal{D}(G)$ for all $G \in \Sigma$; that is, we must demonstrate that if $G \in \Sigma$ and if $A \in \mathcal{U}_{n-1,\Omega}$, then there exists $D \in \mathcal{D}(G)$ such that $D \subseteq A$. So, let $G \in \Sigma$ and let $A \in \mathcal{U}_{n-1,\Omega}$. Then $A = \Omega \setminus \{\omega_0\}$ for some $\omega_0 \in \Omega$. Since $G \in \Sigma$, by Lemma 16, there exists a minimal dominating set $D_0$ of $G$ not containing $\omega_0$. Thus, $D_0 \subseteq \Omega \setminus \{\omega_0\}$. So we can set $D = D_0$.

Now, we will prove that if $\mathcal{H}$ is a domination completion of $\mathcal{U}_{n-1,\Omega}$, then there exists $G \in \Sigma$ such that $\mathcal{D}(G) \leqslant \mathcal{H}$. So, let $\mathcal{H}$ be a domination completion of $\mathcal{U}_{n-1,\Omega}$. Then $\mathcal{U}_{n-1,\Omega} \leqslant \mathcal{H}$ and there is a graph $G_{\mathcal{H}}$ with vertex set $\Omega$ such that $\mathcal{H} = \mathcal{D}(G_{\mathcal{H}})$. Notice that if $\mathcal{U}_{n-1,\Omega} \leqslant \mathcal{D}(G_{\mathcal{H}})$, then $G_{\mathcal{H}}$ has no isolated vertices, (because otherwise the isolated vertex $\omega_0$ should be at every minimal dominating set of $G_{\mathcal{H}}$ implying that $\Omega \setminus \{\omega_0\} \in \mathcal{U}_{n-1,\Omega}$ does not contain any minimal dominating set of $G_{\mathcal{H}}$, which is a contradiction). Thus, by Lemma 15, there exists a spanning subgraph $G$ of $G_{\mathcal{H}}$ that is a disjoint union of stars. Since $G$ is a spanning subgraph of $G_{\mathcal{H}}$, by Lemma 17 it follows that $\mathcal{D}(G) \leqslant \mathcal{D}(G_{\mathcal{H}})$. Therefore we conclude that $G \in \Sigma$ and $\mathcal{D}(G) \leqslant \mathcal{H}$.

Finally, it remains to prove that the dominating clutters of distinct disjoint union of stars with vertex set $\Omega$ are either equal or non-comparable. In other words, we must demonstrate that if $\mathcal{D}(G) \leqslant \mathcal{D}(G')$ with $G, G' \in \Sigma$, then $G = G'$. So, let $G, G' \in \Sigma$ with $\mathcal{D}(G) \leqslant \mathcal{D}(G')$. Then, from Lemma 2 and Lemma 18, it follows that $\mathcal{N}[G'] = \mathrm{tr}(\mathcal{D}(G')) \leqslant \mathrm{tr}(\mathcal{D}(G)) = \mathcal{N}[G]$. By applying Lemma 14 we get that $\mathcal{N}[G'] = E(G')$ and $\mathcal{N}[G] = E(G)$. Therefore $E(G') \leqslant E(G)$. Hence $E(G') \subseteq E(G)$ because $E(G')$ and $E(G)$ are 2-uniform clutters. At this point observe that the addition of an edge to a graph that is a disjoint union of stars gives rise to a graph not satisfying this property. Therefore we conclude that $E(G) = E(G')$ and, consequently, $G = G'$. □

The following proposition characterizes all graphs that realize a minimal domination completion of $\mathcal{U}_{n-1,\Omega}$. After its proof we present an example of a minimal domination completion $\mathcal{H}_0$ of the uniform clutter $\mathcal{U}_{n-1,\Omega}$ whenever $n = 8$, as well as the description of all the graph realizations of $\mathcal{H}_0$ (the example is illustrated in Figure 1).

**Proposition 20.** *Let $G$ be a graph with vertex set $\Omega$ that is a disjoint union of stars $S_1, \ldots, S_r$, and let $G'$ be a graph with vertex set $\Omega$. Then $\mathcal{D}(G) = \mathcal{D}(G')$ if and only if $G'$ is any graph that can be obtained from $G$ in the following way: choosing a set $C = \{c_1, \ldots, c_r\}$ formed by exactly one center $c_i$ of each star $S_i$ and adding to $G$ any set of edges joining vertices of $C$.*

*Proof.* Let $G'$ be a graph with vertex set $\Omega$. By applying Lemma 2 and Lemma 14 we get that $\mathcal{D}(G') = \mathcal{D}(G)$ if and only if $\mathcal{N}[G'] = \mathcal{N}[G]$ if and only if $\mathcal{N}[G'] = E(G)$. It is not hard to prove that $\mathcal{N}[G'] = E(G)$ if and only if the following two conditions are satisfied: $E(G) \subseteq E(G')$, and for every edge $\{x, y\} \in E(G)$ either $x$ or $y$ has degree 1 in $G'$. Therefore we conclude that $\mathcal{D}(G') = \mathcal{D}(G)$ if and only if $G'$ is obtained from $G$ by adding edges joining vertices of a set containing exactly one center of each star of $G$. □

**Example 3.** Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. By Theorem 19, the minimal domination completions of $\mathcal{U}_{7,\Omega}$ are the clutters of the form $\mathcal{D}(G)$ where $G$ is a disjoint union of stars with vertex set $\Omega$. It is straightforward to prove that there are 5041 such graphs $G$, all of them providing different domination clutters. Therefore, $|\mathcal{D}om(7, \Omega)| = 5041$. One of these graphs $G$ is the graph $G_0$ obtained as the disjoint union of 3 stars, two of them isomorphic to $K_2$ and the

