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Power–Factor Compensation is Equivalent to Cyclodissipassivation

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Abstract— **In a previous work [1] we have identified the key role played by the concept of *cyclodissipativity* in the solution of the power–factor–compensation problem for electrical circuits with general nonlinear loads and operating in nonsinusoidal regimes. Namely, we have shown that a necessary condition for a (shunt) compensator to improve the power transfer is that the overall system satisfies a given cyclodissipativity property. In this work, we extend the results of [1] proving that cyclodissipativity is actually *necessary and sufficient* for power–factor improvement. We prove in this way that cyclodissipativity provides a rigorous mathematical framework useful to analyze and design power–factor compensators. Moreover, we give an energy equalization interpretation of the power–factor–compensation problem.**

I. INTRODUCTION

Optimizing energy transfer from an ac source to a load is a classical problem in electrical engineering. In practice, the efficiency of this transfer is typically reduced due to the phase shift between voltage and current at the fundamental frequency. The phase shift arises largely due to energy flows characterizing electric motors that dominate the aggregate load. The *power factor*, defined as the ratio between the real or active power (average of the instantaneous power) and the apparent power (the product of rms values of the voltage and current), then captures the energy–transmission efficiency for a given load. The standard approach to improving the power factor is to place a compensator between the source and the load. To design the compensator it is typically assumed that the equivalent source consists of an ideal generator having zero Thevenin impedance and producing a fixed, purely sinusoidal voltage, see [2]. If the load is linear time invariant (LTI), the resulting steady–state current is a shifted sinusoid, and the power factor is the cosine of the phase–shift angle. Power–factor compensation is then achieved by modifying the circuit to reduce the phase shift between the source voltage and the current.

In the LTI sinusoidal case, a fundamental energy–equalization mechanism underlies the phase–shifting action of power–factor compensation. Indeed, it can be shown that the power factor is improved if and only if the difference between the average electric and magnetic energies stored in the circuit is reduced. The optimal power factor is achieved when electric and magnetic energies are equal, which occurs when the impedance seen from the source behaves like a resistor for the source frequency. Unfortunately, standard textbook presentations [2]–[4] do not explain the power–factor compensation in terms of energy equalization, but rather rely on an axiomatic definition of *reactive* power, which in the LTI sinusoidal case, turns out to be proportional to the energy difference mentioned above, and thus reactive–power reduction is tantamount to energy equalization.

In this work, we prove that a necessary and sufficient condition for power factor improvement is is that the overall system satisfies a given cyclodissipativity property [5]. In the spirit of standard passivation [6], this result leads naturally to a formulation of the power–factor–compensation problem as one of rendering the load cyclodissipative. We prove in this way that cyclodissipativity provides a rigorous mathematical framework useful to analyze and design power factor compensators for general nonlinear loads operating in nonsinusoidal regimes.

II. POWER FACTOR COMPENSATION

We consider the classical scenario of energy transfer from an n–phase ac generator to a load as depicted in Figure 1. Throughout this article, lower case boldface letters denote column vectors, while upper case boldface letters denote matrices. The voltage and current of the source are denoted by the column vectors $\boldsymbol{v}_s, \boldsymbol{i}_s \in \mathbb{R}^n$, while the load is described by a possibly nonlinear, time–varying n–port system $\boldsymbol{\Sigma}$. We formulate the power–factor–compensation problem as follows:

C.1) $\boldsymbol{v}_s \in \mathcal{V}_s \subseteq \mathcal{L}_2^n[0,T) := \{\boldsymbol{x} : [0,T) \to \mathbb{R}^n \ : \ \|\boldsymbol{x}\|^2 := \frac{1}{T}\int_0^T |\boldsymbol{x}(\tau)|^2 \ d\tau < \infty\}$, where $\|\cdot\|$ is the *rms value* and $|\cdot|$ is the Euclidean norm. Depending on the context, the set $\mathcal{V}_s$ may be equal to $\mathcal{L}_2^n[0,T)$ or it may consist of a single periodic signal $\boldsymbol{v}_s(t) = \boldsymbol{v}_s(t+T)$ or a set of sinusoids with limited harmonic content, for example, $v_s(t) = V_s \sin\omega_0 t$, where $\omega_0 \in [\omega_0^m, \omega_0^M] \subset [0,\infty)$.

C.2) The power–factor–compensation configuration is depicted in Fig. 2, where $\mathbf{Y}_c, \mathbf{Y}_\ell : \mathcal{V}_s \to \mathcal{L}_2^n[0,T)$ are the admittance operators of the compensator and the load, respectively. That is, $\mathbf{Y}_c : \boldsymbol{v}_s \mapsto \boldsymbol{i}_c$ and $\mathbf{Y}_\ell : \boldsymbol{v}_s \mapsto \boldsymbol{i}_\ell$, where $\boldsymbol{i}_c, \boldsymbol{i}_\ell \in \mathbb{R}^n$ denote the compensator and load currents, respectively. In the simplest LTI case the operators $\mathbf{Y}_c, \mathbf{Y}_\ell$ can be described by their admittance transfer matrices, which we denote by $\hat{\mathbf{Y}}_c(s), \hat{\mathbf{Y}}_\ell(s) \in \mathbb{R}^{n\times n}(s)$, where $s \in \mathbb{C}$.

