MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Activities and Coefficients of the Tutte polynomial

Author: Sahar Hakim

Advisor: Anna de Mier

Department: Mathematics

Academic year: 2018 - 2019

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

Activities and Coefficients of the Tutte Polynomial

Sahar Hakim

Supervised by Anna de Mier

January, 2019

Acknowledgement

I would like to express my gratitude to my supervisor Anna de Mier for the useful comments, remarks and engagement through the learning process of this master thesis. Furthermore I would like to thank my professor Oriol Serra for introducing me to the topic and to my supervisor as well for the support on the way. I would like to thank my family, whose love and guidance are with me in whatever I pursue.

Abstract

Matroids are combinatorial objects that capture abstractly the essence of dependence. The Tutte polynomial, defined for matroids and graphs, has a numerous amount of information about these structures. Tutte defined activities for graphs, generalized later for matroids by Crapo. In this thesis, we study these activities and we use them to interpret the coefficients of the Tutte polynomial.

Keywords

Matroids, Tutte Polynomial, Discrete Mathematics, Graphs, Combinatorics, Activities, Tutte coefficients

Contents

1. Introduction

In Combinatorics, researchers work on studying the properties of combinatorial objects as graphs and matroids.

W.T.Tutte was one of these researchers who devoted his time and work in studying and finding invariants that give more information about graphs mainly. In 1954, he defined the dichromatic polynomial [14], known later as Tutte polynomial, with two variables x and y denoted $T(G; x, y)$ for graphs, which was generalized later to matroids by Crapo [5].

The importance of this polynomial comes from the information it contains about a graph or a matroid. At the beginning, the Tutte polynomial was conceived as a generalization of the chromatic polynomial, but more properties and information can be found using this polynomial about connectivity, bases, rank and many others.

The Tutte polynomial has many equivalent definitions as we will see in Chapter 3. One of these definitions is as the generating function of spanning trees according to "Tutte activities". Studying these activities and how they can help us understand the coefficients of the Tutte polynomial was the motive of this thesis.

Brylawski discovered that the coefficients of the Tutte polynomial $T(M; x, y)$ of a matroid M satisfy some linear relations [3].These relations were a good start for us to study the activities of a matroid.

Studying the Tutte polynomial is still a very active area of research. One of these results is about the derivatives as generating function of Tutte activities [10], other was about finding new formulas for Tutte's coefficients [8]. These particular results were interesting to this work.

This thesis is structured in five chapters. After this introduction, in the second chapter we will introduce matroids, define them and give the most important properties they have. The third chapter will be about the Tutte polynomial. We will see the different equivalent definitions as well as some interesting results found in this area of research.

Chapter 4 consists of two sections and it will be about interpreting Tutte coefficients. The first section is about some specific Tutte coefficients. We will give first few simple known results, then we will discuss about the relation between Tutte coefficients and parallel and series classes in a matroid. These relations were found recently for graphs [8] and we worked out in applying them for matroids and adding to the given results. The second section will be about a theorem linking matroid connectivity and the coefficient of x, where we have also our contribution in proving it using the activities.

Chapter 5 is devoted to Brylawski's equations. We will discuss the proof of two cases of this equation. The proof of the second equation in this section is a result of our work on this thesis. Previous proofs were presented for the same theorem but they were quite involved [3].

If this thesis was to be continued, the natural way is to continue with the general form of Brylawski's results found in Chapter 5, as we discussed one case which we think can be generalized with further studies.

We assume the reader has basic knowledge in graph theory. Basic definitions can be found in [7] and [16].

2. Matroids

Matroid theory is a combinatorial theory in discrete mathematics that abstracts the notion of linear independence in vector spaces. It is a branch that has deep connections with linear algebra, graph theory and many other fields. Applications of matroids were found in geometry, topology, combinatorial optimization, network theory and coding theory.
Matroids were first defined by Whitney (1935) [17] through studying the fundamental properties of dependence that are common to graphs and matrices. Van der Waerden (1937) [15], working independently in the same area, was able to distinguish some properties that are common to algebraic and linear independence as well.

In this chapter, we will give some basic definitions and properties of matroids.
Often, we shall want to add a single element e to a set X or to remove e from X. In such cases, as a convention in matroid theory, we shall abbreviate $X \cup \{e\}$ and $X - \{e\}$ to $X \cup e$ and $X - e$, respectively.

The main reference that we followed in this chapter is [13].

2.1 Basic Definitions

A characteristic of matroids is that they can be defined in many different equivalent ways. In this section, we will introduce some of these ways. As a basic definition, we will use the independent sets to define the matroids.

Definition 2.1. A **matroid** M is an ordered pair $(E, \mathcal{I})$, consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E such that

1. $\phi \in \mathcal{I}$.
2. If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ then $I_2 \in \mathcal{I}$.
3. If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members of $\mathcal{I}$ are the **independent sets** of M and E is the **ground set** of M.

As mentioned above, matroids can be defined in other equivalent ways, using circuits or bases among others. We will define circuits and bases and see their properties.

Definition 2.2. A minimal dependent set in a matroid M is called a **circuit** of M. The set of circuits of M is denoted $\mathcal{C}(M)$.

Proposition 2.3. *The set $\mathcal{C}$ of circuits of a matroid M has the following properties*

1. $\phi \notin \mathcal{C}$;
2. *If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$; and*
3. *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

Proof. The proof of the first and the second properties is trivial. For the third property, assume that the set $(C_1 \cup C_2) - e$ does not contain a circuit. Then $(C_1 \cup C_2) - e$ is an independent set. The set $C_2 - C_1$ is non-empty. Indeed if $C_2 - C_1$ is empty, then as we have that $e \in C_1 \cap C_2$, then $C_1 = \{e\}$ and $C_1 \subseteq C_2$; hence by the second property $C_1 = C_2$ which is not true since C_1 and C_2 are distinct. So let f be an element in $C_2 - C_1$. As C_2 is a minimal dependent set, $C_2 - f \in \mathcal{I}$. Take now a set I of $C_1 \cup C_2$ such that I is maximal independent set containing $C_2 - f$. As C_1 is a circuit, we have an element g of C_1 which is not in I. The elements f and g are distinct. So,

$$|I| \leq |(C_1 \cup C_2) - \{f, g\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - e|$$

Now we apply the third axiom of Definition 2.1 on the independent sets I and $(C_1 \cup C_2) - e$. Then there exist element x of $[(C_1 \cup C_2) - e] - I$ such that $I \cup x$ is independent. This is a contradiction as I is maximal. □

We can notice that, for a matroid M, if we have the set $\mathcal{I}$, then we can determine $\mathcal{C}(M)$. Similarly, $\mathcal{I}$ can be determined from $\mathcal{C}(M)$. We remark that it can be proved that a collection of subsets of a set E satisfying the three properties in Proposition 2.3 is the collection of circuits of a matroid of ground set E.

We define the bases of a matroid as

Definition 2.4. A **base** of M is a maximal independent set of M.

Studying the bases of a matroid, we can find this fundamental property about the size of these bases.

Theorem 2.5. *If B_1 and B_2 are bases of M, then*

$$|B_1| = |B_2|.$$

Proof. As B_1 and B_2 are bases, then they are independent, so they are members of $\mathcal{I}$. Suppose that $|B_1| < |B_2|$. Then by the third axiom of Definition 2.1, there exist an element e of $B_2 - B_1$ such that $B_1 \cup e \in \mathcal{I}$. This is a contradiction as B_1 is maximal. Hence $|B_1| \geq |B_2|$. Similarly, $|B_2| \geq |B_1|$. As a result, $|B_1| = |B_2|$. □

Proposition 2.6. *The set $\mathcal{B}$ of bases of a matroid M has the following properties:*

1. *$\mathcal{B}$ is non-empty; and*

2. *If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 - B_2$, then there exists an element $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.*

Proof. The first property is trivial, as $\mathcal{I}$ is non-empty, then $\mathcal{B}$ is non-empty. For the second property, consider the two sets $B_1 - x$ and B_2. Both sets are independent. As $|B_1| = |B_2|$ by Theorem 2.5, then $|B_1 - x| < |B_2|$. By axiom 3 of Definition 2.1, there is an element y of $B_2 - (B_1 - x)$ such that $(B_1 - x) \cup y \in \mathcal{I}$. We have $y \in B_2 - B_1$. The set $(B_1 - x) \cup y$ is independent, thus $(B_1 - x) \cup y \subseteq B_1'$, where B_1' is a base of M. Again by Theorem 2.5, $|B_1'| = |B_1|$, but also, $|B_1| = |(B_1 - x) \cup y|$. This implies that $(B_1 - x) \cup y = B_1'$. So $(B_1 - x) \cup y$ is a base of M. □

In a similar approach, if we know the independent sets of a matroid M, we can determine the set of bases $\mathcal{B}$. And knowing $\mathcal{B}$ can lead us to know the set $\mathcal{I}$ of M. This shows the equivalence between the three concepts of independent sets, circuits and bases. We can also prove that a collection of subsets of a set E satisfying the two properties in Proposition 2.6 is the collection of bases of a matroid of ground set E.

