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ON THE INTEGRABILITY OF POLYNOMIAL VECTOR FIELDS IN THE PLANE BY MEANS OF PICARD-VESSIOT THEORY

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Abstract. We study the integrability of polynomial vector fields using Galois theory of linear differential equations when the associated foliations is reduced to a Riccati type foliation. In particular we obtain integrability results for some families of quadratic vector fields, Liénard equations and equations related with special functions such as Hypergeometric and Heun ones. The Poincaré problem for some families is also approached.

Introduction. Given a *polynomial differential system* in $\mathbb{C}^2$,

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

with $P, Q \in \mathbb{C}[x,y]$, we consider its associated differential vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{2}$$

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whose integral curves are intimately related to the solutions of system (1). These solutions, taken as curves on the plane and leaving for a while its time-dependence, constitute its so-called *foliation* and satisfy the first order differential equation

$$y' = \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{3}$$

This expression (3) is often written as a Pfaff equation

$$\Omega = 0, \tag{4}$$

where $\Omega = Q(x,y)dx - P(x,y)dy$ is the corresponding differential 1-form. The connection between integral curves of the vector field X and solutions of $\Omega = 0$ is clear:

- geometrically, it is given by $\Omega \cdot X = 0$, which means that the vector field X is tangent to the leaves of the foliation (the orbits) defined by (4);
- dynamically, the general solution of equation (4), $H(x,y) = ctt$, is given by a *first integral* H of the original vector field X, that is, a non-constant scalar function which remains constant along any of its solutions $(x(t), y(t))$. Since $\Omega \cdot X = 0$, this is equivalent to say that $X(H) = 0$ and to the existence of a suitable scalar function f such that $\Omega = f dH$. In fact, $1/f$ is an integrating factor of the field X, or of the 1-form Ω. Later we will say something more about the geometrical meaning of the integrating factors (Remark 1.3).

From a geometrical point of view, we focus our attention on *invariant algebraic curves*, that is, polynomial integral curves of the vector field X. This is the natural framework where Darboux Theory can be applied. Dynamically speaking, we restrict ourselves to first integrals $H(x,y)$ which are *Liouvillian*, i.e., written as a combination of algebraic functions, quadratures and exponential of quadratures in $\mathbb{C}(x,y)$, the field of rational functions on x, y. As we will see later, Galois Theory provides very useful and powerful tools to approach it.

Two classical problems remain still open for complex polynomial fields:

(i) Concerning the existence of invariant algebraic curves of system (1) or, equivalently, of algebraic solutions of the foliation equation (3).
(ii) About the existence of Liouvillian first integrals for systems (1) (or, in other words, to determine when the general solution of equation (3) is Liouvillian).

For general polynomial vector fields, problems (i) and (ii) are very difficult and we are still far from obtaining an effective method to decide whether a given arbitrary polynomial field has or not an invariant curve or admits a Liouvillian first integral. In fact, problem (i) is equivalent to the (also unsolved) classical *Poincaré problem*, which seeks for a bound of the degree of the invariant algebraic curves as a function of the degree of the vector field (or of the associated foliation defined by (3)). It is known that Darboux Theory and adjacent results, as the ones due to Prelle–Singer and Singer [53, 55], provide connections between this two problems.

In this work we are concerned with the study of the (Darboux, Galois) integrability of some families of equations of type (4) inside the complex analytic category, that is, when the original vector field X defining Ω is complex polynomial or can be reduced to it. Precisely, we restrict ourselves to those systems which can be reduced to a Riccati type equation

$$v' = a_0(x) + a_1(x)v + a_3(x)v^2, \tag{5}$$

a_1, a_2 and a_3 being rational functions with complex coefficients. For Riccati equations there is a very nice theory of integrability in the context of the Galois theory of its associated second order linear differential equation. This is, in our opinion, a natural framework where several known results concerning integrability of Riccati equations (5) should be considered. Kovacic provided in 1986 (see [35]) an effective algorithm which allows to decide whether an equation(5) has got an algebraic solution or not. And, additionally, a theorem of Liouville (see [36]) proved that the existence of an algebraic solution is exactly the definition of the integrability for (5) in the context of the Galois theory for linear differential equations. Thus, for foliations of type (5) problems (i) and (ii) are equivalent and Kovacic algorithm becomes an extremely powerful tool to approach them.

In some sense, this work can be considered as a very particular case of the Malgrange approach to the Galois theory of codimension-1 foliations [44, 45, 14], that is, for Riccati codimension-1 foliations on the complex plane. Our target is not to obtain general theoretical classification results, but several effective criteria of integrability for such foliations. We will provide integrability criteria for some families of polynomial quadratic vector fields and some Liénard equations involving special functions, allowing this to recover previous results established by several authors. The family of Liénard type equations

$$yy' = (a(2m+k)x^{2k} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1},$$

with a, b, c, m, k complex parameters appears in [51, §1.3.3, Exercise 11]. Therefore a little erratum is presented in the previous equation, which has been noticed and corrected in [8] as

$$yy' = (a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1},$$

and in where is proved that when this equation comes from a polynomial vector field, the constants m and k must be integer numbers satisfying $m \geq k+1 > 0$. Aims of this paper include to solve completely the integrability problem for this family of Liénard type equations, as well to approach the Poincaré problem for some particular families of systems (see Proposition 3 and Theorem 3.1).

The paper is structured as follows. To make it as self-contained as possible, we introduce in Section 1 a basic background about Galois theory of linear differential equations and Darboux theory of integrability of polynomial vector fields. In Section 2 we remind some useful properties concerning Riccati equations and Section 3 is devoted to applications. For completeness we include two Appendices about Kovacic algorithm and some special functions.

1. Two notions of integrability for planar polynomial vector fields.

1.1. Darboux theory of Integrability.

We give a very brief overview of Darboux's integrability ideas [25], his terminology and some essential results.

Let us consider a vector field (2) and an irreducible polynomial $f \in \mathbb{C}[x, y]$. The curve $f = 0$ is called an *invariant algebraic curve* of vector field (2) if it satisfies

$$\dot{f}|_{f=0} = 0.$$

This condition is equivalent to the existence of a polynomial $K \in \mathbb{C}[x, y]$, called *cofactor*, such that

$$X(f(x,y)) = P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = K(x,y)f(x,y),$$

or, equivalently, that

$$\frac{X(f)}{f} = X(\log(f)) = K. \tag{6}$$

From this expression it follows that the curve $f = 0$ is formed by leaves and critical points of the vector field $X = (P, Q)$ defined by (2). If the polynomial system (1) has degree d, that is $d = \max\{\deg P, \deg Q\}$, then we have that $\deg K \leq d - 1$, independently of the degree of the curve $f(x, y) = 0$. From definition (6) it follows that if the cofactor K vanishes identically then the polynomial f is a first integral of the vector field X. In terms of the associated foliation, this invariant curve $f = 0$ is a particular solution of $y' = Q/P$ and $Qdx - Pdy = 0$.

An analytic $\mathbb{C}$-valued non-constant function μ is called an *integrating factor* of system (1) if μX is Hamiltonian. This means that

$$\frac{X(\mu)}{\mu} = X(\log(\mu)) = -\nabla \cdot X,$$

where $\nabla \cdot X = (\partial P/\partial x) + (\partial Q/\partial y)$ is the *divergence* of the vector field $X = (P, Q)$. In case that the domain of definition of X is simply connected, a fist integral H of X can be obtained from

$$H(x, y) = \int \mu(x, y) P(x, y)\, dy + \varphi(x),$$

where $\varphi(x)$ is determined by the condition $\partial H/\partial y = -\mu Q$ (see also Remark 2).

To ensure the existence of a first integral for a system (1) is, in general, a very difficult problem. In [25], Darboux introduced a method to detect and construct first integrals using invariant algebraic curves. Namely, he proved that any planar polynomial differential system of degree d having, at least, $d(d+1)/2$ invariant algebraic curves, admits a first integral or an integrating factor which can be obtained from them. Jouanolou [32] proved an stronger version of Darboux Theorem, namely, that if such a system has $d(d+1)/2 + 2$ algebraic invariant curves then it admits, at least, a rational first integral. Darboux's original ideas have been also improved by taking into account the multiplicity of the invariant algebraic curves (see [24] for more details). Related to them some other invariant objects have been introduced (see [19]). They are the so-called *exponential factors*: given $h, g \in \mathbb{C}[x, y]$ relatively prime, the function $F = \exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if there exists a polynomial $\widetilde{K} \in \mathbb{C}[x, y]$ (also called *cofactor*) that satisfies the equation

$$\frac{X(F)}{F} = X\left(\frac{g}{h}\right) = \widetilde{K}. \tag{7}$$

It is known (see [19]) that if h is not a constant polynomial then $h = 0$ is an invariant algebraic curve of (1) of cofactor K_h satisfying that $X(g) = gK_h + h\widetilde{K}$.

The following theorem (coming originally from Darboux) shows how the construction of first integrals and integrating factors of (2) can be carried out from its invariant algebraic curves.

Theorem 1.1. *Let consider a planar polynomial system* (1) *of degree* m*, having*

- p *invariant algebraic curves* $f_i = 0$ *with cofactors* K_i*, for* $i = 1, \ldots, p$ *and*
- q *exponential factors* $F_j = \exp(g_j/h_j)$ *with cofactors* $\widetilde{K}_j$*,* $j = 1, \ldots, q$*.*

Then the following assertions hold:

(a) *There exist constants* $\lambda_i, \widetilde{\lambda}_j \in \mathbb{C}$ *not all vanishing such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \widetilde{\lambda}_j \widetilde{K}_j = 0,$$

if and only if the multivalued function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\widetilde{\lambda}_1} \dots F_q^{\widetilde{\lambda}_q}, \tag{8}$$

is a (Darboux) first integral of system (1).

(b) *There exist constants* $\lambda_i, \widetilde{\lambda}_j \in \mathbb{C}$ *not all vanishing such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \widetilde{\lambda}_j \widetilde{K}_j + div\, X = 0,$$

if and only if the function defined by (8) *is a (Darboux) integrating factor of* X.

For more recent versions of Theorem 1.1 see [42, 43] and for some generalizations see [12, 39].

Functions of the form (8) are called *Darboux functions*. We say that the polynomial system (1) is *Darboux integrable* if it admits a first integral or an integrating factor which is given by a Darboux function.

Remark 1. Prelle and Singer [53] showed that if system (1) admits an *elementary first integral* then it admits an integrating factor which is the n-th root of a rational function (a slightly improved version of this result can be found in [38, Corollary 4]). Later, Singer in [55] showed that if system (1) admits a Liouvillian first integral then it has an integrating factor which is given by a Darboux function. This is an important argument to motivate sentences like "*Darboux functions capture Liouvillian integrability*" or "*Liouvillian first integrals are either Darboux first integrals or integrals coming from a Darboux integrating factor*".

Given a polynomial system (1) of degree m, the computation of all its invariant algebraic curves becomes a complicated problem since nothing is known a priori about the maximum degree of these curves. This makes necessary to impose additional conditions either on the structure of the system (1) or on the nature of such curves (see for instance, [13, 15, 23, 50] or references therein). This difficulty has motivated the study of different types of *inverse problems* of the Darboux theory of integrability, see [48] and see also [20, 21, 22, 23, 40].

We finish this subsection with a remark about a geometrical meaning of the integrating factor, pretty known to people coming from the Sophus Lie mathematical community. It is an established fact in Fluid Dynamics that integrating factors arise as a density in stationary planar regimes: the equation $\nabla \cdot (\mu X) = 0$ is the continuity equation for the field of velocities X, with density function $\mu = \mu(x, y)$ (considered in the context of the symplectic geometry and Hamiltonian dynamics).

