

# POWER TO INVEST

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## ABSTRACT

In this post recession time it is important to measure the possibilities offered by a society in relation to investments. To do that, we consider an investment schema  $\mathcal{I} = \langle R; R_1, \dots, R_n \rangle$  where  $R$  is a lower bound on the desired return and the  $R_i$ 's are the return of the assets (to invest in). We introduce *the power to invest*, denoted by  $\text{Power}(\mathcal{I})$ , a measure of the capability of the schema to fulfilling the requirement  $R$ . The power to invest is inspired in the Coleman's power of a collectivity to act. We consider the angel-daemon approach to uncertainty and extend it to investment schemas. The approach tries to tune cases in-between the worst and the best scenarios and analyze them through game theory. We show how to use the power to invest to asses uncertainty in such situations and develop several examples.

## 1. INTRODUCTION

In 1952 Harry Markowitz introduced the mean-variance approach [6]. Consider a set  $N = [n] = \{1, \dots, n\}$  of assets with expected returns  $R_1, \dots, R_n$ . A portfolio provides positive weights  $(w_1, \dots, w_n) \in \Delta_n$ , that is  $\sum_{1 \leq j \leq n} w_j = 1$ , for assets. The expected return (the mean) is  $\mathbb{E}(I) = \sum_{1 \leq j \leq n} w_j R_j$ . The variance is  $\mathbb{V}\text{ar}(I) = \sum_{1 \leq j, k \leq n} w_j w_k \sigma_j \sigma_k \rho_{j,k}$  where  $\sigma_i$  is the standard deviation for asset  $i$  and  $\rho_{i,j}$  is the correlation coefficient for assets  $i$  and  $j$ . The Markowitz approach consider points  $(\mathbb{E}(I), \mathbb{V}\text{ar}(I))$  for different portfolios and weights. In 1971 James Coleman [2] introduced the formal definition of the power of a collectivity to act. In this paper we adapt this idea to provide investors with information about his degree of freedom to choice. In [4, 3, 5] we model uncertainty trough strategic situations in-between the worst and the best scenarios. Here we use the variance to define such scenarios. This strategic approach is used to extend the power to invest to take into account uncertainty. In this way, the power to invest can be used to study the behaviour, as a whole, of the investing capabilities of a given market.

## 2. POWER TO INVEST

For  $n$  assets with returns  $R_1, \dots, R_n$ , we roughly identify possible investments by the subsets of  $N$ . In order to associate a unique return to any  $I \subseteq N$  we have to take

in addition a probability distribution on the elements in  $I$ . To exemplify our approach we consider in the remaining of the paper the uniform distribution which grants a maximal variety and diversification among investments. Given an investment  $I = (i_1, \dots, i_k) \subseteq N$ , under equiprobable weights, the expected return is

$$\mathbb{E}(I) = \frac{1}{\#I} \sum_{i_j \in I} R_{i_j}$$

An *investment schema* is a tuple  $\mathcal{I} = \langle R; R_1, \dots, R_n \rangle$  where  $R > 0$  is the minimal acceptable return for any investment<sup>1</sup>. Given an investment schema  $\mathcal{I}$ , an investment  $I$  is *feasible* (or acceptable) for  $\mathcal{I}$  iff  $\mathbb{E}(I) \geq R$ . The set of all *feasible investments* is given by  $F(\mathcal{I}) = \{I \mid \mathbb{E}(I) \geq R\}$ . The empty investment  $I = \emptyset$  is never a feasible investment because  $\mathbb{E}(I) = 0$  (there is not possible to get a return if there is no investment at all). Therefore,  $\#F(\mathcal{I}) \leq 2^n - 1$ . We define the *power to invest* of  $\mathcal{I}$  as

$$\text{Power}(\mathcal{I}) = \frac{1}{2^n - 1} \#F(\mathcal{I})$$

The power to invest provides a rough estimation of the capabilities to invest in an environment described by  $\mathcal{I}$  and the associated probability distributions. It measures the dynamicity of the society to fulfil  $R$ . For a moderate  $R$ , in an active society, there should be many different ways to get a return  $R$ . Observe that the size of  $F(\mathcal{I})$  is a measure of this fact. As  $0 \leq \#F(\mathcal{I}) \leq 2^n - 1$  it holds  $0 \leq \text{Power}(\mathcal{I}) \leq 1$ . The power to invest gives precise mathematical meaning to some basic facts. Let  $\mathcal{I} = \langle R; R_1, \dots, R_n \rangle$  and  $\mathcal{I}' = \langle R'; R'_1, \dots, R'_n \rangle$  with  $R' \geq R$ .