At this point we can define loops and isthmuses. A **loop** of a matroid M is a circuit of cardinality one. And an **isthmus** of M is an element that is contained in all the bases.

To clarify the concepts of independent sets, circuits and bases, consider the following example

Example 2.7. Given a graph $G = (V, E)$. Let $\mathcal{I}$ be the set of acyclic subsets of E. Then $(E, \mathcal{I})$ is a matroid denoted $M(G)$ and is called the **cycle matroid** of G. For instance, for the graph in Fig. 1, we have

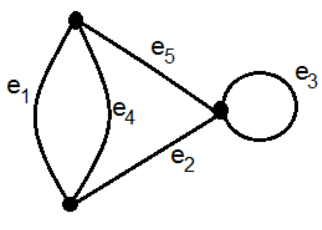

Figure 1: A graph G

The ground set of $M(G)$ is $E = \{e_1, e_2, e_3, e_4, e_5\}$.
The set of independent sets of $M(G)$ is
$\mathcal{I} = \{\phi, \{e_1\}, \{e_2\}, \{e_4\}, \{e_5\}, \{e_1, e_2\}, \{e_1, e_5\}, \{e_2, e_4\}, \{e_2, e_5\}, \{e_4, e_5\}\}$.

The set of circuits of $M(G)$ is
$\mathcal{C} = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_4, e_5\}\}$.
The set of bases of $M(G)$ is
$\mathcal{B} = \{\{e_1, e_2\}, \{e_1, e_5\}, \{e_2, e_4\}, \{e_2, e_5\}, \{e_4, e_5\}\}$.

□

In a graphic matroid, we can observe the following:

- The bases of $M(G)$ are the spanning trees of G if G is connected.
- The circuits of $M(G)$ are the cycles of G.

A matroid can also be obtained from a matrix. As a matter of fact, the name matroid comes from studying the independence of the columns in a matrix.

Proposition 2.8. *Let E be the set of column labels of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labelled by X is a set and is linearly independent in the vector space $V(m, \mathbb{F})$. Then $(E, \mathcal{I})$ is a matroid.*

Proof. Clearly $\mathcal{I}$ is non-empty and every subset of every member of $\mathcal{I}$ is also in $\mathcal{I}$. Let I_1 and I_2 be members of $\mathcal{I}$ with $|I_1| < |I_2|$. We need to show that there is an element e in $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$. Let W be the subspace of $V(m, \mathbb{F})$ spanned by $I_1 \cup I_2$. Then dim W, the dimension of W, is at least $|I_2|$. Now suppose $I_1 \cup e$ is a member of $\mathcal{I}$ for all e in $I_2 - I_1$. Then W is contained in the span of I_1. Thus $|I_2| \leq dimW \leq |I_1|$; this is a contradiction, since we stated that $|I_1| < |I_2|$. So $I_2 - I_1$ contains an element e such that $I_1 \cup e \in \mathcal{I}$.

□

A matroid obtained from a matrix A, denoted $M(A)$, is called a **vector matroid** of A. This is a particular example of such matroid.

Example 2.9. Let A be the following matrix over the field $\mathbb{R}$ of real numbers.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Then $(E, \mathcal{I})$ is a matroid where $E = \{1, 2, 3, 4, 5\}$ and
$\mathcal{I} = \{\phi, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$.
The set of circuits $\mathcal{C}$ is $\mathcal{C} = \{\{3\}, \{1, 4\}, \{1, 2, 5\}, \{2, 4, 5\}\}$.
And the set of bases $\mathcal{B}$ is $\mathcal{B} = \{\{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$.

Notice that, looking back at example 2.7, there is a bijection ψ from $\{1, 2, 3, 4, 5\}$ to $\{e_1, e_2, e_3, e_4, e_5\}$ defined by $\psi(i) = e_i$.

A set X is a circuit, a base or an independent set in $M(A)$ if and only if $\psi(X)$ is a circuit, a base, or an independent set in $M(G)$ respectively.
Thus $M(A)$ and $M(G)$ have the same structure. □

In example 2.7, $\{e_1, e_4\}$ is a circuit of size two. This motivates the following definitions.

Definition 2.10. Let M be a matroid. If f and g are elements of M such that $\{f, g\}$ is a circuit, then f and g are **parallel**.
A **parallel class** of M is a maximal subset X of $E(M)$ such that any two distinct members of X are parallel and no member of X is a loop.
A parallel class is **trivial** if it contains just one element.

Definition 2.11. A matroid M is said to be **simple** if it has no circuits of 1 or 2 elements.

The **rank** of a matroid M, denoted $r(M)$, is the size of a base B of M. For X a set of M, $r(X)$ is equal to the cardinality of the largest independent set of X.
We can express bases, independent sets and circuits in terms of the rank function.

Proposition 2.12. *Let M be a matroid with rank function r and suppose that $X \subseteq E(M)$. Then*

1. *X is independent if and only if $|X| = r(X)$;*
2. *X is a basis if and only if $|X| = r(X) = r(M)$;*
3. *X is a circuit if and only f X is non-empty and, for all x in X, $r(X - x) = |X| - 1 = r(X)$.*

The proof of the previous proposition is trivial.

The **nullity** of a set X in a matroid M, denoted $n(X)$, is defined by $|X| - r(X)$.

There are many representations for matroids as we have seen. One of these representations is using **points and lines** as in Figure 2, where points represent elements, and a set is independent in the matroid if it is affinely independent in the drawing. We have, any two points are independent unless they are placed next to each other, and three points are independent if they are not collinear, and so on.

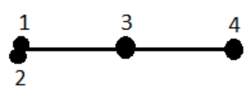

Figure 2: A matroid M whose independent sets are all sets with two or less elements except for {1,2}

2.2 Duality

One attractive feature of matroid theory is that every matroid M has a dual M^*. We will define the dual of a matroid and see some of its basic properties in this section.

Theorem 2.13. *Let M be a matroid and $\mathcal{B}^*(M)$ be $\{E(M) - B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid on $E(M)$.*

In order to prove this theorem, we need the following

Lemma 2.14. *The set $\mathcal{B}$ of bases of a matroid M has the following property*
If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_2 - B_1$, then there is an element y of $B_1 - B_2$ such that $(B_1 - y) \cup x \in \mathcal{B}$.

Proof. of Lemma 2.14. As B_1 is a base and $x \notin B_1$ then $B_1 \cup x$ contains a unique circuit C. As C is dependent and B_2 is independent, $C - B_2$ is non-empty. Moreover $(B_1 - y) \cup x$ is independent since it does not contain C. As $|(B_1 - y) \cup x| = |B_1|$, then $(B_1 - y) \cup x$ is a base. □

Proof. of Theorem 2.13.

As $\mathcal{B}(M)$ is non-empty, $B^*(M)$ is non-empty. So $B^*(M)$ satisfies the first property of Proposition 2.6.

Now suppose B_1^* and B_2^* are in $\mathcal{B}^*$ and $x \in B_1^* - B_2^*$. Let $B_1 = E - B_1^*$ and $B_2 = E - B_2^*$. Then B_1 and B_2 are in $\mathcal{B}$.