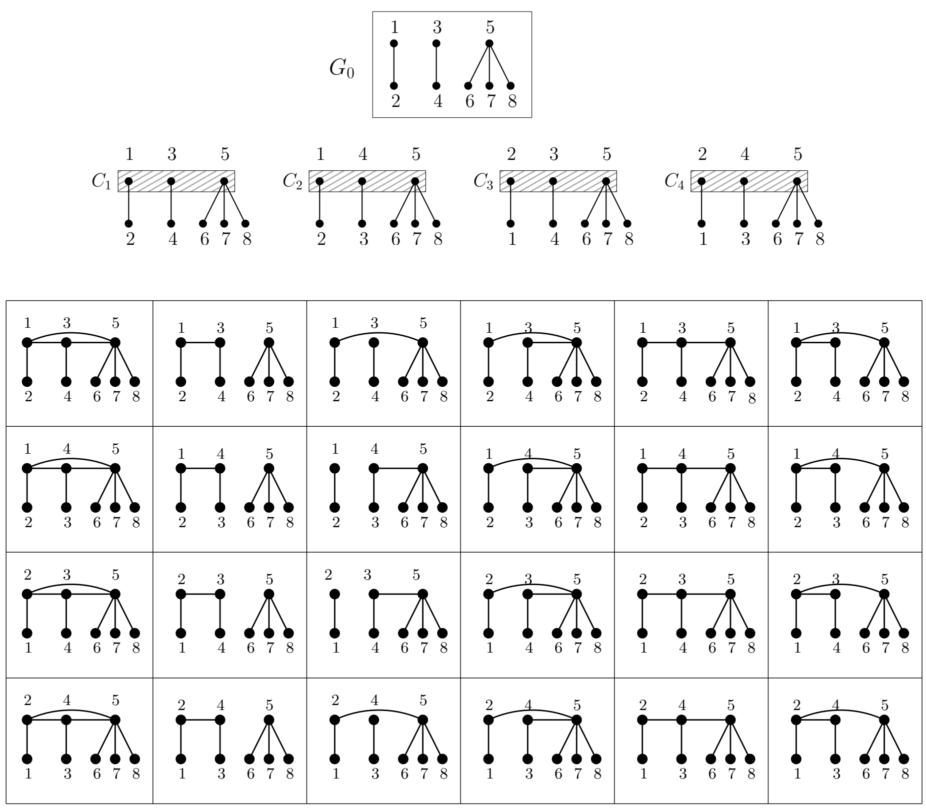

Figure 1: The clutter $\mathcal{H}_0 = \mathcal{D}(G_0)$ is a minimal domination completion of the uniform clutter $\mathcal{U}_{7,\Omega}$ where $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The graph $G_0$ together with the 24 graphs obtained by adding edges joining vertices of $C_i$ ($i = 1, 2, 3, 4$) gives rise to all the 25 graph realizations of the domination clutter $\mathcal{H}_0$ of Example 3.

other one, isomorphic to $K_{1,3}$; namely, the graph $G_0$ with edge set $E(G_0) = \{\{1,2\},\{3,4\},\{5,6\},\{5,7\},\{5,8\}\}$. From Lemma 16, this graph $G_0$ has the following $2^3 = 8$ minimal dominating sets $\mathcal{D}(G_0) = \{\{1,3,5\},\{1,3,6,7,8\},\{1,4,5\},\{1,4,6,7,8\},\{2,3,5\},\{2,3,6,7,8\},\{2,4,5\},\{2,4,6,7,8\}\}$. So, the clutter $\mathcal{H}_0 = \mathcal{D}(G_0)$ is a minimal domination completion of $\mathcal{U}_{7,\Omega}$. In order to obtain all the graph realizations of $\mathcal{H}_0$, we apply Proposition 20. In this case we have four possibilities for the set $C$ containing exactly one center of each star. Concretely $C$ is either $C_1 = \{1,3,5\}$, or $C_2 = \{1,4,5\}$, or $C_3 = \{2,3,5\}$, or $C_4 = \{2,4,5\}$. The graphs $G'$ such that its collection of minimal dominating sets is $\mathcal{D}(G') = \mathcal{D}(G_0)$ are obtained by fixing one of the sets $C_i$ and adding edges joining vertices of $C_i$. It is easy to check that there are exactly 24 different graphs $G' \neq G_0$ obtained in this way (see Figure 1).

To conclude this subsection we present an upper bound on the decomposition parameter $\mathfrak{D}(n-1,\Omega)$ of the uniform clutter $\mathcal{U}_{n-1,\Omega}$ where $|\Omega| = n$ (Proposition 21). It is worth noting that an exhaustive analysis of all possible cases shows that the equality holds whenever $2 \leq n \leq 5$. However, it remains an open problem to determine if the equality holds for $n \geq 6$.

**Proposition 21.** *Let $\Omega$ be a finite set of size $n \geq 3$. Then $\mathfrak{D}(n-1,\Omega) \leq n-1$; that is, there are $n-1$ minimal domination completions $\mathcal{H}_1,\dots,\mathcal{H}_{n-1}$ of $\mathcal{U}_{n-1,\Omega}$ such that $\mathcal{U}_{n-1,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1}$.*