C.3) The power factor compensator is lossless, that is,

$$\langle \boldsymbol{v}_s, \mathbf{Y}_c \boldsymbol{v}_s \rangle = 0, \quad \forall\ \boldsymbol{v}_s \in \mathcal{V}_s, \tag{1}$$

where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \frac{1}{T}\int_0^T \boldsymbol{x}^\top(t)\ \boldsymbol{y}(t)\ dt$ is the inner product in $\mathcal{L}_2^n[0,T)$.

We make the following fundamental assumption throughout the work:

Assumption 1: The source is ideal, in the sense that $\boldsymbol{v}_s$ remains unchanged for all loads $\boldsymbol{\Sigma}$.

The standard definition of power factor [3] is given as follows:

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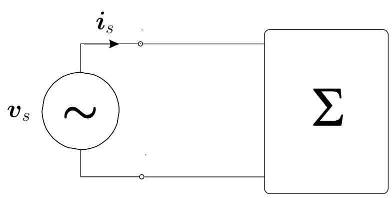

Fig. 1. Circuit schematic of a polyphase ac system.

Definition 1: The power factor of the source is defined by

$$\mathrm{PF} := \frac{\langle \boldsymbol{v}_s, \boldsymbol{i}_s \rangle}{\|\boldsymbol{v}_s\| \|\boldsymbol{i}_s\|}, \tag{2}$$

where $P := \langle \boldsymbol{v}_s, \boldsymbol{i}_s \rangle$ is the active (real) power and the product $S := \|\boldsymbol{v}_s\| \|\boldsymbol{i}_s\|$ is the apparent power.

From (2) and the Cauchy–Schwarz inequality it follows that $P \leq S$. Hence $PF \in [-1, 1]$ is a dimensionless measure of the energy–transmission efficiency. Indeed, under Assumption 1, the apparent power S is the highest average power delivered to the load among all loads that have the same rms current $\|\boldsymbol{i}_s\|$. The apparent power equals the active power if and only if $\boldsymbol{v}_s$ and $\boldsymbol{i}_s$ are collinear. If this is not the case, $P < S$ and compensation schemes are introduced to maximize power factor.

Definition 2: Power–factor improvement is achieved with the compensator $\mathbf{Y}_c$ if and only if

$$\mathrm{PF} > \mathrm{PF}_u := \frac{\langle \boldsymbol{v}_s, \boldsymbol{i}_\ell \rangle}{\|\boldsymbol{v}_s\| \|\boldsymbol{i}_\ell\|}, \tag{3}$$

where PF_u denotes the uncompensated power factor, that is, the value of PF with $\mathbf{Y}_c = 0$.

Remark 1: We assume that all signals in the system are periodic, with fundamental period T and belong to the space $\mathcal{L}_2^n[0, T)$. However, as becomes clear below, all derivations remain valid if we replace $\mathcal{L}_2^n[0, T)$ by the set of square–integrable functions $\mathcal{L}_2^n[0, \infty)$. Hence, periodicity is not essential for our developments. Restricting our analysis to $\mathcal{L}_2^n[0, T)$ captures the practically relevant scenario in which, for most power–factor–compensation problems of interest, the system operates in a periodic, though not necessarily sinusoidal, steady state.

Remark 2: Assumption 1 is tantamount to saying that the source has no impedance, which is justified by the fact that most ac power devices are designed and operated in this manner. For ease of presentation and without loss of generality, we also assume $\langle \boldsymbol{v}_s, \boldsymbol{i}_s \rangle \geq 0$, which indicates that real (active) power is always delivered from the source to the load.

Remark 3: The role of power factor as an indicator of energy–transmission efficiency is usually explained in textbooks as follows [3]. In view of periodicity we can express the qth phase component of the terminal variables in terms of their (exponential) Fourier series as $v_{s_q}(t) = \sum_{k=-\infty}^{\infty} \hat{V}_{s_q}(k) \exp(jk\omega_0 t)$, where $\omega_0 := 2\pi/T$ is the fundamental frequency and, for integers k, $\hat{V}_{s_q}(k) := \frac{1}{T}\int_0^T v_{s_q}(t) \exp(-jk\omega_0 t)dt$, are the Fourier coefficients of the qth phase element of the voltage, also called spectral lines or harmonics. Similar expressions are obtained for the qth phase components of the current vector $\boldsymbol{i}_s$. Because the product of sinusoidal variables of different frequencies integrated over a common period is zero, only components of $\boldsymbol{v}_s$ and $\boldsymbol{i}_s$ that are of the *same frequency* contribute to the average power P. However, if the current is distorted, the rms value of $\boldsymbol{i}_s$ can exceed the rms value of the sum of the current components in phase with the voltage. In this case, the source may not deliver its rated power, although it may deliver its rated rms current.

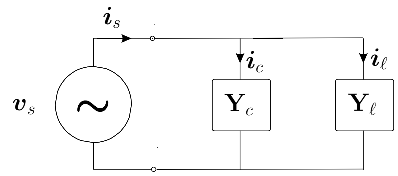

Fig. 2. Typical compensation configuration.

III. A CYCLODISSIPATIVITY CHARACTERIZATION OF POWER–FACTOR COMPENSATION

In this section we prove that power factor is improved if and only if the compensated system satisfies a given cyclodissipativity property. A corollary of this result is the (operator theoretic) characterization of all compensators that improve the power factor. Finally, we show that, as in the LTI sinusoidal case, a phase–shifting interpretation of power factor compensation is possible. To formulate our results we need the following.

