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Morse Theory and Floer Homology

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To my advisors Eva and Cédric, who have not just cared for this thesis, but also for my career. I want to thank them for their counsel and for their efforts.

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Abstract

Morse homology studies the topology of smooth manifolds by examining the critical points of a real-valued function defined on the manifold, and connecting them with the negative gradient of the function. Rather surprisingly, the resulting homology is proved to be independent of the choice of the real-valued function and metric defining the negative gradient. This leads to a topological lower bound on the number of critical points.

In the 1980s, the construction of Morse homology served as a prototype to define a homology spanned by 1-periodic Hamiltonian diffeomorphisms on symplectic manifolds. The resulting homology, introduced by Andreas Floer, spectacularly revolutionized the area of symplectic topology and leaded to a proof of the famous Arnold conjecture. Floer theory still is the subject of a lot of active and exciting research and is nowadays an essential technique in symplectic topology.

Keywords: Homology, Morse theory, Floer theory, periodic orbits, fixed points, Arnold conjecture.

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Contents

Introduction

One of the usual questions regarding smooth manifolds is the existence of topological invariants. In particular, one often wants to know which properties of a manifold depend only on the underlying topology.

For instance, the De Rham cohomology measures the obstruction to the integrability of closed forms in a manifold. This means, for $\alpha \in \Omega^k(M)$ with $d\alpha = 0$, to what extent the equation

$$d\beta = \alpha$$

is not solvable.

Morse theory constructs a topological invariant based on the critical points of certain (regular enough, as we will explain later) smooth functions $f : M \to \mathbb{R}$. As a starting example, we may think of a torus, vertically embedded inside $\mathbb{R}^3$, as shown in Figure 1.

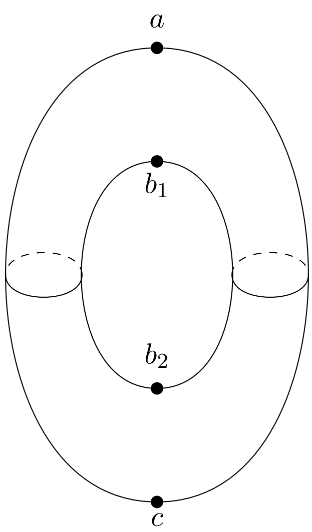

Figure 1: The torus embedded in $\mathbb{R}^3$.

The points marked in the figure are the critical points of a function defined on the torus, namely the height function. It is worth, however, not to think of them as critical points yet. Rather, imagine that the torus is fixed in a space that is gradually flooded by water. Let us imagine that the water level is rising, so it covers first the point c, then b_2, then b_1 and finally a. Let us think about how the topology of the part of the torus that is submerged in water changes over time:

- Before reaching the point c there is no part of the torus under the water level.

- After the water crosses the point c, the region underwater is homeomorphic to a disk. In particular, it is a contractible space.
- Between the points c and b_2, the topology of the torus does not change.
- After the water passes through the point b_2, the topology becomes more interesting. The region under the water looks like a disk to which we have added a strip of the torus: it is homotopic to a disk with a 1 dimensional cell attached. Another way to see it is that it is homeomorphic to an open cylinder.
- After the water covers the point b_1, we have added another strip to the manifold. The resulting space is homotopic to the cylinder of the last step with a 1 dimensional cell attached to it.
- After the water covers the point a the whole torus is underwater, so we have recovered the whole manifold. Specificaly, what we add from the last step is a disk, this means, a 2 dimensional cell.

In this examplewe have constructed a cell skeleton of $\mathbb{T}^2$ using the intuition of the water rising. Nonetheless, this intuition can be formalized, as we stated earlier, by studying the critical points of the height function restricted to the torus. From this formalization, we can retrieve the homology of $\mathbb{T}^2$, which is

$$H_k(\mathbb{T}^2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0 \text{ or } 2 \\ \mathbb{Z}_2^2 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can see that this homology coincides with the cellular homology of the manifold (in fact, this is always the case).

Morse theory, then, relates the topology of the manifold with the critical points of a function defined over it. The truly unexpected result about this relationship is that it does not depend on the particular function that we are choosing to define the complex: the resulting homology is invariant. In consequence, we get some information on the number of critical points of any function over the manifold (assuming that it satisfies some regularity condition). In particular, the most remarkable result are the Morse inequalities, that yield a lower bound for the number of critical points.

The main focus of this master thesis is not to understand what is the Morse homology, but rather to see how it is constructed, this means, what are the steps that one has to take to define it. We do it this way because these steps are the unifying thread with the second part of the thesis, namely Floer homology.

Floer homology was developed by Andreas Floer in the middle 80's in order to prove the Arnold conjecture. This conjecture states that there is a number of periodic orbits of a Hamiltonian system under certain conditions, and is both interesting from the point of view of symplectic geometry and the study of dynamical systems.

The insight that Floer provided was that all the ideas that allow us to define the Morse homology can be applied in a different (infinite dimensional) context to construct a homology that studies the 1-periodic orbits of Hamiltonian systems defined over the manifold. Moreover, as it was the case with Morse homology, this construction depends only on the topology of the underlying manifold and not on the structure required to retrieve the particular periodic orbits, this means, it is a topological invariant. This leads to the interesting conclusion that the (Hamiltonian) dynamics that can be defined over a manifold are constrained by its topology.

This master thesis starts with a complete description of Morse theory in Chapter 1. After that, we talk about symplectic geometry and the Arnold conjecture in more detail in Chapter 2, in order to give the motivation for the study of the Floer homology. In Chapter 3 we present the basic constructions of the theory, focusing on the Floer equation. Finally, we give the ideas of how the Arnold conjecture is proved, and talk about some other results that can be proved with the tools that we provide in this thesis.

Chapter 1

Fundamentals on Morse homology

In this chapter we explain the basic constructions on Morse theory. Our aim is not to achieve a deep understanding of it, but to get familiar with the way to prove the results and the kind of tools that are used. We do so because Floer homology is constructed using ideas analogous to the ones presented here (in a more complex setting), so it is convenient to be familiar with Morse theory before getting started with Floer homology.

This chapter is based on the introduction of Morse homology that can be found in the first chapters of [AD14]. There is also an excellent presentation in [Mil63], which has also been used.

1.1 Morse functions

Let us consider a differentiable manifold M without boundary, and let $f : M \to \mathbb{R}$ be a smooth function.

Definition 1.1 A point $p \in M$ is called a **critical point** of the function f if the tangent map $df_p : \mathrm{T}_p M \longrightarrow \mathrm{T}\,\mathbb{R} \cong \mathbb{R}$ is zero. In this case, we say that $f(p)$ is a **critical value**.

To classify the critical points of a function, we will be interested in the directions in which it is convex and the ones in which it is concave. The best way to study the convexity of a function systematically is its Hessian, or second derivative. However, we find ourselves with a constraint: it is not possible to define the Hessian of a function on a smooth manifold in all points in a way that it is independent of the choice of the coordinate system. However, it is possible to do it at each critical point, as we will show.

Definition 1.2 Let p be a critical point of a function $f \in \mathcal{C}^\infty(M)$. The **Hessian** of f at p is the bilinear map

$$\begin{array}{rccc} \mathrm{H}_p[f]: & \mathrm{T}_p M \times \mathrm{T}_p M & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & v(X_u(f)) \end{array},$$

where X_u is a vector field extending $u \in \mathrm{T}_p M$ locally.

To define the extension of a vector to a vector field, we use this lemma

Lemma 1.3 *Let $u \in \mathrm{T}_p M$. There is a vector field $X_u \in \mathfrak{X}(M)$ such that $X_u(p) = u$.*

This lemma can be proved in a straightforward way using a suitable bump function defined inside a local chart of p in M.

Lemma 1.4 *The Hessian* $\mathrm{H}_p[f]$ *is well defined (this means, it does not depend on the choice of* X_u*), and it is a bilinear and symmetric map.*

Proof. First, we show that it is symmetric, from where it will be clear that it is well defined:

$$\mathrm{H}_p[f](v,w) - \mathrm{H}_p[f](w,v) = v(X_w(f)) - w(X_v(f)) = X_v|_p (X_w(f)) - X_w|_p (X_v(f)) =$$

$$= [X_v, X_w]|_p (f) = d_p f \cdot [X_v, X_w] = 0,$$

where the last term is zero because $d_p f = 0$, as p is a critical point for f. Thus, if we choose extensions X_v and X_w for v and w (respectivelly), then $\mathrm{H}_p[f](v,w) = \mathrm{H}_p[f](w,v)$.

Looking again at the definition,

$$\mathrm{H}_p[f](v,w) = v(X_w(f)),$$

it is obvious that this does not depend on the extension X_v that we choose for v, as in the expression only depends on $X_v(p) = v$ regardless of the extension. On the other hand, as we just proved, $\mathrm{H}_p[f](v,w) = \mathrm{H}_p[f](w,v)$, so, applying the same argument, the Hessian does not depend on the extension chosen for w. This proves that the Hessian is well defined.

Finally, the Hessian is bilinear, because

$$H_p[f](\alpha u + \beta v, w) = (\alpha u + \beta v)(X_w(f)) =$$

$$= \alpha u(X_w(f)) + \beta v(X_w(f)) = \alpha H_p[f](u,w) + \beta H_p[f](v,w),$$

and the same argument applies to the second component by symmetry. □

Notice that the last proof depends entirely on the fact that p is a critical point. This means, in general it is not possible to prove that $\mathrm{H}_p[f]$ is well defined when p is not a critical point.

Remark 1.5 The local form of the Hessian of a function coincides with the Hessian of the local representation of the function in a chart. If $(x_1, ..., x_n)$ is a local chart centered in a critical point $p \in M$ and $\tilde{f}$ is the local representation of f in this chart, then the local expression of $\mathrm{H}_p[f]$ is precisely the matrix

$$\widetilde{\mathrm{H}}_p[f] := \left(\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(0) \right)_{i,j}.$$

As we said before, we are interested in the Hessian of a function at a critical point to, in some way, count the number of directions in which the function is convex. To do so, we need the concepts of index and non-degeneracy.