Remark 2. Let $\mu = \mu(x, y)$ be an integrating factor of the vector field (2) defined in some domain of the plane. Then the vector field X is a hamiltonian vector field with respect to the symplectic form $\omega = \mu dx \wedge dy$ (this form degenerates only at the zeros of μ). In fact from $i_X \omega = dH$, we obtain

$$-\mu Q = \omega\left(X, \frac{\partial}{\partial x}\right) = \frac{\partial H}{\partial x}, \qquad \mu P = \omega\left(X, \frac{\partial}{\partial y}\right) = \frac{\partial H}{\partial y}.$$

Hence, the vector field (2) can be rewritten

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} = \frac{1}{\mu}\left(\frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}\right),$$

which is Hamiltonian with Hamilton function the first integral H. It is straightforward to verify that the symplectic form ω is invariant under the action of the flow of X, i.e.,

$$L_X\omega = di_X\omega + i_X d\omega = di_X\omega = ddH = 0.$$

From this point of view, the dynamics of the vector fields in the plane can be formally considered as an Ergodic theory problem: the existence of an invariant measure, the one defined by the associated integrating factor of the flow.

1.2. **Picard-Vessiot theory.** Picard-Vessiot theory is the Galois theory of linear differential equations. We will just remind here some of its main definitions and results but we refer the reader to [54] for a wide theoretical background, while [3, 4] contain short summaries of this theory including applications in non-integrability of Hamiltonian systems.

We start recalling some basic notions on algebraic groups and, afterwards, Picard-Vessiot theory will be introduced.

An algebraic group of matrices 2×2 is a subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$ defined by means of algebraic equations in its matrix elements and in the inverse of its determinant. That is, there exists a set of polynomials $P_i \in \mathbb{C}[x_1,\ldots,x_5]$, for $i \in I$, such that $A \in \mathrm{GL}(2,\mathbb{C})$ given by

$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

belongs to G if and only if $P_i\left(x_{11}, x_{12}, x_{21}, x_{22}, (\det A)^{-1}\right) = 0$ for all $i \in I$ and where $\det A = x_{11}x_{22} - x_{21}x_{12}$. It is said that G is an algebraic manifold endowed with a group structure.

Recall that a group G is called *solvable* if and only if there exists a chain of normal subgroups

$$e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G,$$

satisfying that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$.

It is well known that any algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. In particular, the *identity connected component* G^0 of G is defined as the largest connected algebraic subgroup of G containing the identity. In case that $G = G^0$ we say that G is a *connected group*. Moreover, if G^0 is solvable we say that G is *virtually solvable*.

The following result provides the relation between virtual solvability of an algebraic group and its structure.

Theorem 1.2 (Lie-Kolchin)**.** *Let $G \subseteq \mathrm{GL}(2,\mathbb{C})$ be a virtually solvable group. Then, G^0 is triangularizable, that is, it is conjugate to a subgroup of upper triangular matrices.*

Now, we briefly introduce Picard-Vessiot Theory.

First, we say that $(\mathcal{K}, \ ')$ - or, simply, $\mathcal{K}$ - is a *differential field* if $\mathcal{K}$ is a commutative field of characteristic zero, depending on x and $'$ is a derivation on $\mathcal{K}$ (that is, satisfying that $(a+b)' = a' + b'$ and $(a\cdot b)' = a'\cdot b + a\cdot b'$ for all $a, b \in \mathcal{K}$). We denote by $\mathcal{C}$ the *field of constants of* $\mathcal{K}$, defined as $\mathcal{C} = \{c \in \mathcal{K} \mid c' = 0\}$.

We will deal with second order linear homogeneous differential equations, that is, equations of the form

$$y'' + b_1 y' + b_0 y = 0, \quad b_1, b_0 \in \mathcal{K}, \tag{9}$$

and we will be concerned with the algebraic structure of their solutions. Moreover, along this work, we will refer to the current differential field as the smallest one containing the field of coefficients of this differential equation.

Let us suppose that y_1, y_2 is a basis of solutions of equation (9), i.e., y_1, y_2 are linearly independent over $\mathcal{C}$ and every solution is a linear combination over $\mathcal{C}$ of these two. Let $\mathcal{L} = \mathcal{K}\langle y_1, y_2\rangle = \mathcal{K}(y_1, y_2, y_1', y_2')$ be the differential extension of $\mathcal{K}$ such that $\mathcal{C}$ is the field of constants for $\mathcal{K}$ and $\mathcal{L}$. In this terms, we say that $\mathcal{L}$, the smallest differential field containing $\mathcal{K}$ and $\{y_1, y_2\}$, is the *Picard-Vessiot extension* of $\mathcal{K}$ for the differential equation (9).

The group of all the differential automorphisms of $\mathcal{L}$ over $\mathcal{K}$ that commute with the derivation $'$ is called the *Galois group* of $\mathcal{L}$ over $\mathcal{K}$ and is denoted by $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$. This means, in particular, that for any $\sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{K})$, $\sigma(a') = (\sigma(a))'$ for all $a \in \mathcal{L}$ and that $\sigma(a) = a$ for all $a \in \mathcal{K}$. Thus, if $\{y_1, y_2\}$ is a fundamental system of solutions of (9) and $\sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{K})$ then $\{\sigma y_1, \sigma y_2\}$ is also a fundamental system. This implies the existence of a non-singular constant matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

This fact can be extended in a natural way to a system

$$\sigma \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(y_1') & \sigma(y_2') \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} A_\sigma,$$

which leads to a faithful representation $\mathrm{Gal}(\mathcal{L}/\mathcal{K}) \to \mathrm{GL}(2, \mathbb{C})$ and makes possible to consider $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ as a subgroup of $\mathrm{GL}(2, \mathbb{C})$ depending (up to conjugacy) on the choice of the fundamental system $\{y_1, y_2\}$.

One of the fundamental results of the Picard-Vessiot Theory is the following theorem (see [33, 36]).

Theorem 1.3. *The Galois group* $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ *is an algebraic subgroup of* $\mathrm{GL}(2, \mathbb{C})$.

We say that equation (9) is *integrable* if the Picard-Vessiot extension $\mathcal{L} \supset \mathcal{K}$ is obtained as a tower of differential fields $\mathcal{K} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L}$ such that $\mathcal{L}_i = \mathcal{L}_{i-1}(\eta)$ for $i = 1, \dots, m$, where either

(*i*) η is *algebraic* over $\mathcal{L}_{i-1}$, that is η satisfies a polynomial equation with coefficients in $\mathcal{L}_{i-1}$.
(*ii*) η is *primitive* over $\mathcal{L}_{i-1}$, that is $\eta' \in \mathcal{L}_{i-1}$.
(*iii*) η is *exponential* over $\mathcal{L}_{i-1}$, that is $\eta'/\eta \in \mathcal{L}_{i-1}$.

Usually in terms of Differential Algebra's terminology we say that equation (9) is integrable if the corresponding Picard-Vessiot extension is *Liouvillian*. Moreover, the following theorem holds.

Theorem 1.4 (Kolchin)**.** *Equation* (9) *is integrable if and only if* $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ *is virtually solvable, that is, its identity component* $(\mathrm{Gal}(\mathcal{L}/\mathcal{K}))^0$ *is solvable.*

For instance, for the case $a = 0$ in equation (9), i.e. $y'' + by = 0$, it is very well known [33, 36, 54] that $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$ (remind that $A \in \mathrm{SL}(2,\mathbb{C}) \Leftrightarrow A \in \mathrm{GL}(2,\mathbb{C})$ and $\det A = 1$). For a more detailed study see the Appendix A.

2. Some remarks about Riccati equation.

2. Some remarks about Riccati equation. Ricatti equation is probably one of the most studied equations in Dynamical Systems. Its rôle in the study of the Darboux and Picard-Vessiot integrability leads us to devote this section to some of its properties. Even though these results are known, their proofs have been included for completeness. We divide these properties in two types: the first one (see Subsection 2.1) concerning transformations leading a general second order differential equation into a Riccati equation (written in the so-called reduced form). Remind that this becomes the starting point of the celebrated Kovacic algorithm (see Appendix A). A second type, more Darboux-like, that studies first integrals and integrating factors for a Riccati equation (Subsection 2.2).

2.1. Transformations related to Riccati equations. It is known that any linear second order differential equation can be led into a general Riccati equation through a classical logarithmic change of variable (see, for instance, [31, 51]). The following proposition recall it and summarises some other related transformations.

Proposition 1. *Let $\mathcal{K}$ be a differential field and let consider functions $a_0(x)$, $a_1(x)$, $a_2(x)$, $r(x)$, $\rho(x)$, $b_0(x)$, $b_1(x)$ belonging to $\mathcal{K}$ that, for simplicity, will be denoted without their explicit dependence on x. Consider now the following forms associated to any second order differential equation (ode) and Riccati equation:*

(i) *Second order ode (in general form):*
$$y'' + b_1 y' + b_0 y = 0. \tag{10}$$

(ii) *Second order ode (in reduced form):*
$$\xi'' = \rho\xi. \tag{11}$$

(iii) *Riccati equation (in general form):*
$$v' = a_0 + a_1 v + a_2 v^2, \quad a_2 \neq 0. \tag{12}$$

(iv) *Riccati equation (in reduced form):*
$$w' = r - w^2, \tag{13}$$

Then, there exist transformations $\mathcal{T}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{R}$ leading some of these equations into the other ones, as showed in the following diagram:

$$\begin{array}{ccc} v' = a_0 + a_1 v + a_2 v^2 & \xrightarrow{\ \mathcal{T}\ } & w' = r - w^2 \\ \Big\downarrow{\scriptstyle \mathcal{B}} & & \Big\downarrow{\scriptstyle \mathcal{R}} \\ y'' + b_1 y' + b_0 y = 0 & \xrightarrow{\ \mathcal{S}\ } & \xi'' = \rho\xi. \end{array}$$

The new independent variables are defined by means of

$$\mathcal{T}: \ v = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right) - \frac{1}{a_2} w, \qquad \mathcal{B}: \ v = -\frac{1}{a_2}\frac{y'}{y},$$

$$\mathcal{S}: \ y = \xi \mathrm{e}^{-\frac{1}{2}\int b_1 \, dx}, \qquad \mathcal{R}: \ w = \frac{\xi'}{\xi},$$

and the functions r, ρ, b_0 and b_1 are given by

$$r = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right), \tag{14}$$

$$\alpha = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right), \quad \beta = -\frac{1}{a_2}, \tag{15}$$

$$b_1 = -\left(a_1 + \frac{a_2'}{a_2}\right), \quad b_0 = a_0 a_2, \tag{16}$$

$$\rho = r = \frac{b_1^2}{4} + \frac{b_1'}{2} - b_0. \tag{17}$$

Proof. The proof is quite standard.

$[\mathcal{T}]$: Applying the change $v = \alpha + \beta w$ we get the equation

$$\alpha' + \beta' w + \beta w' = a_0 + a_1\alpha + a_1\beta w + a_2\alpha^2 + 2a_2\alpha\beta w + a_2\beta^2 w^2$$

that, regrouping terms, leads to

$$w' = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right) + \left(a_1 + 2a_2\alpha - \frac{\beta'}{\beta}\right) w + a_2\beta w^2.$$

Since $a_2 \neq 0$ we can take $\beta = -1/a_2$ and, therefore, $a_2\beta = -1$. Having this into account, the value of α satisfying that the coefficient in w vanishes is given by

$$\alpha = \frac{1}{2a_2}\left(\frac{\beta'}{\beta} - a_1\right).$$

The expressions for α, β and r follow straightforwardly,

$$r = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right), \quad \alpha = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right), \quad \beta = -\frac{1}{a_2}.$$

Moreover, it is clear that α, β and r belong to $\mathcal{K}$.