Definition 3: The n–port system of Fig. 2 is cyclodissipative with respect to the supply rate $w(\boldsymbol{v}_s, \boldsymbol{i}_s)$, where $w : \mathcal{V}_s \times \mathcal{L}_2^n[0, T) \to \mathbb{R}$, if and only if

$$\int_0^T w(\boldsymbol{v}_s(t), \boldsymbol{i}_s(t))dt > 0 \tag{4}$$

for all $(\boldsymbol{v}_s, \boldsymbol{i}_s) \in \mathcal{V}_s \times \mathcal{L}_2^n[0, T)$.

Proposition 1: Consider the system of Fig. 2 with fixed $\mathbf{Y}_\ell$. The compensator $\mathbf{Y}_c$ improves the *power factor* if and only if the system is cyclodissipative with respect to the supply rate

$$w(\boldsymbol{v}_s, \boldsymbol{i}_s) = (\mathbf{Y}_\ell \boldsymbol{v}_s + \boldsymbol{i}_s)^\top (\mathbf{Y}_\ell \boldsymbol{v}_s - \boldsymbol{i}_s). \tag{5}$$

Proof: From Kirchhoff's current law, $\boldsymbol{i}_s = \boldsymbol{i}_c + \boldsymbol{i}_\ell$, the relation $\boldsymbol{i}_c = \mathbf{Y}_c \boldsymbol{v}_s$, and the lossless condition (1) we have that $\langle \boldsymbol{v}_s, \boldsymbol{i}_s \rangle = \langle \boldsymbol{v}_s, \boldsymbol{i}_\ell \rangle$. Consequently, (2) becomes $\mathrm{PF} = (\langle \boldsymbol{v}_s, \boldsymbol{i}_\ell \rangle)/(\|\boldsymbol{v}_s\| \|\boldsymbol{i}_s\|)$, and (3) holds if and only if

$$\|\boldsymbol{i}_s\|^2 < \|\mathbf{Y}_\ell \boldsymbol{v}_s\|^2 \tag{6}$$

where we use $\boldsymbol{i}_\ell = \mathbf{Y}_\ell \boldsymbol{v}_s$. Finally, note that (4) with (5) is equivalent to (6), which yields the desired result. ∎

Corollary 1: Consider the system of Fig. 2. Then $\mathbf{Y}_c$ improves the power factor for a given $\mathbf{Y}_\ell$ if and only if $\mathbf{Y}_c$ satisfies

$$2\langle \mathbf{Y}_\ell \boldsymbol{v}_s, \mathbf{Y}_c \boldsymbol{v}_s \rangle + \|\mathbf{Y}_c \boldsymbol{v}_s\|^2 < 0, \quad \forall\, \boldsymbol{v}_s \in \mathcal{V}_s. \tag{7}$$

Dually, given $\mathbf{Y}_c$, the power factor is improved for all $\mathbf{Y}_\ell$ that satisfy (7).

Proof: Substituting $\boldsymbol{i}_s = (\mathbf{Y}_\ell + \mathbf{Y}_c)\boldsymbol{v}_s$ in (6) yields (7) ∎

To provide a phase–shift interpretation of power–factor compensation, Fig. 3 depicts the vector signals $\boldsymbol{v}_s, \boldsymbol{i}_s, \boldsymbol{i}_\ell$, and $\boldsymbol{i}_c$, where the angles θ and θ_u are understood in the sense of the inner product, as defined below. Note that the lossless condition (1) imposes $\langle \boldsymbol{i}_c, \boldsymbol{v}_s \rangle = 0$. Replacing $\boldsymbol{i}_\ell = \mathbf{Y}_\ell \boldsymbol{v}_s$ and $\boldsymbol{i}_c = \mathbf{Y}_c \boldsymbol{v}_s$ in the power–factor–improvement condition (7) yields

$$\|\boldsymbol{i}_c\|^2 + 2\langle \boldsymbol{i}_c, \boldsymbol{i}_\ell \rangle < 0, \tag{8}$$

which is equivalent to $\|\boldsymbol{i}_c\| < 2\Delta$, where the distance Δ is defined by $\Delta := -\langle \boldsymbol{i}_\ell, \boldsymbol{i}_c \rangle / \|\boldsymbol{i}_c\| > 0$. On the other hand, it is clear from Fig. 3 that $\|\boldsymbol{i}_c\| < 2\Delta$ if and only if $\theta < \theta_u$. The

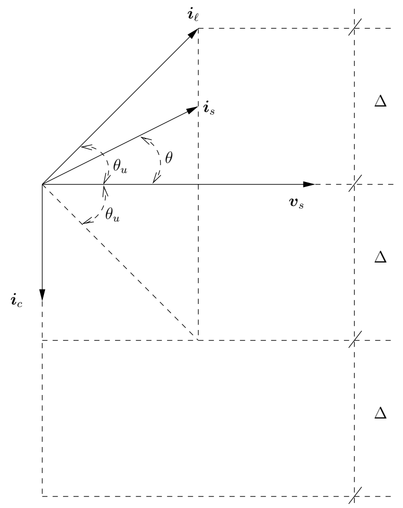

Fig. 3. Phase–shift interpretation of the power factor compensation.