Definition 1.6 We define the

- **Index** of p as the dimension of the maximal subspace $V \subset \mathrm{T}_p M$ such that $\mathrm{H}_p[f]|_V$ is negative definite.

- **Nullity** of p as the dimension of the null-space of $\mathrm{H}_p[f]$, this means, the maximal subspace $N \subset \mathrm{T}_p M$ such that $\mathrm{H}_p[f](N, \cdot) = 0$.
- **Non-degenerate critical points** of f as the points p that have nullity 0, this means, that the local representation of $\mathrm{H}_p[f]$ has maximal rank in any local chart.

Notice that all the definitions that we just gave are independent of the choice of coordinates, because the index and nullity of a matrix are independent of the basis chosen to represent it, so they are also invariant under any change of coordinates.

Therefore, it makes sense to classify the critical points of a manifold according to their index. This is the principle from which the Morse theory is derived.

Definition 1.7 We say that a function $f \in \mathcal{C}^\infty(M)$ is a **Morse function** if all its critical points are non-degenerate. If f is a Morse function, we denote

$$\mathrm{Crit}(f) = \{p \in M \mid df_p = 0\},$$

$$\mathrm{Crit}_k(f) = \{p \in \mathrm{Crit}(f) \mid p \text{ has index } k\}.$$

The first interesting thing to study about the Morse functions is their behaviour in a neighbourhood of a critical point:

Proposition 1.8 (Morse Lemma): *Let $p \in \mathrm{Crit}_k(f)$. Then there is a local coordinate system $(U, (y_1, ..., y_n))$ centered on p (this means, with $y_i(p) = 0\ \forall i$) such that*

$$f|_U = f(p) - y_1^2 - ... - y_k^2 + y_{k+1}^2 + ... + y_n^2.$$

Proof. We begin by looking at a local expression $\tilde{f}$ of f which can be derived from the fundamental theorem of calculus,

$$\tilde{f}(x) = \tilde{f}(0) + \int_0^1 \frac{df(tx_1, ..., tx_n)}{dt} dt = f(p) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial \tilde{f}}{\partial x_i}(tx_1, ..., tx_n) dt.$$

If we take $g_i(x) = \displaystyle\int_0^1 \frac{\partial \tilde{f}}{\partial x_i}(tx_1, ..., tx_n)dt$, we can write $\tilde{f}$ as

$$\tilde{f}(x) = f(p) + \sum_{i=1}^n x_i g_i(x).$$

As $g_i(0) = \frac{\partial \tilde{f}}{\partial x_i}(0) = 0$, we can apply the same process for each i, so there are functions h_{ij} such that

$$g_i(x) = \sum_{j=1}^n x_j h_{ij}(x),$$

$$\tilde{f}(x) = f(p) + \sum_{i,j=0}^n x_i x_j h_{ij}(x).$$

These functions satisfy that

$$h_{ij}(0) = \frac{1}{2} \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(0),$$

and

$$h_{ij} = h_{ji}.$$

Then, we can apply inductivelly a change of coordinates, maybe shrinking the domain of the chart at each transformation. We describe the idea for each step:

Suppose that we are in the following situation: there is a local coordinate system $(U_1, (u_1, ..., u_n))$ (with $U_1 \subseteq U$) such that

$$f = f(p) \pm u_1^2 \pm ... \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_i H_{ij}(u),$$

where $(H_{ij})_{i,j}$ form a symmetric matrix and $(H_{ij}(0))_{i,j}$ form a non-degenerate matrix. Let us suppose that $H_{rr}(0) \neq 0$ (if it is not the case, we can apply a linear change of coordinates to ensure it). Take $S(u) = \sqrt{|H_{rr}(u)|}$, which will be a non-vanishing positive function of u in a neighbourhood $U_2 \subset U_1$ of 0. Thus, we can introduce the new local coordinates $(v_1, ..., v_n)$ on U_2 as

$$v_i = u_i \quad \text{for } i \neq r,$$

$$v_r(u) = S(u) \left[u_r + \sum_{i>r} u_i \frac{H_{ir}(u)}{H_{rr}(u)} \right].$$

Using the inverse function theorem we conclude that $(v_1, ..., v_r)$ form an invertible and smooth set of coordinates on a neighbourhood of the origin, $U_3 \subset U_2$. Also, it can be seen that

$$f(v) = f(p) + \sum_{i \leq r} \left(\pm v_i^2 \right) + \sum_{i,j>r} v_i v_j G_{ij}(v),$$

where G_{ij} are symmetric and form a non-degenerate matrix at $v = 0$.

Therefore, after we apply these steps n times we can construct the coordinate system in some neighbourhood U of p satisfying the claimed properties. □

Corollary 1.9 *The non-degenerate critical points of a differentiable function are isolated.*

Remark 1.10 This last corollary is, of course, false for degenerate critical points. Think, for instance, of the function of $\mathbb{R}^2$ defined by $f(x, y) = x^2 + y^2 + 2xy$ in the canonical coordinates. This function has a line of critical points, the one defined by $x + y = 0$, which has no isolated points.

After studying some nice properties about Morse functions, the natural question that arises is if they do actually exist, as the regularity that we are imposing in the definition may be too restrictive. Even if they exist, the Morse functions could be rare in a manifold (for instance, it might be the case that almost all smooth functions have only degenerate critical points). However, in the appendix we provide a positive solution to both questions. In particular:

- We can construct a Morse function in any smooth manifold M. This is proved in A.3.
- Any smooth function in M can be approximated, in the $\mathcal{C}^\infty$ sense, by Morse functions, so Morse functions are actually generic (this means, dense) in $\mathcal{C}^\infty(M)$. This is proved in A.5.

1.2 Applications to topology

In this section we are going to review some theorems on how to characterize the topology of a manifold using a Morse function or, more specifically, the critical points of a Morse function.

First of all, let us reduce to the simplest case, when a part of a manifold does not contain critical points.

Let M be a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function. For $x \in \mathbb{R}$, let $M_x := \{p \in M \mid f(p) \leq x\}$.

Proposition 1.11 *Consider $a, b \in \mathbb{R}$ such that $f^{-1}([a,b]) \subset M$ is compact and does not contain any critical point. Then, M_a is a deformation retract of M_b and, moreover, $M_a \cong M_b$, this means, they are diffeomorphic.*

Proof. Let $W \subset M$ the open set of non-critical points of f, and consider g a Riemannian metric on M. We are going to use g to construct a vector field that yields an appropriate flow, which will serve to construct our desired diffeomorphism.

Take $X = \frac{1}{\|\mathrm{grad} f\|^2}\mathrm{grad} f \in \mathfrak{X}(W)$, and let $\gamma : I \to M$ be a maximal integral curve of X. As a consequence of the definition, we have that

$$\frac{d}{dt}f(\gamma(t)) = df(\gamma(t)) \cdot \gamma'(t) = df(\gamma(t)) \cdot X(\gamma(t)) =$$

$$g\left(\mathrm{grad} f(\gamma(t)), X(\gamma(t))\right) = \frac{1}{\|\mathrm{grad} f(\gamma(t))\|^2} g(\mathrm{grad} f(\gamma(t)), \mathrm{grad} f(\gamma(t))) = 1.$$

Therefore, if we assume $0 \in I$, we have that $f(\gamma(t)) = f(\gamma(0)) + t$.

Let $K = f^{-1}([a,b]) \subset W$, which is compact by hypothesis. Take the initial condition $\gamma(0) \in f^{-1}(a)$. There are two cases in which we can find ourselves: either $\gamma(t) \in K\ \forall t \in I, t > 0$, or the solution goes out of K after some time:

- If $\gamma(t) \in K\ \forall t > 0$, then the solution is defined inside a compact set. Therefore, it is defined for all positive time, so $[0, +\infty) \subset I$. In particular, the solution is defined in $[0, b-a]$.
- If there is $s \in I, s > 0$ such that $\gamma(s) \notin K$, then $b < f(\gamma(s)) = f(\gamma(0)) + s = a + s$, so $s > b - a$. Therefore, $[0, b-a] \subset I$.

Moreover, we can extend X to the whole manifold without losing the properties that we just announced. Take a bump function $\psi : M \to \mathbb{R}$, such that

1. $\psi|_K = 1$.
2. Its support is contained in W.

Then, we can construct the vector field Y on the whole manifold by

$$Y = \begin{cases} \psi(x)X(x) & \text{if } x \in W \\ 0 & \text{otherwise} \end{cases},$$

and it coincides with X at K, so all the results that we proved remain true. Let φ^t be the flow of Y. If necessary, we can shrink the support of ψ to guarantee that the flow is defined up to time $b-a$, so φ^{b-a} is a well defined diffeomorphism on M that sends M_a onto M_b. This concludes the proof that $M_a \cong M_b$.

To prove that M_a is a deformation retract of M_b, consider the collection of maps

$$r: \quad M_b \times [0,1] \quad \longrightarrow \quad M_b$$

defined by

$$r(x,t) = \begin{cases} x & \text{if } f(x) \le a \\ \varphi^{t(a-f(x))}(x) & \text{if } a \le f(x) \le b \end{cases},$$

which induce the desired retraction. □

There is an immediate corollary to this proposition, that is also a classic theorem on differential geometry.

Corollary 1.12 (Reeb's theorem): *Let M be a compact smooth manifold. Suppose that there is a Morse function $f: M \to \mathbb{R}$ that has only two critical points. Then, M is homeomorphic to a sphere.*

Remark 1.13 A smooth function over a compact manifold must always have at least two critical points (the maximum and the minimum) because of Weierstrass theorem. The theorem, in a way, strengthens this assertion: the *only* (up to homeomorphism) manifolds that admit functions with only two critical points are the spheres.

Proof. Normalize the function so that $f(M) = [0,1]$. By the Morse lemma (1.8), there is $\varepsilon > 0$ small enough so that $f^{-1}([0,\varepsilon]) = M_\varepsilon$ and $f^{-1}([1-\varepsilon,1])$ are diffeomorphic to disks in $\mathbb{R}^n$ (with n the dimension of the manifold).

By the proposition (1.11) that we just proved, we know that M_ε is diffeomorphic to $M_{1-\varepsilon}$, so it is also an open disk. Therefore, M is diffeomorphic to two disks glued together by their boundaries.

