On quasi-static approximations in linear thermoelastodynamics

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Abstract

The validity of the coupled and uncoupled quasi-static approximations is considered for the initial boundary value problem of linear thermoelasticity subject to homogeneous Dirichlet boundary conditions, and for solutions and their derivatives that are mean-square integrable. Essential components in the proof, of independent interest, are conservation laws and associated estimates for the exact and approximate systems.

Keywords: thermoelastodynamics, coupled quasi-static approximations, uncoupled quasi-static approximations, mean-square estimates.

Introduction

Quasi-static approximations to equations of motion, according to Boley and Weiner [2, p.54], were originally proposed by Duhamel in 1837. These approximations suppose that the acceleration and therefore inertia is of an order of magnitude smaller than either the strain or velocity and consequently may be neglected. The strain and velocity, however, retain dependence upon the time variable which is now treated as a parameter.

Various factors may cause the inertia either to be sufficiently small for all time, or eventually to become small in finite or infinite time. Causes include viscous or thermal damping, and energy dissipation due to shock waves. Other causes are time evolving boundary conditions and source terms, while clearly initial data can affect how the inertia behaves. Indeed, for the dissipative system of thermoelasticity, Boley and Weiner [2, Sect 2.5] use a half-space thermoelastodynamic problem to motivate quasi-static approximations.

A precise analysis conducted by Dafermos [7, 8], also for linear thermoelastodynamics, establishes global existence and uniqueness of a generalised and classical solution. Moreover, the analysis establishes that the temperature gradient and specific entropy asymptotically decay to zero with respect to time, and that the displacement converges to zero except for certain regions and boundary conditions when convergence is to an undamped oscillation. The inertia also fails to converge to zero in these exceptional circumstances. Lebeau and Zuazua [18] further develop the analysis by showing that the energy decays at a uniform exponential rate except on convex and certain other regions. A unified account may be found in the book by Jiang and Racke [16]. These conclusions are important in the discussion of quasi-static approximations.

Verification of the quasi-static approximation is provided for particular problems. Scaling arguments based on selected parameters are used to establish the relative magnitudes of the non-dimensionalised inertia, displacement, velocity, and other dependent variables. An obvious example is the derivation of Stokes flow in fluid dynamics described, for example, in [6]. In viscoelasticity, Saccomandi and co-workers have examined shearing motions in various viscous elastic materials; (see [19], [20], [14], and [22].) These investigations identify a boundary layer in which the inertia, although decaying, is not initially negligible. In linear thermoelasticity, justification of quasi-static approximations to the three-dimensional isotropic initial boundary value problem is investigated by Eshan and Weinacht [12,13]. The technique of singular perturbations is employed to extract respective orders of magnitude from series expansions. In particular, the inertia possesses uniform exponential decay. In other treatments, the inertia is controlled to zero by appropriately prescribed source terms and boundary conditions; see, for example, studies of the one-dimensional isotropic initial boundary value problem by Day [9–11], where convergence to zero but not the rate is obtained.

Dependence upon asymptotic behaviour to justify the quasi-static approximation is perhaps impractical due to the comparatively large time that must elapse before the inertia becomes sufficiently small to be neglected. It is preferable that conditions are obtained under which the approximation becomes valid immediately, or within a short time, after motion has commenced. Specifically, it is important to establish how inertia is affected by initial conditions. This aspect is a principal concern of the present study.

Conditions for the validity of quasi-static approximations to general systems appear not to have been rigorously defined in the literature. A possible procedure for such an investigation includes the proof of the following three essential steps in which the measures are not necessarily the same:

- The difference between solutions to the exact and approximate problems depends in suitable measure upon the inertia.

- The inertia becomes uniformly spatially negligible in finite time compared to the displacement, velocity, and temperature.

- The inertia depends continuously upon the data.

This proposed programme for a general system is too broad to be comprehensively undertaken here. Instead, attention is confined to the classical theory of linear three-dimensional nonhomogeneous anisotropic compressible thermoelasticity. Only the effect of initial conditions is treated and in this respect it is convenient to suppose homogeneous Dirichlet boundary conditions and frequently also vanishing source terms. Other types of homogeneous boundary conditions may be easily accommodated in the treatment. Both the coupled and uncoupled quasi-static approximations are considered. Temperature in the uncoupled approximation is independent of the displacement and velocity, but these quantities are influeneced by the temperature through its presence in the mechanical equations of motion as a pseudo-body force.

Continuous dependence of the solution upon initial data is implied by Steps 1 and 3, but may be established directly as shown, for example, in Section . Here, however,to illustrate the above scheme such dependence is obtained by the intermediate step of dependence upon inertia.

Besides exploring the validity of the quasi-static approximations, we devote a considerable proportion of the paper to deriving conservation laws and upper bounds for solutions to the quasi-static approximations. These results, of independent interest, are subsequently required to justify the approximations.

The overall method employed is different to those already mentioned and consists of comparatively elementary arguments that involve well-known inequalities.

Next Section describes the exact initial boundary value problem and disposes of notation. Positive-definite assumptions are stated and a basic inequaltiy derived. Conservation laws for the exact system, constructed in the third Section, enable continuous dependence of the inertia upon initial data to be obtained subject to the solution possessing sufficient differentiablity. Quasi-static approximations are formulated in the fourth Section. First subsection of the fourth Section concerns the coupled quasi-static approximation and derives various upper bound estimates subject to homogeneous Dirichlet boundary conditions but non-zero body force and heat supply. Continuous dependence of the solution on initial data in mean square measure is an immediate consequence. When the source terms vanish, it is further concluded that the velocity dominates the temperature in appropriate mean square measure. Nevertheless, it is also proved that displacement, velocity, and temperature all depend continuously upon the initial temperature. Second subsection of the fourth Section considers the uncoupled quasi-static approximation and, subject to homogeneous Dirichlet boundary conditions and zero source terms, demonstrates that the mean squares of temperature and therefore displacement exponentially vanish irrespective of the mechanical initial data. Indeed it is shown that mechanical initial data cannot be arbitrarily chosen. Dependence upon the inertia in both quasi-static approximations is established in the fitfh Section by determining the error between the exact and approximate solutions. The difference solution satisfies homogeneous boundary conditions and vanishing source terms but not initial conditions. Dependence of both displacement and temperature is in mean-square space-time measure.

A classical solution is assumed to globally exist The comma notation to denote partial differention is adopted together with the convention of summation over repeated suffixes apart from the indices t and η reserved for time variables. Other notation is introduced as required. There is no typographical distinction between scalar, vector, and tensor quantities.

Notation and other preliminaries

Let $\Omega \subset \mathbb{R}^n, n = 1, 2, 3$, a bounded region of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, be occupied by a classical linear nonhomogeneous anisotropic compressible thermoelastic solid in motion subject to specified source terms, initial data and boundary conditions. We consider the three-dimensional problem $n = 3$, although the treatment described is easily adapted to the case $n = 1, 2$. Let a spatial point in Ω or on its boundary be given by the vector position x and let the time variable be denoted by $t \geq 0$.