- When the minimal return is low, the power to invest is high. This is translated as follows, when  $R \leq \min_{i \in N} R_i$ ,  $\text{Power}(\mathcal{I}) = 1$ .
- When the minimal return is high, the power is low. When  $R$  is too high it could be impossible to fulfill it. Formally,  $\text{Power}(\mathcal{I})$  is zero when  $R > \max_{i \in N} R_i$ .
- When the minimal return increases, the power to invest cannot increase. That is, when  $R' > R$  and  $R_i = R'_i$ ,  $1 \leq i \leq n$ ,  $\text{Power}(\mathcal{I}) \geq \text{Power}(\mathcal{I}')$ .
- When productivity (and returns) increases globally, the power to invest cannot decrease. This translates into, when  $R'_i > R_i$  and  $R = R'$ ,  $1 \leq i \leq n$ ,  $\text{Power}(\mathcal{I}) \leq \text{Power}(\mathcal{I}')$ .

Let us consider some highly stylized investment schemes. First of all consider a case where all the assets have the same return. To denote  $n$  assets all of them with the same return  $R$  we write  $n:R$ . Then, we note  $\mathcal{I} = \langle R'; R, \dots, R \rangle = \langle R'; n:R \rangle$ . Consider an  $I$  containing  $k$  assets,  $0 < k \leq n$ . We shorten  $I = (k:R)$ . As  $\mathbb{E}(I) = (kR)/k = R$ ,

<sup>1</sup>The notion of investment schema is inspired from *weighted voting games* [7].

independently of the value of  $k$  (while  $k > 0$ ),  $\#F(\mathcal{I}) = 2^n - 1$  if  $R' \leq R$  and 0 otherwise. Then  $\text{Power}(\mathcal{I}) = 1$  if  $R' \leq R$  and 0 otherwise.

Let us consider another stylized investment schema  $\mathcal{I} = \langle R, n_1:R_1, n_2:R_2 \rangle$ . Assume without loss of generality that  $R_1 \geq R_2$ . Only the case  $R_1 \geq R \geq R_2$  is interesting because, when  $R \leq R_2$  the power is 1 and when  $R_1 < R$  the power is 0. Any investment  $I = (k_1:R_1, k_2:R_2)$  with  $k_1 + k_2 > 0$  verifies

$$\mathbb{E}(I) = \frac{k_1}{k_1 + k_2} R_1 + \frac{k_2}{k_1 + k_2} R_2$$

Feasible investments are defined by pair  $(k_1, k_2)$  in the following set

$$\mathcal{R} = \{(k_1, k_2) \mid (k_1 + k_2)R \leq k_1 R_1 + k_2 R_2, k_1 \leq n_1, k_2 \leq n_2, k_1 + k_2 > 0\}$$

A pair  $(k_1, k_2) \in \mathcal{R}$  allows for  $\binom{n_1}{k_1} \binom{n_2}{k_2}$  different feasible investments. Therefore,

$$\text{Power}(\langle R, n_1 : R_1, n_2 : R_2 \rangle) = \frac{1}{2^{n_1+n_2} - 1} \sum_{(k_1, k_2) \in \mathcal{R}} \binom{n_1}{k_1} \binom{n_2}{k_2}$$

Taking a numerical example  $\mathcal{I} = \langle 0.10, 7:0.15, 4:0.05 \rangle$ . The restriction  $(k_1 + k_2)R \leq k_1 R_1 + k_2 R_2$  gives  $0.10(k_1 + k_2) \leq 0.15k_1 + 0.05k_2$ . Multiplying by 100 dividing by 5 and regrouping, we get  $k_1 \geq k_2$ . Therefore,  $\mathcal{R} = \{(k_1, k_2) \mid k_1 \geq k_2, k_1 \leq 7, k_2 \leq 4, k_1 + k_2 > 0\}$  and  $\text{Power}(\mathcal{I}) = 0.818$ .