*Proof.* Let $\Omega = \{\omega_1,\dots,\omega_n\}$. For $1 \leq i \leq n-1$, let $\mathcal{H}_i$ be the domination clutter $\mathcal{H}_i = \mathcal{D}(S_i)$ where $S_i$ is the star with center $\omega_i$ and isomorphic to $K_{1,n-1}$. By Theorem 19, the clutters $\mathcal{H}_1,\dots,\mathcal{H}_{n-1}$ are minimal domination completions of $\mathcal{U}_{n-1,\Omega}$. Let us show that $\mathcal{U}_{n-1,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1}$. It is clear that $\mathcal{H}_i = \{D_{i,1}, D_{i,2}\}$ where $D_{i,1} = \{\omega_i\}$ and $D_{i,2} = \Omega \setminus \{\omega_i\}$. On the one hand, the elements of $\mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1}$ have size at least $n-1$ and hence the inequality $\mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1} \leqslant \mathcal{U}_{n-1,\Omega}$ holds. On the other hand, if $A \in \mathcal{U}_{n-1,\Omega}$, then $A = \Omega \setminus \{\omega\}$ for some $w \in \Omega$, and thus we get that: $A = D_{1,1} \cup \cdots \cup D_{n-1,1}$ if $w = w_n$; whereas $A = D_{i_0,2} \cup \left(\cup_{i \neq i_0} D_{i,1}\right)$ if $w = w_{i_0} \neq w_n$. So, the inequality $\mathcal{U}_{n-1,\Omega} \leqslant \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1}$ also holds. Therefore, since $\leqslant$ is a partial order, we conclude that $\mathcal{U}_{n-1,\Omega} = \mathcal{H}_1 \sqcap \cdots \sqcap \mathcal{H}_{n-1}$. □

*4.3. Minimal domination completions of $\mathcal{U}_{r,\Omega}$ whith $|\Omega| \leq 5$*

The aim of this subsection is to determine the set $\mathcal{D}om(r,\Omega)$ of the minimal domination completions of the $r$-uniform clutter $\mathcal{U}_{r,\Omega}$ where $\Omega$ is a finite set of size $|\Omega| = n \leq 5$ and $1 \leq r \leq n$. From Proposition 5 and Theorem 9 we get that $\mathcal{D}om(r,\Omega) = \{\mathcal{U}_{r,\Omega}\}$ if and only if $(r,n) \neq (2,3),(2,5),(3,4),(3,5),(4,5)$. In addition, the results of the the preceding subsections provide a complete description of the set $\mathcal{D}om(r,\Omega)$ whenever $(r,n) = (2,3),(2,5),(3,4),(4,5)$. Therefore, it only remains to determine the set of minimal domination completions $\mathcal{D}om(r,\Omega)$ whenever $(r,n) = (3,5)$.

This subsection deals with this issue. Namely, the goal of this subsection is to prove that, for $r = 3$ and $n = 5$ the uniform clutter $\mathcal{U}_{r,\Omega}$ has 22 minimal domination completions: 12 of the form $\mathcal{D}(G)$ with $G$ isomorphic to a cycle

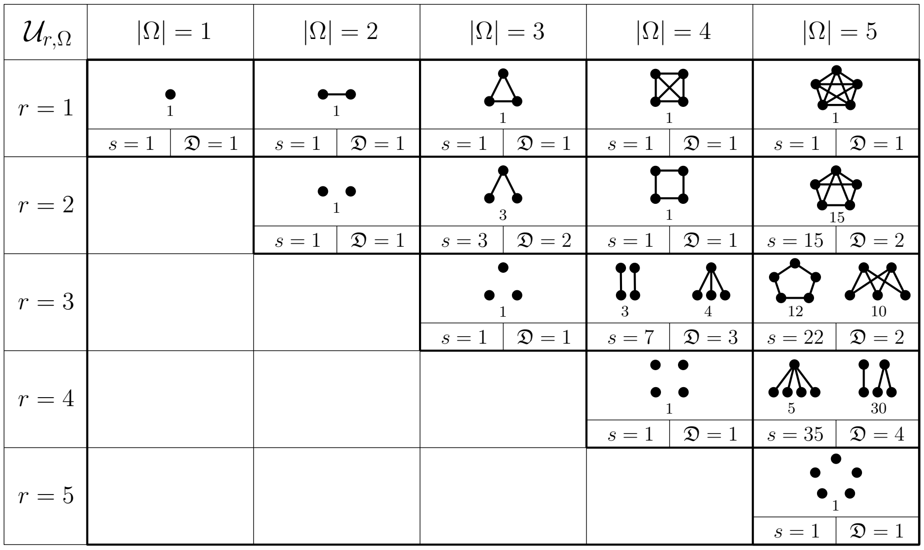


Figure 2: For $1 \leq n = |\Omega| \leq 5$, the minimal domination completions of $\mathcal{U}_{r,\Omega}$ are the domination clutters $\mathcal{D}(G')$, where $G'$ is a graph isomorphic to a graph $G$ in the figure. The number below each graph $G$ denotes the number of different clutters $\mathcal{H} \in \mathrm{Dom}(r, \Omega)$ with $\mathcal{H} = \mathcal{D}(G')$ for some graph with $G'$ isomorphic to $G$. In addition, in each case, the total number $s = |\mathcal{D}om(r, \Omega)|$ of minimal domination completions of $\mathcal{U}_{r,\Omega}$, and the decomposition parameter $\mathfrak{D} = \mathfrak{D}(r, \Omega)$ are given.

$C_5$, and 10 of the form $\mathcal{D}(G)$ with $G$ isomorphic to the complete bipartite graph $K_{2,3}$. This result is stated in Theorem 22. This and all the other results about the minimal domination completions of the uniform clutters $\mathcal{U}_{r,\Omega}$ where $1 \leq r \leq |\Omega| \leq 5$, are summarized in Figure 2 (in each case, the graphs $G$ in the figure provide the realization of all the minimal domination completions $\mathcal{H}$ of $\mathcal{U}_{r,\Omega}$; that is, $\mathcal{H} \in \mathcal{D}om(r, \Omega)$ if and only if $\mathcal{H} = \mathcal{D}(G')$ for some graph $G'$ isomorphic to a graph $G$ in the figure).