The temperature $\phi(x,t)$ and Cartesian components $w_i(x,t)$ of the displacement vector $w(x,t)$ for $(x,t) \in \Omega \times [0,T)$ satisfy the coupled system (see e.g., [2], [5])

$$\rho(x) w_{i,tt}(x,t) = \left(c_{ijkl}(x) w_{k,l}(x,t)\right)_{,j} - \left(\beta_{ij}(x)\phi(x,t)\right)_{,j} + \rho(x) F_i(x,t), \tag{1}$$

$$a(x)\phi_{,t}(x,t) = -\beta_{ij}(x) w_{i,jt}(x,t) + \left(\kappa_{ij}(x)\phi_{,i}(x,t)\right)_{,j} + \tilde{r}(x,t), \tag{2}$$

where the mass density $\rho(x)$ and specific heat $a(x)$ satisfy $0 < \rho_0 \leq \rho(x) \leq \bar{\rho}$, and $0 < a_0 \leq a(x) \leq \bar{a}$ for constants ρ_0, $\bar{\rho}$, a_0, $\bar{a}$. The body-force components per unit mass are denoted by $F_i(x,t)$, while $\tilde{r}(x,t)$ is the heat supply. The maximal interval of existence is denoted by $[0,T)$, which under the assumption of global existence becomes the half interval $[0.\infty)$. It is supposed that the solutions to (1) and (2) are sufficiently smooth for the equations to be valid at $t = 0$. The symmetry of the thermoelastic coupling tensor β, which in terms of the components is expressed by

$$\beta_{ij}(x) = \beta_{ji}(x), \qquad x \in \Omega, \tag{3}$$

is inherited from that of the stress tensor. The elastic moduli $c_{ijkl}(x)$ are functions of x alone and possess the symmetries

$$c_{ijkl}(x) = c_{jikl}(x) = c_{klij}(x). \qquad x \in \Omega; \tag{4}$$

while components of the thermal conductivity tensor $\kappa(x)$ possess the symmetry

$$\kappa_{ij}(x) = \kappa_{ji}(x), \qquad x \in \Omega. \tag{5}$$

In addition to the symmetry (3), it is supposed that the thermoelastic coupling tensor is bounded as follows

$$\beta^2 = \max_{x\in\Omega} [\beta_{ij}\beta_{ij} + \beta_{ij,j}\beta_{ik,k}], \tag{6}$$

for some given positive constant β.

We investigate the effect of initial conditions on the behaviour of the inertia and consequently only homogeneous Dirichlet boundary conditions are considered. Thus, it is assumed that

$$w_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T), \tag{7}$$

$$\phi(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T), \tag{8}$$

Other homogeneous boundary conditions are easily accommodated within the analysis.

Initial conditions are given by

$$w_i(x,0) = w_i^{(0)}(x), \qquad w_{i,t}(x,0) = w_i^{(1)}(x), \qquad x \in \Omega, \tag{9}$$

$$\phi(x,0) = \phi^{(0)}(x), \qquad x \in \Omega, \tag{10}$$

for specified functions $w_i^{(0)}(x)$, $w_i^{(1)}(x)$, and $\phi^{(0)}(x)$.

The second law of thermodynamics implies that

$$\kappa_{ij}(x)\xi_i\xi_j \geq 0, \qquad \forall\, \xi_i, \qquad x \in \Omega,$$

which for later purposes is strengthened to the positive-definite condition

$$\kappa_{ij}(x)\xi_i\xi_j \geq \kappa_0\xi_i\xi_i, \qquad \forall\, \xi_i, \qquad x \in \Omega, \tag{11}$$

for positive constant κ_0.

It is also expedient to suppose that the elasticities are positive- definite in the sense that there exists a positive constant c_0 such that

$$c_{ijkl}\psi_{ij}\psi_{kl} \geq c_0\psi_{ij}\psi_{ij}, \qquad \forall\, \psi_{ij} = \psi_{ji}, \qquad x \in \Omega. \tag{12}$$

No thermodynamic justification exists for this assumption. An appeal to Lyapunov stability is doubtful, if not spurious, and motivation based on static stability is often tautologous. Moreover, the assumption may be violated in the linearised theory of small superposed upon large deformations. Properties of suitably constrained solutions to the system (1)-(10) in the absence of assumption (12) are separately investigated in a forthcoming publication [17].

As already mentioned, for sufficiently smooth solutions, it is supposed that equations (1) and (2) remain valid at $t = 0$. While the smoothness assumption is not required for many of the subsequent calculations, its adoption imposes additional compatibility on the data besides that required for existence of classical solutions. In consequence, the data must be such that

$$w_i^{(0)}(x) = \phi^{(0)}(x) = 0, \qquad x \in \partial\Omega.$$

Furthermore, the validity of (1) and (2) at $t = 0$ implies the relations

$$\begin{aligned}
\rho w_{i,tt}(x,0) &= \left(c_{ijkl}(x)w_{k,l}^{(0)}(x)\right)_{,j} - \left(\beta_{ij}(x)\phi^{(0)}(x)\right)_{,j} + \rho(x)F_i(x,0), \quad x \in \Omega, &(13)\\
a(x)\phi_{,t}(x,0) &= -\beta_{ij}(x)w_{i,j}^{(1)} + \left(\kappa_{ij}(x)\phi_{,i}^{(0)}\right)_{,j} + \tilde{r}(x,0), \qquad x \in \Omega, &(14)
\end{aligned}$$

which determines initial values of the acceleration and rate of change of temperature in terms of Cauchy data (9) and (10).

An equivalent form of the heat equation (2), of subsequent use, is derived as follows. Set

$$\Phi(x,t) = \int_0^t \phi(x,\eta)\,d\eta,$$

and integrate (2) with respect to time to obtain

$$a(x)\Phi_{,t}(x,t) = -\beta_{ij}(x)w_{i,j}(x,t) + \left(\kappa_{ij}(x)\Phi_{,i}\right)_{,j} + r(x,t) + H(x), \qquad (x,t) \in \Omega \times [0,T), \quad (15)$$

where

$$r(x,t) = \int_0^t \tilde{r}(x,\eta)\, d\eta,$$

and

$$\begin{aligned} H(x) &= a\phi(x,0) + \beta_{ij}w_{i,j}(x,0) \\ &= a\phi^{(0)}(x) + \beta_{ij}w_{i,j}^{(0)}(x), \qquad x \in \Omega, \end{aligned}$$

is the initial value of the entropy.

We state two inequalities required later. The first is the Poincaré inequality

$$\lambda \int_\Omega v_i v_i \, dx \le \int_\Omega v_{i,j} v_{i,j} \, dx,$$

for vector functions with components $v_i(x)$ that vanish on $\partial\Omega$. The constant λ is the first eigenvalue of the fixed membrane problem for Ω.

The second is Korn's inequality valid for differentiable functions that vanish on $\partial\Omega$. It is given by (see, e.g., Gurtin [15])

$$\int_\Omega v_{i,j} v_{i,j} \, dx \le 2 \int_\Omega \left(v_{i,j} + v_{j,i}\right)\left(v_{i,j} + v_{j,i}\right) \, dx. \quad (16)$$

On combining these inequalities with the positive-definite condition (12) and symmetries

(4), we are led to a third inequality also of later use:

$$
\begin{aligned}
\lambda \int_\Omega v_i v_i \, dx &\leq \int_\Omega v_{i,j} v_{i,j} \, dx \\
&\leq 2 \int_\Omega \left(v_{i,j} + v_{j,i}\right) \left(v_{i,j} + v_{j,i}\right) \, dx \\
&\leq \frac{2}{c_0} \int_\Omega c_{ijkl} \left(v_{i,j} + v_{j,i}\right) \left(v_{k,l} + v_{l,k}\right) \, dx \\
&\leq \frac{8}{c_0} \int_\Omega c_{ijkl} v_{i,j} v_{k,l} \, dx. \qquad (17)
\end{aligned}
$$

Conservation law for the exact system

The initial boundary value problem (1)-(12) subject to homogeneous Dirichlet boundary conditions posseses the well-known conservation law

$$
E(t) + 2 \int_0^t \int_{\Omega(\eta)} \kappa_{ij} \phi_{,i} \phi_{,j} \, dx d\eta = E(0) + 2 \int_0^t \int_{\Omega(\eta)} \left(F_i w_{i,\eta} + \bar{r}\phi\right) \, dx d\eta, \qquad (18)
$$

where

$$
E(t) = \int_{\Omega(t)} \left(\rho w_{i,t} w_{i,t} + c_{ijkl} w_{i,j} w_{k,l} + a\phi^2\right) \, dx,
$$

and $\Omega(t)$ indicates that terms in the corresponding integrand are evaluated at time t.