equivalence between power–factor improvement and $\theta < \theta_u$ follows directly from the fact that

$$\theta := \cos^{-1} PF, \quad \theta_u := \cos^{-1} PF_u, \tag{9}$$

Notice that these functions are well defined and, furthermore, because of the unidirectional energy–transfer assumption, it follows that $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\theta_u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark 4: Readers familiar with the power–factor–compensation problem may find the statements above to be self–evident. Indeed, under Assumption 1, power–factor improvement is equivalent to reduction of the rms value of the source current. Now, using $\boldsymbol{i}_s = \boldsymbol{i}_c + \boldsymbol{i}_\ell$ to compute the rms value of $\boldsymbol{i}_s$ yields

$$\|\boldsymbol{i}_s\|^2 = \|\boldsymbol{i}_\ell\|^2 + \|\boldsymbol{i}_c\|^2 + 2\langle \boldsymbol{i}_c, \boldsymbol{i}_\ell \rangle. \tag{10}$$

It is clear from (10) that a necessary and sufficient condition for reducing $\|\boldsymbol{i}_s\|$ from its uncompensated rms value $\|\boldsymbol{i}_\ell\|$ is precisely (8), which, as shown in Proposition 1 is equivalent to power–factor improvement.

Remark 5: Definition 3 of cyclodissipativity is not standard, but captures the essence of the property introduced in [5], [7] for systems with a state realization. In other words, a system is cyclodissipative if it cannot create "generalized energy" over closed paths. In our case, these paths are defined for port signals, while these paths are typically associated with state trajectories. The system might, however, produce energy along some initial portion of a closed path; if so, the system would not be dissipative. Clearly, every dissipative system is cyclodissipative, stemming from the fact that in the latter case we restrict the set of inputs of interest to those inputs that generate periodic trajectories, a feature that is intrinsic in the version of the power–factor–compensation problem we are considering.

IV. POWER FACTOR COMPENSATION IN THE LTI SCALAR SINUSOIDAL CASE

We now specialize the above derivations to the case in which $n = 1$, $v_s(t) = V_s \sin \omega_0 t$, where $\omega_0 \in [\omega_0^m, \omega_0^M] \subset [0, \infty)$, and the scalar LTI stable operators Y_ℓ, Y_c are described by their admittance transfer functions $\hat{Y}_\ell(j\omega_0)$ and $\hat{Y}_c(j\omega_0)$, respectively. In this case, the steady–state source current is $i_s(t) = I_s \sin(\omega_0 t + \theta)$, where $I_s := V_s|\hat{Y}_\ell(j\omega_0) + \hat{Y}_c(j\omega_0)|$ and $\theta := \measuredangle\{\hat{Y}_\ell(j\omega_0) + \hat{Y}_c(j\omega_0)\}$. Simple calculations confirm that θ and the uncompensated angle $\theta_u := \measuredangle\{\hat{Y}_\ell(j\omega_0)\}$ coincide with (9). We also have the following simple property.

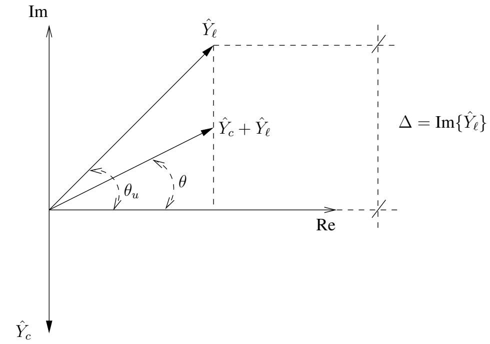

Fig. 4. Power factor compensation in the LTI case.

Lemma 1: The scalar LTI operator Y_c is lossless if and only if $Re\{\hat{Y}_c(j\omega)\} = 0$ for all $\omega \in [0, \infty)$.

Proof: From Parseval's theorem we have

$$\begin{aligned} \langle v_s, Y_c v_s \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}_s(-j\omega)\hat{Y}_c(j\omega)\hat{V}_s(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\{Y_c(j\omega)\}|\hat{V}_s(j\omega)|^2 d\omega, \end{aligned}$$

where, to obtain the second identity, we use the fact that $\mathrm{Im}\{\hat{Y}_c(j\omega)\}$ is an odd function of ω. ■

Proposition 2: In the LTI scalar sinusoidal case, the *power factor* is improved if and only if

$$\frac{Im\{\hat{Y}_\ell(j\omega_0)\}}{Im\{\hat{Y}_c(j\omega_0)\}} < -\frac{1}{2}, \quad \forall\, \omega_0 \in [\omega_0^m, \omega_0^M]. \tag{11}$$

Proof: In this case, the signal space of Fig. 3 can be replaced by the complex plane with the admittances' frequency responses taking the place of the signals, as indicated in Fig. 4. Notice that, because of Lemma 1, $Y_c(j\omega_0)$ is purely imaginary. From basic geometric considerations, we see that $\theta < \theta_u$ if and only if (11) holds. ■

Remark 6: The equivalence between power–factor improvement and $\theta < \theta_u$ is a restatement of the fact that energy–transmission efficiency is improved by reducing the phase shift between the source voltage and current waveforms, a statement that can be found in standard circuits textbooks. However, the explicit characterization (11) does not seem to be widely known.

Remark 7: The action of a power–factor compensator is explained above without resorting to the axiomatic definition of complex power used in textbooks to introduce the notion of reactive power. In contrast with our geometric perspective of power–factor compensation, this mathematical construction cannot easily be extended to the nonlinear nonsinusoidal case. Furthermore, the mathematical background used in the above derivations is elementary.