**Theorem 22.** *Let $\Omega$ be a finite set of size $|\Omega| = 5$. Let $\mathcal{C}_5$ and $\mathcal{K}_{2,3}$ be the families of graphs with vertex set $\Omega$, where the graphs of $\mathcal{C}_5$ are exactly those isomorphic to the cycle $C_5$, whereas the graphs of $\mathcal{K}_{2,3}$ are all those isomorphic to the complete bipartite graph $K_{2,3}$. The following statements hold:*

1. *The minimal domination completions of the uniform clutter $\mathcal{U}_{3,\Omega}$ are the domination clutters $\mathcal{H}$ of the form $\mathcal{H} = \mathcal{D}(G)$ where the graph $G$ is isomorphic to either a cycle $C_5$ or to a complete bipartite graph $K_{2,3}$; that is, $\mathcal{D}om(3, \Omega) = \{\mathcal{D}(G) : G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}\}$.*

2. *The uniform clutter $\mathcal{U}_{3,\Omega}$ has 22 minimal domination completions; that is, $|\mathcal{D}om(3, \Omega)| = 22$.*

3. *The uniform clutter $\mathcal{U}_{3,\Omega}$ has decomposition parameter $\mathfrak{D}(3, \Omega) = 2$; that*

*is, there exist minimal domination completions $\mathcal{H}, \mathcal{H}'$ of $\mathcal{U}_{3,\Omega}$ such that $\mathcal{U}_{3,\Omega} = \mathcal{H} \sqcap \mathcal{H}'$.*

The rest of this subsection is devoted to prove this theorem. From now on we set $\Omega = \{1, 2, 3, 4, 5\}$.

First observe that if $G \in \mathcal{C}_5$, then $\mathcal{D}(G)$ contains the five pairs of non-adjacent vertices; while if $G \in \mathcal{K}_{2,3}$, then $\mathcal{D}(G)$ contains both stable sets and the 6 pairs of adjacent vertices (see Figure 3). Using these facts it is easy to check that if $G, G' \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$, then $\mathcal{D}(G) = \mathcal{D}(G')$ if and only if $G = G'$. Therefore, $|\{\mathcal{D}(G) : G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}\}| = |\mathcal{C}_5| + |\mathcal{K}_{2,3}| = 12 + 10 = 22$. Thus, the statement (2) of the theorem follows from the first one.


$\mathcal{D}(G) = E(K_\Omega) \setminus E(G)$
$= \{\{i_1, i_3\}, \{i_1, i_4\}, \{i_2, i_4\}, \{i_2, i_5\}, \{i_3, i_5\}\}$

$\mathcal{D}(G) = E(G) \cup \{V_1, V_2\}$
$= \{\{i_1, i_2\}, \{i_1, i_4\}, \{i_3, i_2\}, \{i_3, i_4\}, \{i_5, i_2\}, \{i_5, i_4\}, \{i_1, i_3, i_5\}, \{i_2, i_4\}\}$

Figure 3: Minimal dominating sets of a graph $G$ with vertex set $V(G) = \{i_1, i_2, i_3, i_4, i_5\}$, isomorphic to $C_5$ (left) and isomorphic to $K_{2,3}$ (right).

Now let us demonstrate the third statement of the theorem. The statement (1) will be proved after doing this.

Recall that $\mathcal{U}_{3,\Omega}$ is not a domination clutter (Proposition 5). So, $\mathfrak{D}(3, \Omega) \geq 2$. The inequality $\mathfrak{D}(3, \Omega) \leq 2$ follows from Proposition 23. This proposition shows all the ways to obtain $\mathcal{U}_{3,\Omega}$ as $\mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$, when $G_1, G_2 \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$.

**Proposition 23.** *Let $G_1, G_2 \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$. Then $\mathcal{U}_{3,\Omega} = \mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$ if and only if $G_1, G_2 \in \mathcal{C}_5$ and $E(G_1) \cup E(G_2) = E(K_\Omega)$.*

*Proof.* First consider the case $G_1, G_2 \in \mathcal{C}_5$ with $E(G_1) \cup E(G_2) = E(K_\Omega)$. In such a case $E(G_1) \cap E(G_2) = \emptyset$, and so we get that $\mathcal{D}(G_1) = E(G_2)$ and $\mathcal{D}(G_2) = E(G_1)$. Thus, every set $A_1 \cup A_2$, with $A_1 \in \mathcal{D}(G_1)$ and $A_2 \in \mathcal{D}(G_2)$, has size 3 or 4. It is straightforward to check that every element of $\mathcal{U}_{3,\Omega}$ can be obtained as $A_1 \cup A_2$ where $A_1 \in \mathcal{D}(G_1)$ and $A_2 \in \mathcal{D}(G_2)$. Therefore, $\mathcal{U}_{3,\Omega} = \mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$.

The proof of the proposition will be completed by showing that, in any other case, there exists $A_0 \in \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$ with $|A_0| = 2$. Indeed, if there exists $A_0 \in \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$, then $A_0 \in \mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$, and so $\mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$ has at least an element of size $|A_0|$. Hence, if $|A_0| = 2$, then $\mathcal{U}_{3,\Omega} \neq \mathcal{D}(G_1) \sqcap \mathcal{D}(G_2)$.