Let φ denote the gluing map between the boundaries of the disks, so we may write $M \cong \mathbb{D}^n \cup_\varphi \mathbb{D}^n$. Then, if we denote the standard sphere by $\mathbb{S}^n = \mathbb{D}^n \cup_{\mathrm{Id}} \mathbb{D}^n$ (this means, the standard sphere is the result of gluing two disks together with a trivial gluing map), we can construct an explicit homeomorphism

$$h: \quad \mathbb{S}^n \cong \mathbb{D}^n_1 \cup_{\mathrm{Id}} \mathbb{D}^n_2 \quad \longrightarrow \quad \mathbb{D}^n_1 \cup_\varphi \mathbb{D}^n_2$$

by

$$h(z) = \begin{cases} z & \text{if } z \in \mathbb{D}^n_1 \\ \|z\|\varphi\left(\frac{z}{\|z\|}\right) & \text{if } z \in \mathbb{D}^n_2 \setminus \{0\} \\ 0 & z = 0 \in \mathbb{D}^n_2 \end{cases}.$$

Therefore, $M \cong \mathbb{S}^n$. □

Now we can move to a more interesting result, that describes the topology of a manifold when crossing a critical point:

Theorem 1.14 *Let $p \in M$ be a non-degenerate critical point of a smooth function $f: M \to \mathbb{R}$. Let k be its index, and let $c = f(p)$. Take $\varepsilon > 0$ small enough such that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and does not contain any critical point different of p.*

Then, $M_{c+\varepsilon} \simeq M_{c-\varepsilon} \cup \mathbb{D}^k$, this means, $M_{c+\varepsilon}$ is homotopically equivalent to $M_{c-\varepsilon}$ with a k-cell adjointed.

Proof. First of all we will construct a function F with $F < f$ in a neighbourhood of p and such that it coincides with f elsewhere.

Let $(U, u_1, ..., u_n)$ be a Morse chart centered on p, so the function f is

$$f|_U = c - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 + \cdots + u_n^2.$$

We can take a chart and ε such that the image of U under the chart contains the closed ball $B = \{(u_1, ..., u_n) \in \mathbb{R}^n \mid u_1^2 + \cdots + u_n^2 \leq 2\varepsilon\}$. On the other hand, consider the k-disk inside B,

$$e^k = \{(u_1, ..., u_n) \in \mathbb{R}^n \mid u_1^2 + \cdots + u_k^2 \leq \varepsilon \text{ and } u_{k+1} = \cdots = u_n = 0\}.$$

We will use e^k to denote both the disk as we have defined it and its preimage by the chart $(u_1, ..., u_n)$. Thus, e^k is precisely a cell attached to $M_{c-\varepsilon}$, because $M_{c-\varepsilon} \cap e^k = \partial e^k$. What we need to prove, then, is that $M_{c-\varepsilon} \cup e^k$ is a deformation retract of $M_{c+\varepsilon}$.

Let

$$\xi = u_1^2 + \cdots + u_k^2$$
$$\eta = u_{k+1}^2 + \cdots + u_n^2$$

so that

$$f|_U = c - \xi + \eta.$$

We will use this decomposition to define a function F as we said before. To this end, take $\mu : [0, +\infty) \to \mathbb{R}$ a smooth function such that

1. $\mu(0) > \varepsilon$.
2. $\mu(t) = 0 \ \forall t \geq 2\varepsilon$.
3. $-1 < \frac{d\mu}{dt}(t) \leq 0 \ \forall t \in (0, +\infty)$.

Then, let F be defined by

$$F(q) = \begin{cases} f(q) & \text{if } q \notin U \\ f(q) - \mu(\xi(q) + 2\eta(q)) & \text{if } q \in U \end{cases},$$

so

$$F|_U = c - \xi + \eta - \mu(\xi + 2\eta).$$

F is well defined, because U contains the closed ball $\{\xi + \eta \leq 2\varepsilon\}$.

Let us break the rest of the proof in 4 steps.

Step 1: See that $F^{-1}((-\infty, c+\varepsilon]) = M_{c+\varepsilon} \ \big(= f^{-1}((-\infty, c+\varepsilon])\big)$.

Notice, first of all, that F and f coincide outside of the region $E := \{\xi + 2\eta \leq 2\varepsilon\}$, so it suffices to show that $F^{-1}((-\infty, c+\varepsilon]) \cap E = f^{-1}((-\infty, c+\varepsilon]) \cap E$. But notice that, if $q \in E$,

$$F(q) \leq f(q) = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta = c + \frac{1}{2}(\xi + 2\eta) \leq c + \varepsilon,$$

so, in fact, $E \subset F^{-1}((-\infty, c+\varepsilon])$, and $F^{-1}((-\infty, c+\varepsilon]) \subset f^{-1}((-\infty, c+\varepsilon])$. Therefore, $F^{-1}((-\infty, c+\varepsilon]) = f^{-1}((-\infty, c+\varepsilon])$.

Step 2: See that F and f have exactly the same critical points.

We can express F as a function of ξ and η, so we can say that

$$dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta.$$

Where

1. $\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$ by the definition of μ.
2. $\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 0$ by the definition of μ.
3. $d\xi$ and $d\eta$ are simultaneously zero if and only if $\xi = \eta = 0$, this means, at p.

Therefore, the only critical point of F in U is p, so F and f have exactly the same critical points.

Step 3: See that $F^{-1}((-\infty, c-\varepsilon])$ is a deformation retract of $f^{-1}((-\infty, c+\varepsilon])$.

The fact that $F \leq f$, together with the claim proved in Step 1, allows us to deduce that $F^{-1}([c-\varepsilon, c+\varepsilon]) \subset f^{-1}([c-\varepsilon, c+\varepsilon])$. In particular, this means that $F^{-1}([c-\varepsilon, c+\varepsilon])$ is compact. Moreover, notice that the critical value of F at p satisfies that

$$F(p) = c - \mu(0) < c - \varepsilon,$$

because of the definition of μ. Therefore, $p \notin F^{-1}([c-\varepsilon, c+\varepsilon])$, so this set does not contain critical points. With all these considerations, we see that we are under the hypothesis of proposition 1.11. This means that $F^{-1}((-\infty, c-\varepsilon])$ is a deformation retract of $F^{-1}((-\infty, c+\varepsilon]) = f^{-1}((-\infty, c+\varepsilon]) = M_{c+\varepsilon}$ (and, in fact, they are diffeomorphic).

Let $H = \overline{F^{-1}((-\infty, c-\varepsilon]) \backslash M_{c-\varepsilon}}$. With this notation, $F^{-1}((-\infty, c-\varepsilon]) = M_{c-\varepsilon} \cup H$.

Actually, we have seen that $M_{c+\varepsilon} \simeq M_{c-\varepsilon} \cup H$.

Step 4: See that $M_{c-\varepsilon} \cup e^k$ is a deformation retract of $M_{c-\varepsilon} \cup H$.

In the variables (ξ, η) the disk e^k in M can be expressed as $e^k = \{q \in U \mid \xi(q) \leq \varepsilon \text{ and } \eta(q) = 0\}$. We claim that $e^k \subset H$.

First of all, $e^k \subset F^{-1}((-\infty, c-\varepsilon])$. This can be seen because, if $q \in e^k$,

$$F(q) = c - \xi(q) - \mu(\xi(q)) \leq c - \mu(0) \leq c - \varepsilon.$$

In the first inequality we have used the fact that $\xi \geq 0$ and that μ is a decreasing function. In the second inequality we used that $\mu(0) \geq \varepsilon$.

On the other hand, $f(q) = c - \xi \geq c - \varepsilon$, with an equality only when $\xi = \varepsilon$, this means, at ∂e^k. Therefore, as we claimed, $e^k \subset H$.

Now we can construct a retraction of $M_{c-\varepsilon} \cup H$ onto $M_{c-\varepsilon} \cup e^k$. Let us call it r_t. Let r_t be the identity ouside of U for all t, and separate $U \cap (M_{c-\varepsilon} \cup H)$ in three regions:

$$C_1 = \{q \mid \xi(q) \leq \varepsilon\},$$

$$C_2 = \{q \mid \varepsilon \leq \xi(q) \leq \eta(q) + \varepsilon\},$$

$$C_3 = \{q \mid \eta(q) + \varepsilon \leq \xi(q) \Leftrightarrow f(q) \leq c - \varepsilon\}.$$

We will construct r_t separately on each of these three regions, and we will prove that it is the desired retraction.

- r_t on C_1. We define

$$r_t(u_1, ..., u_k, u_{k+1}, ..., u_n) = (u_1, ..., u_k, tu_{k+1}, ..., tu_n),$$

or, equivalently, $r_t(\xi, \eta) = (\xi, t^2\eta)$. It is clear that r_1 is the identity, and that r_0 is a projection onto e^k. Moreover, $F(r_t(q)) \leq c - \varepsilon$, because $\frac{\partial F}{\partial \eta} > 0$.

- r_t on C_2. We define

$$r_t(u_1, ..., u_n) = (u_1, ..., u_k, s_t u_{k+1}, ..., s_t u_n),$$

or, as before, $r_t(\xi, \eta) = (\xi, s_t^2 \eta)$. We define

$$s_t = t + (1 - t)\sqrt{\frac{\xi - \varepsilon}{\eta}}.$$

It is clear that r_1 is the identity. On the other hand, notice that

$$f(r_0(q)) = f(\xi, s_0^2\eta)c - \xi + s_0^2\eta = c - \xi + \xi - \varepsilon = c - \varepsilon,$$

so r_0 maps all of C_2 onto the boundary of $M_{c-\varepsilon}$.

- On C_3, we let $r_t = \text{Id}$ for all t. When $\xi - \varepsilon = \eta$, it coincides with the last definition.

We need to check that r_t is continuous. In particular, we need to check it when $\xi \to \varepsilon$ and $\eta \to 0$. First of all, notice that

- when $\xi = \varepsilon$, $s_t = t$,
- when $\xi - \varepsilon = \eta$, $s_t = 1$.