$[\mathcal{B}]$: Imposing $\alpha = 0$ and taking $\beta = -1/a_2$ in transformation $\mathcal{T}$ we have $v = -w/a_2$ and we obtain the Riccati equation

$$w' = -a_0 a_2 + \left(a_1 + \frac{a_2'}{a_2}\right) w - w^2.$$

Performing now the change of variables $w = (\log y)'$ (or, equivalently, $v = -a_2 y'/y$) we obtain the differential equation $y'' + b_1 y' + b_0 y = 0$ with

$$b_1 = -\left(a_1 + \frac{a_2'}{a_2}\right), \quad b_0 = a_0 a_2.$$

Obviously, b_0 and b_1 belong to $\mathcal{K}$.

$[\mathcal{S}]$: The change of variable $y = \mu\xi$, with $\mu = \mu(x)$ and $\xi = \xi(x)$, lead us to

$$\xi'' + \left(2\frac{\mu'}{\mu} + b_1\right)\xi' + \left(\frac{\mu''}{\mu} + b_1\frac{\mu'}{\mu} + b_0\right)\xi = 0.$$

In order to obtain the equation $\xi'' = \rho\xi$ we need to impose

$$2\frac{\mu'}{\mu} + b_1 = 0, \qquad \frac{\mu''}{\mu} + b_1\frac{\mu'}{\mu} + b_0 = -\rho,$$

which gives rise to

$$\mu = \mathrm{e}^{-\frac{1}{2}\int \mathrm{b}_1}, \qquad \rho = \frac{\mathrm{b}_1^2}{4} + \frac{\mathrm{b}_1'}{2} - \mathrm{b}_0.$$

Moreover, it is straightforward to check that $\rho \in \mathcal{K}$.

$[\mathcal{R}]$: This is a particular case of transformation $[\mathcal{B}]$ with the choice $a_0 = r$, $a_1 = 0$ and $a_2 = -1$.

□

From this Lemma, it follows that the function v is algebraic over $\mathcal{K}$ if and only if the function w is also algebraic over $\mathcal{K}$. Furthermore, in such case, the degree over $\mathcal{K}$ of both functions v and w is the same.

It is known that a Riccati equation (12) has an algebraic solution over $\mathcal{K}$ if and only if the differential equation (10) is integrable in a Picard-Vessiot sense. In this situation we say that the Riccati equation is integrable over $\mathcal{K}$. We notice that Kovacic algorithm (see Appendix A) starts from an equation in form (11).

2.2. **Integrating factor and first integrals for Riccati vector fields.** We briefly show some relations between the existence of invariant curves of a certain type of vector fields and the integrability, via Kovacic algorithm (see Appendix A), of its associated Riccati foliation. It includes important suggestions and remarks from one of the referees of the original manuscript.

From Singer [55] (see Remark 1) we know that if a planar polynomial vector field (2) admits a Liouvillian first integral then it has also an integrating factor given by a Darboux function. However, few results are known about the relation between the existence of an algebraic invariant curve of a general planar vector field and the Liouvillian integrability of its foliation. Let us focus in the case of Riccati vector fields.

Thus, consider a family of planar vector fields of the form

$$X = (p(x) - q(x)w^2)\frac{\partial}{\partial w} + q(x)\frac{\partial}{\partial x}, \tag{18}$$

with $p(x), q(x) \in \mathbb{C}[x]$ complex polynomials. Introducing an independent variable t, usually called *time*, we can associate to them the following system of differential equations

$$\begin{array}{rcl} \dot{w} & = & p(x) - q(x)w^2, \\ \dot{x} & = & q(x), \end{array}$$

where we denote by $\dot{} = d/dt$. Its foliation, governed by the equation

$$w' = \frac{dw}{dx} = \frac{p(x) - q(x)w^2}{q(x)} = \frac{p(x)}{q(x)} - w^2, \tag{19}$$

is a Riccati equation given in reduced form $w' = r(x) - w^2$ with $r = p/q \in \mathbb{C}(x)$. If we denote by $f = w^m + \sum_{i=1}^{m} a_{m-i}(x)w^{m-i}$ an irreducible polynomial, it is known that $f = 0$ is an algebraic invariant curve of (18) if and only if its roots $w(x)$ are (algebraic) solutions of this Riccati equation. Furthermore (see, for instance [50, 55, 59] and references therein), it is also known that X admits a rational first integral if and only if all solutions of the Riccati equation are algebraic.

Remark 3. A direct computation shows that if $f = 0$ is an invariant algebraic curve of X, that is satisfies $X(f) = Kf$, then the cofactor K has the form $q(x)(-mw + a_{m-1}(x))$. In [27, 57] is proved that this is equivalent to the fact that $-a_{m-1}(x)$ is the logarithmic derivative of a function $g(x)$, i.e. $a_{m-1}(x) = -g'(x)/g(x)$, where $g(x)$ is an exponential solution of the m-th symmetric power of $y'' - r(x)y$. In that case, the coefficients $a_{m-2}(x), \ldots, a_0(x)$ are given by the Kovacic recursion (see [27, 35, 57]). From the expression above for the cofactor K it follows the existence of a rational first integral only if there exist two invariant algebraic curves with the same degree m. Such a rational first integral could be obtained from computing an integral dependence between their degrees in such a form that no term in w remains and the remaining term in x is an exact derivative.

Next lemma shows that the integrability of this "Riccati foliation" is closely related to the existence of an algebraic invariant curve of its vector field (18). A similar approach for this problem can be found in [29, 30].

Lemma 2.1. *Let consider a vector field X in (18), with $p(x), q(x) \in \mathbb{C}[x]$ and its associated (reduced) Riccati equation (19). Therefore,*

1. *If $w_1(x)$ is a solution of equation (19) then the associated vector field X has an integrating factor μ_1 given by*
$$\mu_1(x) = \frac{\mathrm{e}^{-2\int \mathrm{w_1(x)\,dx}}}{q(x)(w - w_1(x))^2}.$$
2. *Moreover, if $w_1(x)$ is algebraic of order 2 with $f_2(x, w_1(x)) = 0$ and $f_2(x, w) = w^2 + a_1(x)w + a_0(x)$, then X has an integrating factor μ_2 given by*
$$\mu_2(x) = \frac{\mathrm{e}^{\int \mathrm{a_1(x)\,dx}}}{q(x) f_2(x, w_1(x))}.$$

Proof. It is straightforward to check that if $w_1(x)$ is a solution of $w' = p/q - w^2$ then it holds $X(f_1) = K_1 f_1$ with $f_1(x, w) = -w + w_1(x) = 0$ and $K_1 = -q(w + w_1(x))$. Indeed,

$$\begin{aligned} X(f_1) = \quad & (p - qw^2)\tfrac{\partial}{\partial w}(-w + w_1) + q\tfrac{\partial}{\partial x}(-w + w_1) = \\ & -p + qw^2 + qw_1' = -p + qw^2 + p - qw_1^2 = \\ & q(w^2 - w_1^2) = -q(w + w_1)(-w + w_1). \end{aligned}$$

Additionally, for $f_2 = q(x)$ we have that $X(f_2) = q'q = K_2 f_2$ with $K_2 = q'$. Besides, $F_1(x) = \mathrm{e}^{-\int \omega_1(x)\mathrm{d}x}$ satisfies $X(F_1) = L_1 F_1$ with $L_1 = -q\omega_1$. Since X has divergence $\mathrm{div}X = -2q(x)w + q'(x)$ and $-2K_1 - K_2 + 2L_1 + \mathrm{div}X = 0$ it follows that vector field (18) admits the integrating factor

$$\mu_1(w, x) = \frac{F_1^2}{f_1^2 f_2} = \frac{e^{-2\int w_1(x)dx}}{q(x)(w - w_1(x))^2},$$

as it was claimed. Item 2 happens in case 2 of Kovacic algorithm. This implies that $-a_1$ is the logarithmic derivative of $\sqrt{g(x)}$ with $g(x) \in \mathbb{C}(x)$.

□

The important fact is that, conversely, Picard-Vessiot theory and in particular, Kovacic algorithm, supply information about first integrals and integrating factors of the equation $w' = r(x) - w^2$ from the knowledge of some of its solutions, w_1, w_2, w_3. Indeed, from the first three cases in Kovacic algorithm [35] (the integrable ones) one obtains the following types of first integrals (see Weil [59] and Żołądek [62]).

Proposition 2. *Under the same hypotheses of Lemma 2.1, we describe the first integrals following the cases of Kovacic algorithm:*

1. *Case 1: rational solutions. One has the following two possibilities:*

- *If equation (19) has a unique rational solution $w_1(x)$ then X admits the first integral*

$$H_1 = -\frac{\mathrm{e}^{2\int \mathrm{w_1(x)\,dx}}}{w - w_1(x)} + \int \mathrm{e}^{-2\int \mathrm{w_1(x)\,dx}}\,\mathrm{dx}.$$

- *If (19) has two distinct rational solutions $w_1(x), w_2(x)$ then X admits the first integral*

$$H_1 = \frac{w - w_1(x)}{w - w_2(x)}\ \mathrm{e}^{\int \mathrm{w_2(x) - w_1(x)\,dx}}.$$

In both cases H_1 is Liouvillian in x and rational in w.

2. *Case 2: quadratic solutions. Assume that equation (19) admits a quadratic solution $w_1(x)$, that is, $w_1(x)$ being root of a quadratic polynomial $f_2 = w^2 + a_1(x)w + \frac{1}{2}(a_1(x)^2 - a_1'(x) - 2r(x))$, where $r = p/q$. Then X admits a first integral of hyperelliptic type, given either by*

$$H_2 = \frac{w + a_1(x) + w_1(x)}{w - w_1(x)}\ \mathrm{e}^{-\int \sqrt{\Delta(\mathrm{x})}\,\mathrm{dx}}$$

(Δ denotes the discriminant of f_2) or via the integrating factor method by

$$H_2 = \int \frac{\mathrm{e}^{\int \mathrm{a_1(x)\,dx}}}{f_2(x,w)}\ w + \varphi(x),$$

with φ given by the condition $\partial H_2/\partial x + \mu_2(p(x) - q(x)w^2) = 0$.

3. *Case 3: algebraic solutions. If equation (19) admits more than two algebraic solutions then all the solutions in (19) are algebraic and X admits a rational first integral.*

Proof.

1. *Case 1.1.* Using the expression μ_1 in Lemma 2.1 for the integrating factor, one has

$$H_1 = \int \frac{\mathrm{e}^{-2\int \mathrm{w_1(x)\,dx}}}{(w - w_1(x))^2}\ dw + \varphi(x) = -\frac{\mathrm{e}^{-2\int \mathrm{w_1(x)\,dx}}}{w - w_1(x)} + \varphi(x).$$

The function $\varphi(x)$ is determined by the condition $\partial H_1/\partial x = -\mu_1(p - qw^2)$, giving rise to $\varphi'(x) = \exp(-2\int w_1(x)\,dx)$.

Case 1.2. Using again Lemma 2.1, it is possible to build two integrating factors μ_1, μ_2 so that μ_1/μ_2 is the first integral.