Uniqueness of the initial boundary value problem immediately follows from the conservation law (18) and the positive-definite assumptions (11) and (12). Thus, assume the existence of a non-trivial solution subject to homogeneous initial and boundary data and zero source terms. A contradiction is then clearly obtained from (18). It is unnecessary, however, to assume condition (12). Uniqueness in the thermoelastic initial boundary value problem holds subject only to a positive definite symmetric heat conduction tensor (11), and the major symmetry of the elastic moduli (see [3, 4]):

$$
c_{ijkl} = c_{klij}.
$$

Here, however, the conservation law (18) is used to derive continuous dependence of the inertia upon initial data. An application of Schwarz's inequality to the right side leads to

$$I'(t) \leq E(0) + 2\left[J(t)I(t)\right]^{1/2}, \tag{19}$$

where a superposed prime indicates differentiation with respect to the argument of the function, and

$$\begin{aligned} I(t) &= \int_0^t E(\eta)d\eta, &(20)\\ J(t) &= \int_0^t \int_{\Omega(\eta)} \left(\rho F_i F_i + a_0^{-1}\bar{r}^2\right)\, dx d\eta, &(21)\end{aligned}$$

Young's inequaltiy followed by integration with respect to time leads to the estimate

$$I(t) \leq \frac{e^{\gamma t} - 1}{\gamma} E(0) + \frac{t e^{\gamma t}}{\gamma} J(t), \tag{22}$$

where γ is an arbitrary positive constant. At any given finite time t, we set $\gamma t = 1$, and (22) simplifies to

$$I(t) \leq te\left(E(0) + tJ(t)\right).$$

Substitution in (19) then leads to an estimate for $E(t)$ explicitly given by

$$\begin{aligned} \int_{\Omega(t)} \left(\rho w_{i,t} w_{i,t} + c_{ijkl} w_{i,j} w_{k,l} + a\phi^2\right)\, dx &\leq E(0) + 2\left[J(t)et\left(E(0) + tJ(t)\right)\right]^{1/2} \\ &\leq 2E(0) + (1+e)\, tJ(t) &(23)\end{aligned}$$

which consequently not only provides bounds for

$$\int_{\Omega(t)} \rho w_{i,t} w_{i,t}\, dx, \qquad \int_{\Omega(t)} a\phi^2\, dx,$$

but also by inequality (17) for

$$\int_{\Omega(t)} w_i w_i\, dx.$$

A corresponding continuous dependence estimate for the inertia may be deduced on noting that the time derivative of the solution is also a solution to the system under consideration so that

$$\int_{\Omega(t)} \rho w_{i,tt} w_{i,tt}\, dx \leq 2E_1(0) + (1+e)\, t J_1(t), \tag{24}$$

where

$$J_1(t) = \int_0^t \int_{\Omega(\eta)} \left(\rho F_{i,\eta} F_{i,\eta} + a_0^{-1} \bar{r}^2_{,\eta}\right)\, dx d\eta,$$

and

$$E_1(0) = \int_{\Omega(0)} \left(\rho w_{i,tt} w_{i,tt} + c_{ijkl} w_{i,jt} w_{k,lt} + a\phi^2_{,t}\right)\, dx$$

is known from (13) and (14) in terms of the Cauchy initial data (9) and (10)

It is evident that the bounds (23) and (24) are effective for all time provided the source terms ensure the asymptotic behaviour

$$J(t) = O(t^{-2}), \qquad J_1(t) = O(t^{-2}), \qquad \text{as } t \to \infty.$$

The absence of source terms reduces (18) to

$$E(t) + 2\int_0^t \int_{\Omega(\eta)} \kappa_{ij} \phi_{,i} \phi_{,j}\, dx d\eta = E(0),$$

and in consequence we deduce from the conservation law corresponding to (18) that

$$\int_{\Omega(t)} \rho\, w_{i,tt} w_{i,tt}\, dx \leq E_1(0).$$

Continuous dependence on initial data is easily concluded.

The calculations of this Section assume the solution $(w_i(x,t), \phi(x,t))$ is sufficiently smooth for the energies $E(t)$, $E_1(t)$ to exist for $t \in [0, T)$. Continuous dependence does not necessarily hold when, for example, initial data has lost smoothness and $E(0)$ is no longer bounded.

Quasi-static approximations

The initial boundary value problem (1)-(12) can be difficult to solve. In these cicumstances, the problem is customarily replaced by either of two approximations in each of which the inertia is discarded in (1). Strains, velocity and thermal terms and their dependence upon time, however, are retained. Expressed otherwise, neglect of the inertia $\rho w_{i,tt}(x,t)$ in (1) causes only a small error in the solution $(w_i(x,t), \phi(x,t))$.

Precise conditions for the approximations to be valid appear to be seldom comprehensively stated, let alone proved, in the literature, although certain particular problems have been thoroughly studied including those cited in the introduction. Conditions under which the approximations hold are not entirely obvious as testified by several elementary examples and by systems that are metastable.

This Section is devoted to stating the approximations for coupled and uncoupled systems. Continuous dependence estimates and consequent error estimates are considered in later Sections

Coupled quasi-static approximation

In the coupled quasi-static approximation, the time rate of change of the displacement is not deleted from the heat conduction equation (2). In consequence, the equations are coupled and there is mutual interaction between the displacement $v_i(x,t)$ and temperature $\psi(x,t)$. The appropriate equations are given by

$$\left(c_{ijkl}v_{k,l} - \beta_{ij}\psi\right)_{,j} + \rho F_i = 0, \qquad (x,t) \in \Omega \times [0,T), \tag{25}$$

$$-\beta_{ij}v_{i,jt} + \left(\kappa_{ij}\psi_{,i}\right)_{,j} + \bar{r} = a\psi_{,t}, \qquad (x,t) \in \Omega \times [0,T), \tag{26}$$

where, after a time integration, the last relation may be written alternatively as

$$a\Psi_{,t}(x,t) = -\beta_{ij}v_{i,j} + \left(\kappa_{ij}\Psi_{,i}\right)_{,j} + r + H^{(1)}(x), \qquad (x,t)\in\Omega\times[0,T), \tag{27}$$

with

$$\begin{aligned}\Psi(x,t) &= \int_0^t \psi(x,\eta)\,d\eta, \qquad r(x,t) = \int_0^t \bar{r}(x,\eta)\,d\eta,\\ H^{(1)}(x) &= a\psi(x,0) + \beta_{ij}v_{i,j}(x,0), \qquad x\in\Omega. \end{aligned} \tag{28}$$

The same homogeneous boundary conditions as in (7) and (8) are adopted for the coupled quasi-static approximation; that is, we assume

$$\begin{aligned} v_i(x,t) &= 0, \qquad (x,t)\in\partial\Omega\times[0,T),\\ \psi(x,t) &= 0, \qquad (x,t)\in\partial\Omega\times[0,T), \end{aligned}$$

For the moment, it is convenient to suppose that initial data

$$v_i(x,0), \qquad \psi(x,0), \qquad x\in\Omega$$

are prescribed separately to the Cauchy data (9) and (10) specified in the exact problem.