Thus, the only points where it is not clear if r_t is continuous are those such that $\xi = \varepsilon$ and $\eta = 0$. In particular, we are to check the continuity in the region C_2. In this case, however, we have that

$$\xi - \varepsilon \leq \eta \Rightarrow 0 \leq \frac{\xi - \varepsilon}{\eta} \leq 1,$$

so s_t stays bounded in the whole C_2. Moreover, for each $i > k$, the coordinate u_i is mapped as $u_i \mapsto s_t u_i$. In addition, $|u_i| \leq \eta$. Taking all of this into account, we deduce that

$$0 \leq |s_t u_i| \leq s_t \eta \xrightarrow[\eta \to 0, \xi \to \varepsilon]{} 0,$$

so, in particular, $s_t u_i \xrightarrow[\eta \to 0, \xi \to \varepsilon]{} 0$, as we wanted to see.

Thus, r_t is continuous, so it is a retraction from $M_{c-\varepsilon} \cup H$ onto $M_{c-\varepsilon} \cup e^k$. This concludes the proof. □

1.3 The Morse complex

With all the concepts introduced in Section 1.1, we are able to construct the Morse complex. From now on, consider M to be a compact smooth manifold.

Definition 1.15 A **complex** over a ring R is a sequence of modules $\{C_k\}_{k\in\mathbb{N}}$ over R and a sequence of morphisms $\partial_k : C_k \to C_{k-1}$ such that $\partial_{k-1} \circ \partial_k = 0$.

In the case of the complexes defined in this thesis, the ring that we are taking is $\mathbb{Z}_2$. In this Section we are defining the modules of the Morse complex, and the differential (which is the usual name for the maps $\partial_\bullet$) will be defined properly later.

Definition 1.16 The k-th group of the Morse complex of the manifold M with the function f is the free module over $\mathbb{Z}_2$ generated by the critical points of index k of the function f:

$$C_k(M, f) := \langle \mathrm{Crit}_k(f)\rangle_{\mathbb{Z}_2}.$$

Remark 1.17 The number of critical points of f in M has to be finite, because critical points are isolated (as we commented in corollary 1.9) and M is compact. Therefore, the module $C_k(M, f)$ is finitely generated.

Example 1.1 Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the 2-sphere seen as a submanifold of $\mathbb{R}^3$, and consider the height function $h(x, y, z) = z$ in $\mathbb{R}^3$, but restricted to $\mathbb{S}^2$. Then, it can be checked easily that the only critical points of h are the north and south poles, $N = (0, 0, 1)$ and $S = (0, 0, -1)$, and that the index of h is respectively 2 and 0. Therefore, the Morse complex is isomorphic to

$$C_k(\mathbb{S}^2, h) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

Let us take a moment to outline the key steps to define the Morse complex. It is worth stopping for a global picture before continuing, because these are the same steps that we will have to follow to define the Floer complex. Some of the points have already been discussed:

1. Identify the object to study and its regularity conditions. In our case, the critical points of a Morse function, which has been studied in Section 1.1.

2. Define a way to graduate the objects to study (in order to define the graduation in the complex). In our case, the index of the critical points, as has also been discussed in Section 1.1.

3. Find a way to "connect" the objects being studied. In our case, this will be accomplished using the flow of a certain vector field that we will call pseudogradient, in Section 1.4.

4. Use the connection that we just defined to provide a differential map ∂ in the complex. We will tackle this issue in Section 1.5.

5. Prove that $\partial^2 = 0$, so we have indeed a complex and it defines an homology.

6. Prove that the homology is well defined. In our case, this means proving that the homology depends only on the manifold being studied, and neither on the particular Morse function used to define the Morse complex nor on the pseudogradient that we choose. This is proved in Section 1.7.

7. Understand the resulting homology. This could imply proving classical results for the Morse homology as the Künneth formula, the Poincaré duality, and the functoriality of the homology.

Going a little ahead of ourselves, we can already present the principal result derived from Morse theory: the Morse inequalities:

Theorem 1.18 *Let f be a Morse function, and let $c_k = \#\text{Crit}_k(f)$. Then:*

- *The alternating sum of c_k equals the Euler characteristic of the manifold:*

$$\sum_k (-1)^{-k} c_k = \chi(M).$$

- *The number of critical points of f is bounded from below by the sum of dimensions of the homology groups:*

$$\sum_k c_k \geq \sum_k \text{dim} H_k(M).$$

- *(**Morse Inequalities**): If we take $\beta_k = \text{dim} H_k(M)$ the k-th Betti number, then*

$$c_k \geq \beta_k.$$

In particular, this theorem shows that Morse functions (or, more precisely, the critical points of Morse functions) are constrained by the topology of the manifold in which we define them.

1.4 Pseudogradients and the Smale condition

1.4.1 Pseudogradients

In this section we will focus on how we can connect the points of $\text{Crit}(f)$ (for a Morse function) between them. The way to do this, in $\mathbb{R}^n$, would be the gradient of the function, this means,

$$\text{grad} f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right).$$

This vector field can be intrinsecally defined on a manifold M using a Riemannian metric g as the only vector field that satisfies that, for any vector field $Y \in \mathfrak{X}(M)$,

$$g(\text{grad} f, Y) = df \cdot Y.$$

For our purposes, we do not need to use precisely the gradient of a function, and sometimes it might not satisfy the properties that we need for the flow, so we introduce the more general concept of **pseudogradient**.

Definition 1.19 Let f be a Morse function defined on M. Then, a vector field $X \in \mathfrak{X}(M)$ is a **pseudogradient** adapted to f if it satisfies the two conditions

1. For every $p \in M$ we have $(df)_p \cdot X_p \leq 0$, and the equality holds if and only if x is a critical point for f.

2. For each critical point of f, X coincides with the negative gradient defined with the canonical metric defined in $\mathbb{R}^n$ on the domain of the Morse chart.

Remark 1.20 If (M, g) is a manifold with a Riemannian metric, then the vector field $-\text{grad} f$ defined using g is a pseudogradient.

As with Morse functions, one might ask if pseudogradients do exist. As before, the answer is affirmative:

Proposition 1.21 *Let M be a compact and smooth manifold, and f a Morse function on M. Then, there exists a pseudogradient field X adapted to f.*

Proof. Let $c_1, ..., c_r$ be the critical points of f (there are finitely many as we saw in remark 1.17), and let $(U_1, h_1), ..., (U_r, h_r)$ be Morse charts in the neighbourhoods of these points. We assume that the images of these charts, which we will denote $\Omega_1, ..., \Omega_r$, are disjoint. We can add more charts to get a finite open cover of M, $\{\Omega_j\}_{1 \leq j \leq m}$. We can also refine the open cover so that each critical point c_i is contained only in its associated open set Ω_i.

For each Ω_i, let X_i be the push-forward of the negative gradient of $\tilde{f}$ in U_i by h_i. Also, take a partition of unity associated to $\{\Omega_j\}_{1 \leq j \leq m}$, $\{\varphi_j\}_{1 \leq j \leq m}$. We will have that $\varphi_i(c_i) = 1$ for all i because of the refinement we just applied to the open cover of M. Then, we can define the vector fields

$$\widetilde{X}_j(x) = \begin{cases} \varphi_j(x) X_j & \text{if } x \in \Omega_j \\ 0 & \text{otherwise} \end{cases},$$

and

$$X = \sum_{j=1}^{m} \widetilde{X}_j.$$

In this case, the two conditions to be a pseudogradient adapted to f are satisfied:

1. If we compute
$$df_x \cdot X_x = \sum_{j=1}^{m} df_x \cdot \widetilde{X}_{j,x} \leq 0,$$
and the inequality is an equality iff $\varphi_j(x) X_j(x) = 0$ for every j, so that either x is a critical point, or $\varphi_j(x) = 0$ for all j, which is impossible since $\{\varphi_j\}_{1 \leq j \leq m}$ form a partition of unity.

2. Let c_i be a critical point. By construction, X coincides with the image of the negative gradient with the canonical metric over $U_i \bigcap \left(\bigcup_{i \neq j} U_j \right)$, which, also by construction, contains a neighbourhood of c_i.

□

Going back to the question of how to connect two critical points, let us define the stable and unstable manifolds:

Definition 1.22 Let a be a critical point of f, and X a pseudogradient adapted to f. We denote by φ^s the flow of X. We define the **stable manifold** of a as

$$W^s(a) = \left\{ x \in M \;\middle|\; \lim_{s \to +\infty} \varphi^s(x) = a \right\},$$

and its **unstable manifold** as

$$W^u(a) = \left\{ x \in M \;\middle|\; \lim_{s \to -\infty} \varphi^s(x) = a \right\}.$$

These are, actually, smooth manifolds, and it can be shown that they are diffeomorphic to open disks. Moreover, we have that

$$\dim W^u(a) = \text{codim} W^s(a) = \text{Ind}(a).$$

Finally, we want to see that the flow of a pseudogradient does actually connect critical points. To this end, we have the following result:

Proposition 1.23 *Let M a compact manifold, f a Morse function and $\gamma : \mathbb{R} \to M$ a trajectory of a pseudogradient field X. Then, there are critical points a and b such that*

$$\lim_{s \to -\infty} \gamma(s) = a \ , \ \textit{and} \ \lim_{s \to +\infty} \gamma(s) = b.$$

Proof. Let us prove this result in the case that $s \to +\infty$, as the case for $-\infty$ is proved the same way. Suppose that the result is false, so there is a trajectory γ such that it has no limit to $+\infty$. Then, each time that γ enters a Morse neighbourhood for any critical point c_i, Ω_i, it must leave sometime later, because f is strictly decreasing along the trajectories of a pseudogradient. Thus, there is a time $s_0 > 0$ such that $\forall s > s_0$,

$$\gamma(s) \in M \setminus \bigcup_i \Omega_i,$$

so $df(\gamma(s)) \cdot X(s) \leq -\varepsilon_0$ for some $\varepsilon_0 > 0$ and for any $s > s_0$.

Therefore, for any $s > s_0$,

$$f(\gamma(s)) - f(\gamma(s_0)) = \int_{s_0}^{s} \frac{d(f \circ \gamma)}{dt} dt = \int_{s_0}^{s} df(\gamma(t)) \cdot X(\gamma(t)) dt \leq -\varepsilon_0 (s - s_0).$$

This leads to the conclusion that $\lim_{s \to +\infty} f(\gamma(s)) = -\infty$, which is impossible since M is compact. □

1.4.2 The Smale condition

The notion of connecting critical points between them needs to be complemented by the fact that the intersection of the unstable and stable manifolds is good enough, in the sense that it is regular. In particular, the notion that we need is the one of transversality.