In addition, $B_1^* - B_2^* = B_1^* \cap (E - B_2^*) = (E - B_1) \cap B_2 = B_2 - B_1$. By Lemma 2.14, as $x \in B_2 - B_1$, there is an element y of $B_1 - B_2$ such that $(B_1 - y) \cup x \in \mathcal{B}$. We can see that y is an element of $B_2^* - B_1^*$ and that $E - ((B_1 - y) \cup x) \in \mathcal{B}^*$. But

$E - ((B_1 - y) \cup x = ((E - B_1) - x) \cup y = (B_1^* - x) \cup y$. And we get that $\mathcal{B}^*(M)$ satisfies the second property of Proposition 2.6.

As a result, $\mathcal{B}^*(M)$ in the set the bases of a matroid on E. □

This matroid defined by $\mathcal{B}^*(M)$ is called the **dual** of M and denoted by M^*. We have $\mathcal{B}(M^*) = \mathcal{B}^*(M)$.

The bases of M^* are called **cobases** of M. In a similar convention, the circuits of M^* are called **cocircuits** of M. For instance, we can see these definitions and elementary relationships.

Definition 2.15. Let M be a matroid with rank r. Let X be a set of M. The **closure** of X in M is defined by

$$cl_M(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

A subset $X \subseteq E$ is **closed** if $cl_M(X) = X$. Closed sets are called **flats.**
The **hyperplanes** of M are the elements of

$$\mathcal{H}(M) = \{X \subseteq E : cl_M(X) = X \text{ and } r(X) = r(M) - 1\}.$$

Proposition 2.16. *Let M be a matroid on a set E and suppose $X \subseteq E$. Then*

1. *X is a hyperplane if and only if $E - X$ is a cocircuit;*
2. *X is a circuit if and only if $E - X$ is a cohyperplane.*

It is interesting to observe that in case we have a planar graph G where G^* is its dual, then $M(G)^* = M(G^*)$.

Let M be a matroid and B be a basis of M. For an element e in $E(M) - B$, $B \cup e$ contains a unique circuit denoted $C(e, B)$. And for an element f in B, there is a unique cocircuit disjoint with $B - f$ denoted $C^*(f, B)$ and is given by $C^*(f, B) = \{x \in E : B \cup f - x \in \mathcal{B}\}$.

Definition 2.17. $C(e, B)$ is called the **fundamental circuit** of e with respect to B. And $C^*(f, B)$ is called the **fundamental cocircuit** of f with respect to B.

One elementary relationship between circuits and cocircuits in a matroid is the following.

Proposition 2.18. *Let M be a matroid and B be a base of M. Take $f \in B$, then*

$$e \in C^*(f, B) \text{ if and only if } f \in C(e, B).$$

At this point it is interesting to define the notions of deletion and contraction in a matroid.

Let M be a matroid $(E, \mathcal{I})$ and e be an element of E. Let $\mathcal{I}' = \{I \subseteq E - \{e\} : I \in \mathcal{I}\}$. It is easy to check that $(E - \{e\}, \mathcal{I}')$ is a matroid. We denote this matroid by $M \setminus e$ and we call it the **deletion** of e from M. Now take the matroid $(E - \{e\}, \mathcal{I}")$ where $\mathcal{I}" = \{I \subseteq E - \{e\} : I \cup \{e\} \in \mathcal{I}\}$. Again this is a matroid denoted M/e and called the **contraction** of e from M.

Here we can note that the matroid M/e is the matroid $(M^* \setminus e)^*$. And we have the following proposition

Proposition 2.19. *Let M be a matroid and E be its ground set. Let $e \in E$. If e is a loop or an isthmus of M then,*

$$M \setminus e = M/e.$$

We define a series class in M as a parallel class in M^*.

Proposition 2.20. *Let M be a matroid. We have*

$$\overline{\mathrm{p}}(M) = \overline{\mathrm{s}}(M^*).$$

where $\overline{\mathrm{p}}(M)$ is the number of non-trivial parallel classes of M and $\overline{\mathrm{s}}(M^*)$ is the number of non-trivial series classes of M^*.

2.3 Connectivity

We will discuss in this section the concept of connectivity of matroids which comes from the concept of 2-connected graphs.

Recall that a graph $G = (V, E)$ is said to be connected if there is a path between every pair of vertices of G.

Definition 2.21. A connected graph G is called 2-connected if, for every vertex $x \in V(G)$, $G - x$ is connected.

Given two matroids M_1 and M_2 on disjoint sets E_1 and E_2, we can get a new matroid by just putting M_1 and M_2 together.

Definition 2.22. Let M_1 and M_2 be matroids on disjoint ground sets E_1 and E_2, respectively. Define the direct sum $M_1 \oplus M_2$ to be the matroid on the ground set $E = E_1 \cup E_2$, having as independent sets all sets of the form $I_1 \cup I_2$, where $I_1 \subseteq E_1$ is independent in M_1 and $I_2 \subseteq E_2$ is independent in M_2.

Pictorially, the direct sum is generated by placing the two matroids side and side. An example of the sum of $M(G) \oplus M(G)$ defined in Example 2.7 can be seen in Figure 3.

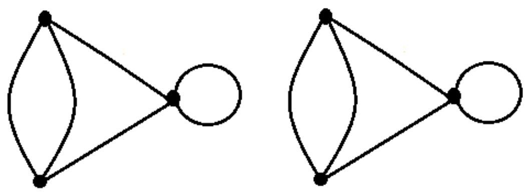

Figure 3: The direct sum of $M(G) \oplus M(G)$

A matroid is said to be **connected** if it has exactly one connected component. In other words,

Definition 2.23. A matroid M is connected if and only if M cannot be written as a direct sum of smaller matroids.

We can see the relationship between connectivity of graphs and matroids in the following proposition.

Proposition 2.24. *Let G be a loopless graph without isolated vertices and suppose that $|V(G)| \geq 3$. Then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.*

3. The Tutte Polynomial

The Tutte polynomial is one of the important and rich polynomials defined for graphs and matroids, and gives many information about the nature of these graphs and matroids. It was defined by Tutte in 1954 who called the two-variable polynomial $T(G; x, y)$ the **Dichromate** of the graph G, but with time it was called the **Tutte polynomial of** G (or M for matroids).

We will define the Tutte polynomial for matroids in general. The Tutte polynomial of a graph is the Tutte polynomial of its cycle matroid.

The main references we followed in this chapter are [1] and [11].

3.1 Definitions and Properties

The Tutte polynomial has three different equivalent definitions. We will take the definition using the internal and external activities of a matroid as our main definition. For that, we will start by defining these activities.Then, the other two definitions will be given and a particular example will show the equivalence between the three forms. The equivalence of the three forms can be found in [11]. At the end of this section we will look at particular values of the Tutte polynomial.

Let M be a matroid and let $<$ be a linear order on the ground set E. For a basis B, define the sets:

$EA(B) = \{e \in E - B : min(C(e, B)) = e\}$
$EP(B) = \{e \in E - B : min(C(e, B)) \neq e\}$
$IA(B) = \{i \in B : min(C^*(i, B)) = i\}$
$IP(B) = \{i \in B : min(C^*(i, B)) \neq i\}$

The elements of $EA(B)$ and $EP(B)$ are called externally active and externally passive with respect to B, respectively. And the elements of $IA(B)$ and $IP(B)$ are called internally active and internally passive with respect to B, respectively.

Let us examine the activities in the following example

Example 3.1. Let M be the matroid in Figure 4.

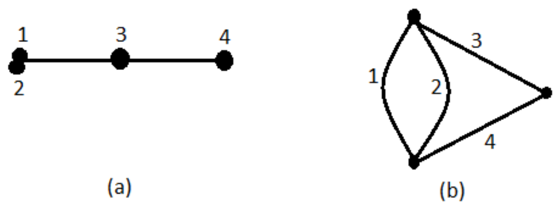

Figure 4: Matroid M defined (a) by points and lines or (b) by a graph

One of the bases of M is $B = \{2, 3\}$. To find the internal active elements of B, we need to find the fundamental cocircuits of 2 and 3.

The fundamental cocircuit of 2 is $C^*(2, B) = \{1, 2, 4\}$. As 2 is not the smallest element in $C^*(2, B)$, 2 is internally passive.