We establish various relations satisfied by solutions to the coupled quasi-static approximation and first derive a conservation law analogous to (18) that leads to separate upper bounds for mean square integrals of the temperature and displacement. Multiplication of (25) by v_i and integration by parts yields

$$\begin{aligned}\int_{\Omega(t)} c_{ijkl}v_{i,j}v_{k,l}\,dx &= \int_{\Omega(t)}\beta_{ij}\psi v_{i,j}\,dx + \int_{\Omega(t)}\rho F_i v_i\,dx,\\ &\le \left[\frac{16}{\lambda c_0}\int_{\Omega(t)}\left(\lambda\beta_{ij}\beta_{ij}\psi^2 + \rho^2F_iF_i\right)dx\right]^{1/2}\left[\frac{c_0}{16}\int_{\Omega(t)}\left(v_{i,j}v_{i,j} + \lambda v_iv_i\right)dx\right]^{1/2}\\ &\le \left[\frac{16}{\lambda c_0}\left(\lambda\beta^2\int_{\Omega(t)}\psi^2\,dx + \bar{\rho}\int_{\Omega(t)}\rho F_iF_i\,dx\right)\right]^{1/2}\left[\int_{\Omega(t)}c_{ijkl}v_{i,j}v_{k,l}\,dx\right]^{1/2}\\ &\le \frac{16}{\lambda c_0}\left(\lambda\beta^2\int_{\Omega(t)}\psi^2\,dx + \bar{\rho}\int_{\Omega(t)}\rho F_iF_i\,dx\right) \end{aligned} \tag{29}$$

where (17) is used. The estimate indicates, as expected, that the mean square of displacement is controlled by the evolving temperature and body force.

To obtain the second conservation law, we set

$$
\begin{aligned}
V(t) &= \int_0^t \int_{\Omega(\eta)} c_{ijkl} v_{i,j} v_{k,l}\, dx d\eta, \\
S(t) &= \int_0^t \int_{\Omega(\eta)} a\psi^2\, dx d\eta, \\
E_2(0) &= \int_{\Omega(0)} \left(c_{ijkl} v_{i,j} v_{k,l} + a\psi^2 - 2\rho F_i v_i \right)\, dx,
\end{aligned}
$$

Multiplication of (25) by $v_{i,t}$ added to (26) multiplied by ψ and an integration by parts leads to

$$
\begin{aligned}
(V' + S') &+ 2\int_0^t \int_{\Omega(\eta)} \kappa_{ij} \psi_{,i} \psi_{,j}\, dx d\eta \\
&= 2\int_0^t \int_{\Omega(\eta)} \left(\rho F_i v_{i,\eta} + \bar{r}\psi \right)\, dx d\eta + (V'(0) + S'(0)) \\
&= 2\int_{\Omega(t)} \rho F_i v_i\, dx + 2\int_0^t \int_{\Omega(\eta)} \left(\bar{r}\psi - \rho F_{i,\eta} v_i \right)\, dx d\eta + E_2(0) \\
&\leq \frac{1}{\alpha_1} \int_{\Omega(t)} v_i v_i\, dx + \frac{1}{\alpha_2} \int_0^t \int_{\Omega(\eta)} v_i v_i\, dx d\eta + \frac{1}{\alpha_3} \int_0^t \int_{\Omega(\eta)} a\psi^2\, dx d\eta \\
&\quad + D(t) + E_2(0) \\
&\leq \frac{8}{\alpha_1 \lambda c_0} V'(t) + \frac{8}{\alpha_2 \lambda c_0} V(t) + \frac{1}{\alpha_3} S(t) + D(t) + E_2(0),
\end{aligned}
\tag{30}
$$

where, as already mentioned, a superposed prime denotes differentiation with respect to the time variable, Young's inequality and (17) are repeatedly used, α_i, $i = 1, 2, 3$, are arbitrary positive constants to be chosen, and

$$
D(t) = \alpha_1 \bar{\rho} \int_{\Omega(t)} \rho F_i F_i\, dx + \alpha_2 \bar{\rho} \int_0^t \int_{\Omega(\eta)} \rho F_{i,\eta} F_{i,\eta}\, dx d\eta + \alpha_3 \int_0^t \int_{\Omega(\eta)} a_0^{-1} \bar{r}^2\, dx d\eta.
$$

Select α_i, $i = 1, 2, 3$, to satisfy

$$
\left(1 - \frac{8}{\alpha_1 \lambda c_0} \right) = \epsilon > 0, \qquad \frac{8}{\alpha_2 \epsilon \lambda c_0} = \frac{1}{\alpha_3} = \gamma_1,
$$

and rewrite (30) as

$$\frac{d}{dt}\left(\exp\left(-\gamma_1 t\right)\right)[\epsilon V + S]) \leq \exp\left(-\gamma_1 t\right)\left[D(t) + E_2(0)\right].$$

Upon integration with respect to time, we conclude that

$$\epsilon V(t) + S(t) \leq \exp\left(\gamma_1 t\right)\left(\int_0^t D(\eta)\, d\eta + \gamma_1^{-1} E_2(0)\right). \tag{31}$$

In particular, put $\epsilon = 1/2$, and insert the estimate (31) into (30) to obtain

$$\int_{\Omega(t)} c_{ijkl} v_{i,j} v_{k,l}\, dx + 2\int_{\Omega(t)} a\psi^2\, dx \leq 2\left[D(t) + E_2(0)\right] + 2\left[\exp\left(\gamma_1 t\right)\left(\gamma_1 \int_0^t D(\eta)\, d\eta + E_2(0)\right)\right]. \tag{32}$$

Uniqueness of both the displacement and temperature may be immediately inferred from either (31) or (32) subject to sufficiently smooth initial values of $\psi(x, 0)$ and $v_i(x, 0)$. See also Remark 1

A further conservation law is obtained on multiplying (27) by ψ and integrating by parts to give

$$\int_{\Omega(t)} a\psi^2\, dx = -\int_{\Omega(t)} \beta_{ij}\psi v_{i,j}\, dx - \int_{\Omega(t)} \kappa_{ij}\Psi_{,i}\Psi_{,jt}\, dx + \int_{\Omega(t)} r\psi\, dx + \int_{\Omega(t)} H^{(1)}\psi\, dx.$$

Addition of the last equation to (25) after multiplication by v_i and integration with respect to time yields

$$\begin{aligned}
\int_0^t \int_{\Omega(\eta)} \left(c_{ijkl} v_{i,j} v_{k,l} + a\psi^2\right)\, dx d\eta \quad &+ \quad \frac{1}{2}\int_{\Omega(t)} \kappa_{ij}\Psi_{,i}\Psi_{,j}\, dx \\
&= \quad \int_0^t \int_{\Omega(\eta)} \left(H^{(1)}\psi + \rho F_i v_i + r\psi\right)\, dx d\eta \\
&\leq \quad \left[a_0^{-1} t \int_{\Omega(0)} H^{(1)2}\, dx + J(t)\right]^{1/2} \times \\
&\quad\;\; \times \left[\int_0^t \int_{\Omega(\eta)} \left(\rho v_i v_i + 2a\psi^2\right)\, dx d\eta\right]^{1/2}.
\end{aligned} \tag{33}$$

where $J(t)$ is defined in (21). Consider the second term on the right, and introduce the notation

$$\begin{aligned} V_1(t) &= \int_0^t \int_{\Omega(\eta)} c_{ijkl} v_{i,j} v_{k,l} \, dx d\eta, \\ S_1(t) &= \int_0^t \int_{\Omega(\eta)} a\psi^2 \, dx d\eta. \end{aligned}$$