2. *Case 2.* The first formula follows from applying case 1.2 and using that w_1, w_2 are roots of the polynomial f_2. For the second one, it follows from Lemma 2.1 and the integrating factor $\mu_2 = \frac{\mathrm{e}^{\int \mathrm{a_1(x)\,dx}}}{q(x) f_2}$. Note that the shape of polynomial f_2 is given by the Kovacic algorithm.
3. *Case 3.* This is an standard result in Darboux Theory. The process to construct such a rational first integral will be illustrated with some examples in Section 3.4b). The underlying idea is, as mentioned above, to look for two algebraic invariant curves and to seek for a dependence relation between their cofactors.

□

For a recent approach, using these ideas, to the study of families of polynomial vector fields of Schrödinger type see [7]. The following result characterises the rational integrability of the polynomial vector fields that we are considering in this work.

Corollary 1. *The Galois group of* (11) *is finite if and only if its corresponding planar polynomial vector field has a rational first integral.*

Here is the proof suggested us by the referee, shorter and covering one missing case. Proof. Assume that the Galois group of $\xi'' = \rho\xi$ is finite. Then all solutions of this equation are algebraic so, in particular, all solutions of the Riccati equation (19) are algebraic and hence the associated vector field X admits a rational first integral. Conversely, if X admits a rational first integral then all solutions of the Riccati equation are algebraic. It follows from the classifications of the Kovacic algorithm that the differential Galois group is then finite. □

Remark 4. The following proof provides an explicit formula for the first integral in two cases of Kovacic algorithm. It does not apply for Case 2, when the Galois group is a finite dihedral group.

Proof. Notice that $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ is finite if and only if we fall in case 3 of Kovacic Algorithm or in case 1 of the type

$$\mathrm{Gal}(\mathcal{L}/\mathcal{K}) = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1 \right\}.$$

Hence, only remains to study this last case (the cyclic one). Let ξ_1, ξ_2 be solutions of $\xi'' = \rho\xi$. Then there exists $g \in \mathbb{C}(x)$ such that $\xi_1 = g^{\frac{1}{n}}$ and $\xi_2 = g^{-\frac{1}{n}}$. We define $\omega_1 = \xi_1'/\xi_1$, $\omega_2 = \xi_2'/\xi_2$, and we obtain

$$\omega_1 = \frac{1}{n}\frac{g'}{g}, \qquad \omega_2 = -\frac{1}{n}\frac{g'}{g}.$$

Thus, the corresponding vector field (11) admits the first integral

$$H_1(\omega, x) = \frac{-\omega + \omega_2}{-\omega + \omega_1} e^{\int (\omega_2 - \omega_1)dx} = \frac{-\omega - \dfrac{1}{n}\dfrac{g'}{g}}{-\omega + \dfrac{1}{n}\dfrac{g'}{g}}\; g^{\frac{-2}{n}},$$

and so also admits the first integral

$$H(\omega, x) = H_1(\omega, x)^n = \frac{1}{g^2}\left(\frac{-ng\omega - g'}{-ng\omega + g'}\right)^n \in \mathbb{C}(\omega, x),$$

which completes the proof. □

Remark 5. Let $P \in \mathbb{C}[x]$ be a polynomial of odd degree. It is known that the planar polynomial vector field $\dot{x} = 1,\ \dot{y} = P(x) + y^2$ (with associated foliation $y' = P(x) + y^2$) is not integrable, that is, it has no invariant curves, since it falls in case 4 of Kovacic algorithm [35].

2.3. **Riccati foliations.** Let us recall some well-known geometrical properties of Riccati foliations defined by planar polynomials vector fields (see, for instance, [37]). Although we are not using these properties along the paper, we include them for completeness.

Let

$$\Omega = q(x)dy - (p_1(x) + p_2(x)y + p_3(x)y^2)dx, \quad p_i, q \in \mathbb{C}[x] \tag{20}$$

be the 1-form defining a Riccati foliation on the complex plane. Since the equation $\Omega = 0$ is the projective version of the corresponding second order linear differential equation defined over the vector bundle $\mathbf{P}^1 \times \mathbb{C}^2$ (i.e., the fibre $\mathbb{C}^2$ is projectivized to $\mathbf{P}^1$, see subsection 2.1), the Riccati equation $\Omega = 0$ is defined in a natural way over $(x, y) \in \mathbf{P}^1 \times \mathbf{P}^1$.

The singular points of $\Omega = 0$ are the zeros of the polynomial $q(x)$ and, possibly, the point at infinity $x = \infty \in \mathbf{P}^1$. Moreover, these singular points are exactly the poles of the coefficients of the associated second order linear differential equation. We define $d := max(deg(p_1), deg(p_2), deg(p_3), deg(q) - 2)$. Thus the point $x = \infty$ is a singular point if and only if $deg(q)$ is less than $d + 2$. In Kovacic algorithm, which applies to the reduced form of the second order linear differential equation (see Appendix A), this set of singular points is denoted by Γ. Therefore it seems natural to call it in the same way also here, that is, $\Gamma = \{x_1, ..., x_r\}$. Thus the Riccati foliation is *holomorphic* on $(\mathbf{P}^1 - \Gamma) \times \mathbf{P}^1$, because the singular points of the associated linear differential equation are exactly the set Γ, given by the singularities of their coefficients. We notice that the ("singular") sets $\{x_i\} \times P^1$ are invariant by the foliation and are usually called *invariant fibres* because they are already fibres of the fibration $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$, $(x, y) \mapsto x$. This fibration is transversal to the Riccati foliation since fibres $T(x) := \pi^{-1}(x) = \{x\} \times \mathbf{P}^1$, with x non singular, are *global* and transversal to the foliation, i.e., transversal to all the leaves. Over any of these transversals the holonomy group of the foliation is defined as a representation

$$\pi_1(\mathbf{P}^1 - \Gamma, x_0) \to \mathrm{Diff}(T(x_0)),$$

(where $\mathrm{Diff}(T(x_0))$ is the group of diffeomorphisms on the transversal), given by lifting the loops in the fundamental group to the leaves of the foliation, this is, by solving the Riccati equation with initial conditions and final points on the transversal $T(x_0)$. As the Riccati equation is the projectivization of a second order linear differential equation, the holonomy group must be the projectivization of the monodromy group of the linear second order equation acting on the vector space fibre of the meromorphic vector bundle $\mathbf{P}^1 \times \mathbb{C}^2$, $\{x\} \times \mathbb{C}^2 \approx \mathbf{C}^2$. By fixing a base of fundamental solutions, this can be considered as the space of solutions of the linear differential equation. Hence, as the monodromy group is represented by the linear group $\mathrm{GL}(2, \mathbb{C})$, the holonomy group is represented by the projective linear group $\mathrm{PGL}(2, \mathbb{C})$, the Möbius transformations

$$\pi_1(\mathbf{P}^1 - \Gamma, x_0) \to \mathrm{PGL}(2, \mathbb{C}).$$

The Riccati foliations are the most well-known class of a family of foliations, the projective foliations, with holonomy group represented in the projective group. For Riccati foliations, the holonomy group is contained in the Galois group of the foliation either in the Malgrange approach [14, 44, 45] or in the Lie-Vessiot-Kolchin approach [10, 11]. In fact, if the singular points of the associated linear differential equations are singular regular ones, then the Zariski adherence of the holonomy group is the Galois group of the foliation, because in this case the Zariski adherence of the monodromy group of the linear differential equation is the Galois group of the associated linear differential equation.

The critical points of the associated vector field, i.e., zeroes of $q(x)$ and of $p_1(x)+p_2(x)y+p_3(x)y^2$ are obviously contained in the invariant fibres. For general Riccati foliations there are two critical points on the invariant fibre for each point in Γ. For some special Riccati foliations there are no critical points. This is the case, for example, of Riccati foliations given in reduced form by $\Omega = q(x)dy - (p(x) + q(x)y^2)dx$ (with p and q relatively primes), corresponding to the field (18).

3. **Applications.** In this section we analyse some examples involving integrability and non-integrability of some families of Riccati planar vector fields or planar vector fields whose foliation can be reduced into a Riccati form.

3.1. **Quadratic polynomials fields.** The study of the integrability of the quadratic polynomial vector field

$$\begin{aligned} \dot{x} &= a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}, \\ \dot{y} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00}, \end{aligned}$$

with $a_{ij}, b_{i,j} \in \mathbb{C}$ is, in its general form, a hard problem. One of its possible approaches is the so-called linear-quadratic case, when one of the two components is a polynomial of degree one. In [41, Prop.3] it is proved that its study around a finite equilibrium point (the origin) can be reduced to consider two families of systems. Using the notation introduced therein, we refer to these families as (S1)-type,

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned} \tag{S1}$$

and (S2)-type,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \tag{S2}$$

In [41], the authors prove that the linear-quadratic systems having a global analytic first integral are those satisfying:

(a_1) $b_{02} = \lambda = 0$.
(b_1) $b_{02} = 0$ and $\lambda = -p/q \in \mathbb{Q}^-$,

in the case of (S1)-type systems and

(a_2) $b_{20} = b_{02} = \lambda = 0$ and $\varepsilon b_{11} \neq 0$.
(b_2) $b_{20} = b_{11} = \lambda = 0$ and $\varepsilon b_{02} \neq 0$.
(c_2) $b_{11} = \lambda = 0$ and $b_{20} \neq 0$,

for (S2)-type systems. Furthermore, they also provide the explicit form of the corresponding first integrals. It is important to notice that all of them are of Darboux type and, therefore, Liouvillian.

Our aim in this example is to show that these results can be recovered using arguments coming from the Galois theory of linear differential equations. We start first with the (S1)-case, whose associated foliation is given by the Riccati equation:

$$\frac{dy}{dx} = (\varepsilon + b_{20}x) + \left(\frac{\lambda + b_{11}x}{x}\right) y + \frac{b_{02}}{x}y^2. \tag{21}$$

By Lemma 1 this equation can be transformed into the reduced form $w' = r(x) - w^2$, with

$$r(x) = \frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2-1}{4x^2}, \quad \kappa = \frac{1}{\sqrt{b_{11}^2 - 4b_{20}b_{02}}}\left(b_{02}\varepsilon + \frac{b_{11}}{2}(1-\lambda)\right), \quad \mu = \frac{\lambda}{2}, \tag{22}$$

provided $b_{11}^2 - 4b_{20}b_{02} \neq 0$, and into the form $\xi'' = r(x)\xi$. This equation is a Whittaker equation (see Appendix B) to which one can apply the Martinet-Ramis Theorem (see theorem B.2). This Theorem asserts that such Whittaker equation is integrable if and only if at least one of the following conditions is verified:

$$\pm\kappa \pm \mu \in \frac{1}{2} + \mathbb{N},$$

or, equivalently (and more suitable for the expressions derived of κ and μ)

$$2\,(\kappa \pm \mu) \in 2\mathbb{Z} + 1.$$

In our case one has that

$$2\,(\kappa \pm \mu) = \frac{2b_{20}\varepsilon + b_{11}(1-\lambda)}{\sqrt{b_{11}^2 - 4b_{20}b_{02}}} \pm \lambda,$$

so for (S1)-type systems conditions (a_1) and (b_1) read, respectively, $2(\kappa \pm \mu) = 1 \in 2\mathbb{Z}+1$ and $2(\kappa+\mu) = (1+(p/q)) + (-p/q) = 1 \in 2\mathbb{Z}+1$. Therefore Galois Theory recovers the integrability result asserted in [41, Thm.1].