Remark 8: For clarity the above analysis is restricted to the scalar case, that is, $n = 1$. Similar derivations can easily be carried out for n–phase systems. For instance, if $\hat{\mathbf{Y}}_c(s)$ is diagonal, power–factor improvement is equivalent to

$$\left[\mathrm{Im}\{\hat{Y}_c(j\omega_0)\}\right]^{-1} \mathrm{Im}\{\hat{Y}_\ell(j\omega_0)\} < -\frac{1}{2} I_n, \ \forall\ \omega_0 \in [\omega_0^m, \omega_0^M].$$

V. POWER–FACTOR COMPENSATION WITH LTI CAPACITORS AND INDUCTORS

Corollary 1 identifies all load admittances for which the source power factor is improved with a given compensator, namely, those load admittances that satisfy inequality (7). In this section we further explore this condition for LTI capacitive and inductive compensation. For simplicity we assume throughout the section that the system is single phase, that is $n = 1$, but the load is possibly *nonlinear*.

Proposition 3: Consider the system of Fig. 2 with $n = 1$ and fixed LTI capacitor compensator with admittance $\hat{Y}_c(s) = C_c s$, where $C_c > 0$. The following statements are equivalent:

(i) There exist $C_{max} > 0$ such that the load is cyclodissipative with supply rate

$$w_C(\dot{v}_s, i_\ell) = -2 i_\ell \dot{v}_s - C_{max}\dot{v}_s^2. \tag{12}$$

(ii) For all C_c satisfying $0 < C_c < C_{max}$, the power factor is improved.

Proof: Assume (i) holds. Integrating $w_C(\dot{v}_s, i_\ell)$ and using Definition 3 yields the cyclodissipation inequality

$$2\langle i_\ell, \dot{v}_s\rangle + C_{max}\|\dot{v}_s\|^2 \leq 0. \tag{13}$$

Note that (13) implies that $2\langle i_\ell, C_c\dot{v}_s\rangle + \|C_c\dot{v}_s\|^2 \leq 0$ for all $0 < C_c < C_{max}$. The latter is the condition for power–factor improvement (7) for the case at hand. The converse proof is established by reversing these arguments. ■

A similar proposition can be established for inductive compensation. In contrast with the upper bound given for C_c, a lower bound on the inductance L_c is imposed. Furthermore, an assumption on v_s is needed to ensure absolute integrability of the supply rate.

Proposition 4: Consider the system of Fig. 2 with $n = 1$ and a fixed LTI inductor compensator with admittance $\hat{Y}_c(s) = \frac{1}{L_c s}$, where $L_c > 0$. Assume v_s has no dc component. The following statements are equivalent:

(i) The load is cyclodissipative with supply rate

$$w_L(z, i_\ell) = -2L_{min} i_\ell z - z^2, \tag{14}$$

for some constant $L_{min} > 0$ and $\dot{z} = v_s$.

(ii) For all $L_c > L_{min}$, the power factor is improved.

Proposition 3 (resp., 4) states that the power factor of a load can be improved with a capacitor (resp., inductor) if and only if it is cyclodissipative with supply rate (12) [resp., (14)]. This result constitutes an extension, to the nonlinear nonsinusoidal case, of the definition of the inductive (resp., capacitive) loads.

VI. ENERGY EQUALIZATION AND POWER–FACTOR COMPENSATION

We now explore connections between LTI LC power–factor compensation and energy equalization, where the latter is understood in the sense of reducing the difference between the stored magnetic and electrical energies of the circuit. We study conditions for load cyclodissipativity, which is established in Propositions 3 and 4 as equivalent to power–factor improvement. Results on cyclodissipativity of nonlinear RLC circuits are summarized in [8]. It is shown in [9] that general n–port nonlinear RL (respectively, RC) circuits with convex energy functions are cyclodissipative with supply rate $i_\ell \dot{v}_s$ (respectively, $v_s \frac{d}{dt} i_\ell$). In [10] a similar property is established for RLC circuits, which is a slight variation of the result given below.

In this section we also prove a one–to–one correspondence between cyclodissipativity and energy equalization for scalar circuits with linear inductors and capacitors and nonlinear resistors. Then, we identify a class of nonlinear RLC circuits for which a large (quantifiable) difference between the average electrical and magnetic energies implies power–factor compensation. Finally, we show by example, that this relation may not hold for time–varying linear circuits.

A. Equivalence for Circuits with Linear Inductors and Capacitors

The class of RLC circuits that we consider as load models consists of interconnections of possibly nonlinear lumped dynamic elements (n_L inductors, n_C capacitors) and static elements (n_R resistors). Capacitors and inductors are defined by the physical laws and constitutive relations [11]

$$\boldsymbol{i}_C = \dot{\boldsymbol{q}}_C,\ \boldsymbol{v}_C = \nabla H_C(\boldsymbol{q}_C),\ \boldsymbol{v}_L = \dot{\boldsymbol{\phi}}_L,\ \boldsymbol{i}_L = \nabla H_L(\boldsymbol{\phi}_L), \tag{15}$$