Definition 1.24 Let M a smooth manifold and $S, T \subset M$ smooth submanifolds. We say that S and T **intersect transversally** if, $\forall p \in L \bigcap T$,

$$\mathrm{T}_p M = \mathrm{T}_p S + \mathrm{T}_p T.$$

Notice that the sum in the definition does not need to be direct, so it is possible that $\dim(S) + \dim(T) \geq \dim(M)$.

If S and T intersect transversally, we denote it as $S \pitchfork T$.

It can be proved that, if S and T intersect transversally, then:

1. $S \cap T$ is a submanifold of M.
2. $\dim(M) = \dim(S) + \dim(T) - \dim(S \cap T)$.

Definition 1.25 We say that a pseudogradient X adapted to f satisfies the **Palais-Smale condition** if, for any $a, b \in \mathrm{Crit}(f)$,

$$W^u(a) \pitchfork W^s(b).$$

Remark 1.26 If X satisfies the Palais-Smale condition, then for any pair of critical points a and b, we have that

$$\dim(W^u(a) \cap W^s(b)) = \mathrm{Ind}(a) - \mathrm{Ind}(b).$$

Let $\mathcal{M}(a,b) := W^u(a) \cap W^s(b)$. It is the space covered by the trajectories going from a to b. The group $\mathbb{R}$ acts freely and transitivelly on $\mathcal{M}(a,b)$ by

$$t \cdot p = \varphi_X^t(p).$$

Definition 1.27 Let $\mathcal{L}(a,b)$ denote the **space of trajectories** between a and b defined as

$$\mathcal{L}(a,b) := \mathcal{M}(a,b) \,/\mathbb{R}\,.$$

Remark 1.28 As a consequence of formula 1.26, the dimension of the space of trajectories is

$$\dim(\mathcal{L}(a,b)) = \mathrm{Ind}(a) - \mathrm{Ind}(b) - 1.$$

Corollary 1.29 *If* $\mathrm{Ind}(a) - \mathrm{Ind}(b) = 1$*, then the set* $\mathcal{L}(a,b)$ *is discrete.*

1.4.3 Critical points of compact 1-dimensional manifolds

Here, we are going to use what we learned about Morse functions and pseudogradients to prove a classification theorem for compact 1-dimensional manifolds that will be important in the following section.

First of all, we need to define the notion of incoming vector field of a manifold with boundary.

Definition 1.30 Consider M a manifold with boundary, and let X be a vector field defined on M. We say that X is **incoming** on the boundary of M if, for any chart (U,φ) covering a part of ∂M, the local representation of X is

$$\widetilde{X} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

with a_i functions such that $a_i(\varphi^{-1}(q)) < 0 \ \forall q \in \partial M$.

Lemma 1.31 *For any manifold with boundary M, we can construct a pair (f, X) of Morse function and pseudogradient adapted to it such that X is incoming.*

Theorem 1.32 *Let M be a compact and connected manifold of dimension 1. Then, M is diffeomorphic to $\mathbb{S}^1$ if it has no boundary, and to $[0,1]$ otherwise.*

Proof. First of all, construct a pair (f, X) of Morse function and incoming pseudogradient adapted to f as indicated in lemma 1.31. As M is 1-dimensional, the critical points of f can only be local minima or maxima. Let $c_1, ..., c_k$ be the minima of f.

By definition, the stable manifold of c_i, $W^s(c_i)$, is an open interval embedded in M. Let A_i be the closure of this stable manifold. Then, we are adding the starting points of the trajectories of $W^s(c_i)$. There are two options:

- They are both local maxima (they may or may not coincide).
- At least one of them is a boundary point of M, so the points must be distinct.

Also, note that $M = \bigcup_i A_i$. Therefore, if $x \in M$ there are two options: either $x \in W^s(c_i)$ for some minimum c_i, or x is a local maximum, but then it belongs to the closure of a stable manifold.

If $k = 1$, the theorem is already proved, so consider $k \geq 2$. As M is connected, there is some $i \geq 2$ such that $A_1 \cap A_i \neq \emptyset$. The intesection may only contain local maxima that are not boundary points, because they are points from which the trajectories reach different minimum points. Thus, $\partial M \cap (A_1 \cap A_i) = \emptyset$. There are two options:

- The intesection contains two points. Then, they are both local maxima, $A_1 \cup A_i \cong \mathbb{S}^1$ and we are done.
- The intersection contains a single point. Then, $A_1 \cup A_i \cong [0,1]$. If $A_1 \cup A_i = M$, we are done.

Otherwise, we can replace A_1 by $A_1 \cup A_i$ in the collection of closed sets $\{A_i\}_i$, and remove A_i from this list. Then, we can repeat the same argument with the collection $\{A_1, ..., A_{i-1}, A_{i+1}, ..., A_k\}$, which contains $k-1$ sets. Therefore, we may apply the process a finite number of times, and end up with one of the cases above. □

1.5 The differential on the Morse complex

Continuing with the program sketched in Section 1.3, we are to define a differential on the Morse complex using a pseudogradient adapted to our Morse function f and satisfying the Palais-Smale property, as explained in Section 1.4. More precisely, we are going to use the flow of our pseudogradient to define the differential, and then to prove that $\partial^2 = 0$.

1.5.1 Definition of the differential

Let f a Morse function, and X a pseudogradient adapted to it and satisfying the Smale condition. Then, we can define for each pair of critical points $a, b \in \mathrm{Crit}(f)$, the manifold $\mathcal{L}(a, b)$. Recall that, when $\mathrm{Ind}(a) \leq \mathrm{Ind}(b)$, this manifold is empty. Moreover, if $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 1$, it is discrete. This is because

$$\dim \mathcal{L}(a, b) = \mathrm{Ind}(a) - \mathrm{Ind}(b) - 1.$$

In this Section we are going to show that, in fact, $\mathcal{L}(a, b)$ must be finite when $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 1$, so it makes sense to define

$$n_X(a, b) := \#\mathcal{L}(a, b) \pmod 2,$$

which denotes the number of trajectories of the flow of X which go from a to b (in infinite time). We are taking it modulo 2 because our homology groups are defined over $\mathbb{Z}_2$. It is posible to define the Morse homology over $\mathbb{Z}$ also, but then orientations must be introduced. To keep it simple, we restrict ourselves to the case with $\mathbb{Z}_2$.

Definition 1.33 The **differential of the Morse complex** with function f and pseudogradient X can be defined over the generators of $C_k(M, f, X)$ (this means, $a \in \mathrm{Crit}_k(f)$) as

$$\partial_X(a) = \sum_{b \in \mathrm{Crit}_{k-1}(f)} n_X(a, b)b.$$

Our aim in this Section will be to prove that it is well defined (this means, that $n_X(a, b)$ is well defined), and to show that $\partial_X^2 = 0$. To do that, we will need to study the space of broken trajectories.

1.5.2 The space of broken trajectories. Compactness

Consider $a, b \in \mathrm{Crit}(f)$ with $\mathrm{Ind}(a) > \mathrm{Ind}(b)$. We already know that $\mathcal{L}(a, b)$ (the space of the trajectories from a to b) is a smooth manifold. However, we are interested in study its compactification, in one hand to prove that $n_X(a, b)$ is well defined, and on the other hand to prove that $\partial_x^2 = 0$.

Instead of studying how to compactify the space $\mathcal{L}(a, b)$, we will introduce a candidate of its compactification, the space of broken trajectories, containing $\mathcal{L}(a, b)$. Then, we will define a meaningful topology on this space. Finally, we are going to show that this topology makes this space compact.

Definition 1.34 The **space of broken trajectories** from a to b is the disjoint union

$$\overline{\mathcal{L}}(a, b) := \bigcup_{c_1, \ldots, c_{m-1} \in \mathrm{Crit}(f)} \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{m-1}, b),$$

where the union spans all the tuples of critical points, regardless of the size of the tuple. In particular, $\mathcal{L}(a, b) \subset \overline{\mathcal{L}}(a, b)$.

It is important to notice that the only tuples of critical points contributing to the union are the ones such that $\mathrm{Ind}(a) > \mathrm{Ind}(c_1) > ... > \mathrm{Ind}(c_{m-1}) > \mathrm{Ind}(b)$.

Remark 1.35 When $\mathrm{Ind}(a) - \mathrm{Ind}(b) = 1$, we have that $\overline{\mathcal{L}}(a, b) = \mathcal{L}(a, b)$.

Remark 1.36 Suppose that $\mathrm{Ind}(a) - \mathrm{Ind}(b) = 2$, and take $k = \mathrm{Ind}(a)$. Then,

$$\overline{\mathcal{L}}(a, b) = \mathcal{L}(a, b) \bigcup \left[\bigcup_{c \in \mathrm{Crit}_{k-1}} \mathcal{L}(a, c) \times (c, b) \right].$$

We can see that in the union above, the dimension of $\mathcal{L}(a, b)$ is 1, whereas $\dim\mathcal{L}(a, c) = \dim\mathcal{L}(c, b) = 0$ for all $c \in \mathrm{Crit}_{k-1}(f)$. Thus, we are adding some points to a 1-dimensional manifold.

With this definition, we can introduce a topology on the space of broken trajectories.

For each $p \in \text{Crit}(f)$, let $\Omega(p) \subset M$ denote the domain of the Morse chart centered in p. If necessary, shrink some of these domains to guarantee that $\Omega(p) \cap \Omega(q) = \emptyset$ for each pair $p \neq q \in \text{Crit}(f)$.