Therefore, we must demonstrate that there exists $A_0 \in \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$ with $|A_0| = 2$. We distinguish three cases: whenever $G_1, G_2 \in \mathcal{C}_5$ and $E(G_1) \cup$

$E(G_2) \neq E(K_\Omega)$; whenever $G_1, G_2 \in \mathcal{K}_{2,3}$; and whenever $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$. If $G_1, G_2 \in \mathcal{C}_5$ and $E(G_1) \cup E(G_2) \neq E(K_\Omega)$, then there exists $\{x, y\} \in E(K_\Omega) \setminus (E(G_1) \cup E(G_2))$. In this case $\mathcal{D}(G_i) = E(K_\Omega) \setminus E(G_i)$. So, we can set $A_0 = \{x, y\} \in \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$. Now let us assume that $G_1, G_2 \in \mathcal{K}_{2,3}$. Then $|E(G_1)| = |E(G_2)| = 6$. So $E(G_1) \cap E(G_2) \neq \emptyset$ and thus there exists $\{x, y\} \in E(G_1) \cap E(G_2)$. Since $E(G_i) \subseteq \mathcal{D}(G_i)$, in this case the subset $A_0 = \{x, y\}$ satisfies the required conditions. Finally, suppose that $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$. Then $|E(K_\Omega) \setminus E(G_1)|$=5 and $|E(G_2)| = 6$. So there exists $\{x, y\} \in (E(K_\Omega) \setminus E(G_1)) \cap E(G_2) \subseteq \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$, and thus the proof is completed by setting $A_0 = \{x, y\}$. □

At this point, the proof of Theorem 22 will be completed by proving the first statement. Observe that the proof of this statement is a consequence of the following three facts: first, the clutters $\mathcal{D}(G)$, where $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$, are domination completions of $\mathcal{U}_{3,\Omega}$; second, any pair of different such clutters are non-comparable; and third, $\mathcal{D}om(3, \Omega) \subseteq \{\mathcal{D}(G) : G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}\}$. Indeed, from these three facts it easily follows that $\mathcal{D}om(3, \Omega) = \{\mathcal{D}(G) : G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}\}$.

We demostrate the first two facts in Propositions 24 and 25, respectively. The third fact is a consequence of Proposition 29, whose proof is involved and requires three technical lemmas concerning the size of the elements of the minimal domination completions of $\mathcal{U}_{3,\Omega}$ and their transversal (Lemmas 26, 27 and 28).

**Proposition 24.** *If $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$, then $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G)$.*

*Proof.* From Lemma 6, we must demonstrate that if $A$ is a subset of $\Omega$ of size 3, then there exists $D \in \mathcal{D}(G)$ such that $D \subseteq A$. This is clear if $G \in \mathcal{C}_5$, because in such a case every set of three vertices of $G$ contains a pair of two non-adjacent vertices, that are a minimal dominating set of $G$. Now let assume that $G \in \mathcal{K}_{2,3}$. In this case the result follows by taking into account that the stable set of size 3 is a minimal dominating set of $G$, and that every other set of three vertices contains two adjacent vertices. So any subset of size three contains a minimal dominating set of $G$. □

**Proposition 25.** *Let $G_1, G_2 \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$. If $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.*

*Proof.* First, suppose that $G_1, G_2 \in \mathcal{C}_5$. Then the clutters $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ contain both exactly five elements of size 2. In such a case it is clear that if $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.

Now assume that $G_1, G_2 \in \mathcal{K}_{2,3}$. Then the clutters $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ contain both exactly one element of size 3 and 7 elements of size 2. Therefore, from $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$ we deduce that the 7 elements of size 2 must be the same. In addition, since there is no inclusion relation between the elements of a clutter, the element of size 3 must be also the same. Hence we conclude that $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.

The proof will be completed by showing that the inequality $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$ is not possible neither in the case $G_1 \in \mathcal{K}_{2,3}$ and $G_2 \in \mathcal{C}_5$, nor in the case $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$. If $G_1 \in \mathcal{K}_{2,3}$ and $G_2 \in \mathcal{C}_5$, then $\mathcal{D}(G_1)$ contains 7 elements of size 2, while $\mathcal{D}(G_2)$ contains only 5 elements of size 2. Thus, in such a case, the inequality $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$ is not possible. Finally, assume $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$. If $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$, then the 5 elements of size 2 of $\mathcal{D}(G_1)$ must be in $\mathcal{D}(G_2)$; that is, $\mathcal{D}(G_1) \subseteq \mathcal{D}(G_2)$. But the five elements of $\mathcal{D}(G_1)$ are the 5 pairs of non-adjacent vertices of $G_1$, and they correspond to the five pairs of edges of another cycle of order 5. Nevertheless, it is not possible to define a cycle of order 5 with 5 elements of size 2 of $\mathcal{D}(G_2)$. Therefore, $\mathcal{D}(G_1) \leqslant \mathcal{D}(G_2)$ is not possible in that case. This completes the proof of the proposition. $\square$

**Lemma 26.** *Let $\mathcal{H}$ be a clutter. If $\mathcal{U}_{3,\Omega} \leqslant \mathcal{H}$, then $|X| \geq 3$ for every $X \in \mathrm{tr}(\mathcal{H})$.*

*Proof.* On the contrary, assume that there exists $X \in \mathrm{tr}(\mathcal{H})$ such that $|X| \leq 2$. In such a case, consider a subset $A \subseteq \Omega$ of size $|A| = 3$ satisfying $A \cap X = \emptyset$. Since $|A| = 3$, hence $A \in \mathcal{U}_{3,\Omega}$. Therefore there exists $B \in \mathcal{H}$ contained in $A$ because $\mathcal{U}_{3,\Omega} \leqslant \mathcal{H}$. Hence $B \cap X \subseteq A \cap X$, and so $B \cap X = \emptyset$. This leads us to a contradiction because $B \in \mathcal{H}$ and $X \in \mathrm{tr}(\mathcal{H})$. $\square$

**Lemma 27.** *Let $\mathcal{H}$ be a minimal domination completion of $\mathcal{U}_{3,\Omega}$. If there exists $A \in \mathcal{H}$ such that $|A| = 3$, then $\mathcal{H} = \mathcal{D}(G)$ for some graph $G \in \mathcal{K}_{2,3}$.*

*Proof.* It is enough to prove that $\mathcal{D}(G) \leqslant \mathcal{H}$ for some $G \in \mathcal{K}_{2,3}$, because by Proposition 24, $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G)$, and so, the minimality of $\mathcal{H}$ implies that $\mathcal{D}(G) = \mathcal{H}$.