where $\boldsymbol{i}_C, \boldsymbol{v}_C, \boldsymbol{q}_C \in \mathbb{R}^{n_C}$ are the capacitor currents, voltages, and charges, $\boldsymbol{i}_L$, $\boldsymbol{v}_L$, $\boldsymbol{\phi}_L \in \mathbb{R}^{n_L}$ are the inductor currents, voltages, and flux–linkages, $H_L : \mathbb{R}^{n_L} \to \mathbb{R}$ is the magnetic energy stored in the inductors, $H_C : \mathbb{R}^{n_C} \to \mathbb{R}$ is the electric energy stored in the capacitors, and ∇ is the gradient operator. We assume that the energy functions H_L and H_C are twice differentiable. For linear capacitors and inductors, H_L and H_C are given by $H_C(\boldsymbol{q}_C) = \frac{1}{2}\boldsymbol{q}_C^\top \mathbf{C}^{-1}\boldsymbol{q}_C$ and $H_L(\boldsymbol{\phi}_L) = \frac{1}{2}\boldsymbol{\phi}_L^\top \mathbf{L}^{-1}\boldsymbol{\phi}_L$, respectively, where $\mathbf{L} \in \mathbb{R}^{n_L \times n_L}$ and $\mathbf{C} \in \mathbb{R}^{n_C \times n_C}$ are positive definite. For simplicity we assume that $\mathbf{L}$ and $\mathbf{C}$ are diagonal. Finally, the circuit has n_{R_L} current–controlled resistors, which are described by their characteristic functions $\hat{v}_{R_i}(i_{R_i})$, $i = 1, \ldots, n_{R_L}$, while the n_{R_C} voltage–controlled resistors are described by $\hat{i}_{R_i}(v_{R_i})$, $i = 1, \ldots, n_{R_C}$.

Proposition 5: Consider the system of Fig. 2 with $n = 1$, $v_s \in \mathcal{L}_2[0, T)$, a (possibly nonlinear) RLC load with time–invariant resistors, and fixed LTI capacitor compensator with admittance $\hat{Y}_c(s) = C_c s$, where $0 < C_c < C_{max}$. Then the following statements hold:

1. The power factor is improved if and only if

$$\langle \boldsymbol{v}_L, \nabla^2 H_L \boldsymbol{v}_L\rangle - \langle \boldsymbol{i}_C, \nabla^2 H_C \boldsymbol{i}_C\rangle \geq C_{max}\omega_0^2 \sum_{k=1}^{\infty} k^2 |\hat{V}_s(k)|^2, \tag{16}$$

where $\hat{V}_s(k)$ is the kth spectral line of $v_s(t)$.

2. If the inductors and capacitors are linear (16) reduces to

$$\sum_{k=1}^{\infty} k^2 \left[\sum_{q=1}^{n_L} L_q |\hat{I}_{L_q}(k)|^2 - \sum_{q=1}^{n_C} C_q |\hat{V}_{C_q}(k)|^2\right] \geq \frac{C_{max}}{2}\sum_{k=1}^{\infty} k^2 |\hat{V}_s(k)|^2, \tag{17}$$

where C_q, L_q are the qth capacitance and inductance, and $\hat{V}_{C_q}(k)$, $\hat{I}_{L_q}(k)$ are the spectral lines of the corresponding capacitor voltage and inductor current.

3. If, in addition, $v_s(t) = V_s \sin\omega_0 t$ then (16) becomes

$$H_{L_{av}}(\omega_0) - H_{C_{av}}(\omega_0) \geq \frac{C_{max}}{8} V_s^2,$$

where $H_{C_{av}}(\omega_0) := \sum_{q=1}^{n_C} \frac{1}{4} C_q |\hat{V}_{C_q}(1)|^2$ and $H_{L_{av}}(\omega_0) := \sum_{q=1}^{n_L} \frac{1}{4} L_q |\hat{I}_{L_q}(1)|^2$ are, respectively, the average electric and magnetic energy stored in the load.

Proof: Applying Tellegen's theorem [11] to the RLC load yields $i_\ell \dot{v}_s = \boldsymbol{i}_R^\top \dot{\boldsymbol{v}}_R + \boldsymbol{i}_L^\top \dot{\boldsymbol{v}}_L + \boldsymbol{i}_C^\top \dot{\boldsymbol{v}}_C$, which upon integration yields

$$\begin{aligned}\langle i_\ell, \dot{v}_s\rangle &= \langle \boldsymbol{i}_R,\dot{\boldsymbol{v}}_R\rangle+\langle \boldsymbol{i}_L,\dot{\boldsymbol{v}}_L\rangle+\langle \boldsymbol{i}_C,\dot{\boldsymbol{v}}_C\rangle\\ &= -\left\langle \frac{d}{dt}i_L,v_L\right\rangle+\langle \boldsymbol{i}_C,\dot{\boldsymbol{v}}_C\rangle\\ &= -\langle \nabla^2 H_L \boldsymbol{v}_L, \boldsymbol{v}_L\rangle + \langle \boldsymbol{i}_C, \nabla^2 H_C \boldsymbol{i}_C\rangle,\end{aligned}$$

where the second identity uses the fact that, along periodic trajectories, $\langle \boldsymbol{i}_R, \dot{\boldsymbol{v}}_R\rangle = 0$ for time–invariant resistors. The last identity follows from the constitutive relations (15). The proof of the first claim is completed by replacing the expression above in (13) and using

$$\|\dot{f}\|^2 = \langle \dot{f}, \dot{f}\rangle \triangleq 2\omega_0^2 \sum_{n=1}^{\infty} n^2 |\hat{F}(n)|^2 \tag{18}$$

to compute $\|\dot{v}_s\|^2 \triangleq 2\omega_0^2 \sum_{n=1}^{\infty} n^2|\hat{V}(n)|^2$.