We have

$$\begin{aligned} \left[\int_0^t \int_{\Omega(\eta)} \left(\rho v_i v_i + 2a\psi^2 \right) \, dx d\eta \right] &\leq 2 \int_0^t \int_{\Omega(\eta)} \left(\frac{\bar{\rho}}{2} v_i v_i + a\psi^2 \right) \, dx d\eta \\ &\leq 2\gamma_2 \left(V_1(t) + S_1(t) \right), \qquad (34) \end{aligned}$$

where (17) has been employed and

$$\gamma_2 = \max\left(\frac{4\bar{\rho}}{\lambda c_0}, 1 \right).$$

It immediately follows from (34) and (33) that

$$V_1(t) + S_1(t) \leq 2\gamma_2 \left[a_0^{-1} t \int_{\Omega(0)} H^{(1)2} \, dx + J(t) \right] \qquad (35)$$

and by Young's inequality that

$$\int_{\Omega(t)} \kappa_{ij} \Psi_{,i} \Psi_{.j} \, dx \leq \gamma_2 \left[a_0^{-1} t \int_{\Omega} H^{(1)2} \, dx + J(t) \right] \qquad (36)$$

Improved estimates are obtained by insertion of the bound (34) into (33). Young's inequality leads to

$$V_1 + S_1(t) + \frac{1}{2} \int_{\Omega(t)} \kappa_{ij} \Psi_{,i} \Psi_{,j} \, dx \leq \frac{\alpha_4}{2} \left[a_0^{-1} t \int_{\Omega(0)} H^{(1)2} \, dx + J(t) \right] + \frac{\gamma_2}{\alpha_4} \left[V_1(t) + S_1(t) \right], \quad (37)$$

and on choosing the arbitrary positive constant α_4 to satisfy

$$\alpha_4 = 2\gamma_2$$

we conclude that (37) becomes

$$V_1(t) + S_1(t) + \int_{\Omega(t)} \kappa_{ij} \Psi_{,i} \Psi_{,j} \, dx \leq 2\gamma_2 \left[a_0^{-1} t \int_{\Omega(0)} H^{(1)2} \, dx + J(t) \right]. \tag{38}$$

The estimates (35), (36), and (38) represent the required further conservation laws for the coupled quasi-static approximation subject to homogeneous Dirichlet boundary conditions.

Remark 1. *The assumption that* (25) *is valid at* $t = 0$ *implies that the initial values of* $v_i(x, 0)$ *and* $\psi(x, 0)$ *cannot be independently prescribed. For example, when* $\psi(x, 0) = 0$*, under the stated conditions,* $v_i(x, 0)$ *is uniquely determined from the data. In particular, when* $F_i = r = 0$ *and the boundary conditions are homogeneous, we have* $v_i(x, 0) = 0$ *and consequently* $H^{(1)}(x)$ *vanishes. It follows from inequality* (38) *that only the trivial solution exists when* $\psi(x, 0) = 0$.

Vanishing source terms in addition to homogeneous Dirichlet boundary conditions also are sufficient for the velocity to dominate both the temperature and displacement. The assertion is proved by first noting that from (26) we obtain

$$
\begin{aligned}
\int_{\Omega(t)} a\psi\psi_{,t}\,dx &= \int_{\Omega(t)} \left[\left(\kappa_{ij}\psi_{,i}\right)_{,j}\psi - \beta_{ij}v_{i,jt}\psi\right]\,dx \\
&= \int_{\Omega(t)} \left[\beta_{ij}v_{i,t}\psi_{,j} + \beta_{ij,j}v_{i,t}\psi - \kappa_{ij}\psi_{,i}\psi_{,j}\right]\,dx \\
&\leq \left[\int_{\Omega(t)} \left(\beta_{ij}\beta_{ij} + \lambda^{-1}\beta_{ik,k}\beta_{im,m}\right) v_{l,t}v_{l,t}\,dx \int_{\Omega(t)} \left(\psi_{,i}\psi_{,i} + \lambda\psi^2\right),dx\right]^{1/2} \\
&\quad - \int_{\Omega(t)} \kappa_{ij}\psi_{,i}\psi_{,j}\,dx \\
&\leq \left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\left[\int_{\Omega(t)} v_{i,j}v_{i,j}\,dx \int_{\Omega(t)} \psi_{,i}\psi_{,i}\,dx\right]^{1/2} - \int_{\Omega(t)} \kappa_{ij}\psi_{,i}\psi_{,j}\,dx \\
&\leq \left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\left[\frac{\alpha_5}{2}\int_{\Omega(t)} v_{i,j}v_{i,j}\,dx + \frac{1}{2\alpha_5}\int_{\Omega(t)} \psi_{,i}\psi_{,i}\,dx\right] - \int_{\Omega(t)} \kappa_{ij}\psi_{,i}\psi_{,j}\,dx \\
&\leq \left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\frac{\alpha_5}{2}\int_{\Omega(t)} v_{i,j}v_{i,j}\,dx \\
&\quad + \left(\left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\frac{1}{2\alpha_5} - \kappa_0\right)\int_{\Omega(t)} \psi_{,i}\psi_{,i}\,dx. \qquad (39)
\end{aligned}
$$

Select the positive constant α_5 to satisfy

$$
-\gamma_3 \equiv \left(\left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\frac{1}{2\alpha_5} - \kappa_0\right) < 0
$$

so that (39) may be rewritten as

$$
G'(t) + 2\left(\frac{\lambda\gamma_3}{\bar{a}}\right)G(t) \leq \alpha_5\left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\int_{\Omega(t)} v_{i,t}v_{i,t}\,dx
$$

where

$$
G(t) = \int_{\Omega(t)} a\psi^2(x,t)\,dx.
$$

Integration of the last inequality leads to the required continuous dependence estimate

$$
G(t) \leq \exp\left(-\frac{2\gamma_3\lambda t}{\bar{a}}\right)G(0) + \alpha_5\left[2\beta^2\left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2}\int_0^t\int_{\Omega(\eta)} v_{i,\eta}v_{i,\eta}\,dxd\eta. \qquad (40)
$$

On the other hand, commencing from (25), we have the relations

$$\begin{aligned}
\int_{\Omega(t)} c_{ijkl} v_{i,j} v_{k,l}\, dx &= \int_{\Omega(t)} \beta_{ij} \psi v_{i,j}\, dx \\
&\leq \left[\beta^2 a_0^{-1} G(t) \int_{\Omega(t)} v_{i,j} v_{i,j}\, dx\right]^{1/2} \\
&\leq \left[\frac{8\beta^2}{a_0 c_0} G(t) \int_{\Omega(t)} c_{ijkl} v_{i,j} v_{k,l}\, dx\right]^{1/2} \\
&\leq \frac{8\beta^2}{a_0 c_0} G(t) \\
&\leq \frac{8\beta^2}{a_0 c_0} \left[\exp\left(-\frac{2\gamma_3 \lambda t}{\bar{a}}\right) G(0)\right. \\
&\quad \left. + \alpha_5 \left[2\beta^2 \left(\frac{1+\lambda}{\lambda}\right)\right]^{1/2} \int_0^t \int_{\Omega(\eta)} v_{i,\eta} v_{i,\eta}\, dx d\eta\right], \qquad (41)
\end{aligned}$$

upon substitution from (40) and recalling (17). A continuous dependence estimate for the displacement is derived from (41) and (17).

We conclude that the evolving velocity and the initial value of the temperature, both in mean square measure, control both the temperature and displacement.