Now we compute the power to invest of data taken from the 1959 foundational work of H. Markowitz [6]. We consider the years 1943 and 1944 with really impressive returns (World War II) and the average over 18 years (1937-54) with more moderate (average) returns.

Year	1 Am.T	2 A.T. & T.	3 U.S.S.	4 G.M.	5 A.T.& Sfe	6 C.C	7 Bdn.	8 Frstn	9 S.S
1943	0.428	0.300	0.149	0.225	0.313	0.351	0.341	0.580	0.639
1944	0.192	0.103	0.260	0.290	0.637	0.233	0.227	0.473	0.282
1937-54	0.066	0.062	0.146	0.173	0.198	0.055	0.128	0.190	0.116

We consider the associated investment schemas:

$$\mathcal{I}_{1943}(R) = \langle R; 0.428, 0.300, 0.149, 0.225, 0.313, 0.351, 0.341, 0.580, 0.639 \rangle$$

$$\mathcal{I}_{1944}(R) = \langle R; 0.192, 0.103, 0.260, 0.290, 0.637, 0.233, 0.227, 0.473, 0.282 \rangle$$

$$\mathcal{I}_{1937-54}(R) = \langle R; 0.066, 0.062, 0.146, 0.173, 0.198, 0.055, 0.128, 0.190, 0.116 \rangle$$

The power to invest, for different values of  $R$  is given in the following table.

$R$	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\text{Power}(\mathcal{I}_{1943}(R))$	1.0	0.996	0.878	0.287	0.019	0.003	0.0
$\text{Power}(\mathcal{I}_{1944}(R))$	1.0	0.962	0.497	0.048	0.003	0.001	0.0
$\text{Power}(\mathcal{I}_{1937-54}(R))$	0.906	0.0	0.0	0.0	0.0	0.0	0.0

According to this table, in 1943 there was more power to invest than in 1944. On the average returns the power is impressively smaller.

### 3. UNCERTAINTY

Returns are strongly volatile. Markowitz [6] studied the volatility of Am.T from 1937 to 1954. The expected return was  $R_{\text{Am.T}} = 0.066$  and  $\sigma_{\text{Am.T}} = 0.231$ . Therefore, using the variance to measure the spread of a perturbation it can strongly affect the return  $R'_{\text{Am.T}} = R_{\text{Am.T}} - \sigma_{\text{Am.T}} = -0.165$ ,  $R''_{\text{Am.T}} = R_{\text{Am.T}} + \sigma_{\text{Am.T}} = 0.297$ .

To deal with the volatility in the case of the power to invest we adapt the notion of uncertainty profile [3] to this framework. A *uncertainty profile* is a tuple  $\mathcal{U} = \langle \mathcal{I}, \mathcal{A}, \mathcal{D}, \delta_\alpha, \delta_\delta, b_\alpha, b_\delta \rangle$  where  $\mathcal{I} = \langle R; R_1, \dots, R_n \rangle$  is an investment schema;  $\mathcal{A}, \mathcal{D} \subseteq [n]$  are the *sets of assets* whose returns may be subject to angelic and daemonic perturbations, respectively;  $\delta_\alpha : \mathcal{A} \rightarrow \mathbb{R}$  and  $\delta_\delta : \mathcal{D} \rightarrow \mathbb{R}$  represent the *strength* of the potential return's perturbations;  $b_\alpha, b_\delta \in \mathbb{N}$  are such that  $b_\alpha \leq \#\mathcal{A}$  and  $b_\delta \leq \#\mathcal{D}$  and they represent the *spread* of the angelic and daemonic perturbations. Based in mean-variance approach [6] we take  $\delta_\alpha(i) = \sigma_i$  and  $\delta_\delta(i) = -\sigma_i$ . The perturbation is exerted though joint actions  $(a, d)$ , for  $a \subseteq \mathcal{A}$ ,  $d \subseteq \mathcal{D}$  with  $\#a = b_\alpha$  and  $\#d = b_\delta$ . The consequence is a perturbed investment schema  $\mathcal{I}[a, d] = \langle R; R'_1, \dots, R'_n \rangle$  defined as  $R'_i = R_i + x_a(i)\sigma_i - x_d(i)\sigma_i$ , where  $x_a(i) = 1$  if  $i \in a$ ; 0 otherwise, and  $x_d(i) = 1$  if  $i \in d$ ; 0 otherwise.