The second and third claims are established as follows. From linearity of capacitors and inductors we have

$$\begin{aligned}\langle i_\ell, \dot{v}_s\rangle &= -\langle \mathbf{L}^{-1}\boldsymbol{v}_L, \boldsymbol{v}_L\rangle + \langle \boldsymbol{i}_C, \mathbf{C}^{-1}\boldsymbol{i}_C\rangle\\ &= -\langle \mathbf{L}^{-1}\dot{\boldsymbol{\phi}}_L, \dot{\boldsymbol{\phi}}_L\rangle + \langle \dot{\boldsymbol{q}}_C, \mathbf{C}^{-1}\dot{\boldsymbol{q}}_C\rangle\\ &= 2\omega_0^2 \sum_{k=1}^{\infty} k^2 \left[\sum_{q=1}^{n_C} C_q |\hat{V}_{C_q}(k)|^2 - \sum_{q=1}^{n_L} L_q |\hat{I}_{L_q}(k)|^2\right]\end{aligned}$$

where (15) is used for the second identity and equation (18) to compute the last line. Claim 3 follows by taking one spectral line and using the classical definition of averaged energy stored in linear inductors and capacitors [11]. ■

Results analogous to Proposition 5 can be established for inductive compensation checking the key cyclodissipation inequality $\langle i_\ell, z\rangle + \frac{1}{2L_m}\|z\|^2 \leq 0$, which stems from (14). Simple calculations show that the latter is equivalent to

$$\langle \boldsymbol{q}_C, \nabla H_C\rangle - \langle \boldsymbol{\phi}_L, \nabla H_L\rangle \geq \frac{1}{2L_m}\|z\|^2, \tag{19}$$

which in the LTI sinusoidal case becomes

$$H_{C_{av}}(\omega_0) - H_{L_{av}}(\omega_0) \geq \frac{V_s^2}{8\omega_0^2 L_{min}}. \tag{20}$$

Inequalities (17) and (20) reveal the energy–equalization mechanism of power–factor compensation in the LTI scalar sinusoidal case, that is, power–factor improvement with a capacitor (respectively, inductor) is possible if and only if the average magnetic (respectively, electrical) energy dominates the average electrical (respectively, magnetic) energy. Claim 2 shows that this interpretation of power–factor compensation remains valid when the source is an arbitrary periodic signal and the resistors are nonlinear, by viewing, in a natural way, $L_q|\hat{I}_{L_q}(k)|^2$ and $C_q|\hat{V}_{C_q}(k)|^2$ as the magnetic and electric energies of the kth harmonic for the qth inductive and capacitive element, respectively.

Remark 9: Claim 3 of Proposition 5 is established in [12] using the relation between the impedance of an LTI RLC circuit, $\hat{Z}_\ell(s) = \hat{V}_s(s)/\hat{I}_\ell(s)$, and the averaged stored energies

$$\hat{Z}_\ell(j\omega) = \frac{1}{|\hat{I}_\ell(j\omega)|^2}\left\{2P_{av}(\omega) + 4j\omega\left[H_{L_{av}}(\omega) - H_{C_{av}}(\omega)\right]\right\}, \tag{21}$$

where $P_{av}(\omega) = \frac{1}{2}\sum_{q=1}^{n_R} R_q |\hat{I}_q(j\omega)|^2$ is the power dissipated in the resistors. The expression (21) appears in equation 5.6 of Chapter 9 of [11]. Indeed, applying Parseval's theorem to the cyclodissipation inequality (13), we obtain the equivalences

$$\langle i_\ell, \dot{v}_s\rangle + \frac{C_{max}}{2}\|\dot{v}_s\|^2 \leq 0$$

if and only if

$$\begin{cases} \text{Re}\{j\omega \hat{Z}_\ell(j\omega)\}|\hat{I}_\ell(j\omega)|^2 + \frac{C_{max}\omega^2}{2}V_s^2 \leq 0\\ 4\omega^2\left[H_{C_{av}}(\omega) - H_{L_{av}}(\omega)\right] + \frac{C_{max}\omega^2}{2}V_s^2 \leq 0. \end{cases} \tag{22}$$

Remark 10: Simple calculations show that (11) of Proposition 2 with $\hat{Y}_c(s) = C_{max}s$ is equivalent to (22). Indeed, it is easy to prove that

$$\text{Re}\{j\omega \hat{Z}_\ell(j\omega)\} = \omega|\hat{Z}_\ell(j\omega)|^2 \text{Im}\{\hat{Y}_\ell(j\omega)\}.$$

Replacing the latter, together with $|\hat{V}_s(j\omega)|^2 = |\hat{Z}_\ell(j\omega)|^2|\hat{I}_\ell(j\omega)|^2$, in (22) yields $\text{Im}\{\hat{Y}_\ell(j\omega)\} < -\frac{C_{max}\omega}{2}$, which is the expression obtained in (11) for capacitive compensation (See Fig. 4).

B. Necessity of Energy Equalization for Nonlinear RLC Loads

The presence of the energy functions in (16) and (19), which hold for nonlinear RLC loads, suggests that energy equalization is related with power–factor compensation for more general loads. Indeed, Proposition 6 establishes that a sufficiently large difference between magnetic and electrical energies is necessary for capacitive power–factor compensation. The proof of this result, which is technical and thus is outside the scope of this article follows from the arguments used in [10]. The dual result for inductive power–factor compensation is also true, but is omitted for brevity.