Let $\lambda \in \overline{\mathcal{L}}(a,b)$. Then, $\lambda = (\lambda_1, ..., \lambda_m)$ for some trajectories λ_i. Consider $a = c_0, c_1, ..., c_{m-1}, c_m = b$ the critical points connected by these trajectories. Let U^-_{i-1} be a neighbourhood of the exit point of λ_i from the set $\Omega(c_{i-1})$ taken to lie in a level set of f. Similarly, take U^+_i a neighbourhood of the entry point of λ_i into the set $\Omega(c_i)$, also lying in a level set of f. Then, we have a family of sets, which we will denote by

$$\mathcal{W}(\lambda; U^-_0, U^\pm_1, ..., U^\pm_{m-1}, U^+_m).$$

Thus, we get a set of families, spanning all the broken trajectories $\lambda \in \overline{\mathcal{L}}(a,b)$ and the neighbourhoods $U^\pm_i$ that we just described. With all of this, we can define a topology on $\overline{\mathcal{L}}(a,b)$:

Definition 1.37 We say that a broken trajectory $\mu = (\mu_1, ..., \mu_k)$ **belongs** to a open neighbourhood $\mathcal{W}(\lambda, \mathbf{U})$ and denote it by $\mu \in \mathcal{W}(\lambda, \mathbf{U})$, if there are $0 < i_0 < ... < i_{k-1} < i_k = m$ such that

- $\mu_j \in \mathcal{L}(c_{i_j}, c_{i_{j+1}})$ $\forall j < k$.
- For all j, μ_j exits the chart $\Omega(c_{i_j - 1})$ through some point in the interior of $U^-_{i_j - 1}$, and enters the chart $\Omega(c_{i_j})$ through some point in the interior of $U^+_{i_j}$.

This way, the $\mathcal{W}(\lambda, \mathbf{U})$ form a fundamental system of open neighbourhoods to define the topology of $\overline{\mathcal{L}}(a,b)$.

Remark 1.38 We can see that $\mu \in \mathcal{W}(\lambda, \mathbf{U})$ implies that $k \leq m$, this means, μ has less or equal components than λ. Equivalently, μ passes through the same number of critical points that λ or less.

This topology establishes that a broken trajectory μ is "close" to λ if all the critical points connected by μ are also connected by λ and if μ leaves $\Omega(c_{i_j})$ and enters $\Omega(c_{i_{j+1}})$ sufficiently close to λ.

Remark 1.39 This topology coincides with the topology of $\mathcal{L}(a,b)$ as a manifold.

Finally, we can prove the central theorem of this Section:

Theorem 1.40 *The space* $\overline{\mathcal{L}}(a,b)$ *is compact.*

Proof. We will prove that $\overline{\mathcal{L}}(a,b)$ is sequentially compact, this means, that for any sequence $(l_n)_n$ we can extract a subsequence that is convergent to some $l \in \overline{\mathcal{L}}(a,b)$.

First of all, let $(l_n)_n \subset \mathcal{L}(a,b)$. The general case will follow from this one.

Let l^-_n denote the exit point of the trajectory l_n from $\Omega(a)$. As $l^-_n \in \partial\Omega(a)$, which is compact, we can extract a subsequence such that $l^-_{n_k} \xrightarrow[k\to\infty]{} a^-$, with $a^- \in \partial\Omega(a)$ and $a^- \in W^u(a)$.

Let $\gamma(t) = \varphi^t_X(a^-)$ be the trajectory of X through a^-. By Proposition 1.23, $c_1 = \lim_{t\to+\infty} \gamma(t)$ is a critical point, and $\gamma \in \mathcal{L}(a, c_1)$. Let d^+ denote the entry point of γ into

$\Omega(c_1)$. By the theorem of dependence of solutions of differential equations on the initial conditions, l_n must also (at least for n large enough) enter $\Omega(c_1)$ through a point d_n^+. We have that $\lim_{n\to\infty} d_n^+ = d^+$ by the following lemma:

Lemma 1.41 *Let $x \in M\backslash\text{Crit}(f)$, and $(x_n)_n$ a sequence that tends to x. Let y_n a sequence of points such that y_n and x_n belong to the same trajectory of X for each n, and y belonging to the same trajectory as x. Moreover, suppose that $f(y_n) = f(y)$. Then,*

$$\lim_{n\to\infty} y_n = y.$$

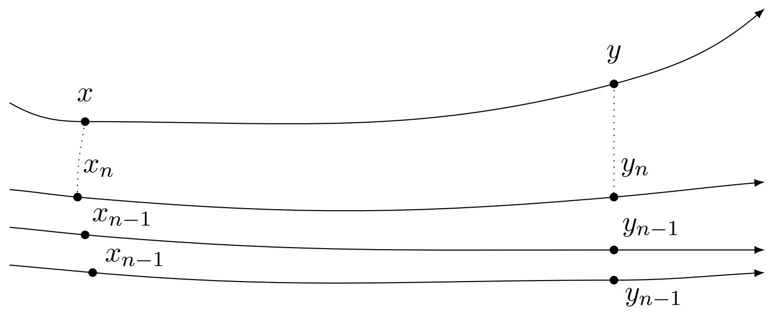

Figure 1.1: Illustration of Lemma 1.41.

Proof. Consider U an open subset of $M\backslash\text{Crit}(f)$ such that it contains x, y, x_n and y_n (at least for n sufficiently large). On U we can define the vector field

$$Y = -\frac{1}{df(X)}X.$$

Let ψ^t be its flow. Y and X are colinear at each point, so they have the same trajectories. In addition,

$$f(\psi^t(z)) = f(z) - t.$$

Therefore, we can express y_n as

$$y_n = \psi^{f(x_n)-f(y_n)}(x_n) = \psi^{f(x_n)-f(y)}(x_n),$$

and

$$y = \psi^{f(x)-f(y)}(x).$$

So $\lim_{n\to\infty} y_n = y$. □

If $c_1 = b$ then $\lim_{n\to\infty} l_n = \gamma \in \mathcal{L}(a,b)$, so $(l_n)_n$ has a convergent subsequence. This means that we need to check what happens for $c_1 \neq b$.

The points d_n^+ do not belong to $W^s(c_1)$ (because, otherwise, we would have that $l_n \in \mathcal{L}(a, c_1)$, which contradicts our hypothesis). Therefore, l_n exits $\Omega(c_1)$ through a point d_n^-. As before, we can extract a subsequence such that $\lim_{n\to\infty} d_n^- = d^-$.

We claim that $d^- \in W^u(c_1)$. Suppose that it is not the case. Let μ denote the trajectory of X passing through d^-. If $d^- \notin W^u(c_1)$, there must be a point d_* such that $f(d_*) = f(d_n^+)$.

By the lemma that we just proved, $d_n^+ \xrightarrow[n\to\infty]{} d_*$, so $d_* = d^+$. Therefore, $d_* \in W^s(c_1)$. However, this is a contradiction, because μ passes through d_* and exits $\Omega(c_1)$ through d^-. Therefore, we have proved that $d^- \in W^u(c_1)$.

If we repeat this process a finite number of times (as there is a finite number of critical points), we will be able to construct a subsequence $(l_n)_n$ and a broken trajectory $\lambda = (\lambda_1, ..., \lambda_m)$ such that $\lim_{n\to\infty} l_n = \lambda$, as in Figure 1.5.2.

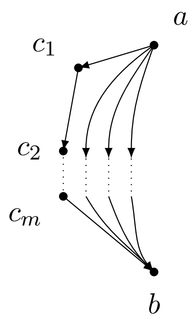

Figure 1.2: The trajectories l_n tend to a broken trajectory.

Let us consider the general case of a sequence of $\overline{\mathcal{L}}(a,b)$. We can extract a subsequence $(l_n)_n$ such that there are critical points $c_1, ..., c_{m-1}$ with $l_n \in \mathcal{L}(a,c_1) \times \cdots \times \mathcal{L}(c_{m-1}, b)$. We can do this because the union

$$\overline{\mathcal{L}}(a,b) = \bigcup \mathcal{L}(a,c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b)$$

has a finite number of terms, so a infinite number of terms of the sequence $(l_n)_n$ must lie in some of the terms.

Thus, $l_n = (l_n^1, ..., l_n^{m-1}) \in \mathcal{L}(a,c_1) \times \cdots \times \mathcal{L}(c_{m-1}, b)$. If we apply the previous argument to each of the $(l_n^i)_n$, we can construct a subsequence and a broken trajectory μ such that $l_n \xrightarrow[n\to\infty]{} \mu$. □

From here we can start deducing the results that we announced at the beginning of this Section:

Corollary 1.42 *If Ind(a) − Ind$(b) = 1$, the set $\mathcal{L}(a,b)$ is finite.*

Proof. On one hand, $\dim\mathcal{L}(a,b) = 0$, so it is a discrete set. On the other hand, $\mathcal{L}(a,b) = \overline{\mathcal{L}}(a,b)$, so it is compact. We conclude that it must be finite. □

With this, we have shown that ∂_X is well defined. To prove that $\partial_X^2 = 0$ is more delicate:

Theorem 1.43 *If $a, b \in \mathrm{Crit}(f)$ and Ind(a) − Ind$(b) = 2$, then $\overline{\mathcal{L}}(a,b)$ is a compact manifold with boundary. In addition, the set*

$$\bigcup_{c\in\mathrm{Crit}(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b)$$

is the boundary of $\overline{\mathcal{L}}(a,b)$.

This theorem is proved in Appendix A.2. Assuming that it is true, by the theorem 1.32 we have that each connected component of $\overline{\mathcal{L}}(a,b)$ must be diffeomorphic to $\mathbb{S}^1$ or to $[0,1]$. From this, we are able to prove the following result:

Corollary 1.44 *The differential of the Morse complex satisfies that $\partial_X^2 = 0$.*

Proof. Take $a \in \mathrm{Crit}_k(f)$. Then,

$$\partial_X(\partial_X a) = \sum_{b\in \mathrm{Crit}_{k-2}(f)} \sum_{c\in \mathrm{Crit}_{k-1}(f)} n_X(a,c) n_X(c,b) b =$$

$$= \sum_{b\in \mathrm{Crit}_{k-1}(f)} \left(\sum_{c\in \mathrm{Crit}_{k-2}(f)} n_X(a,c) n_X(c,b) \right) b.$$

However, we can see that

$$\sum_{c\in \mathrm{Crit}_{k-2}(f)} n_X(a,c) n_X(c,b) = \# \left\{ \bigcup_{c\in \mathrm{Crit}(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b) \right\} \pmod 2.$$

As we just said, the term on the right is the boundary of a 1-dimensional compact manifold with boundary (not necessarily connected), so it has an even number of points. Therefore, its cardinal modulo 2 must be 0. □

1.6 Examples of the Morse complex

In this section we are going to provide some examples of the Morse complexes constructed in various manifolds these examples are meant to illustrate the properties of the Morse homology, as well as give some sense to how the theory is developed.