The fundamental cocircuit of 3 is $C^*(3, B) = \{3, 4\}$. As 3 is the smallest element in $C^*(3, B)$, 3 is internally active. To see which elements are now externally active, we need to find the fundamental circuits of 1 and 4.

The fundamental circuit of 1 is $C(1, B) = \{1, 2\}$. As 1 is the smallest element in $C(1, B)$, 1 is externally active.

And finally the fundamental circuit of 4 is $C(4, B) = \{2, 3, 4\}$. So 4 is externally passive.

□

The following is an easy but useful observation

Corollary 3.2. *The element '1' will always be active and the largest element will always be inactive unless it is a loop or an isthmus.*

We can now give the definition the Tutte polynomial

Definition 3.3 (Internal and External activity)**.** Let $M(E, \mathcal{I})$ be a matroid. For any non-negative integers i, j, let t_{ij} be the number of basis of M with exactly i internally active elements and j externally active elements. Define the Tutte polynomial of M as

$$T(M; x, y) = \sum_{i,j \geq 0} t_{ij} x^i y^j$$

Let us calculate the Tutte polynomial for matroid M given in Example 3.1. We need to find the set of bases $\mathcal{B}$ of M. $\mathcal{B} = \{\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$.

For each base, we need to find its internally and externally active elements.

Base $\{1,3\}$ has two internally active elements, 1 and 3; and no externally active elements.
Base $\{2,3\}$ has one internally active element,3; and one externally active element, 1, as we have seen in Example 3.1.
Base $\{1,4\}$ has one internally active element, 1; and no externally active elements.
Base $\{2,4\}$ has no internally active elements and one externally active element, 1.
Base $\{3,4\}$ has no internally active elements and two externally active elements, 1 and 2.

Then

$$T(M;x,y) = 1 \cdot x^2y^0 + 1 \cdot x^1y^1 + 1 \cdot x^1y^0 + 1 \cdot x^0y^1 + 1 \cdot x^0y^2 = x^2 + xy + x + y + y^2$$

The Tutte polynomial can be also defined using the rank and nullity of sets of E, the ground set of the matroid M. We have

Theorem 3.4 (Rank-Nullity). *Let $M(E,\mathcal{I})$ be a matroid. The Tutte polynomial of M can be written as:*

$$T(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}$$

We can notice here that the equivalence of these two forms guarantees that the coefficients of the Tutte polynomial in Definition 3.3 is independent to the choice of order.

For the same matroid in Example 3.1, let us calculate the Tutte polynomial using Rank-Nullity. To visualize the results we will put the results in the following Table 1.
Note that the rank of E is $r(E) = 2$.

Table 1: Expansions of the Tutte polynomial of Example 3.1

A	$r(A)$	$r(E)-r(A)$	$\|A\|-r(A)$	$(x-1)^{r(E)-r(A)}(y-1)^{\|A\|-r(A)}$
ϕ	0	2	0	$(x-1)^2$
{1}	1	1	0	$x-1$
{2}	1	1	0	$x-1$
{3}	1	1	0	$x-1$
{4}	1	1	0	$x-1$
{1,2}	1	1	1	$(x-1)(y-1)$
{1,3}	2	0	0	1
{1,4}	2	0	0	1
{2,3}	2	0	0	1
{2,4}	2	0	0	1
{3,4}	2	0	0	1
{1,2,3}	2	0	1	$y-1$
{1,2,4}	2	0	1	$y-1$
{1,3,4}	2	0	1	$y-1$
{2,3,4}	2	0	1	$y-1$
{1,2,3,4}	2	0	2	$(y-1)^2$

The Tutte polynomial is the sum of the last column of Table 1.
The result of this sum is

$$T(M;x,y) = x^2 + x + y + xy + y^2$$

which is the same result calculated earlier.

In the third equivalent definition of the Tutte polynomial we use a linear recursion relation of deleting and contracting elements which are neither loops nor isthmuses.

Theorem 3.5 (Deletion-Contraction). *Let M be a matroid and e be an element of E.*

$$T(M;x,y) = \begin{cases} xT(M \setminus e;x,y) & \text{if } e \text{ is an isthmus} \\ yT(M/e;x,y) & \text{if } e \text{ is a loop} \\ T(M \setminus e;x,y) + T(M/e;x,y) & \text{if } e \text{ is neither an isthmus nor a loop} \end{cases}$$

If E is empty, then $T(M;x,y) = 1$

We will consider the same Example 3.1 to find the Tutte polynomial using deletion and contraction. The following Figure 5 illustrates how the recursion is followed.

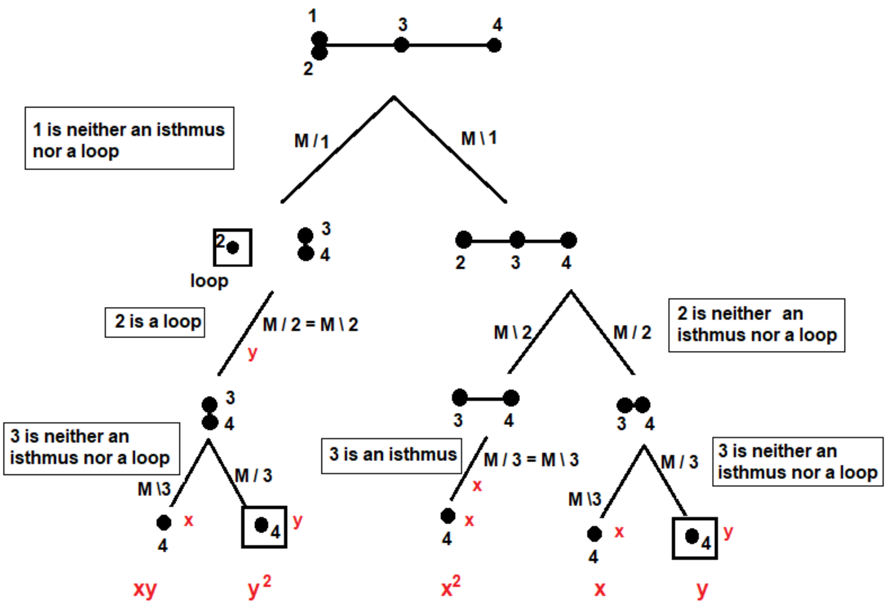

Figure 5: The deletion/contraction recursion of Matroid M

We can notice that the Tutte polynomial is the sum of the last row in Figure 5.
So

$$T(M;x,y) = xy + y^2 + x^2 + x + y$$

which is the result we were expecting.

Note that the result is invariant under the order in which we contract or delete, but indeed the order in which contraction or deletion are applied significantly affects the size of the computation tree.

There are several nice values of the Tutte polynomial. We can see some of them in the following theorem

Theorem 3.6. *For a matroid M and a graph G we have the following*

1. *$T(M;1,1)$ counts the number of bases of M (spanning trees in a connected graph);*
2. *$T(M;2,1)$ counts the number of independent sets of M (forests in G);*
3. *$T(M;1,2)$ counts the number of spanning sets of M (spanning subgraphs of G).*

These results follows easily from Theorem 3.4.

We have some useful properties of the Tutte polynomial, we will state the two most interesting ones that we used later in our research.

Theorem 3.7. *Let M be a matroid given as $M = M_1 \oplus M_2$. Then*

$$T(M; x, y) = T(M_1; x, y)T(M_2; x, y)$$

The proof of Theorem 3.7 directly follows from Definition 2.22 and Theorem 3.4.

Theorem 3.8. *Let M be a matroid. Then,*

$$T(M^*; x, y) = T(M; y, x)$$

Note that for a matroid M of rank r, and M^* its dual of rank r^*, $r^*(A) = |A| - r(E) - r(E \setminus A)$. The proof then directly follows from Theorem 3.4.

As we see, in order to find the Tutte polynomial of the dual matroid M^*, it is enough to exchange the positions of x and y in the Tutte polynomial of the matroid M.

3.2 Derivatives of matroid Tutte polynomial

In this section, we will refer to results found in [10] by Las Vergnas. We will see that in an ordered matroid the partial derivative $\frac{\partial^{i+j} T}{\partial x^i \partial y^j}$ of the Tutte polynomial is $i!j!$ times the generating function of activities of subsets with corank i and nullity j. We will see that there are four equivalent forms to find these derivatives.
Some basic definitions and notations are needed.