Let us consider an example. The following table is adapted from Table 8.1 in[1].

	1	2	3	4	5
	BKE	FCEL	GG	OII	SEB
$R_i$	0.027	0.068	0.021	0.018	0.010
$\sigma_i$	0.15	0.270	0.189	0.164	0.124

For instance, given  $\mathcal{I} = \langle R, 0.027, 0.068, 0.021, 0.018, 0.010 \rangle$  and the perturbation  $(a, d) = (\{\text{BKE}\}, \{\text{GG}\}) = \{\{1\}, \{3\}\}$  we have

$$\mathcal{I}[a, d] = \langle R; R_1 + \sigma_1, R_2, R_3 - \sigma_3, R_4, R_5 \rangle = \langle R; 0.177, 0.068, -0.168, 0.018, 0.010 \rangle$$

We are interested to know how perturbations affect the power of an investment. Given  $\mathcal{U} = \langle \mathcal{I}, \mathcal{A}, \mathcal{D}, \delta_\alpha, \delta_\delta, b_\alpha, b_\delta \rangle$ , the associated *angel/daemon* (or  $\alpha/\delta$ ) game is  $G(\mathcal{U}) = \langle \{\alpha, \delta\}, A_\alpha, A_\delta, u_\alpha, u_\delta \rangle$ . Game  $G(\mathcal{U})$  has two players: the angel  $\alpha$  and the daemon  $\delta$ . The player's actions are  $A_\alpha = \{a \subseteq \mathcal{A} \mid \#a = b_\alpha\}$  and  $A_\delta = \{d \subseteq \mathcal{D} \mid \#d = b_\delta\}$ . For  $(a, d) \in A_\alpha \times A_\delta$  utilities are  $u_\alpha(a, d) = \text{Power}(\mathcal{I}[a, d])$  and  $u_\delta(a, d) = -u_\alpha(a, d)$ .

For instance, consider  $\mathcal{U}_1$  with  $R = 0.015$ ,  $\mathcal{A} = \mathcal{D} = \{\text{GG}, \text{OII}\} = \{3, 4\}$  and  $b_\alpha = b_\delta = 1$ . The  $\alpha/\delta$  game  $G(\mathcal{U}_1)$  has  $A_\alpha = A_\delta = \{\{\text{GG}\}, \{\text{OII}\}\} = \{\{3\}, \{4\}\}$ .

This game is represented, in tabular form ( $\alpha$  is the row player and  $\delta$  is the column player) where only  $u_\alpha$  appears, as

	{GG}	{OII}
{GG}	0.935	0.709
{OII}	0.580	0.935

Notice that, in an  $\alpha/\delta$  game the set of strategy profiles is  $A_\alpha \times A_\delta$ . Angel and daemon choices of actions can be done probabilistically. Mixed strategies for  $\alpha$  and  $\delta$  are probability distributions  $\alpha : A_\alpha \rightarrow [0, 1]$  and  $\beta : A_\delta \rightarrow [0, 1]$  respectively. A *mixed strategy* is a tuple  $(\alpha, \beta)$  such that  $u_p(\alpha, \beta) = \sum_{(a,d) \in A_\alpha \times A_\delta} \alpha(a)u_p(a, d)\beta(d)$  for  $p \in \{\alpha, \delta\}$ . Given  $u_\alpha(a, d) = \text{Power}(\mathcal{I}[a, d])$  it makes sense to extend Power to mixed strategies defining

$$\text{Power}(\mathcal{I}(\alpha, \beta)) = u_\alpha(\alpha, \beta) = \sum_{(a,d) \in A_\alpha \times A_\delta} \alpha(a)(\text{Power}(\mathcal{I}[a, d]))\beta(d)$$