1.6.1 The sphere

Example 1.2 Consider the 2 dimensional sphere embedded inside $\mathbb{R}^3$ and the height function $h(x,y,z) = z$ (restricted to $\mathbb{S}^2$). As we saw in 1.12, the spheres are the only compact manifolds without boundary that admit only two critical points, and the height function is the prime example of this.

In this case (see figure 1.3), the complex is easy to construct. The only critical points of h are S (of index 0) and N of index 2, and the vector field $-\mathrm{grad}h$ (defined using the euclidean metric) satisfies the Smale condition, so we can use it to define the differential of the complex. However, we just commented that

$$C_k(\mathbb{S}^2, h) = \begin{cases} \mathbb{Z}_2 S & \text{if } k = 0 \\ \mathbb{Z}_2 N & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases},$$

so $\partial_k = 0$ for all k. Therefore, the homology that results is the one that we might expect:

$$H_k(\mathbb{S}^2, h) = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

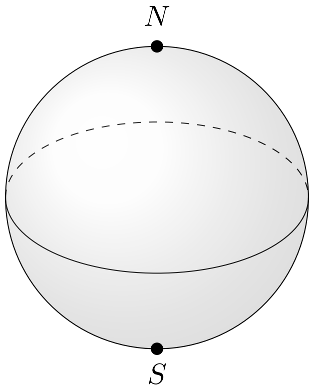

Figure 1.3: The critical points of h on $\mathbb{S}^2$.

Example 1.3 Let us think about a different function on the sphere. This function, f is such that it induces the dynamics shown in Figure 1.4. In particular, it has the following critical points:

- The local maxima a_1, a_2 and a_3, which are critical points of index 2.
- The critical points b_1 and b_2 of index 1.
- The absolute minimum c, of index 0.

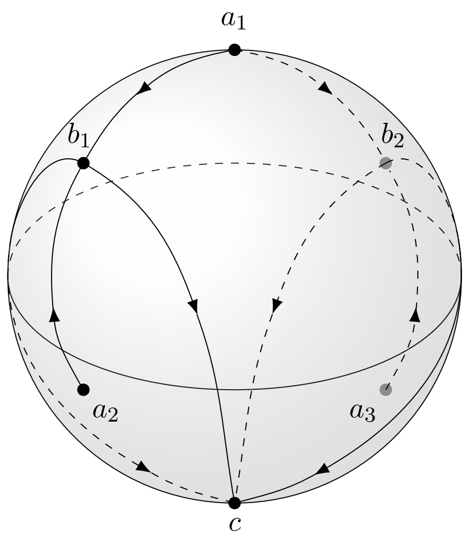

Figure 1.4: The critical points and some trajectories of function f over $\mathbb{S}^2$.

In Figure 1.4 we sketch the dynamics induced by this function by plotting the trajectories that connect critical points of consecutive indices (one has to imagine that all the points of $\mathbb{S}^2$ not belonging to any of these trajectories belong to a trajectory connecting a critical point of index 2 to c).

Therefore, we already know that

$$C_k(\mathbb{S}^2, f) = \begin{cases} a_1\mathbb{Z}_2 \oplus a_2\mathbb{Z}_2 \oplus a_3\mathbb{Z}_2 & \text{if } k = 2 \\ b_1\mathbb{Z}_2 \oplus b_2\mathbb{Z}_2 & \text{if } k = 1 \\ c\mathbb{Z}_2 & \text{if } k = 0 \end{cases},$$

this means, $C_2(\mathbb{S}^2, f) \cong \mathbb{Z}_2^3$, $C_1(\mathbb{S}^2, f) \cong \mathbb{Z}_2^2$ and $C_0(\mathbb{S}^2, f) \cong \mathbb{Z}_2$.

Moreover, we can compute the differential over the generators of each group of the complex:

$$\partial_2 a_1 = b_1 + b_2 \,,\ \partial_2 a_2 = b_1 \,,\ \partial_2 a_3 = b_2,$$

$$\partial_1 b_1 = 2c = 0 \,,\ \partial_1 b_2 = 2c = 0.$$

From this, we deduce that $\partial_1 = 0$, and

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

This means, $\operatorname{Im}(\partial_2) = C_1(\mathbb{S}^2, f)$, and $\operatorname{Ker}(\partial_2) = \langle a_1 + a_2 + a_3 \rangle_{\mathbb{Z}_2}$. This gives us a complete description of the homology groups:

$$H_0(\mathbb{S}^2, f) = C_0(\mathbb{S}^2, f) \,/\operatorname{Im}(\partial_1) = \mathbb{Z}_2[c],$$

$$H_1(\mathbb{S}^2, f) = \operatorname{Ker}(\partial_1) \,/\operatorname{Im}(\partial_2) = C_1(\mathbb{S}^2, f) \,/C_1(\mathbb{S}^2, f) = 0,$$

$$H_2(\mathbb{S}^2, f) = \operatorname{Ker}(\partial_2) \,/0 = \mathbb{Z}_2[a_1 + a_2 + a_3].$$

In particular, it is clear that $H_k(\mathbb{S}^2, h) \cong H_k(\mathbb{S}^2, f)$, suggesting that (as we will prove in Section 1.7) the homology of a manifold does not depend on the function used to define it.

1.6.2 The torus

Example 1.4 Consider the torus embedded in $\mathbb{R}^3$ as shown in figure 1.5.

It can be checked that it is a Morse function, so it provides the intuition of how to recover the cell decomposition of $\mathbb{T}^2$ from the critical points of the height function h, as seen in Theorem 1.14. However, its negative gradient does not satisfy the Smale condition. This can be seen, for instance, because $\dim(W^U(b_1) \cap W^S(b_2)) = 1$, so $W^U(b_1) \not\pitchfork W^S(b_2)$. This prevents us to use it to define the differential of the Morse complex on $\mathbb{T}^2$.

Nevertheless, this problem can be solved taking a different pseudogradient adapted to h. Otherwise, we can choose a different embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$, yielding a tilted torus, and taking the height function the same way, as shown in Figure 1.6.

Example 1.5 In the case of the tilted torus, all the intersections of stable and unstable manifolds are transversal, so we can use the negative gradient of the height function h' as pseudogradient to define the differential on the complex.

In particular, looking at figure 1.6, we see that

$$C_k(\mathbb{T}^2, h') = \begin{cases} a\mathbb{Z}_2 & \text{if } k = 2 \\ b_1\mathbb{Z}_2 \oplus b_2\mathbb{Z}_2 & \text{if } k = 1 \\ c\mathbb{Z}_2 & \text{if } k = 0 \end{cases}.$$

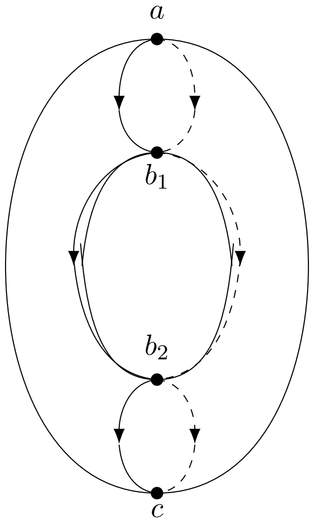

Figure 1.5: The critical points of h in $\mathbb{T}^2$.

Moreover, we can compute the differentials of the complex by looking at the picture. From there we deduce that

$$\partial_2 a = 2b_1 + 2b_2 = 0,$$
$$\partial_1 b_1 = \partial_1 b_2 = 2c = 0,$$

so both ∂_2 and ∂_1 are 0. Therefore, we deduce that the Morse homology of the torus is

$$H_k(\mathbb{T}, h') \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 0 \text{ or } 2 \\ \mathbb{Z}_2^2 & \text{if } k = 1 \end{cases}.$$

1.6.3 Projective spaces

In order to construct the Morse complex over the real projective spaces, we are going to do first the case of the projective plane $\mathbb{P}^2_{\mathbb{R}}$ and then generalize the solution for any projective space $\mathbb{P}^n_{\mathbb{R}}$.

Example 1.6 Let $\mathbb{P}^2_{\mathbb{R}} = \mathbb{S}^2/\sim$, where $\sim$ identifies antipodal points. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x, y, z) = y^2 + 2z^2$, and let $f = F|_{\mathbb{S}^2}$. It is clear that, if $z \sim z'$ then $f(z) = f(z')$, so f is well defined as a smooth function from $\mathbb{P}^2_{\mathbb{R}}$ to $\mathbb{R}$. Moreover, we will show that it is a Morse function, and use it to construct the Morse complex, by studying it in an open cover of charts of $\mathbb{P}^2_{\mathbb{R}}$:

1. Consider the chart of points with $x \neq 0$. Then, $x = \sqrt{1 - y^2 - z^2}$, with $y^2 + z^2 < 1$. In this case, the local form of f in this chart coincides with the original formula,

$$\tilde{f}(y, z) = y^2 + 2z^2,$$

so $d\tilde{f}(y, z) = 2ydy + 4zdz$. Thus,

$$\mathrm{H}_{(0,0)}[\tilde{f}] = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Thus, the point $a = [1 : 0 : 0]$ is a non-degenerate critical point of index 0.

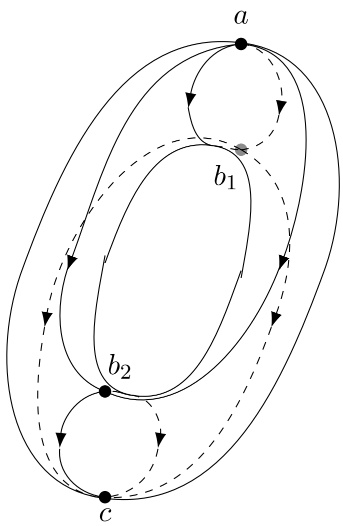

Figure 1.6: The critical points of the tilted torus.