Let us consider now a (S2)-type system, namely,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned}$$

with foliation given by the differential equation

$$\frac{dy}{dx} = (\lambda + b_{11}x) + \left(\varepsilon x + b_{20}x^2\right)\frac{1}{y} + b_{02}y. \tag{23}$$

This equation falls in one of the following situations:

(*i*) $\lambda = b_{11} = 0$ yields to a Bernoulli equation

$$\frac{dy}{dx} = \left(\varepsilon x + b_{20}x^2\right)\frac{1}{y} + b_{02}y,$$

which corresponds to cases (b_2) and (c_2).

(*ii*) If $\varepsilon = b_{20} = 0$ we obtain the linear equation (and, of course, integrable in a Liouville sense)

$$\frac{dy}{dx} = (\lambda + b_{11}x) + b_{02}y.$$

This possibility is not taken into account by Llibre and Valls [41] since this equation is not, strictly speaking, in Riccati form.

(*iii*) $b_{20} = b_{02} = \lambda = 0$ and $\varepsilon b_{11} \neq 0$ (case (a_2)) gives rise to $dy/dx = b_{11}x + \varepsilon x y^{-1}$, which is a separable equation (and a Bernoulli as well) and whose solutions are all Liouvillian.

(*iv*) If $b_{02} = 0$ we obtain a Liénard equation,

$$y\frac{dy}{dx} = (\lambda + b_{11}x)\,y + \left(\varepsilon x + b_{20}x^2\right),$$

that will be considered more deeply in a forthcoming section.

3.2. **Families of orthogonal polynomials.** Recall that the Hypergeometric equation, including confluences, is a particular case of the differential equation

$$y'' + \frac{L}{Q}y' + \frac{\lambda}{Q}y = 0, \quad \lambda \in \mathbb{C}, \quad L = a_0 + a_1x, \quad Q = b_0 + b_1x + b_2x^2. \qquad (24)$$

It is well known (see, for example, [18]) that classical orthogonal and Bessel polynomials are solutions of equation (24) for suitable values of a_j, b_j and λ. Namely,

- Hermite H_n,
- Chebyshev of first kind T_n,
- Chebyshev of second kind U_n,
- Legendre P_n,
- Laguerre L_n,
- associated Laguerre $L_n^{(m)}$,
- Gegenbauer $C_n^{(m)}$,
- Jacobi $\mathcal{P}_n^{(m,\nu)}$
- Bessel B_n,

where

Family	Q	L	λ
H_n	1	$-2x$	$2n$
T_n	$1-x^2$	$-x$	n^2
U_n	$1-x^2$	$-3x$	$n(n+2)$
P_n	$1-x^2$	$-2x$	$n(n+1)$
L_n	x	$1-x$	n
$L_n^{(m)}$	x	$m+1-x$	n
$C_n^{(m)}$	$1-x^2$	$-(2m+1)x$	$n(n+2m)$
$\mathcal{P}_n^{(m,\nu)}$	$1-x^2$	$\nu-m-(m+\nu+2)x$	$n(n+1+m+\nu)$
B_n	x^2	$2(x+1)$	$-n(n+1)$

Integrability conditions and solutions of equation (24) can be obtained applying Kovacic algorithm (Case 1 of the algorithm). Besides, they can also be achieved via Kimura and Martinet-Ramis Theorems and the *parabolic cylinder equation* (see [27, 34, 46]).

As a consequence, from this table, we obtain the following result.

Proposition 3. *We consider Q, L and λ as in the previous table. Then, for any $\mu \neq 0$, the planar quadratic polynomial vector field*

$$\begin{aligned} \frac{dv}{dt} &= \frac{\lambda}{\mu}Q + (Q'-L)v + \mu v^2, \\ \frac{dx}{dt} &= Q, \end{aligned} \qquad (25)$$

has invariant algebraic curves of the form $\mu v + \dfrac{Q(x)\mathcal{P}_n'(x)}{\mathcal{P}_n(x)} = 0$ where $\mathcal{P}_n(x)$ is any orthogonal polynomial associated to λ, Q, L and $n \in \mathbb{N}$.

Proof. The associated (Riccati) foliation of system (25) is written into the form

$$\frac{dv}{dx} = \frac{\lambda}{\mu} + \frac{Q'-L}{Q}v + \frac{\mu}{Q}v^2. \qquad (26)$$

Performing the change $\tilde{v} = \mu v$ equation (26) becomes

$$\frac{d\tilde{v}}{dx} = \lambda + \frac{Q' - L}{Q}\tilde{v} + \frac{1}{Q}\tilde{v}^2,$$

and can be led into the form (24) through the transformation $\tilde{v} = -Qy'/y$ (see transformation $\mathcal{B}$ of Proposition 1). For fixed λ, Q and L, let $\mathcal{P}_n$ be any orthogonal polynomial solution of equation (24). Then $v_n = -(Q\mathcal{P}_n')/(\mu\mathcal{P}_n)$ is a rational solution of equation (26). Therefore the curve $v - v_n = 0$ is an invariant curve of the vector field (25) for any $n \in \mathbb{N}$. □

According to Proposition 1 equation (26) can be reduced to the form $\xi'' = \rho\xi$ with

$$\rho = \frac{1}{2}\left(\frac{L}{Q}\right)' - \frac{\lambda}{Q} + \left(\frac{L}{2Q}\right)^2,$$

and $\xi = \mathcal{P}_n e^{\int \frac{L}{2Q}}$ is a solution for any $n \in \mathbb{N}$. Additionally, we notice that we fall in Case 1 of Kovacic algorithm.

3.3. **Liénard equation.** Let us consider first order differential equations whose associated foliation can be expressed into the Liénard form

$$yy' = f(x)y + g(x), \tag{27}$$

with $y = y(x)$ and rational functions $f(x)$ and $g(x)$. We are concerned with the problem of obtaining criteria on $f(x)$ and $g(x)$ such that equation (27) can be led into a Riccati equation.

This is a difficult problem and, as far as the authors know, only partial answers have been given to it. In what follows we give some examples of such results coming from the handbook [51] and papers [16, 17].

A first example is given by the 5-parametric family (1.3.3.11 in [51]), which has been corrected in [8] as

$$yy' = (a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}, \tag{28}$$

a, b, c, m, k being complex parameters. In order that (28) come from a polynomial vector field we have that m and k must be integer numbers satisfying $m \geq k+1 > 0$ (see [8]). The change $w = x^k$, $y = x^m(z + ax^k + bx^{-k})$ leads (28) into the Riccati form

$$(-mz^2 + 2abm - c)w'(z) = bk + kzw + akw^2, \tag{29}$$

whose associated second order linear equation is a Riemann equation. More precisely, by Lemma 1, it can be written as a Legendre equation

$$(1-t^2)u''(t) - 2tu'(t) + \left(\nu(\nu+1) - \frac{\mu^2}{1-t^2}\right)u(t) = 0, \tag{30}$$

with

$$\mu = -\frac{m+k}{2m},$$

and ν being a solution of

$$\nu^2 + \nu + \frac{m^2 - k^2}{4m^2} - \frac{abk^2}{mc - 2abm^2} = 0.$$

The difference of exponents in (30) corresponds to $(\mu, \mu, 2\nu - 1)$. Thus, we are under the hypotheses of Kimura's Theorem (see Appendix B.1.1).

Proposition 4. *Legendre equation* (30) *is integrable if and only if, either*

1. $\mu \pm \nu \in \mathbb{Z}$ *or* $\nu \in \mathbb{Z}$, *or*
2. $\pm\mu$, $\pm\mu$, $\pm(2\nu+1)$ *belong to one of the following seven families*

Case	$\mu \in$	$\nu \in$	$\mu+\nu \in$
(a)	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{C}$	
(b)	$\mathbb{Z}\pm\frac{1}{3}$	$\frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$	$\mathbb{Z}+\frac{1}{6}$
(c)	$\mathbb{Z}\pm\frac{2}{5}$	$\frac{1}{2}\mathbb{Z}\pm\frac{1}{5}$	$\mathbb{Z}+\frac{1}{0}$
(d)	$\mathbb{Z}\pm\frac{1}{3}$	$\frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$	$\mathbb{Z}+\frac{1}{10}$
(e)	$\mathbb{Z}\pm\frac{1}{5}$	$\frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$	$\mathbb{Z}+\frac{1}{10}$
(f)	$\mathbb{Z}\pm\frac{2}{5}$	$\frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$	$\mathbb{Z}+\frac{1}{6}$

Proof. In Kimura's Theorem, the difference of exponents μ, μ and $2\nu+1$ correspond to the possibilities listed above. Indeed, they are cases (i), $(ii.1)$, $(ii.3)$, $(ii.11)$, $(ii.12)$, $(ii.13)$ and $(ii.15)$ of Kimura's table (see Appendix B.1.1). For the case (i) we have

- $\mu+\mu+2\nu+1 \in 2\mathbb{Z}+1 \Rightarrow \mu+\nu \in \mathbb{Z}$,
- $-\mu+\mu+2\nu+1 \in 2\mathbb{Z}+1 \Rightarrow \nu \in \mathbb{Z}$,
- $\mu-\mu+2\nu+1 \in 2\mathbb{Z}+1 \Rightarrow \nu \in \mathbb{Z}$,
- $\mu+\mu-2\nu-1 \in 2\mathbb{Z}+1 \Rightarrow \mu-\nu \in \mathbb{Z}$.

The rest of the cases can be proven in a similar way.

(ii.1) We see that $\pm\mu \in \frac{1}{2}+\mathbb{Z}$ and $\pm(2\nu+1) \in \mathbb{C}$ and therefore $\mu \in \mathbb{Z}+\frac{1}{2}$ and $\nu \in \mathbb{C}$.

(ii.3) We consider that $\pm\mu = l+\frac{2}{3}$, $\pm\mu = m+\frac{1}{3}$, $\pm(2\nu+1) = q+\frac{1}{3}$, being $l,m,q \in \mathbb{Z}$. Take for instance $\mu \in \mathbb{Z}\pm\frac{1}{3}$ and $\nu \in \frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$. Furthermore, $l+m+q$ must be even and in consequence we obtain $\mu+\nu \in \mathbb{Z}+\frac{1}{6}$.

(ii.11) We have that $\pm\mu = l+\frac{2}{5}$, $\pm\mu = m+\frac{2}{5}$, $\pm(2\nu+1) = q+\frac{2}{5}$ with $l,m,q \in \mathbb{Z}$. For example we consider $\mu \in \mathbb{Z}\pm\frac{2}{5}$ and $\nu \in \frac{1}{2}\mathbb{Z}\pm\frac{1}{5}$. Moreover $l+m+q$ must be even and therefore we have that $\mu+\nu \in \mathbb{Z}+\frac{1}{10}$.

(ii.12) Now we consider $\pm\mu = l+\frac{2}{3}$, $\pm\mu = m+\frac{1}{3}$, $\pm(2\nu+1) = q+\frac{1}{5}$, being $l,m,q \in \mathbb{Z}$. We take for instance $\mu \in \mathbb{Z}\pm\frac{1}{3}$ and $\nu \in \frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$. Additionally $l+m+q$ must be even and so $\mu+\nu \in \mathbb{Z}+\frac{1}{10}$.

(ii.13) Let be $\pm\mu = l+\frac{2}{3}$, $\pm\mu = m+\frac{1}{3}$, $\pm(2\nu+1) = q+\frac{1}{5}$ with $l,m,q \in \mathbb{Z}$. Consider for example $\mu \in \mathbb{Z}\pm\frac{1}{5}$ and $\nu \in \frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$. Furthermore, $l+m+q$ must be even and therefore $\mu+\nu \in \mathbb{Z}+\frac{1}{10}$.