Dependence upon velocity is now removed. For this purpose, we seek a bound for the mean-square velocity occurring in (41) subject to zero source terms that mplies $J(t) = 0$. In consequence, for sufficient differentiability, (38) leads to

$$\begin{aligned}
\int_0^t \int_{\Omega(\eta)} v_{i,\eta} v_{i,\eta}\, dx d\eta &\leq \frac{8}{\lambda c_0} \int_0^t \int_{\Omega(\eta)} c_{ijkl} v_{i,j\eta} v_{k,l\eta}\, dx d\eta \\
&\leq \frac{16\gamma_2}{\lambda a_0 c_0} t \int_{\Omega(0)} H^{(2)2}\, dx
\end{aligned}$$

where, by analogy with (28), we have

$$\begin{aligned}
H^{(2)}(x) &= a\psi_{,t}(x,0) + \beta_{ij} v_{i,jt}(x,0) \\
&= \left[-\beta_{ij} v_{i,jt}(x,0) + \left(\kappa_{ij} \psi_{,i}(x,0)\right)_{,j}\right] + \beta_{ij} v_{i,jt}(x,0) \\
&= \left(\kappa_{ij} \psi_{,i}(x,0)\right)_{,j},
\end{aligned}$$

which is known from the initial data $\psi(x, 0)$.

We conclude that the displacement, velocity, and temperature are each continuously dependent upon the initial temperature and therefore are uniquely defined by this initial value.

Remark 2. *When source terms are absent and* (25) *holds at* $t = 0$ *a non-zero initial temperature is compatible with vanishing initial displacement* $v_i(x, 0)$ *provided the initial temperature satisfies*

$$\psi(x, 0) = \tilde{\beta}_{ij} c_{ij} = f(x) \tag{42}$$

where $\tilde{\beta}_{ij}$ *are conponents of the matrix inverse to* β *and* c_{ij} *are constants satisfying*

$$\tilde{\beta}_{ik} c_{ij} = 0, \qquad k \neq j.$$

In general, $\tilde{\beta}_{ij}$ *and therefore the function* f *defined in* (42) *does not vanish on the boundary and consequently* $c_{ij} = \psi(x, 0) = 0$*. The previous argument implies that* $v_i(x, t) = \psi(x, t) \equiv 0$.

The related property of spatial stability for the coupled quasi-static approximation is studied by Quintanilla [21] by means of differential inequalities. The region Ω is assumed to be a semi-infinite cylinder subject to zero source terms and homogeneous Dirichlet boundary conditions except on the cylinder's base. Conditions on the base at infinity are unspecified.

Uncoupled quasi-static approximation

The uncoupled quasi-static approximation supposes that the heat equation is independent of the velocity gradient, but that the temperature remains coupled to both displacement and velocity as a pseudo body-force in the mechanical equations of motion. The system for the uncoupled quasi-static approximation is accordingly given by

$$\begin{aligned}
\left(c_{ijkl}v_{k,l}\right)_{,j} - \left(\beta_{ij}\psi\right)_{,j} + \rho F_i &= 0, \qquad (x,t) \in \Omega \times [0,T), && (43)\\
\left(\kappa_{ij}\psi_{,i}\right)_{,j} + \bar{r} &= a\psi_{,t}, \qquad (x,t) \in \Omega \times [0,T), && (44)
\end{aligned}$$

where the same notation as in the previous Section is employed without confusion. The integrated form of the thermal conduction equation (44) becomes

$$a\Psi_{,t} = \left(\kappa_{ij}\Psi_{,i}\right)_{,j} + r + H^{(3)}(x), \qquad (x,t) \in \Omega \times [0,T), \tag{45}$$

where

$$H^{(3)}(x) = a\psi(x,0).$$

The displacement and temperature boundary conditions are those given by the homogenous relations (7) and (8). Specification of initial conditions is postponed. Moreover, we suppose for the remainder of this subsection that source terms are zero.

Consequently, (44) multiplied by ψ may be spatially integrated to give

$$\int_{\Omega(t)} a\psi\psi_{,t}\,dx + \int_{\Omega(t)} \kappa_{ij}\psi_{,i}\psi_{,j}\,dx = 0,$$

which after a time integration and use of the positive-definite condition (11) leads to the well-known decay estimate

$$\int_{\Omega(t)} a\psi^2\,dx \le \exp\left(-\frac{2\kappa_0\lambda t}{\bar{a}}\right)\int_{\Omega(0)} a\psi^2\,dx. \tag{46}$$

A related result is that

$$\max_{x\in\Omega}|\psi|$$

monotonically decreases with respect to time $t \geq 0$. See [1, p.149] and the cited references.

Inequality (17) in conjunction with (43) is now used to show that

$$\begin{aligned}
\int_{\Omega(t)} c_{ijkl} v_{i,j} v_{k,l}\, dx &= \int_{\Omega(t)} \beta_{ij} \psi v_{i,j}\, dx \\
&\leq \beta \left[\int_{\Omega(t)} \psi^2\, dx \int_{\Omega(t)} v_{i,j} v_{i,j}\, dx \right]^{1/2} \\
&\leq \beta \left(\frac{8}{c_0} \right)^{1/2} \left[\int_{\Omega(t)} \psi^2\, dx \int_{\Omega(t)} c_{ijkl} v_{i,j} v_{k,l}\, dx \right]^{1/2} \\
&\leq \frac{8\beta^2}{c_0 a_0} \int_{\Omega(t)} a\psi^2\, dx \\
&\leq \exp\left(-\frac{2\kappa_0 \lambda t}{\bar{a}} \right) \frac{8\beta^2}{c_0 a_0} \int_{\Omega(0)} a\psi^2\, dx, \qquad (47)
\end{aligned}$$

where the last inequality follows from (46).

A similar decaying upper bound is valid for the displacement in mean-square measure upon appeal to inequaltiy (17).

It is easily inferred from (46) and (47) that zero initial temperature $\psi(x, 0) = 0$ implies that only the trivial solution $v_i(x, t) = \psi(x, t) = 0$ exists for $(x, t) \in \Omega \times [0, T]$. Remark 1 is also relevant. Furthermore, when (43) holds at $t = 0$ a non-zero initial temperature is compatible with vanishing initial displacement $v_i(x, 0)$ subject to conditions discussed in Remark 2.

Dependence on inertia

Errors that occur when the exact problem is replaced by the respective quasi-static approximations are determined by Steps $1, 2,$ and 3 of the procedure proposed in teh introduction. Dependence of inertia on initial data is established in the third Section. Consequently, we complete the analysis by demonstrating for Step 1 how the difference in the solutions depends upon the inertia. We separately treat the coupled and uncoupled approximations. Since for the problem under consideration motion is governed solely by initial data the decay envisaged in Step 2 is brought about solely by thermal dissipation which is included in our analysis. Recall that we have chosen to exclude nonhomogeneous boundary data from the present study.

Dependence for the coupled quasi-static approximation

Let us set

$$\begin{aligned} u_i(x,t) &= w_i(x,t) - v_i(x,t), \qquad (x,t) \in \bar{\Omega} \times [0,T), \\ \theta(x,t) &= \phi(x,t) - \psi(x,t) \qquad (x,t) \in \bar{\Omega} \times [0,T), \\ \Theta(x,t) &= \Phi(x,t) - \Psi(x,t) \qquad (x,t) \in \bar{\Omega} \times [0,T), \end{aligned}$$

where $\bar{\Omega}$ designates the closure of Ω, and (v_i, ψ) is the solution pair to the coupled quasi-static approximation introduced in Section .