Without loss of generality we may assume that $A = \{1, 2, 3\} \in \mathcal{H}$. As $\mathcal{H}$ is a clutter, every element of $\mathcal{H}$ different from $A$ contains either 4 or 5. Hence, the intersection of $\{1, 4, 5\}$, $\{2, 4, 5\}$ and $\{3, 4, 5\}$ with every element of $\mathcal{H}$ is non-empty. Therefore, $\{1, 4, 5\}$, $\{2, 4, 5\}$ and $\{3, 4, 5\}$ are in $\mathrm{tr}(\mathcal{H})$, because there are no elements of cardinality less or equal than 2 in $\mathrm{tr}(\mathcal{H})$ (Lemma 26). Since $\mathcal{H}$ is a domination clutter, there exists a graph $G_0$ such that $\mathcal{H} = \mathcal{D}(G_0)$, and so $\mathrm{tr}(\mathcal{H}) = \mathcal{N}[G_0]$ (Lemma 2). Therefore, $\{1, 4, 5\}$, $\{2, 4, 5\}$ and $\{3, 4, 5\}$ are the closed neighborhoods for some $x, y, z \in \Omega$; that is, $N_{G_0}[x] = \{1, 4, 5\}$, $N_{G_0}[y] = \{2, 4, 5\}$ and $N_{G_0}[z] = \{3, 4, 5\}$. Observe that at least one of the elements $x, y, z \in \Omega$ is different from 4 and 5. So, without loss of generality we may assume that $x \neq 4, 5$ and so $x = 1$. Thus $\{1, 4, 5\} = N_{G_0}[1]$. Hence $\{1, 4\}, \{1, 5\} \in E(G_0)$, and consequently, $N_{G_0}[4] \neq \{2, 4, 5\}, \{3, 4, 5\}$ and $N_{G_0}[5] \neq \{2, 4, 5\}, \{3, 4, 5\}$. So we conclude that $N_{G_0}[1] = \{1, 4, 5\}$, that $N_{G_0}[2] = \{2, 4, 5\}$, and that $N_{G_0}[3] = \{3, 4, 5\}$. Hence it follows that $F = \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\} \subseteq E(G_0)$. At this point let us consider the subgraph $G$ induced by the edges of $F$. Observe that $G$ is isomorphic to $K_{2,3}$ with stable sets $\{1, 2, 3\}$ and $\{4, 5\}$. So, $G \in \mathcal{K}_{2,3}$. Moreover, the graph $G$ is a spanning subgraph of $G_0$ and hence, from Lemma 17 it follows that $\mathcal{D}(G) \leqslant \mathcal{D}(G_0) = \mathcal{H}$. This completes the proof of the lemma. $\square$

**Lemma 28.** *Let $\mathcal{H}$ be a minimal domination completion of $\mathcal{U}_{3,\Omega}$. If $|A| = 2$ for all $A \in \mathcal{H}$, then $\mathcal{H} = \mathcal{D}(G)$ for some graph $G \in \mathcal{C}_5$.*

*Proof.* Let $G_0$ be a graph with $\mathcal{H} = \mathcal{D}(G_0)$. Reasoning as in the proof of the previous lemma, here it is enough to prove that $\mathcal{D}(G) \leqslant \mathcal{D}(G_0)$ for some $G \in \mathcal{C}_5$.

To prove this inequality we will use the following four facts.

First, notice that $G_0$ has no vertex of degree 4, because otherwise there would be an element in $\mathcal{D}(G_0)$ of size 1.

Second, we claim that if $\{a,b\} \notin \mathcal{H}$, then $\{a,b\} \in E(G_0)$. Let us prove our claim. Suppose to the contrary that $a$ and $b$ are non-adjacent in $G_0$. In such a case, both vertices $a$ and $b$ belong to a minimal dominating set $D$ of $G_0$ (for instance, we can consider a inclusion-maximal independent set $D$ containing $a$ and $b$). But $\mathcal{D}(G_0) = \mathcal{H}$ and, by assumption, all the elements of $\mathcal{H}$ have size 2. Therefore we conclude that $\{a,b\} = D \in \mathcal{D}(G_0) = \mathcal{H}$, which is a contradiction. This completes the proof of our claim.

Next, observe that $\mathcal{N}[G_0] = \mathrm{tr}(\mathcal{D}(G_0)) = \mathrm{tr}(\mathcal{H})$ (Lemma 2). So, by applying Lemma 26 it follows that all the elements of $\mathcal{N}[G_0]$ have at least 3 elements.

Finally, let us show that in fact $\mathcal{N}[G_0]$ has at least one element $X$ of size 3. Suppose on the contrary that it is not true. If $\mathcal{N}[G_0] = \{\Omega\}$, then $G_0$ has at least one vertex of degree 4, which is not possible. So, without loss of generality we may assume that $\Omega \notin \mathcal{N}[G_0]$ and that $\{1,2,3,4\} \in \mathcal{N}[G_0]$. In such a case, all subsets of cardinality 4 must be in $\mathcal{N}[G_0]$, because otherwise there exists $j \in \bigcap_{N \in \mathcal{N}[G_0]} N$, so $\deg_{G_0}(j) = 4$, which is a contradiction. Therefore $\mathcal{N}[G_0]$ contains all the subsets of cardinality 4. But this is not possible, because there is no graph of order 5 with all the vertices of degree 3.

At this point, using the foregoing four facts, we will prove that there exists a graph $G \in \mathcal{C}_5$ such that $\mathcal{D}(G) \leqslant \mathcal{D}(G_0)$.