(ii.15) Consider $\pm\mu = l+\frac{3}{5}$, $\pm\mu = m+\frac{2}{5}$, $\pm(2\nu+1) = q+\frac{1}{3}$, being $l,m,q \in \mathbb{Z}$, take for instance $\mu \in \mathbb{Z}\pm\frac{2}{5}$ and $\nu \in \frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$. Moreover, $l+m+q$ must be even and in consequence $\mu+\nu \in \mathbb{Z}+\frac{1}{6}$.

Finally, observe that the difference exponents μ, μ and $2\nu+1$ do not satisfy the conditions $(ii.2)$, $(ii.4)$, $(ii.5)$, $(ii.6)$, $(ii.7)$, $(ii.8)$, $(ii.9)$, $(ii.10)$ and $(ii.14)$ □

Now we deal with an example from [16, 17]. We consider the equation

$$\frac{dx}{dw} = A(x) + B(x)w, \tag{31}$$

where x is the dependent variable and w is the independent one. Now, for $B \not\equiv 0$, by means of the change of variable

$$w = y - \frac{A}{B},$$

and changing $(w, x) \longrightarrow (x, y)$ (that is, we consider now x as the independent variable and y as the dependent one) equation (31) is transformed into the Liénard equation

$$y\frac{dy}{dx} = \frac{1}{B} + \frac{d}{dx}\left(\frac{A}{B}\right)y, \tag{32}$$

for any functions A and $B \not\equiv 0$. In particular, for

$$A = A(x) = a + bx + cx^2, \quad B = B(x) = \alpha + \beta x + \gamma x^2,$$

equation (31) falls into the Riccati form

$$\frac{dx}{dw} = (a + \alpha w) + (b + \beta w)x + (c + \gamma w)x^2.$$

By Proposition 2.1, applying the transformation $\mathcal{T}$ it follows the reduced Riccati equation (13) and through the transformation $\mathcal{R}$ the normalized second order differential equation $\xi'' = \rho(x)\xi$ with

$$\rho(x) = \frac{\beta^2 - 4\alpha\gamma}{4}x^2 - \frac{2a\gamma + 2\alpha c - b\beta}{2}x - \frac{4ac - b^2}{4} + \frac{b\gamma - \beta c}{2(\gamma x + c)} + \frac{3\gamma^2}{4(\gamma x + c)^2}. \tag{33}$$

Introducing the change $\tau = \gamma x + c$ we get

$$\xi'' = \rho(\tau)\xi, \tag{34}$$

where

$$\begin{aligned}\rho(\tau) \quad = \quad & \frac{\beta^2 - 4\alpha\gamma}{4\gamma^2}\tau^2 - \frac{2a\gamma^2 - 2\alpha c\gamma - \beta b\gamma + \beta^2 c}{2\gamma^2}\tau + \frac{b^2\gamma^2 - 2b\beta c\gamma + \beta^2 c^2}{4\gamma^2} + \\ & \frac{b\gamma - \beta c}{2\tau} + \frac{3\gamma^2}{4\tau^2},\end{aligned}$$

and performing $z = \sqrt[4]{\frac{\beta^2 - 4\alpha\gamma}{4\gamma^2}}\tau$ we arrive to

$$\psi'' = \phi(z)\psi, \quad \phi(z) = z^2 + \delta_1 z + \frac{\delta_1^2}{4} - \delta_2 + \frac{\delta_3}{2z} + \frac{\delta_0^2 - 1}{4z^2}, \tag{35}$$

with δ_i being algebraic functions in a, b, c, α, β and γ. This equation (35) is exactly the *biconfluent Heun equation* whose integrability is analysed in Appendix B.2.

Assuming $\beta = \gamma = 0$ we obtain a Liénard equation which is transformable into a reduced second order differential equation with $r \in \mathbb{C}[x]$ and $\deg(r) = 1$. This means that the equation is not integrable (see [35] and section 3.4). As a particular case, we have a Liénard equation that can be reduced to the Riccati equation given in [51, equation 1.3.2.1],

$$2yy' = (ax + b)y + 1. \tag{36}$$

Moreover, under some restrictions over the parameters of the biconfluent Heun equation one can obtain the Whittaker equation. For instance, the Liénard equation

$$y\frac{dy}{dx} = (\lambda + b_{11}x)\,y + \left(\varepsilon x + b_{20}x^2\right),$$

falls into a Whittaker equation for some special values of the parameters.

Remark 6. We would like to stress the following facts.

(a) It is well-known that via the change $z(x) = \int f(x)dx$ (with inverse $x = x(z)$), the Liénard equation (27) can be led into the equation

$$yy'(z) = y + h(z), \tag{37}$$

with

$$h(z) := \frac{g(x(z))}{f(x(z))}.$$

In a similar way, the change $z(x) = \int g(x)dx$ reduces (27) to

$$yy'(z) = h(z)y + 1, \tag{38}$$

with

$$h(z) := \frac{f(x(z))}{g(x(z))}.$$

However, in general, these transformations do not preserve the differential field of the coefficients. This is why we do not use them to reduce equation (27).

(b) Sometimes equation (27) is called Abel equation of second kind since through the change $y = 1/w$ it is reduced to an Abel equation

$$w' = -f(x)w^2 - g(x)w^3.$$

3.4. **Other families.** Here we consider some special cases of Riccati equations.

a) *Polynomial Riccati equations*
The Riccati equation

$$w' = r(x) - w^2, \quad r(x) \in \mathbb{C}[x]$$

has been studied by several authors (see, for instance [5, 58, 62]). The Galois group of its associated second order linear differential equation is connected and can be either $\mathrm{SL}(2, \mathbb{C})$ or the Borel group, see [1, 2, 5, 6]. In the first case the tangent field associated to its Riccati equation has no invariant curves. In the second one, there is no rational first integral for its vector field. As an example, the reduced form for the *triconfluent Heun equation* is of this type and is given by

$$\xi'' = \rho(x)\xi, \quad \rho(x) = \frac{9x^4}{4} + \frac{3}{2}\delta_2 x^2 - \delta_1 x + \frac{\delta_2^2}{4} - \delta_0. \tag{39}$$

b) *Equations with finite Galois group*

Consider the polynomial Riccati vector field

$$\begin{array}{rcl} \dot{x} & = & -x^2(x-1)^2 \\ \dot{v} & = & a(x-1)^2 + bx^2 + cx(x-1) + x^2(x-1)^2 v^2. \end{array} \tag{40}$$

Its foliation was studied in [60] and is related to the following differential equation

$$\xi'' = -\left(\frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{1-\nu^2+\lambda^2+\mu^2}{4x(x-1)}\right)\xi, \tag{41}$$

where

$$a = \frac{1-\lambda^2}{4}, \quad b = \frac{1-\lambda^2}{4}, \quad c = \frac{1-\lambda^2}{4},$$

being λ, μ and ν the differences of the exponents at 0, 1 and ∞ of the hypergeometric equation, see also Appendix B.

When the equation (41) is integrable their solutions are Legendre functions. For $(\lambda,\mu,\nu)=(\frac{1}{2},\frac{1}{2},\frac{1}{n})$, the solutions of equation (41) are given by

$$\begin{aligned}\xi_1 &= \sqrt[4]{x^2-x}\left(2x-1+2\sqrt{x^2-x}\right)^{\frac{1}{2}\sqrt{\frac{1-2n^2}{n^2}}},\\ \xi_2 &= \sqrt[4]{x^2-x}\left(2x-1+2\sqrt{x^2-x}\right)^{-\frac{1}{2}\sqrt{\frac{1-2n^2}{n^2}}},\end{aligned}$$

and its differential Galois group is the *Dihedral Group* D_n (case 2 of Kovacic algorithm). One can also obtain the tetrahedral, octahedral and icosahedral groups for $(\lambda,\mu,\nu)=(\frac{1}{3},\frac{1}{2},\frac{1}{3})$, $(\lambda,\mu,\nu)=(\frac{1}{3},\frac{1}{2},\frac{1}{4})$ and $(\lambda,\mu,\nu)=(\frac{1}{3},\frac{1}{2},\frac{1}{5})$, respectively (case 3 of Kovacic algorithm).

Related to the vector field (40) we have the vector field X as follows:

$$\begin{aligned}\dot{x} &= -144x(x-1)\\ \dot{v} &= 1-36\nu^2+24(7x-4)v+144x(x-1)v^2.\end{aligned} \tag{42}$$

The vector field X given by (42) is related to the differential equation (see [60])

$$y''+\frac{7x-4}{6x(x-1)}y'-\frac{36\nu^2-1}{144x(x-1)}y=0, \tag{43}$$

where the differential Galois group can be tetrahedral, octahedral and icosahedral depending on the values of ν (indeed, for $1/3, 1/4$ and $1/5$ respectively). Even though explicit solutions of equations (41) and (43) are difficult to get in general, it can be proved the existence of rational first integrals for the associated vector field (42) for values of the parameters $\nu\in\{\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$. For instance, solutions of equation (43) for $\nu=\frac{1}{3}$ are

$$\begin{aligned}y_1 &= \sqrt[4]{-\sqrt[3]{x}+1}\,\sqrt[8]{\frac{2\sqrt{\sqrt[3]{x^2}+\sqrt[3]{x}+1}-\sqrt{3}\,\sqrt[3]{x}-\sqrt{3}}{-\sqrt{3}\,\sqrt[3]{x}-\sqrt{3}-2\sqrt{\sqrt[3]{x^2}+\sqrt[3]{x}+1}}},\\ y_2 &= \sqrt[4]{-\sqrt[3]{x}+1}\,\sqrt[8]{\frac{2\sqrt{\sqrt[3]{x^2}+\sqrt[3]{x}+1}-\sqrt{3}\,\sqrt[3]{x}-\sqrt{3}}{-\sqrt{3}\,\sqrt[3]{x}-\sqrt{3}-2\sqrt{\sqrt[3]{x^2}+\sqrt[3]{x}+1}}}.\end{aligned}$$

In this case, the Galois group is the tetrahedral group. The fourth symmetric power of (43) admits the solutions 1 and $x^{1/3}$. It follows that the associated Riccati vector field X given in (42) admits the algebraic curves

$$f_{4,1}=U^4-6x(x-1)U^2+8x(x-1)^2U-3x^2(x-1)^2,$$

with the cofactor

$$K_{4,1}=48U-480x+192,$$

and

$$f_{4,2}=U^4-(4x-4)U^3+6x(x-1)U^2-4x(x-1)^2U+x(x-4)(x-1)^2,$$

with the cofactor

$$K_{4,2}=-48U-432x+144,$$

being $U=12x(x-1)v$. In this way, the rational first integral of the vector field X (as in (42)) is given by

$$H_{12}=\frac{f_{4,1}^3}{xf_{4,2}^3}.$$

Now, considering the case $\nu = \frac{1}{4}$,we obtain the solutions $y_1 = \dfrac{g_1 g_2}{g_3}$ and $y_2 = \dfrac{g_1}{g_2 g_3}$ where

$$\begin{aligned}
g_1 &= \sqrt[6]{\sqrt[3]{\tfrac{\sqrt{1-x}+1}{\sqrt{1-x}-1}} - 1},\\
g_2 &= \sqrt[8]{-\frac{-2\sqrt{\left(\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}\right)^{2/3}+\sqrt[3]{\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}}+1}+\sqrt{3}\sqrt[3]{\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}}+\sqrt{3}}{\sqrt{3}\sqrt[3]{\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}}+\sqrt{3}+2\sqrt{\left(\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}\right)^{2/3}+\sqrt[3]{\frac{\sqrt{1-x}+1}{\sqrt{1-x}-1}}+1}}},\\
g_3 &= \left(\left(\tfrac{\sqrt{1-x}+1}{\sqrt{1-x}-1}\right)^{2/3} + \sqrt[3]{\tfrac{\sqrt{1-x}+1}{\sqrt{1-x}-1}} + 1\right)^{\frac{1}{12}}.
\end{aligned}$$

In this case, the Galois group is the octahedral group. The 6-th symmetric power of (43) admits the solution 1 and the 8-th symmetric power admits the solution $x^{1/3}$. It follows that the associated Riccati vector field X given by (42) admits the algebraic curves

$$\begin{aligned}
f_6 &= U^6 - 15x(x-1)U^4 + 40x(x-1)^2U^3 - 45x^2(x-1)^2U^2 +\\
&\quad 24x^2(x-1)^3U - (5x-32)x^2(x-1)^3,\\
f_8 &= U^8 - 8(x-1)U^7 + 28x(x-1)U^6 - 56x(x-1)^2U^5 + 14x(5x-8)(x-1)^2U^4 -\\
&\quad 56x^2(x-1)^3U^3 + 28x^2(x+8)(x-1)^3U^2 - 8x^2(x-1)^4(x+32)U +\\
&\quad x^3(x+80)(x-1)^4
\end{aligned}$$

with cofactors

$$\begin{aligned}
K_6 &= 36U - 720x + 288,\\
K_8 &= 48U - 912x + 336,
\end{aligned}$$

respectively, where $U = 24x(x-1)v$.