Subtraction of the respective equations of motion and heat conduction equations shows that the pair (u_i, θ) satisfies the system

$$\left(c_{ijkl}u_{k,l}\right)_{,j} - \left(\beta_{ij}\theta\right)_{,j} = \rho w_{i,tt}, \qquad (x,t) \in \Omega \times [0,T), \tag{48}$$

$$\left(\kappa_{ij}\theta_{,i}\right)_{,j} - \beta_{ij}u_{i,jt} = a\theta_{,t}, \qquad (x,t) \in \Omega \times [0,T), \tag{49}$$

$$\left(\kappa_{ij}\Theta_{,i}\right)_{,j} - \beta_{ij}u_{i,j} + Q(x) = a\Theta_{,t}, \qquad (x,t) \in \Omega \times [0,T), \tag{50}$$

where

$$Q(x) = H(x) - H^{(1)}(x) = a\theta(x,0) + \beta_{ij}u_{i,j}(x,0), \qquad x \in \Omega.$$

Corresponding boundary conditions are given by

$$\begin{aligned} u_i(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T), \\ \theta(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T). \end{aligned}$$

Cauchy initial conditions are specified later.

Set

$$\begin{aligned} V_2(t) &= \int_0^t \int_{\Omega(\eta)} c_{ijkl} u_{i,j} u_{k,l} \, dx d\eta, \\ S_2(t) &= \int_0^t \int_{\Omega(\eta)} a\theta^2 \, dx d\eta. \end{aligned}$$

Multiplication of (48) and (50) respectively by u_i and θ, integration both by parts and with respect to time, after addition of the resulting equations leads to

$$\begin{aligned} V_2(t) + S_2(t) + \frac{1}{2} \int_{\Omega(t)} \kappa_{ij} \Theta_{,i} \Theta_{,j} \, dx &= \int_0^t \int_{\Omega(\eta)} \rho w_{i,\eta\eta} u_i \, dx d\eta + \int_0^t \int_{\Omega(\eta)} Q(x)\theta \, dx d\eta \\ &\leq \left[\frac{8\bar{\rho}}{\lambda c_0} \int_0^t \int_{\Omega(\eta)} \rho w_{i,\eta\eta} w_{i,\eta\eta} \, dx d\eta + a_0^{-1} t \int_{\Omega(0)} Q^2 \, dx \right]^{1/2} \times \\ &\quad \times \left[\frac{\lambda c_0}{8} \int_0^t \int_{\Omega(\eta)} u_i u_i \, dx d\eta + a_0 \int_0^t \int_{\Omega(\eta)} \theta^2 \, dx d\eta \right]^{1/2} \\ &\leq \left[\frac{8\bar{\rho}}{\lambda c_0} \int_0^t \int_{\Omega(\eta)} \rho w_{i,\eta\eta} w_{i,\eta\eta} \, dx d\eta + a_0^{-1} t \int_{\Omega(0)} Q^2 \, dx \right]^{1/2} \times \\ &\quad \times [V_2(t) + S_2(t)]^{1/2} \end{aligned}$$

Young's inequality leads to the required continuous dependence estimate given by

$$V_"(t) + S_2(t) + \int_{\Omega(t)} \kappa_{ij} \Theta_{,i} \Theta_{,j} \, dx \leq \left[\frac{8\bar{\rho}}{\lambda c_0} \int_0^t \int_{\Omega(\eta)} \rho w_{i,\eta\eta} w_{i,\eta\eta} \, dx d\eta + a_0^{-1} t \int_{\Omega(0)} Q^2 \, dx \right]^{1/2} \quad (51)$$

Remark 3. *Observe that* (51) *establishes continuous dependence upon the inertia in mean square measure over space-time. Moreover, when initial data differ between the exact and approximate problems so that $Q(x) \neq 0$, then the error caused by adopting the approximation* *is clearly indicated by* (51). *It is physically reasonable to assume, however, that the initial data remains the same for both the exact problem and its coupled quasi-static approximation. Consequently,* $Q(x) = 0$*, and only inertia affects the error*

Dependence for the uncoupled quasi-static approximation

The notation adopted in the previous subsection is retained. Subtraction of the equations (43) and (45) from (1) and (15) yields

$$\rho w_{i,tt} = (c_{ijkl}u_{k,l})_{,j} - (\beta_{ij}\theta)_{,j} \qquad (x,t) \in \Omega \times [0,T), \tag{52}$$

$$a\Theta_{,t} = (\kappa_{ij}\Theta_{,i})_{,j} - \beta_{ij}u_{i,j} - \beta_{ij}v_{i,j} + Q(x), \qquad (x,t) \in \Omega \times [0,T), \tag{53}$$

where $(v_i,\, \phi)$ is the solution to (43) and (44), and now

$$Q(x) = H(x) - H^{(3)}(x) = a\theta(x,0) + \beta_{ij}w_{i,j}^{(0)}, \qquad x \in \Omega.$$

The difference displacement $u_i(x,t)$ and difference temperature $\theta(x,t)$ satisfy homogeneous boundary conditions and the corresponding source terms vanish.

To derive an estimate for continuous dependence upon inertia in the uncoupled approximation, we treat (52) and (53) by arguments similar to those employed previously. Properties, however, of the solution (v_i, ψ) to the uncoupled quasi-static approximation derived previously are now used. The following relations are obtained:

$$\begin{aligned}
\int_{\Omega(t)} c_{ijkl}u_{i,j}u_{k,l}\,dx - \int_{\Omega(t)} \beta_{ij}\theta u_{i,j}\,dx &= -\int_{\Omega(t)} \rho w_{i,tt}u_i\,dx,\\
\int_{\Omega(t)} a\theta^2\,dx + \int_{\Omega(t)} \kappa_{ij}\Theta_{,i}\Theta_{,jt}\,dx + \int_{\Omega(t)} \beta_{ij}\theta u_{i,j}\,dx &= -\int_{\Omega(t)} \beta_{ij}\theta v_{i,j}\,dx + \int_{\Omega(t)} Q\theta\,dx,
\end{aligned}$$

which on addition leads to

$$
\begin{aligned}
& \int_{\Omega(t)} c_{ijkl}u_{i,j}u_{k,l}\,dx + \int_{\Omega(t)} \kappa_{ij}\Theta_{,i}\Theta_{,jt}\,dx + \int_{\Omega(t)} a\theta^2\,dx \\
= \; & -\int_{\Omega(t)} \rho w_{i,tt}u_i\,dx - \int_{\Omega(t)} \beta_{ij}\theta v_{i,j}\,dx + \int_{\Omega(t)} Q\theta\,dx \\
\leq \; & \left[\frac{8\bar{\rho}}{\lambda c_0}\int_{\Omega(t)} \rho w_{i,tt}w_{i,tt}\,dx + \frac{2\beta^2}{a_0}\int_{\Omega(t)} v_{i,j}v_{i,j}\,dx + \frac{2}{a_0}\int_{\Omega} Q^2\,dx\right]^{1/2} \times \\
& \times \left[\int_{\Omega(t)} c_{ijkl}u_{i,j}u_{k,l}\,dx + \int_{\Omega(t)} a\theta^2\,dx\right]^{1/2}. \qquad (54)
\end{aligned}
$$

Young's inequality and an application of the estimate (47) after rearrangement reduces (54) to

$$
\begin{aligned}
& \int_{\Omega(t)} c_{ijkl}u_{i,j}u_{k,l}\,dx + 2\int_{\Omega(t)} \kappa_{ij}\Theta_{,i}\Theta_{,jt}\,dx + \int_{\Omega(t)} a\theta^2\,dx \\
\leq \; & \frac{8\bar{\rho}}{\lambda c_0}\int_{\Omega(t)} \rho w_{i,tt}w_{i,tt}\,dx + \frac{16\beta^4}{a_0^2 c_0}\exp\left(-\frac{2\kappa_0\lambda t}{\bar{a}}\right)\int_{\Omega(0)} a\psi^2\,dx \\
+ \; & \frac{2}{a_0}\int_{\Omega(0)} Q^2\,dx. \qquad (55)
\end{aligned}
$$