Definition 3.9. Let M be a matroid on a linearly ordered set E. Let $A \subseteq E$. Set

$Ext_M(A) := \{e \in E - A :$ e is the smallest element of some circuit of M contained in $A \cup \{e\}\}$, and let $e_M(A) = |Ext_M(A)|$.
We say that $Ext_M(A)$ is the set of externally active elements of A with respect to M.

We have $Int_M(A) := \{e \in A :$ e is the smallest element of some cocircuit of M contained in $(E - A) \cup \{e\}\}$.
We say that $Int_M(A)$ is the set of internally active elements of A with respect to M.

Remark 3.10. Note that these definitions generalize the activities for bases we have seen earlier.

Theorem 3.11. *Let M be a matroid on a linearly ordered set E, and i, j be non-negative integers. Then*

$$\frac{\partial^{i+j} T(M; x, y)}{\partial x^i \partial y^j} = i!j! \sum_{\substack{A \subseteq E \\ cr_M(A)=i \\ nl_M(A)=j}} x^{i_M(A)} y^{e_M(A)}$$

where $cr_M(A) :=$ the corank of A in M ; $nl_M(A) :=$ the nullity of A in M.

The proof of Theorem 3.11 is done in a more general setting of matroid perspectives. Properties of Dawson partition [10] of the Boolean lattice defined by matroid bases are used to prove this theorem.

We can observe that finding the derivative in Theorem 3.11, setting $x, y = 0$ and diving by $i!j!$ give us the coefficient of the Tutte polynomial t_{ij},

Corollary 3.12. *Let M be a matroid and E be its ground set.*

$$t_{ij} = |A \subseteq E : cr_M(A) = i, nl_M(A) = j, i_M(A) = 0 \text{ and } e_M(A) = 0|$$

Having elements in a set A forming a circuit, we can observe the following

Corollary 3.13. *Let $e_1, e_2, ..., e_k \in A$ such that $\{e_1, e_2, ..., e_k\}$ form a circuit, then e_i is not internally active for all $i = 1, ..., k$. We write $i(e_i) = 0$ for all $i = 1, ..., k$.*

Three other alternative expansions can be deduced from Theorem 3.11 to find the same derivatives.

Corollary 3.14. *We have*

$$\frac{\partial^{i+j} T(M; x, y)}{\partial x^i \partial y^j} = i!j! \sum_{\substack{A \subseteq E \\ i_M(A)=i \\ nl_M(A)=j}} x^{cr_M(A)} y^{e_M(A)}$$

$$\frac{\partial^{i+j} T(M; x, y)}{\partial x^i \partial y^j} = i!j! \sum_{\substack{A \subseteq E \\ cr_M(A)=i \\ e_M(A)=j}} x^{i_M(A)} y^{nl_M(A)}$$

and

$$\frac{\partial^{i+j} T(M; x, y)}{\partial x^i \partial y^j} = i!j! \sum_{\substack{A \subseteq E \\ i_M(A)=i \\ e_M(A)=j}} x^{cr_M(A)} y^{nl_M(A)}$$

4. Interpretations for specific Tutte coefficients

The Tutte Polynomial contains much information about the matroids (or graphs). In this section we will discuss some specific coefficients of Tutte polynomial and interpret them to find more about the nature of matroids. The results in this chapter can be proved using deletion and contraction method and by induction, but our focus is to look in terms of activities of the matroid, especially when there are no such proofs in the literature.

4.1 Interpretation of specific Tutte coefficients

In this section, we will interpret some coefficients of the Tutte polynomial. We will start with some basic known results, then we will discuss about the coefficients $t_{r-1,1}$ and $t_{1,m-r-1}$.

Theorem 4.1. *Let M be a matroid of m elements and rank r. Then*

1. $t_{00} = 0$;
2. $t_{ij} = 0$ *whenever* $i > r$ *or* $j > m - r$;
3. $t_{r0} = 1$ *and* $t_{0,m-r} = 1$, *if M has neither loops nor isthmuses.*

The proof of this theorem follows easily from the Definition 3.3.

For a given matroid $M(E, \mathcal{I})$ of rank $r(M) = r$, we can see a relation between $t_{r-1,1}$ and the parallel classes in M. Using duality we can deduce a relation between $t_{1,m-r-1}$ and the series classes in M.

The results obtained in this section were written for graphs [8]. And the proofs in the original paper are quite involved. We worked on equivalent results for matroids and tried finding more direct and shorter proofs.

Theorem 4.2. *Let M be a matroid with no loops or coloops of rank $r(M) = r$ and E be its ground set. Then*

1. $t_{r-1,1}(M) = \overline{\mathrm{p}}(M)$,
2. $t_{1,m-r-1}(M) = \overline{\mathrm{s}}(M)$.

where $\overline{\mathrm{p}}(M)$ and $\overline{\mathrm{s}}(M)$ are the number of non-trivial parallel and series classes of M respectively.

Proof. 1. M is a matroid on a linearly ordered set E.
Using Theorem 3.11, we have

$$t_{r-1,1}(M) = \#\{A \subseteq E : cr_M(A) = r-1, nl_M(A) = 1, i_M(A) = 0 \text{ and } e_M(A) = 0\}$$

We know

$$cr_M(A) = r(E) - r(A) = r - 1$$

As

$$r(E) = r \text{ then } r(A) = 1$$

And

$$nl_M(A) = |A| - r(A) = 1,$$

then

$$|A| = nl_M(A) + r(A) = 1 + 1 = 2$$

So A are sets of size 2 and rank 1, such that $i(A) = 0$ and $e(A) = 0$.

As M has no loops, the only sets with rank 1 and 2 elements are the sets of two parallel elements. We can write:

$$A = \{s_1, s_2\} \subseteq E \text{ such that } s_1 \text{ and } s_2 \text{ are parallel and } i(A) = e(A) = 0$$

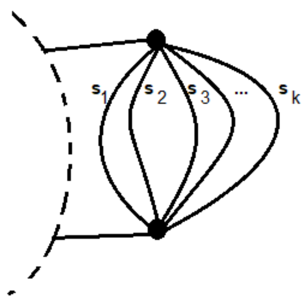

Figure 6: A parallel class having elements s_1 and s_2

As s_1 and s_2 are parallel, they form a circuit then $i(s_1) = i(s_2) = 0$ by Corollary 3.13.
So $i(A) = 0$.

Using Definition 3.9, for A to have external activity zero, any element of $E - A$ having s_1 or s_2 in their circuit must not be the smallest element. Namely the other elements in the parallel class containing s_1 and s_2. See Figure 6 .

So we conclude that s_1 or s_2 must be the smallest elements in their parallel class.

As a result, in each parallel class, choose A to be the set of the two smallest elements.

We get

$$t_{r-1,1}(M) = \overline{\mathrm{p}}(M).$$

2. Dually,

$$r(M^*) = |E| - |B| = m - r$$

So we have using (1),

$$t_{m-r-1,1}(M^*) = \overline{\mathrm{p}}(M^*).$$

But we have by Theorem 3.8,

$$T(M^*; x, y) = T(M; y, x).$$

In particular,

$$t_{m-r-1,1}(M^*) = t_{1,m-r-1}(M).$$

And by Proposition 2.20,

$$\overline{\mathrm{p}}(M^*) = \overline{\mathrm{s}}(M).$$

As a result,

$$t_{1,m-r-1}(M) = \overline{\mathrm{s}}(M).$$

□

The results in Theorem 4.2 can be extended to $t_{r-1,j}$. These results are not in the original paper.

Theorem 4.3. *Let M be a connected matroid on a linearly ordered set E of rank $r(M) = r$. Then*

1. $t_{r-1,j}(M) = \overline{\mathrm{p_j}}(M)$
2. $t_{j,m-r-1}(M) = \overline{\mathrm{s_j}}(M)$

where $\overline{\mathrm{p_j}}(M)$ and $\overline{\mathrm{s_j}}(M)$ are the number of parallel and series classes of M with at least j elements respectively.