We distinguish three cases: $\mathcal{N}[G_0]$ has exactly one element of size 3; $\mathcal{N}[G_0]$ has at least two elements $X$ and $Y$ of size 3 with $|X \cap Y| = 2$; and $\mathcal{N}[G_0]$ has at least two elements $X$ and $Y$ of size 3 with $|X \cap Y| = 1$.

First suppose that $\mathcal{N}[G_0]$ has exactly one element of size 3. Hence, the remaining elements of $\mathcal{N}[G_0]$ have size 4. We may assume that $\{1,2,3\} \in \mathcal{N}[G_0]$ and $N_{G_0}[1] = \{1,2,3\}$. In such a case, $\mathcal{N}[G_0] \subseteq \{\{1,2,3\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$. Since $4,5 \notin N_{G_0}[1]$, we have that $1 \notin N_{G_0}[4]$ and $1 \notin N_{G_0}[5]$. Consequently, $N_{G_0}[4] = N_{G_0}[5] = \{2,3,4,5\}$. Therefore $\{\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,5\}\} \subseteq E(G_0)$. So, $G_0$ contains a spanning subgraph $G$ that is isomorphic to the cycle $C_5$ and, by Lemma 17, $\mathcal{D}(G) \leqslant \mathcal{D}(G_0)$.

Next suppose that $\mathcal{N}[G_0]$ has at least two elements $X$ and $Y$ of size $|X| = |Y| = 3$ with $|X \cap Y| = 2$. Without loss of generality we may assume that $X = \{1,2,3\}$ and that $Y = \{1,2,5\}$. Since $X, Y \in \mathcal{N}[G_0]$, and since $\{4,5\} \cap X = \emptyset$ and $\{3,4\} \cap Y = \emptyset$, we get that $\{4,5\},\{3,4\} \notin \mathrm{tr}(\mathcal{N}[G_0]) = \mathcal{D}(G_0)$. Therefore $\{4,5\},\{3,4\} \notin \mathcal{H}$ and thus, as we have showed before, we conclude that $\{4,5\},\{3,4\} \in E(G_0)$. Hence it follows that $\{\{1,2,3\},\{1,2,5\}\} = \{N_{G_0}[1], N_{G_0}[2]\}$. By symmetry, we may assume that $N_{G_0}[1] = \{1,2,3\}$ and that $N_{G_0}[2] = \{1,2,5\}$. In such a case, $\{\{1,2\},\{1,3\},\{2,5\},\{4,5\},\{3,4\}\} \subseteq E(G_0)$. Hence, $G_0$ contains a spanning subgraph $G$ that is isomorphic to the cycle $C_5$ and, by Lemma 17, $\mathcal{D}(G) \leqslant \mathcal{D}(G_0)$.

Finally, suppose that $\mathcal{N}[G_0]$ has at least two elements $X$ and $Y$ of size $|X| = |Y| = 3$ with $|X \cap Y| = 1$. Without loss of generality we may assume that $X = \{1,2,3\}$ and that $Y = \{3,4,5\}$. Reasoning as in the preceding case, $\{4,5\}$ and $\{1,2\}$ belong to $E(G_0)$. If $N_{G_0}[3] = \{1,2,3\}$, then

$\{3,4,5\}$ must be either $N_{G_0}[4]$ or $N_{G_0}[5]$, obtaining respectively that either $\{3,4\} \in E(G_0)$ or that $\{3,5\} \in E(G_0)$. So, if $N_{G_0}[3] = \{1,2,3\}$, then we get that either $4 \in N_{G_0}[3]$ or $5 \in N_{G_0}[3]$, a contradiction. Therefore we conclude that $N_{G_0}[3] \neq \{1,2,3\}$ and, by symmetry, we get that $N_{G_0}[3] \neq \{3,4,5\}$. Hence, without loss of generality we may assume that $N_{G_0}[1] = \{1,2,3\}$ and that $N_{G_0}[4] = \{3,4,5\}$. At this point recall that the intersection of all the elements of $\mathcal{N}[G_0]$ is empty (because otherwise there would be a vertex $u$ of $G_0$ of degree 4). Set $Z \in \mathcal{N}[G_0]$ such that $3 \notin Z$. If $|Z| = 3$, then either $|X \cap Z| = 2$ or $|Y \cap Z| = 2$, and we proceed as in the preceding case. If $|Z| \neq 3$, then $Z = \{1,2,4,5\}$, and so $Z$ is either $N_{G_0}[2]$ or $N_{G_0}[5]$. In any case, $\{2,5\} \in E(G_0)$. Therefore, $\{\{1,2\},\{1,3\},\{2,5\},\{3,4\},\{4,5\}\} \subseteq E(G_0)$. So, $G_0$ contains a spanning subgraph $G$ that is isomorphic to the cycle $C_5$ and, by Lemma 17, $\mathcal{D}(G) \leqslant \mathcal{D}(G_0)$. □

**Proposition 29.** *Let $\mathcal{H}$ be a domination completion of $\mathcal{U}_{3,\Omega}$. Then there exists a graph $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$ such that $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G) \leqslant \mathcal{H}$.*

*Proof.* Let $\mathcal{H}_0 = \mathcal{D}(G_0)$ be a minimal domination completion of $\mathcal{U}_{3,\Omega}$ such that $\mathcal{U}_{3,\Omega} \leqslant \mathcal{H}_0 \leqslant \mathcal{H}$. By Lemmas 27 and 28, it is enough to show that $\mathcal{H}_0$ has either an element of size 3 or all its elements have size 2. Let us prove it.

First observe that for all $A \in \mathcal{H}_0$, we have $|A| \leq 3$. Indeed, suppose on the contrary that there exists $A \in \mathcal{H}_0$ such that $|A| \geq 4$. If $\{a,b,c,d\} \subseteq A$, then $\mathcal{H}_0$ does not contain any subset of $\{a,b,c\} \in \mathcal{U}_{3,\Omega}$, contradicting that $\mathcal{U}_{3,\Omega} \leqslant \mathcal{H}_0$.