The required continuous dependence estimate is obtained by a time integration of (55). We have

$$
\begin{aligned}
& \int_0^t \int_{\Omega(\eta)} c_{ijkl}u_{i,j}u_{k,l}\,dx d\eta + \int_{\Omega(t)} \kappa_{ij}\Theta_{,i}\Theta_{,j}\,dx + \int_0^t \int_{\Omega(\eta)} a\theta^2\,dx d\eta \\
\leq \; & \frac{8\bar{\rho}}{\lambda c_0}\int_0^t \int_{\Omega(\eta)} \rho w_{i,\eta\eta}w_{i,\eta\eta}\,dx d\eta + \frac{8\bar{a}\beta^4}{\lambda\kappa_0 a_0^2 c_0}\int_{\Omega(0)} a\psi^2\,dx \\
& + \frac{2t}{a_0}\int_{\Omega(0)} Q^2\,dx. \qquad (56)
\end{aligned}
$$

Remark 4. *When initial data in the exact and uncoupled approximation problems are the same then clearly $\theta(x,0) = u_i(x,0) = 0$, and the estimate* (56) *simplifies to dependence solely upon the inertia and initial value of the uncoupled quasi-static temperature.*

Dependence of the inertia upon initial data for the uncoupled quasi-static approximation may be established from estimate (24). Alternatively, but again for zero source terms and homogeneous Dirichlet boundary conditions, under the assumption of sufficiently differentialblity, we may proceed differently. We begin by writing equation (52) in the form

$$\rho u_{i,tt} = \left(c_{ijkl}u_{k,l} - \beta_{ij}\theta\right)_{.j} - \rho v_{i,tt}, \qquad (x,t) \in \Omega \times [0,T), \tag{57}$$

while subtraction of (44) from (2) gives

$$a\theta_{,t} = -\beta_{ij}u_{i,jt} + \left(\kappa_{ij}\theta_{,i}\right)_{,j} - \beta_{ij}v_{i,jt}. \tag{58}$$

Let

$$L(t) = \int_{\Omega(t)} \left(\rho u_{i,t}u_{i,t} + c_{ijkl}u_{i,j}u_{k,l} + a\theta^2\right)\, dx.$$

On combining (57) and (58) and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2}L'(t) \;\; &+ \;\; \int_{\Omega(t)} \kappa_{ij}\theta_{,i}\theta_{,j}\, dx = -\int_{\Omega(t)} \left(\rho v_{i,tt}u_{i,t} + \beta_{ij}v_{i,jt}\theta\right)\, dx \\ &\leq \;\; L^{1/2}(t)\left[\int_{\Omega(t)} \rho v_{i,tt}v_{i,tt}\, dx + a_0^{-1}\beta^2 \int_{\Omega(t)} v_{i,jt}v_{i,jt}\, dx\right]^{1/2}. \end{aligned} \tag{59}$$

Integration of (59) yields

$$L^{1/2}(t) \leq L^{1/2}(0) + \int_0^t \left[\int_{\Omega(\eta)} \rho v_{i,\eta\eta}v_{i,\eta\eta}\, dx + a_0^{-1}\beta^2 \int_{\Omega(\eta)} v_{i,j\eta}v_{i,j\eta}\, dx\right]^{1/2} d\eta. \tag{60}$$

Bounds are now sought for both terms on the right of the last inequality. Consider the first term. We have from (17) and (47) that

$$\begin{aligned} \int_{\Omega(t)} \rho v_{i,tt}v_{i,tt}\, dx \;\; &\leq \;\; \bar{\rho}\int_{\Omega(t)} v_{i,tt}v_{i,tt}\, dx \\ &\leq \;\; \frac{8\bar{\rho}}{\lambda c_0}\int_{\Omega(t)} c_{ijkl}v_{i,jtt}v_{k,ltt}\, dx \\ &\leq \;\; \frac{64\bar{\rho}\beta^2}{\lambda(a_0c_0)^2}\exp\left\{-\frac{2\kappa_0\lambda t}{\bar{a}}\right\}\int_{\Omega(0)} (a\psi_{,tt})^2\, dx. \end{aligned} \tag{61}$$

Moreover, (44) implies for sufficient smoothness that

$$
\begin{aligned}
a\psi_{,tt} &= \left(\kappa_{ij}\psi_{,it}\right)_{,j} \\
&= \left[\kappa_{ij}\left\{a^{-1}\left(\kappa_{pq}\psi_{,p}\right\}_{,q}\right)_{,i}\right]_{,j},
\end{aligned}
$$

which therefore is known from the initial data for ψ provided (44) holds at $t = 0$.

The second term on the right of (60) is treated as follows:

$$
\begin{aligned}
\int_{\Omega(t)} v_{i,jt}v_{i,jt}\,dx &\le \frac{8}{c_0}\int_{\Omega(t)} c_{ijkl}v_{i,jt}v_{k,lt}\,dx \\
&\le \frac{64\beta^2}{(a_0c_0)^2}\exp\left\{-\frac{2\kappa_0\lambda t}{\bar{a}}\right\}\int_{\Omega(0)}(a\psi_{,t})^2\,dx,
\end{aligned}
$$

where (17) and (47) are again used. The initial value of $\psi_{,t}$ is obtained from $\psi(x, 0)$ on employing (44).

Substitution of these estimates in (60) leads to the final bound

$$
\begin{aligned}
L^{1/2}(t) \;\le\; & L^{1/2}(0) + \frac{8\beta\bar{a}}{\lambda\kappa_0c_0a_0}\left(1 - \exp\left\{-\frac{\kappa_0\lambda t}{\bar{a}}\right\}\right)\times \\
&\times\left[\left(\frac{\bar{\rho}}{\lambda}\right)^{1/2}\left(\int_{\Omega(0)} a\psi_{,tt}^2\,dx\right)^{1/2} + \frac{\beta}{a_0^{1/2}}\left(\int_{\Omega(0)}(a\psi_{,t})^2\,dx\right)^{1/2}\right]. \qquad (62)
\end{aligned}
$$

Dependence of inertia on initial data is now obtained on noting that

$$
\int_{\Omega(t)} \rho w_{i,tt}w_{i,tt}\,dx \le 2\int_{\Omega(t)} \rho\left(u_{i,tt}u_{i,tt} + v_{i,tt}v_{i,tt}\right)\,dx.
$$

The integral on the right is evaluated by insertion of the bound (61) for the second term, and adapting (62) to derive a bound for the first term. The respective solutions must be sufficiently differentiable for these operations to be valid.

Concluding remarks

The uncritical application of quasi-static approximations is not uncommon in the literature of both linear and nonlinear systems. There is a comparatively small number of papers that rigorously validate the approximations but only for particular problems. The present study treats one aspect of the general procedure proposed in Section for the initial boundary value problem of classical linear thermoelasticity.

The analysis presented justifies the coupled and uncoupled quasi-static approximations subject to homogeneous Dirichlet boundary conditions and zero source terms by examining the effect of initial conditions. Thermal dissipation and initial data affect the rate of decay of the inertia which must be of a smaller order of magnitude than either the displacement, velocity or temperature for the approxiamtions to be valid.

Other linear and nonlinear theories including coupled linearised systems await detailed investigation. In particular, a discussion of nonhomogeneous time evolving boundary conditions for general systems would be of significant interest. Whether singular perturbations, the application of inequalities or some other approach is required remains open but the three component steps listed in the introduction seem crucial for these developments.

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