Proof. 1. We have

$$t_{r-1,j}(M) = \#\{A \subseteq E : cr_M(A) = r - 1, nl_M(A) = j, i_M(A) = 0 \text{ and } e_M(A) = 0\}.$$

And

$$r(A) = 1 \text{ and } |A| = j + 1.$$

So

$$A = \{s_1, s_2, ..., s_{j+1}\} \subseteq E \text{ such that } s_i \text{ for } i = 1, ..., j + 1 \text{ are parallel, and } i(A) = e(A) = 0.$$

In a similar proof to Theorem 4.2, we get that A is a set of the $j + 1$ smallest elements in a parallel class.
So

$$t_{r-1,j}(M) = \overline{\mathrm{p_j}}(M).$$

2. Analogous to the proof of Theorem 4.2 (2). □

4.2 Connectivity of Matroids

Connectivity is always a property to be studied. By looking only at one coefficient of the Tutte polynomial we can directly find out if a matroid is connected or not. The following theorem was proved using different methods. Here we have our contribution in proving the connectivity theorem using Tutte's activities.

Theorem 4.4. *If M has at least two elements, M is a connected matroid if and only if $t_{10} > 0$.*

Proof. **Sufficient Condition:** $t_{10} > 0 \implies M$ is a connected matroid.

Suppose that M is not connected, then M is the direct sum of at least two connected components M_1 and M_2;

$$M = M_1 \oplus M_2.$$

Using Theorem 3.7, Tutte polynomial of M can be written as

$$T(M; x, y) = T(M_1; x, y)T(M_2; x, y).$$

By the Definition 3.3 of the Tutte polynomial, the terms of the polynomial are of the form $t_{ij}x^i y^j$ for some i, j.
We know that $t_{00} = 0$, so $T(M_1; x, y)$ and $T(M_2; x, y)$, both contain no constant term.
Multiplying both polynomials can never give a term kx for some value $k \neq 0$.
This implies that $T(M; x, y)$ has no kx term of the form x^1y^0. So $t_{10} = 0$. Contradiction.

Necessary Condition: M is a connected matroid $\implies t_{10} > 0$.

Let M be a connected matroid and $B = \{b_1, b_2, ...b_k\}$ be a base for M.

It is enough to prove that there exists an ordering for M such that B has one internally active element and no externally active element.

The proof is constructed in the following way:

For every element in B find its fundamental cocircuit and for every element in $E - B$, label them $\{a_1, a_2, ..., a_{n-k}\}$, find its fundamental circuit.

Take any element in B and label it as 1. Say $b_1 = 1$.

As b_1 is an element of the base, it is internally active by Corollary 3.2.

Label all other elements in the fundamental cocircuit of b_1 as $a_1 = 2, a_2 = 3, ..., a_i = i + 1$.

Using Proposition 2.18 b_1 is in the fundamental circuit of $a_1 = 2, a_2 = 3, ..., a_i = i + 1$.

So none of $a_1, a_2, ..., a_i$ is the smallest element in its fundamental circuit. Hence they are not active.

For every labeled element $a_1, a_2, ...a_i$, consider their fundamental circuit and label the unlabeled elements as $i + 2, i + 3, ...$ in increasing order.

We proceed in the same way until no other element can be labelled.

Suppose there exist an element x which is unlabeled. This means that the set of labeled elements and all elements in their circuits and cocircuits are labeled, and x and all elements in its circuit or cocircuit is unlabeled. Figure 7 .

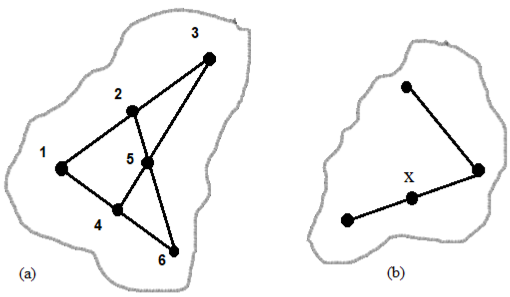

Figure 7: (a)Set of labeled elements and their circuits/cocircuits. (b)Set of unlabeled elements containing element x and its circuit/cocircuit.

Let E_1 be the set of all labeled elements and E_2 be the set of unlabeled elements (contains at least one element x).

And let $\mathcal{I}_1$ be the independent sets in E_1 and $\mathcal{I}_2$ be the independent sets in E_2. Then M is $M_1 \oplus M_2$ where $M_1(E_1, \mathcal{I}_1)$ and $M_2(E_2, \mathcal{I}_2)$. By Definition 2.23, we reach a contradiction as M is connected, so such unlabeled x cannot exist. $\square$

Here we can see the algorithm in a particular example

Example 4.5. Consider matroid M in Figure 8, and take a base $B = \{b, d, e\}$.

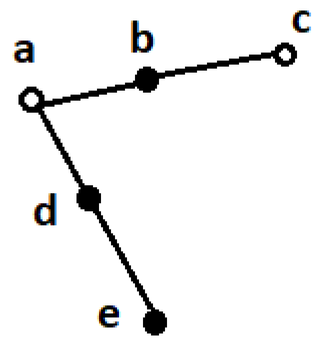

Figure 8: Matroid M with basis B

First, find the circuits and cocircuits of every element.
$C^*(b, B) = \{b, c\}$.
$C^*(d, B) = \{a, c, d\}$.
$C^*(e, B) = \{a, c, e\}$.
$C(a, B) = \{a, d, e\}$.
$C(c, B) = \{b, c, d, e\}$.

Then label any element of B as 1. Take $b = 1$.
$C^*(b, B) = \{1, c\}$. Label $c = 2$.
$C(c, B) = \{1, 2, d, e\}$. Label $d = 3$ and $e = 4$.
$C^*(d, B) = \{a, 2, 3\}$. Label $a = 5$.
$C^*(e, B) = \{5, 2, 4\}$.

As a result we get
$C^*(1, B) = \{1, 2\}$.
$C^*(3, B) = \{5, 2, 3\}$.
$C^*(4, B) = \{5, 2, 4\}$.
$C(5, B) = \{5, 3, 4\}$.
$C(2, B) = \{1, 2, 3, 4\}$.

Notice that 1 is the only active element, and it is internally active. So $t_{10} > 0$

5. Brylawski's Identities for Tutte's Coefficients

In 1972, Brylawski [4] discovered that the coefficients of the Tutte polynomial $T(M;x,y)$ of a matroid M satisfy a collection of linear relations.

Theorem 5.1. *Let M be a simple matroid having m elements none of which is an isthmus. We have*

$$\sum_{i=0}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k-i}{j}t_{ij}=0 \qquad \text{for } k=0,1,\ldots,m-1$$

Brylawski published two different proofs of these identities. Both proofs were done using corank and nullity. In the first [4], he counts the number of flats in a matroid of a given corank and nullity. The second was more general [2], as he used general properties of relations between the Tutte polynomial and corank-nullity generating function.
Here is a simplified version of the first three affine relations of Theorem 5.1.

1. $t_{00}=0$ if $m\geq 1$
2. $t_{10}=t_{01}$ if $m\geq 2$
3. $t_{20}+t_{02}=t_{10}+t_{11}$ if $m\geq 3$

We have already seen earlier that $t_{00}=0$.

In this chapter, we will have two sections. The first one will be the proof of the second identity, and the second will be the proof of the third one.
The proof in Section 5.2 in a fruit of our work on this thesis, as to our knowledge, it is the first to use activities in proving this identity.

5.1 Brylawski's identity $t_{10} = t_{01}$

Let M be a matroid and E its ground set such that $|E| \geq 2$. Define $<$ a linear order on E. We need to prove that

$$t_{10} = t_{01}$$

In words, we need to prove that the number of bases with one internally active element and no externally active elements (t_{10}) is equal to the number of bases with one externally active element and no internally active elements (t_{01}).

Consider all bases of E with one internally active element and no externally active elements. Denote them B_i for $i = 1, ..., k$

Observations:

- The element 1 is always active by Corollary 3.2. So 1 belongs to each base B_i for $i = 1, ..., k$.
- The element 2 is not in B_i for all $i = 1, ..., k$. Indeed, if 2 was an element of B_i for some i, then 2 would be the smallest in its fundamental cocircuit. Hence it would be active.
- Element 2 is not externally active. This means that 2 is not the smallest element in its fundamental circuit. Hence 1 must be an element of $C(2, B_i)$ for all $i = 1, ..., k$.