From the above, it only remains to prove that if $\mathcal{H}_0$ has no elements of size 3, then all its elements have size exactly 2. On the contrary, let us assume that there exists $A \in \mathcal{H}_0$ such that $|A| = 1$. We are going to prove that, in such a case, a contradiction is achieved.

Without loss of generality we may assume that $A = \{5\}$. Hence, $\deg_{G_0}(5) = 4$ because $A \in \mathcal{H}_0 = \mathcal{D}(G_0)$. Let $G_1 = G_0 - 5$ be the graph obtained by deleting the vertex 5 from $G_0$. It is clear that $G_0 = G_1 \vee K_{\{5\}}$. Since $\mathcal{U}_{3,\Omega} \leqslant \mathcal{H}_0$, hence $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G_1 \vee K_{\{5\}})$, and thus, by applying Lemma 12 it follows that $\mathcal{U}_{3,\Omega\setminus\{5\}} = \mathcal{U}_{3,\Omega}[\Omega \setminus \{5\}] \leqslant \mathcal{D}(G_1)$. Let $\mathcal{D}(G')$ be a minimal domination completion of $\mathcal{U}_{3,\Omega\setminus\{5\}}$ such that $\mathcal{U}_{3,\Omega\setminus\{5\}} \leqslant \mathcal{D}(G') \leqslant \mathcal{D}(G_1)$. By using Lemma 3, it is easy to check that $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G' \vee K_{\{5\}})$ and that $\mathcal{D}(G' \vee K_{\{5\}}) \leqslant \mathcal{D}(G_1 \vee K_{\{5\}})$. Therefore, $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G' \vee K_{\{5\}}) \leqslant \mathcal{D}(G_1 \vee K_{\{5\}}) = \mathcal{D}(G_0) = \mathcal{H}_0$. So, $\mathcal{H}_0 = \mathcal{D}(G' \vee K_{\{5\}})$ because $\mathcal{H}_0$ is a minimal domination completion of $\mathcal{U}_{3,\Omega}$. By Theorem 19 we may assume that $G'$ is isomorphic to $K_{1,3}$ or to $2\,K_2$. If $G'$ is isomorphic to $K_{1,3}$, then $\mathcal{H}_0 = \mathcal{D}(G' \vee K_{\{5\}}) = \{\{5\}\} \cup \mathcal{D}(G')$ has an element of size 3, which contradicts our assumption. If $G'$ is isomorphic to $2\,K_2$, then by applying Proposition 20 we get that $\mathcal{D}(G') = \mathcal{D}(G'')$ where $G''$ is a path of order 4 obtained by joining a pair of vertices of the different connected components in $2K_2$. Therefore, $\mathcal{H}_0 = \mathcal{D}(K_{\{5\}} \vee G') = \{\{5\}\} \cup \mathcal{D}(G') = \{\{5\}\} \cup \mathcal{D}(G'') = \mathcal{D}(K_{\{5\}} \vee G'')$. But the graph $K_{\{5\}} \vee G''$ contains a spanning subgraph $G''' \in \mathcal{C}_5$. So, from Lemma 17 and Proposition 24 we get that $\mathcal{U}_{3,\Omega} \leqslant \mathcal{D}(G''') \leqslant \mathcal{D}(K_{\{5\}} \vee G'') = \mathcal{H}_0$. Therefore, $\mathcal{H}_0 = \mathcal{D}(G''')$ because $\mathcal{H}_0$ is a minimal domination completion of $\mathcal{U}_{3,\Omega}$. This leads us to a contradiction since all the dominating sets of $G'''$ have size 2. □

## References


[1] C. Berge. *Hypergraphs: Combinatorics of Finite Sets.* North Holland, Amsterdam, 1989.

[2] E. Boros, V. Gurvich, and I. Zverovich. Neighborhood Hypergraphs of Bipartite Graphs. *J. Graph Theory*, 58:69–95, 2008. DOI 10.1002/jgt.20293.

[3] A. Brandstadt, V. B. Le, and J. P. Spinrad. *Graph Classes: A survey.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[4] G. Chartrand and L. Lesniak. *Graphs and Digraphs, Fourth Ed.*. Chapman&Hall/CRC, Boca Raton, 2005.

[5] P. Duchet. Hypergraphs. In: *Handbook of Combinatorics*, R.L. Graham, M. Grötschel and L. Lovász, Eds., Vol. 1, pages 381–432. The MIT Press-North Holland, Amsterdam-Cambridge, 1995.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[7] S. R. Jayaram. Minimal dominating sets of cardinality two in a graph. *Indian J. pure appl. Math.*, 28(1):43–46, 1997.

[8] M.M. Kanté, V. Limouzy, A. Mary, and L. Nourine. On the enumeration of minimal dominating sets and related notions. *SIAM J. Discrete Math.*, 28(4):1916–1929, 2014. doi:10.1137/120862612.

[9] J. Martí-Farré and A. de Mier. Completion and decomposition of a clutter into representable matroids. *Linear Algebra Appl.*, 472:31–47, 2015.

[10] J. Martí-Farré and A. de Mier. Transformation and decomposition of clutters into matroids. *Advances in Mathematics*, 312, 2017, 286–314, doi:10.1016/j.aim.2017.03.022.

[11] J. Martí-Farré, M. Mora, and J.L. Ruiz. Completion and decomposition of clutters into dominating sets of graphs. *Electronic Notes in Discrete Mathematics*, 49:213–220, 2015.

[12] D.B. West. *Introduction to Graph Theory. Second Ed.* Prentice Hall, Upper Saddle River, 2001.