Proposition 6: Consider a nonlinear topologically complete RLC circuit with a voltage source $\boldsymbol{v}_s \in \mathcal{L}_2^n[0,T)$ in series with inductors and satisfying the following assumptions:

- B.1 The energy functions of the inductors and capacitors are strictly convex.
- B.2 The voltage–controlled resistors are linear and passive.
- B.3 All capacitors have a (voltage–controlled) resistor in parallel and the value of the resistance is sufficiently small.

Then, the circuit is cyclodissipative with supply rate $\frac{d}{dt}\boldsymbol{i}_\ell^\top \boldsymbol{v}_s$. Furthermore if the current–controlled resistors are passive then the circuit is dissipative.

Assumptions B.1 and B.2 are technical conditions needed to construct the virtual storage function. Assumption B.3 ensures that the electrical energy stored in the capacitors is smaller than the magnetic energy stored in the inductors. As shown in [10], the qualifier "sufficiently small" in Assumption B.3 can be explicitly quantified using an upper bound on the resistances. Indeed, since all capacitors have linear resistors in parallel, we have that as the value of the resistances decreases the currents tend to flow through the resistors and the energy stored in the capacitors becomes small. The stored energy tends to zero as the resistances go to zero, which is the limiting case when all of the capacitors are short–circuited.

C. Limits of Energy Equalization Equivalence

Unfortunately, the energy–equalization interpretation of power–factor compensation breaks down even for simple *time–varying* LTI circuits, as shown in the following example taken from [13].

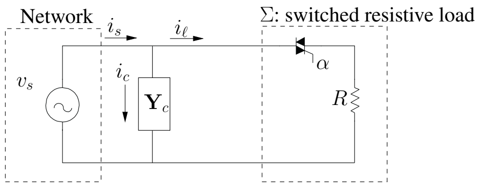

Fig. 5. Circuit with a TRIAC controlled resistive load.

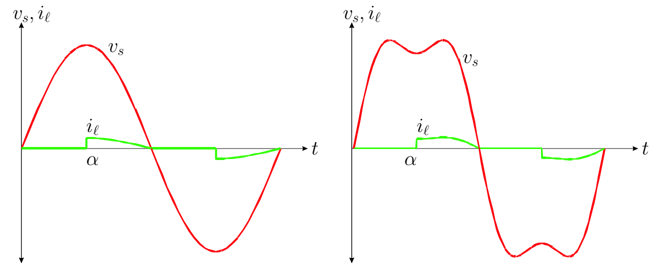

Fig. 6. Voltage and current waveforms for the (uncompensated) circuit with the TRIAC controlled resistive load.

D. Example

Consider the linear time–varying circuit of Fig. 5 with a TRIAC controlled purely resistive load $R = 10\ \Omega$. The TRIAC can be modeled as a switched resistor with characteristic

$$i_\ell(t) = \begin{cases} 0 & \text{if} \quad t \in [\frac{kT}{2}, \frac{kT}{2} + \alpha),\ k = 0, 1, \ldots \\ \frac{v_s(t)}{R} & \text{otherwise.} \end{cases}$$

where $T = 2\pi/\omega_0$ is the fundamental period and $0 \leq \alpha < T/2$ is the TRIAC's firing angle. The uncompensated voltage $v_s(t)$ and current $i_s(t)$ are depicted in Fig. 6 for $v_s(t) = 220\sqrt{2}\sin(\omega_0 t)$ V and $v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 50\sqrt{2}\sin(3\omega_0 t)$ V, with $\omega_0 = 100\pi$ rad/s and $\alpha = T/4 = 0.005$ s. It is important to emphasize that this switched resistor circuit does not contain energy–storage elements. Furthermore, the TRIAC does not satisfy condition $\langle i_R, \dot{v}_R \rangle = \langle i_\ell, \dot{v}_s \rangle = 0$, which is used to establish the proof of Proposition 5.

For the sinusoidal source we obtain $\langle \dot{v}_s, i_\ell \rangle = -48.4 \times 10^4$ V-A/s and $\|\dot{v}_s\| = 6.91 \times 10^4$ V/s, and thus a shunt capacitor with $0 < C_c < 0.202$ mF improves the power factor. The optimal capacitor is $C_\star = 0.101$ mF, which increases the uncompensated power factor $PF_u = 0.7071$ to $PF = 0.7919$.

If $v_s(t)$ is the two–harmonic periodic signal above, we obtain $\langle v_s, \frac{d}{dt} i_\ell \rangle = 28.9 \times 10^4$ V-A/s. Hence the load can be compensated with a capacitor whose optimal value is $C_\star = 0.0413$ mF, yielding $PF = 0.7258$.

VII. CONCLUDING REMARKS

This article advances an analysis and compensator design framework for power–factor compensation based on cyclodissipativity. While we concentrated here on passive shunt compensation, we are certainly aware that current source–based control is an attractive option cases. For these actuators or active filters, which can be modeled by discontinuous differential equation, the control objective is current tracking. See [14] for an introduction and [15] for a modeling procedure consistent with the energy–based approach advocated here. Although nonlinear control strategies have been used for basic topologies [16]–[18], many questions remain unanswered [19]. Another important problem in energy–processing systems with distorted signals is the regulation of harmonic content. Although we have not explicitly addressed this issue here, it is clear that improving the power factor reduces the harmonic distortion; a quantification of this effect is a subject of current research.

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