Consider now all bases of E with one externally active element and no internally active elements. Similar observations can be found regarding elements 1 and 2, having 1 externally active and 2 in every base with 1 in its fundamental cocircuits.
This makes us think of the following bijection:

For every base B_i for $i = 1, ..., k$, exchange the elements 1 and 2. For instance, denote them B_i' for $i = 1, ..., k$. Indeed the B_i's are bases of M as 1 was an element in the fundamental circuit of 2.
Note that 1 was internally active and 2 was not externally active in each B_i for $i = 1, ..., k$.
After exchanging 1 and 2, 1 is externally active, and 2 is not internally active as,by Proposition 2.18 1 is in its fundamental circuit now. See Figure 9.
Observation: For every other element of E

- If 1 was an element in their fundamental circuit, now 2 is an element in their fundamental circuit.
- If 2 was an element in their fundamental cocircuit, now 1 is an element in their fundamental cocircuit.
- If neither 1 nor 2 where elements in their fundamental circuits/cocircuits, then their fundamental circuits/cocircuits remain the same.

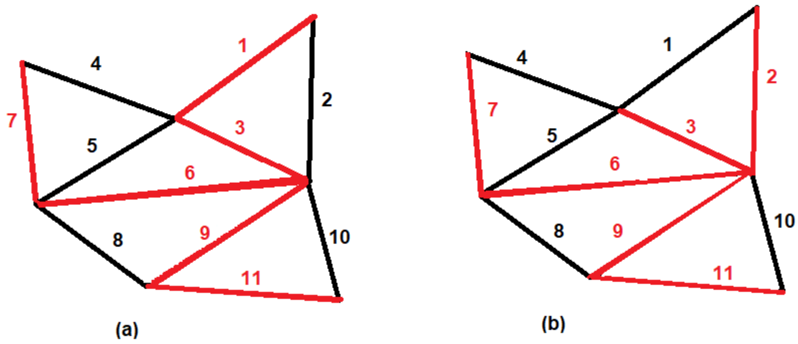

Figure 9: A matroid M (a) before exchanging 1 and 2, (b) after exchanging 1 and 2

Result: As a result, all bases with one externally active element and no internally active elements (t_{01}) can be obtained from the bases with one internally active element and no externally active elements (t_{10}) by exchanging the elements 1 and 2.

Note that the opposite is true. All bases with one internally active elements and no externally active elements (t_{10}) can be obtained from bases with one externally active element and no internally active elements (t_{01}) by exchanging again 1 and 2 with a similar argument.

This gives us the bijection we are looking for. Hence $t_{10} = t_{01}$.

5.2 Brylawski's identity $t_{20} + t_{02} = t_{10} + t_{11}$

Let M be a matroid and E its ground set such that $|E| \geq 3$. Define $<$ a linear order on E.

We need to prove that

$$t_{20} + t_{02} = t_{10} + t_{11}$$

In words, for a given matroid M , the number of bases with two internally active elements and no externally active elements (t_{20}) or with two externally active elements and no internally active elements (t_{02}) is equal to the number of bases with one internally active elements and no externally active elements (t_{10}) or one internally and one externally active elements (t_{11}).

Proof. Our goal is to prove this equality by finding a bijection between the bases mentioned above as we did in the previous part.

Consider all bases with two internally active elements and no externally active elements or with two externally active elements and no internally active elements. We will produce all bases with either one internally and no externally active elements, or with one internally and one externally active elements.

We will study each case alone. Let us start with

Case 1. Two internally active elements and no externally active elements

Observation: Element 1 will always be one of the internally active elements. The second element may be 2 or any other element. We will take two cases depending on the second active element.

Case 1.1. Suppose now the internally active elements are 1 and $i \neq 2$

Observation: As 2 is not internally active, then 2 is not in the base. Indeed, as 1 is an element of the base,if 2 was an element of the base, then 2 would be the smallest element in its fundamental cocircuit. So 2 is definitely not in the base.

As 2 is not in the base and it is not externally active, then 1 is an element in the fundamental circuit of 2.

Exchange 1 and 2. Set $B' = B \setminus 1 \cup 2$. Indeed B' is a base of M as 1 was an element in the fundamental circuit of 2.

We get now that element 1 is now externally active. Indeed 1 is always active.
Element 2 is not internally active as 1 belongs to its fundamental cocircuit by Theorem 2.18.

Let us look at the other elements in E different than 2 and i.

If 1 belonged to their fundamental circuit, now 2 belongs to their fundamental circuit and they stay inactive, and if 2 was an element in their fundamental circuit, now 1 is an element in their fundamental circuit and they stay inactive as well. See Figure 9.

If e was an element in B such that $2 \notin C^*(e, B)$, then after exchanging 1 and 2, $1 \notin C^*(e, B')$. And $C^*(e, B) = C^*(e, B')$.

Similarly, if e was an element in $E - B$ such that $1 \notin C(e, B)$, then after exchanging 1 and 2, $2 \notin C(e, B')$. And $C(e, B) = C(e, B')$.

Notice that i will stay internally active. Indeed, i was internally active so i was the smallest element in $C^*(i, B)$. We can see that element $2 \notin C^*(i, B)$, otherwise i would not be active. Then $1 \notin C^*(i, B')$. So $C^*(i, B) = C^*(i, B')$ and i is internally active.

Result: The element 1 is externally active, and the element i is internally active. Hence we have one internally and one externally active element.

Case 1.2. Suppose now the internally active elements are 1 and 2

The element 3 is not in the base, otherwise it will be active as 1 and 2 belong to the base. This means the elements 1 or 2 or both belong to the fundamental circuit of 3. And by Proposition 2.18, 3 belongs to the fundamental cocircuits of 1 or 2 or both.

If 3 belongs to the fundamental cocircuit of 1 and does not belong to the fundamental cocircuit of 2, then exchange 1 and 3 and we get 1 externally active and 2 will stay internally active. The element 3 will stay inactive as well, as 1 will be in its fundamental cocircuit.
Result: The element 1 is externally active and the element 2 is internally active. Hence, we have one internally and one externally active elements.

If 3 belongs to the fundamental cocircuit of 2 and does not belong to the fundamental cocircuit of 1, then exchange 2 and 3. The element 1 will stay internally active, the element 2 will become externally active, as 1 does not belong to its fundamental circuit, and the element 3 will stay inactive as 2 is in its fundamental cocircuit.
Result: The element 1 is internally active and the element 2 is externally active. Hence, we have one internally and one externally active elements.

If 3 belongs to the fundamental cocircuits of both 1 and 2, then exchange 2 and 3. The element 1 will stay internally active. The element 2 will be not active externally as 1 belongs to its fundamental circuit. Indeed as 3 was an element in the fundamental cocircuit of 1, after exchange, 2 is an element in the fundamental cocircuit of 1 and by Proposition 2.18, 1 is an element in the fundamental circuit of 2. Hence 2 is not active. The element 3 is not active because 2 is in its fundamental cocircuit.
Result: The element 1 is internally active and no externally active elements. Hence we have one internally and no externally active elements.

The reasoning is analogue to the reasoning in section 5.1.

Now we will study the second case

Case 2. Two externally active elements and no internally active elements

Case 2.1. Suppose now the externally active elements are 1 and $i \neq 2$

The element 2 is an element of the base. Otherwise it would be externally active. As 2 is not an active element, 1 belongs to its fundamental cocircuit.

Exchange 1 and 2. Set $B' = B \setminus 2 \cup 1$.

1 belongs to the base B', and is internally active.

2 is externally inactive as 1 belongs to its fundamental circuit.

For other elements different than 2 and i, if 1 or 2 belonged to their fundamental cocircuit or circuit respectively, then after exchanging 1 and 2, 2 or 1 would belong to their fundamental cocircuit or circuit respectively, so they stay inactive.

Otherwise their fundamental circuit or cocircuit remain the same.

Result: The element 1 is internally active, and the element i is externally active. Hence one internally and one externally active element.

Case 2.2. Suppose now the externally active elements are 1 and 2.