Again, we obtain a rational first integral of the vector field X, as in (42) given by

$$H_{24} = \frac{f_6^4}{x f_8^3}.$$

Finally, setting $\nu = \frac{1}{5}$, we get the solutions $y_1 = e^{\int \omega_1}$ and $y_2 = e^{\int \omega_2}$, where $\omega_1 = \omega_1(x)$ and $\omega_2 = \omega_2(x)$ are roots of the polynomial $\sum_{k=0}^{12} a_k(x)\omega^k$ and

$$\begin{aligned}
a_0(x) &= 102400 - 11264x - 11x^2\\
a_1(x) &= -3686400x + 3679200x^2 + 7200x^3\\
a_2(x) &= 479001600x^2 - 476863200x^3 - 2138400x^4\\
a_3(x) &= 30412800000x^2 - 60445440000x^3 + 29652480000x^4 + 380160000x^5\\
a_4(x) &= -821145600000x^3 + 1597384800000x^4 - 731332800000x^5 -\\
&\quad 44906400000x^6\\
a_5(x) &= -3695155200000x^4 + 11085465600000x^5 + 3695155200000x^7 -\\
&\quad 11085465600000x^6
\end{aligned}$$

$$
\begin{aligned}
a_6(x) &= -492687360000000x^4 + +1693612800000000x^5 - 2124714240000000x^6 + \\
&\quad 1139339520000000x^7 - \\
&\quad 215550720000000x^8 \\
a_7(x) &= 8868372480000000x^5 - 35473489920000000x^6 + 53210234880000000x^7 - \\
&\quad 35473489920000000x^8 + \\
&\quad 8868372480000000x^9, \\
a_8(x) &= -249422976000000000x^6 + 997691904000000000x^7 - \\
&\quad 1496537856000000000x^8 + 997691904000000000x^9 - \\
&\quad 249422976000000000x^{10} \\
a_9(x) &= -4434186240000000000x^6 + 22170931200000000000x^7 - \\
&\quad 44341862400000000000x^8 + 44341862400000000000x^9 - \\
&\quad 22170931200000000000x^{10} + 4434186240000000000x^{11} \\
a_{10}(x) &= 39907676160000000000x^7 - 199538380800000000000x^8 + \\
&\quad 399076761600000000000x^9 - 399076761600000000000x^{10} + \\
&\quad 199538380800000000000x^{11} - 39907676160000000000x^{12} \\
a_{11}(x) &= 0 \\
a_{12}(x) &= 2176782336000000000000x^8 - 13060694016000000000000x^9 + \\
&\quad 32651735040000000000000x^{10} - 43535646720000000000000x^{11} + \\
&\quad 32651735040000000000000x^{12} - 13060694016000000000000x^{13} + \\
&\quad 2176782336000000000000x^{14}.
\end{aligned}
$$

Riccati vector field X given in (42) admits the algebraic curves f_{12} with cofactor K_{12} and f_{20} with cofactor K_{20}. The first integral is

$$H_{60} = \frac{f_{12}^5}{x f_{20}^3},$$

where

$$
\begin{aligned}
f_{12} &= U^{12} - 66x(x-1)U^{10} + 440x(x-1)^2U^9 - 1485x^2(x-1)^2U^8 \\
&\quad +3168x^2(x-1)^3U^7 - 660x^2(7x-16)(x-1)^3U^6 + 4752x^3(x-1)^4U^5 + \\
&\quad -495x^3(7x+128)(x-1)^4U^4 + 1760x^3(x+80)(x-1)^5U^3 \\
&\quad -594x^4(x+224)(x-1)^5U^2 + 120(x+512)x^4(x-1)^6U \\
&\quad -(11x^2+11264x-102400)x^4(x-1)^6, \text{ being } U = 60x(x-1)v.
\end{aligned}
$$

Expressions for the algebraic curve f_{20}, of degree 20 given by the Kovacic algorithm, and for the cofactors K_{12} and K_{20} are very large and complicated.

Remark 7. Differential Galois Theory and the Kovacic algorithm show that, whenever we are in Case 3 of the Kovacic algorithm, the rational first integrals will be of the above form.

c) *Lamé families.*

Let us consider, in the Lamé equation

$$\frac{d^2y}{dx^2}+\frac{f'(x)}{2f(x)}\frac{dy}{dx}-\frac{n(n+1)x+B}{f(x)}y=0, \tag{44}$$

the particular case where $f(x)=4x^3-g_2x-g_3$ and with parameters n, B, g_2, g_3 such that the discriminant of f, namely $27g_3^2-g_2^3$, is non-zero (see Appendix B). Performing the transformation

$$v=-\frac{y'}{cy}, \tag{45}$$

with $c=c(x)$ any non-zero arbitrary rational function, it follows the family of Riccati equations associated to the Lamé equation,

$$v'=-\frac{n(n+1)x+B}{cf}-\left(\frac{f'(x)}{2f(x)}+\frac{c'}{c}\right)v+cv^2. \tag{46}$$

Theorem 3.1. *For any fixed n in the Lamé equation (44) and any choice of a rational function c in $\mathbb{C}(x)$, the polynomial vector field X associated to the Ricatti equation family (46) has an invariant algebraic curve. The corresponding first integral of X is not rational. Furthermore, the degree of X is unbounded, depending of the choice of c.*

Proof. For the Lamé case (i.1) of B.3 with $B=B_i$ and from Remark 8 it follows the existence of a polynomial solution of equation (46) - even thought the general solution of this equation is not algebraic. It is also clear that for any fixed value of n we have associated Riccati equations (46) with rational coefficients of arbitrary degree since the non-zero rational function $c=c(x)$ is arbitrary as well.

Thus, the Lamé functions are algebraic solutions of (46) and taking $n\in\mathbb{N}$ we obtain algebraic invariants curves

$$v+\frac{E'(x)}{c(x)E(x)}=0,$$

of arbitrary degree, which leads to the result claimed.

□

Appendix.

Appendix A. **Kovacic Algorithm.** This algorithm is devoted to solve the reduced linear differential equation (RLDE) $\xi''=\rho\xi$ and is based on the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$. For more details see [35]. Although improvements for this algorithm are given in [27, 57], we follow the original version given by Kovacic in [35].

Theorem A.1. *Let G be an algebraic subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *Then one of the following four cases can occur.*

1. *G is triangularizable.*
2. *G is conjugate to a subgroup of infinite dihedral group (also called meta-abelian group) and case 1 does not hold.*
3. *Up to conjugation G is one of the following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold.*

4. $G = \mathrm{SL}(2,\mathbb{C})$.

Each case in Kovacic algorithm is related to each one of the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$ and its associated Riccati equation

$$\theta' = r - \theta^2 = \left(\sqrt{r} - \theta\right)\left(\sqrt{r} + \theta\right), \quad \theta = \frac{\xi'}{\xi},\ r = \rho.$$

According to Theorem A.1 we obtain four cases. Only for cases 1, 2 and 3 one can solve the differential equation RLDE and in case 4 one has not Liouvillian solutions for it. Kovacic algorithm can possibly provide one solution (ξ_1), so the second one (ξ_2) can be got through

$$\xi_2 = \xi_1 \int \frac{dx}{\xi_1^2}. \tag{47}$$

Notations. For the RLDE given by

$$\xi'' = \rho\xi = r\xi, \qquad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x],$$

we use:

1. Denote by Γ' be the set of (finite) poles of r, $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
3. By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
4. By the order of r at ∞, $\circ(r_\infty)$, we mean the order of ∞ as a zero of r. That is $\circ(r_\infty) = deg(t) - deg(s)$.

A.1. **The four cases. Case 1.** In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+,-\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0) If $\circ(r_c) = 0$, then
$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1) If $\circ(r_c) = 1$, then
$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2) If $\circ(r_c) = 2$, and
$$r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then}$$
$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3) If $\circ(r_c) = 2v \geq 4$, and
$$r = (a\,(x-c)^{-v} + ... + d\,(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then}$$
$$\left[\sqrt{r}\right]_c = a\,(x-c)^{-v} + ... + d\,(x-c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$

(∞_1) If $\circ(r_\infty) > 2$, then
$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then
$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3) If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + ... + d)^2 + bx^{v-1} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_\infty = ax^v + ... + d, \quad \text{and} \quad \alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : m = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c\in\Gamma'} \alpha_c^{\varepsilon(c)}, \forall \left(\varepsilon(p)\right)_{p\in\Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\#D > 0$, then for each $m \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon(\infty)\left[\sqrt{r}\right]_\infty + \sum_{c\in\Gamma'}\left(\varepsilon(c)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1}\right).$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m with

$$P_m'' + 2\omega P_m' + (\omega' + \omega^2 - r)P_m = 0.$$

If one successes then $\xi_1 = P_m e^{\int\omega}$ is a solution of the differential equation RLDE. Else, Case 1 cannot hold.

Case 2. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows.

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{4\}$.
(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(c_3) If $\circ(r_c) = v > 2$, then $E_c = \{v\}$.
(∞_1) If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$.
(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(∞_3) If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$.

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : \quad m = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma'} e_c\right), \forall e_p \in E_p,\ p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\#D > 0$, then for each $m \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2}\sum_{c\in\Gamma'} \frac{e_c}{x-c}.$$

Step 3. For each $m \in D$, search a monic polynomial P_m of degree m, such that

$$P_m''' + 3\theta P_m'' + (3\theta' + 3\theta^2 - 4r)P_m' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)P_m = 0.$$

If P_m does not exist, then Case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + P'/P$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\phi' + \phi^2 - 2r\right) = 0.$$

Then $\xi_1 = e^{\int \omega}$ is a solution of the differential equation RLDE.