Let  $\Delta_\alpha$  and  $\Delta_\delta$  denote the set of mixed strategies for  $\alpha$  and  $\delta$ , respectively. A pure strategy profile  $(a, d)$  is a special case of mixed strategy profile  $(\alpha, \beta)$  in which  $\alpha(a) = 1$  and  $\beta(d) = 1$ . A mixed strategy profile  $(\alpha, \beta)$  is a *Nash equilibrium* if for any  $\alpha' \in \Delta_\alpha$  it holds  $u_\alpha(\alpha, \beta) \geq u_\alpha(\alpha', \beta)$  and for any  $\beta' \in \Delta_\delta$  it holds  $u_\delta(\alpha, \beta) \geq u_\delta(\alpha, \beta')$ . A *pure Nash equilibrium*, pne, is a Nash equilibrium  $(a, d)$  where  $a$  and  $d$  are pure strategies. The preceding  $G(\mathcal{U}_1)$  has no pne. Game  $G(\mathcal{U}_1)$  has (mixed) Nash equilibrium given by  $(\alpha, \beta)$  such that  $\alpha = (\alpha(\{3\}), \alpha(\{4\})) = (0.388, 0.612)$  and  $\beta = (\beta(\{3\}), \beta(\{4\})) = (0.611, 0.389)$ . In this case  $\text{Power}(\mathcal{I}(\alpha, \beta)) = 0.796$ .

Given  $\mathcal{U}_2$  with  $R = 0.02$ ,  $\mathcal{A} = \{\text{BKE}, \text{FCEL}\}$ ,  $\mathcal{D} = \{\text{GG}, \text{OII}\}$  and  $b_\alpha = b_\delta = 1$ , the  $G(\mathcal{U}_2)$  is

	{GG}	{OII}
{BKE}	0.516	0.548
{FCEL}	0.580	0.612

The strategy  $(\{\text{FCEL}\}, \{\text{GG}\})$  is the only pne of  $G(\mathcal{U}_2)$  having a power of 0.580.

It is well known that all Nash equilibria of a zero-sum game  $G$  have the same value  $\nu(G)$  corresponding to the utility of the row player [8]. For an  $\alpha/\delta$  game  $G(\mathcal{U})$ ,

$$\nu(G(\mathcal{U})) = \max_{\alpha \in \Delta_\alpha} \min_{\beta \in \Delta_\delta} \text{Power}(\mathcal{I}(\alpha, \beta)) = \min_{\beta \in \Delta_\delta} \max_{\alpha \in \Delta_\alpha} \text{Power}(\mathcal{I}(\alpha, \beta))$$

Considering  $\alpha/\delta$  games we can extend the definition of the power to invest to uncertainty profiles as  $\text{Power}(\mathcal{U}) = \nu(G(\mathcal{U}))$ . In the preceding examples  $\text{Power}(\mathcal{U}_1) = 0.796$  and  $\text{Power}(\mathcal{U}_2) = 0.580$ .

#### 4. CONCLUSIONS AND OPEN TOPICS

We have introduced the notion of power to invest as a measure of the freedom to invest in different assets. We apply this notion to a variety of cases and showing its workability. We considered equiprobable weights but other distributions could also be considered. In particular repeating assets values allow us to model other distributions.

We also adapt the notion of uncertainty profile to deal with the volatility. Using the  $\alpha/\delta$  approach we show that the power to invest is well shaped to deal with uncertainty

profiles. Here the roles of  $\alpha$  and  $\delta$  are symmetrical. The angel increases the return by  $\sigma_i$  and the daemon decreases by  $-\sigma_i$ . Nevertheless, asymmetrical views are also possible (in order to emphasize disasters, we can take  $\delta_\alpha = \sigma_i/2$  and  $\delta_\delta = -2\sigma_i$ ). We have not taken into account correlations. However, correlation can help to design  $\mathcal{A}$  and  $\mathcal{D}$ . Finally, the power to invest is inspired from the Coleman's power to act in cooperative game theory. In game theory, other power indices are possible like Banzhaf index or Shapley-Shubik index. The possible applications of such indices to this setting remains an interesting open problem at the best of our knowledge.

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