The element 3 is in the base, otherwise it will be externally active as 1 and 2 are not in the base.
In a similar observation to Case 1.2, the elements 1 or 2 or both belong to the fundamental cocircuit of 3. And by Proposition 2.18, 3 belongs to the fundamental circuits of 1 or 2 or both.

If 3 belongs to the fundamental circuit of 1 and does not belong to the fundamental circuit of 2, then exchange 1 and 3 and we get 1 internally active and 2 will stay externally active. The element 3 will stay inactive as well.
Result: The element 1 is internally active and the element 2 is externally active. Hence, we have one internally and one externally active elements.

If 3 belongs to the fundamental circuit of 2 and does not belong to the fundamental circuit of 1, then exchange 2 and 3. We get 1 externally active and 2 will become internally active, as 1 is not an element in its fundamental cocircuit.
Result: The element 1 is externally active and the element 2 is internally active. Hence, we have one internally and one externally active elements.

If 3 belongs to the fundamental circuits of both 1 and 2, then exchange 1 and 3. The element 1 will become internally active. The element 2 will become externally not active, as 1 will be an element in its fundamental circuit. The reasoning is the same as in Case 1.2.
Result: The element 1 is internally active and no externally active elements. Hence, we have one internally and no externally active elements.

Conclusion: Having bases with two internally active elements and no externally active elements or with two externally active elements and no internally active elements, we produced bases with one internally active element and one externally active element and bases with one internally active element and no externally active elements.

Note that the opposite is achieved in a similar way. For all the cases with one internally active element and one externally active element, we follow the same strategy as explained above by noticing the position of the element 3.
The critical case here is bases with one internally active element and no externally active elements.
In this case, we know that the internally active element is 1.
We also know that the element 2 is not in the base otherwise it will be internally active. We can have two cases looking at element 3. The first case is having 3 in the base with 2 an element in its fundamental cocircuit, and the second case is having 3 not in the base with 1 an element in its fundamental circuit.

In case the element 3 is in the base, then exchange 3 and 2. We will get, 2 is internally active and 3 will become externally not active, as 2 belongs to its fundamental circuit. The element 1 will stay internally active. As a result, we have two internally active elements, 1 and 2.

In case the element 3 is not in the base, then exchange 1 and 3. We will get, the element 1 as an externally active element. And the element 2 will become externally active element as 1 is not in its fundamental circuit anymore. Element 3 will be internally not active as 1 belongs to its fundamental cocircuit. As a result, we have two externally active elements, 1 and 2.

This gives our bijection. Hence

$$t_{20} + t_{02} = t_{10} + t_{11}$$

□

Here is a particular example to see how the procedure takes place.

Example 5.2. Consider the matroid $M(G)$ of graph G in Figure 10.

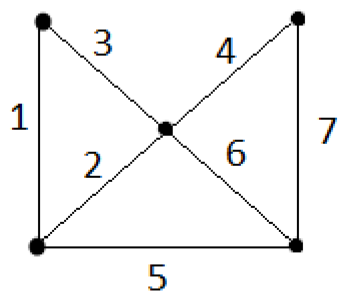

Figure 10: Graph G

Finding all bases with two internal or two external active elements we get the following five results Figure 11.

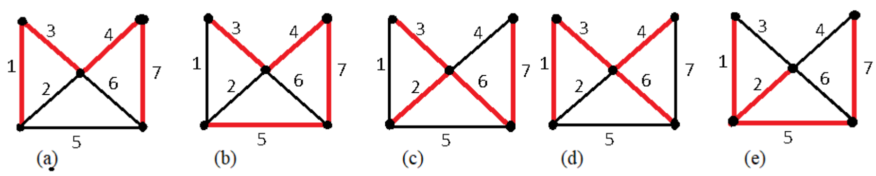

Figure 11: bases of $M(G)$ with 2 internally or 2 externally elements

There are three bases with two internal active elements ($t_{20} = 3$).Two bases with 1 and 4 internal active elements Figure 11a - 11d. One base with 1 and 2 internal active elements Figure 11e.

And two bases with two external active elements ($t_{02} = 2$). One base with 1 and 2 external active elements Figure 11b and one base with 1 and 4 external active elements Figure 11c.

In the 1st base (Figure 11a),the two internally active elements are 1 and 4. Exchange 1 and 2, we get base with 1 externally active and 4 internally active (Figure 12d).

In the 2nd base (Figure 11d),the two internally active elements are 1 and 4. Exchange 1 and 2, we get base with 1 externally active and 4 internally active (Figure 12b). In the 3rd base (Figure 11e),the two internally active elements are 1 and 2. The fundamental cocircuit of 1 is {1,3} and the fundamental cocircuit of 2 is {2,3,4,6}.So 3 is in both cocircuits. Exchange 2 and 3, we get base with 1 internally active and no externally active elements (Figure 12e).

In the 4th base (Figure 11b),the two externally active elements are 1 and 2. The fundamental circuit of 1 is {1,3,4,5,7} and the fundamental circuit of 2 is {2,4,5,7}. The element 3 belongs to the fundamental circuit of 1 and does not belong to the fundamental circuit of 2. Exchange 1 and 3, we get base with 1 internally active and 2 externally active (Figure 12a).

In the 5th base (Figure 11c), the two externally active elements are 1 and 4. Exchange 1 and 2, we get base with 1 internally active and 4 externally active (Figure 12c).

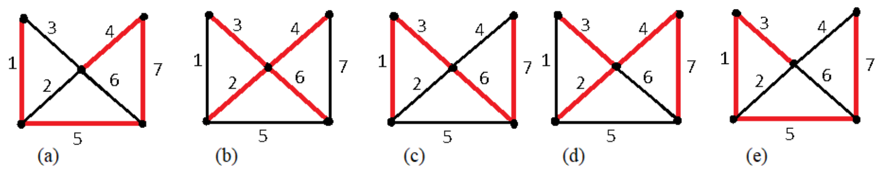

Figure 12: Bases of $M(G)$ with one internally and one externally active elements or one internally and no externally active elements

References

[1] Béla Bollobás. *Modern Graph Theory*, volume 184 of *Graduate texts in mathematics*. New York: Springer, 1998.

[2] Thomas Brylawski. The tutte polynomial part i: General theory. In *Matroid theory and its applications*, pages 125–275. Springer, 2010.

[3] Thomas Brylawski and James Oxley. *The Tutte polynomial and its applications*, volume 40 of *Encyclopedia of Mathematics and its Applications, N. White ed.* Cambridge University Press, 1992.

[4] Thomas H Brylawski. A decomposition for combinatorial geometries. *Transactions of the American Mathematical Society*, 171:235–282, 1972.

[5] Henry H Crapo. The tutte polynomial. *Aequationes Mathematicae*, 3(3):211–229, 1969.

[6] Anna de Mier Vinué. Graphs and matroids determined by their tutte polynomials. *Doctoral program in Applied Mathematics, Universitat Politècnica de Catalunya*, 2003.

[7] Reinhard Diestel. *Graph Theory, ser*, volume 173. 2005.

[8] Helin Gong, Mengchen Li, and Xian'an Jin. Several extreme coefficients of the tutte polynomial of graphs. *arXiv preprint arXiv:1705.10023*, 2017.

[9] Gary Gordon. Linear relations for a generalized tutte polynomial. *arXiv preprint arXiv:1405.3735*, 2014.

[10] Michel Las Vergnas. The tutte polynomial of a morphism of matroids—5. derivatives as generating functions of tutte activities. *European Journal of Combinatorics*, 34(8):1390–1405, 2013.

[11] Bo Lin. An introduction to tutte polynomial. `https://math.berkeley.edu/~linbo/documents/math249/Tutte%20Polynomial.pdf`, 2013.

[12] James Oxley. What is a matroid. *Cubo Matemática Educacional*, 5(3):179–218, 2003.

[13] James G Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, USA, 2nd edition, 2011.

[14] William Thomas Tutte. A contribution to the theory of chromatic polynomials. *Canad. J. Math*, 6(80-91):3–4, 1954.

[15] BL Van der Waerden. *Moderne Algebra, 2nd. ed.* Springer, 1937.

[16] Douglas Brent West. *Introduction to Graph Theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[17] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, 57(3):509–533, 1935.