Case 3. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{12\}$.
(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{6 + k\sqrt{1+4b} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}.$$

(∞) If $\circ(r_\infty) = v \geq 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}, \qquad n \in \{4, 6, 12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{m \in \mathbb{Z}_+ : \quad m = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma'} e_c\right), \forall e_p \in E_p,\ p \in \Gamma\right\}.$$

In this case we start with $n = 4$ to obtain the solution, afterwards $n = 6$ and finally $n = 12$. If $D = \emptyset$, then the differential equation has not Liouvillian solution because it falls in the case 4. Now, if $\#D > 0$, then for each $m \in D$ with its respective n, search a rational function

$$\theta = \frac{n}{12}\sum_{c\in\Gamma'} \frac{e_c}{x-c},$$

and a polynomial S defined as

$$S = \prod_{c\in\Gamma'}(x-c).$$

Step 3. Search for each $m \in D$, with its respective n, a monic polynomial $P_m = P$ of degree m, such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_n = -P,$$

$$P_{i-1} = -SP_i' - \left((n-i)\,S' - S\theta\right)P_i - (n-i)\,(i+1)\,S^2 r P_{i+1},$$

where $i \in \{0, 1 \ldots, n-1, n\}$. If P does not exist, then the differential equation has not Liouvillian solution because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{n} \frac{S^i P}{(n-i)!}\omega^i = 0,$$

then a solution of the differential equation the RLDE is given by

$$\xi = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree n.

Appendix B. Some Special Functions.

B.1. Hypergeometric families.

B.1.1. *Kimura's Theorem.* The hypergeometric (or Riemann) equation is the more general second order linear differential equation over the Riemann sphere with three regular singular singularities. If we place the singularities at $x = 0, 1, \infty$ it is given by

$$\frac{d^2y}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{dy}{dx} + \left(\frac{\alpha\alpha'}{x^2} + \frac{\gamma\gamma'}{(x-1)^2} + \frac{\beta\beta'-\alpha\alpha'\gamma\gamma'}{x(x-1)}\right)y = 0, \tag{48}$$

where (α, α'), (γ, γ'), (β, β') are the exponents at the singular points and must satisfy the Fuchs relation $\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1$.

Now, we will briefly describe Kimura's Theorem that provides necessary and sufficient conditions for the integrability of the hypergeometric equation. Let be $\lambda = \alpha - \alpha'$, $\mu = \beta - \beta'$ and $\nu = \gamma - \gamma'$.

Theorem B.1 (Kimura, [34]). *The hypergeometric equation (48) is integrable if and only if either*

(i) At least one of the four numbers $\lambda + \mu + \nu$, $-\lambda + \mu + \nu$, $\lambda - \mu + \nu$, $\lambda + \mu - \nu$ is an odd integer, or

(ii) The numbers λ or $-\lambda$, μ or $-\mu$ and ν or $-\nu$ belong (in an arbitrary order) to some of the following fifteen families

1	$1/2+l$	$1/2+m$	*arbitrary complex number*	
2	$1/2+l$	$1/3+m$	$1/3+q$	
3	$2/3+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
4	$1/2+l$	$1/3+m$	$1/4+q$	
5	$2/3+l$	$1/4+m$	$1/4+q$	$l+m+q$ *even*
6	$1/2+l$	$1/3+m$	$1/5+q$	
7	$2/5+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
8	$2/3+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
9	$1/2+l$	$2/5+m$	$1/5+q$	$l+m+q$ *even*
10	$3/5+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
11	$2/5+l$	$2/5+m$	$2/5+q$	$l+m+q$ *even*
12	$2/3+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
13	$4/5+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
14	$1/2+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*
15	$3/5+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*

Here l, m and q are integers.

B.1.2. *Confluent hypergeometric.* The *confluent Hypergeometric equation* is a degenerate form of the Hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. The following are two classical forms.

- *Kummer's* form

$$y'' + \frac{c-x}{x}y' - \frac{a}{x}y = 0, \qquad a, c \in \mathbb{C} \tag{49}$$

- *Whittaker's* form

$$y'' = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2}\right) y, \tag{50}$$

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$. Furthermore, using the expression (10), we can see that the Whittaker's equation is the reduced form of the Kummer's equation (49). The Galoisian structure of these equations has been deeply studied in [46, 27].

Theorem B.2 (Martinet & Ramis, [46])**.** *The Whittaker's differential equation (50) is integrable if and only if either, $\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, or $\kappa - \mu \in \frac{1}{2} + \mathbb{N}$, or $-\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, or $-\kappa - \mu \in \frac{1}{2} + \mathbb{N}$.*

The *Bessel's equation* is a particular case of the confluent Hypergeometric equation and is given by

$$y'' + \frac{1}{x} y' + \frac{x^2 - n^2}{x^2} y = 0. \tag{51}$$

Under a suitable transformation, the reduced form of the Bessel's equation is a particular case of the Whittaker's equation (50).

Corollary 2. *The Bessel's differential equation (51) is integrable if and only if $n \in \frac{1}{2} + \mathbb{Z}$.*

B.2. **Heun's families.** The Heun's equation is the generic differential equation with four regular singular points at 0, 1, c and ∞. In its reduced form, the Heun's equation is $y'' = r(x)y$, where

$$r(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-c} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-c)^2}, \tag{52}$$

$$A = -\frac{\alpha\beta}{2} - \frac{\alpha\gamma}{2c} + \frac{\delta\eta h}{c}, \quad B = \frac{\alpha\beta}{2} - \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(h-1)}{c-1},$$

$$C = \frac{\alpha\gamma}{2c} + \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(c-h)}{c(c-1)}, \quad D = \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right), \quad E = \frac{\beta}{2}\left(\frac{\beta}{2} - 1\right),$$

$$F = \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right), \quad \text{with} \quad \alpha + \beta + \gamma - \delta - \eta = 1.$$

To our purposes we write the determinant $\Pi_{d+1}(a, b, u, v, \xi, w)$ as in [27]:

$$\begin{vmatrix} w & u & 0 & 0 & 0 & \dots & 0 \\ d\xi w + 1 & v & 2(u+b) & 0 & 0 & \dots & 0 \\ 0 & (d-1)\xi & w + 2(v+a) & 3(u+2b) & 0 & \dots & 0 \\ 0 & 0 & (d-2)\xi & w + 3(v+2a) & 4(u+3b) & \dots & 0 \\ \vdots & & & & & \dots & \\ 0 & \dots & & \dots & 2\xi & w + (d-1)(v+(d-2)a) & d(u+(d-1)b) \\ 0 & \dots & & \dots & 0 & \xi & w + d(v+(d-1)a) \end{vmatrix}$$

B.2.1. *Biconfluent Heun.* The equation

$$\xi'' = \left(x^2 + \delta_1 x + \frac{\delta_1^2}{4} - \delta_2 + \frac{\delta_3}{2x} + \frac{\delta_0^2 - 1}{4x^2}\right)\xi, \tag{53}$$

is the well known *biconfluent Heun equation* which has been deeply analyzed by Duval and Loday-Richaud in [27, p. 236].

Theorem B.3. [27]. *The biconfluent Heun equation* (53) *has Liouvillian solutions if and only if it falls in Case 1 of Kovacic algorithm and one of the following conditions is fulfilled.*

1. $\delta_0^2 = 1$, $\delta_3 = 0$ *and* $\delta_2 \in 2\mathbb{Z}+1$.
2. $\delta_0^2 = 1$, $\delta_3 \neq 0$ *and* $\delta_2 \in 2\mathbb{Z}^*+1$ *with* $|\delta_2| \geq 3$, *and if* $\varepsilon = \mathsf{sign}\,\delta_2$, *then*
$$\Pi_{(|\delta_2|-1)/2}\left(0,1,2,\varepsilon\delta_1,-2\varepsilon,\varepsilon\delta_1-\frac{\delta_3}{2}\right)=0.$$
3. $\delta_0 \neq \pm 1$, $\pm\delta_0 \pm \delta_2 \in 2\mathbb{Z}^*$ *and if* $\varepsilon_0, \varepsilon_\infty \in \{\pm 1\}$ *are such that* $\varepsilon_\infty\delta_2 - \varepsilon_0\delta_0 = 2d^* \in 2\mathbb{N}^*$ *then*
$$\Pi_{d^*}\left(0,1,1+\varepsilon_0\delta_0,\varepsilon_\infty\delta_1,-2\varepsilon_\infty,\frac{1}{2}(\varepsilon_\infty\delta_1(1+\varepsilon_0\delta_0)-\delta_3)\right)=0.$$

B.3. **Lamé equation.** The algebraic form of the Lamé Equation is [52, 61]

$$\frac{d^2y}{dx^2}+\frac{f'(x)}{2f(x)}\frac{dy}{dx}-\frac{n(n+1)x+B}{f(x)}y=0, \tag{54}$$

where $f(x) = 4x^3 - g_2x - g_3$, with n, B, g_2 and g_3 parameters such that the discriminant of f, $\Delta = 27g_3^2 - g_2^3$, is non-zero and, therefore, it has no multiple roots. This equation is a Fuchsian differential equation with four singular points over the Riemann sphere: the roots e_1, e_2, e_3 of f and the point of the infinity.

The mutually-exclusive known cases of solutions in closed form of the Lamé equation (54) are the following.

(i) The *Lamé-Hermite* case (see [52, 61]). We have $n \in \mathbb{N}$ and arbitrary parameters B, g_2 and g_3.

(ii) The *Brioschi-Halphen-Crawford* case (see [28, 52]). We have $n \in \mathbb{N}$ such that $m := n + \frac{1}{2} \in \mathbb{N}$ and parameters B, g_2 and g_3 satisfying an algebraic condition $Q_m(g_2/4, g_3/4, B) = 0$, where $Q_m \in \mathbb{Z}[g_2/4, g_3/4, B]$ is a polynomial of degree m in B, known as the *Brioschi determinant.*

(iii) The *Baldassarri* case (see [9]). One asks n to satisfy that $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} - \mathbb{Z}$ besides some additional (involved) algebraic restrictions on the other parameters.

It is possible to prove that the only integrable cases of the Lamé equation are cases (i)–(iii) above, see [47]. In (ii) and (iii) the general solution of (54) is algebraic and the Galois group is finite. Case (i) splits in the following two subcases [52, 61].

(i.1) The Lamé case. For a fixed integer n, this equation admits a solution (called *Lamé function*) of the form

$$E(x)=\prod_{i=1}^{3}(x-e_i)^{k_i}P_m(x), \tag{55}$$

being P_m a monic polynomial of degree $m = n/2 - (k_1 + k_2 + k_3)$ and $k_i \in \left\{0, \frac{1}{2}\right\}$, $i = 1, 2, 3$. Since $m \in \mathbb{N}$, eight different possibilities regarding n appear: If n is even, we have $k_1 = k_2 = k_3 = 0$ or just one zero k_i; if n is odd, we could have all non-zero k_i's or combinations with exactly one non-zero k_i. All these possibilities give rise to classes of Lamé functions. Concerning parameter B, it must be one of the $m + 1$ different roots $B_1, \ldots, B_{m+1}$ of certain irreducible polynomial of degree $m + 1$, with all its roots real and simple [28]. Furthermore, the numbers B_i are reals.

(i.2) The Hermite case. Here we are not in case (i.1) and n is an arbitrary natural number. We are also fall in case 1 of Kovacic algorithm, but with a diagonal Galois group.

Remark 8. We notice that the polynomial P_m in (i.1) satisfies a second order linear differential equation similar to the one that appears in the first case of Kovacic algorithm. In fact it is possible to obtain the above passing into normal form and applying Kovacic algorithm. Therefore the second linear independent solution is not algebraic and the associated Riccati equation has no rational first integral.

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