2. Consider the chart of points with $y \neq 0$. As in the last case, in this chart we have that $y = \sqrt{1 - x^2 - z^2}$, with $x^2 + z^2 < 1$. Therefore, the local form of f in this chart is $\tilde{f}(x, z) = 1 - x^2 + z^2$, and $d\tilde{f}(x, z) = -2xdx + 2zdz$. Then,

$$\mathrm{H}_{(0,0)}[\tilde{f}] = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

   Therefore, the point $b = [0 : 1 : 0]$ is a non-degenerate critical point of index 1.

3. Take the chart of points with $z \neq 0$. As before, in this chart we have that $z = \sqrt{1 - x^2 - y^2}$, so the local form of f is $\tilde{f}(x, y) = 2 - y^2 - 2x^2$. Thus, $d\tilde{f}(x, y) = -4xdx - 2ydy$, so

$$\mathrm{H}_{(0,0)}[\tilde{f}] = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}.$$

   Thus, the point $c = [0 : 0 : 1]$ is a non-degenerate critical point of index 2.

As we have checked all the charts in an open covering of $\mathbb{P}^2_{\mathbb{R}}$, we deduce that a, b, c are the only critical points of f, and therefore it is a Morse function. We can, in fact, sketch the dynamics induced by the negative gradient of f in two dimensions, taking the points of the sphere with $z \geq 0$, as shown in Figure 1.7.

From this we deduce that $\partial_2 c = 2b = 0$ and $\partial_2 b = 2a = 0$. Thus, $H_k(\mathbb{P}^2_{\mathbb{R}}, f) \cong \mathbb{Z}_2$ for $k = 0, 1, 2$ (and 0 otherwise).

This process can be generalized to any dimension: let $\mathbb{P}^n_{\mathbb{R}} = \mathbb{S}^n / \sim$, where $\sim$ denotes the antipodal identification, and consider the function in $\mathbb{R}^{n+1}$ defined by

$$F(x_0, ..., x_n) = \sum_{k=1}^{n} k x_k^2,$$

and denote by f its restriction to $\mathbb{S}^n$, and, as before, by abuse of notation, let f denote also the induced map in $\mathbb{P}^n_{\mathbb{R}}$. Then, we can take $n + 1$ charts, where the k-th chart is defined by

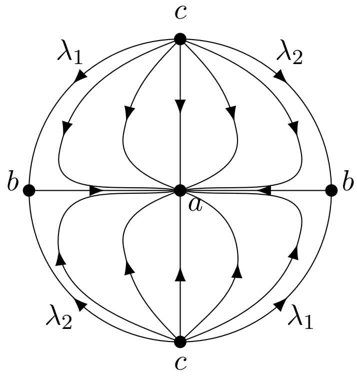

Figure 1.7: The dynamics in the projective plane.

$x_k \neq 0$. The local form of f at this chart is

$$\tilde{f}(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n) = k + \sum_{j=0}^{n} (j-k)x_j^2,$$

so

$$d\tilde{f}(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n) = 2\sum_{j=0}^{n} (j-k)x_j dx_j,$$

and therefore the only critical point at the chart k is $c_k = [e_k]$ (where e_k denotes the k-th vector of the canonical basis of $\mathbb{R}^{n+1}$, starting with 0). The Hessian at this point is

$$\mathrm{H}_{(0,...,0)}[\tilde{f}] = \begin{pmatrix} -2k & 0 & \cdots & 0 \\ 0 & -2(k-1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2(k-n) \end{pmatrix}$$

(omitting the null term).

Therefore, $\chi(c_k) = k$. Moreover, as in the case with $n = 2$, it is possible to see that $\partial_k c_k = 0$, so

$$H_k(\mathbb{P}^n_{\mathbb{R}}, f) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

1.7 The Morse homology is well defined

Our goal in this section is to prove that the Morse homology is independent of the choice of the Morse function used to define it, and of the pseudogradient field used to define the differential. The whole section is dedicated to prove the following theorem:

Theorem 1.45 *Let M be a compact manifold, and $f_0, f_1 : M \to \mathbb{R}$ Morse functions. Let X_0, X_1 be pseudogradients adapted to f_0 and f_1, respectively, satisfying the Palais-Smale condition.*

Then, there exists a morphism of complexes

$$\Phi_* : \quad (C_*(f_0), X_0) \quad \longrightarrow \quad (C_*(f_1), X_1) \tag{1.1}$$

such that it induces an isomorphism in the homology.

Proof. Consider an homopy

$$F: \begin{array}{ccc} M\times[0,1] & \longrightarrow & \mathbb{R} \\ (x,s) & \longmapsto & F_s(x) \end{array}$$

such that

$$\begin{cases} F_s = f_0 & \forall s\in\left[0,\frac{1}{3}\right] \\ F_s = f_1 & \forall s\in\left[\frac{2}{3},1\right] \end{cases}. \tag{1.2}$$

To prove the result we will proceed as follows:

1. Use F to construct a morphism $\Phi^F: \ (C_*(f_0),\partial_{X_0}) \ \to \ (C_*(f_1),\partial_{X_1})$.
2. Prove that if I is the constant homotopy from a map to itself (this means, $I_s(x)=f_0(x)\ \forall(x,s)\in M\times[0,1]$), then $\Phi^I=\mathrm{Id}_{(C_*(f_0),\partial_{X_0})}$.
3. Prove functioriality in the homology: Let f_0,f_1,f_2 be Morse functions in M, and let X_0,X_1,X_2 be pseudogradients adapted to each Morse function and satisfying the Palais-Smale condition. Let F,G,H be homotopies defined as F, this means, satisfying 1.2, with F going from f_0 to f_1, G going from f_1 to f_2, and H going from f_0 to f_2. Then, the morphisms induced in the homology by these homotopies satisfy that $\widetilde{\Phi^G}\circ\widetilde{\Phi^F}=\widetilde{\Phi^H}$.

Proof of (1): Consider an extension of F to $\left[-\frac{1}{3},\frac{4}{3}\right]$ such that

$$\begin{cases} F_s = f_0 & \forall s\in\left[-\frac{1}{3},\frac{1}{3}\right] \\ F_s = f_1 & \forall s\in\left[\frac{2}{3},\frac{4}{3}\right] \end{cases} \tag{1.3}$$

Take $g:\left[-\frac{1}{3},\frac{4}{3}\right]\to\mathbb{R}$ a Morse function such that

1. Its only critical points are 0 (a maximum) and 1 (a minimum).
2. $g'(x)>0$ for $x<0$ and for $x>1$.
3. For all $x\in M$ and $s\in(0,1)$, $\frac{\partial F}{\partial s}(x,s)+g'(s)<0$.

Take $\tilde{F}=F+g$. We are going to use the information about the Morse complex in the manifold $M\times\left[-\frac{1}{3},\frac{4}{3}\right]$ to deduce the desired morphism. Notice that

$$d\tilde{F}=d_xF+\left(\frac{\partial F}{\partial s}+g'\right)ds,$$

so the second term can only be 0 when $s=0$ or $s=1$, because of the properties of g and the fact that F is constant in the direction of s in both points, because of 1.3. By this argument, we deduce that $\mathrm{Crit}(\tilde{F})=\mathrm{Crit}(f_0)\times\{0\}\cup\mathrm{Crit}(f_1)\times\{1\}$.

Moreover, as 0 is the unique maximum of g and 1 is its unique minimum, we deduce that

- $\forall a\in\mathrm{Crit}(f_0),\ \mathrm{Ind}_{\tilde{F}}((a,0))=\mathrm{Ind}_{f_0}(a)+1.$
- $\forall b\in\mathrm{Crit}(f_1),\ \mathrm{Ind}_{\tilde{F}}((b,1))=\mathrm{Ind}_{f_1}(b).$

Take $\tilde{X}$ a pseudogradient field adapted to $\tilde{F}$ such that

1. It coincides with $X_0 - \text{grad}g$ on $M \times \left[-\frac{1}{3}, \frac{1}{3}\right]$.
2. It coincides with $X_1 - \text{grad}g$ on $M \times \left[\frac{2}{3}, \frac{4}{3}\right]$.
3. It satisfies the Palais-Smale condition.
4. It is transversal to the sections $M \times \{s\}$ for $s \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$.

Then, we can deduce that

$$\left(C_*\left(\tilde{F}\Big|_{M\times\left[-\frac{1}{3},\frac{1}{3}\right]}\right), \partial_{\tilde{X}}\right) = \left(C_*(f_0 + g|_{\left[-\frac{1}{3},\frac{1}{3}\right]}), \partial_{X_0-\text{grad}g}\right) = \left(C_{*+1}(f_0), \partial_{X_0}\right),$$

and

$$\left(C_*\left(\tilde{F}\Big|_{M\times\left[\frac{2}{3},\frac{4}{3}\right]}\right), \partial_{\tilde{X}}\right) = \left(C_*(f_1 + g|_{\left[\frac{2}{3},\frac{4}{3}\right]}), \partial_{X_1-\text{grad}g}\right) = \left(C_*(f_1), \partial_{X_1}\right).$$

From this, we deduce that

$$\begin{cases} C_0(\tilde{F}) = C_0(f_1) \\ C_k(\tilde{F}) = C_{k-1}(f_0) \oplus C_k(f_1) \end{cases},$$

so, for $k > 0$,

$$\partial_{\tilde{X}}: \quad C_k(f_0) \oplus C_{k+1}(f_1) \quad \longrightarrow C_{k-1}(f_0) \oplus C_k(f_1) \ ,$$

and

$$\partial_{\tilde{X}} = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1} \end{pmatrix}.$$

In this last equality we already used the map Φ^F, which can be defined over the generators of $C_k(f_0)$ by

$$\Phi^F(a) := \sum_{b\in\text{Crit}_k(f_1)} n_{\tilde{X}}(a,b)b, \tag{1.4}$$

where $n_{\tilde{X}}(a,b)$ denotes the number of trajectories of $\tilde{X}$ connecting a critical point $a \in M \times \{0\} \cap \text{Crit}(\tilde{F})$ to $b \in M \times \{1\} \cap \text{Crit}(\tilde{F})$, modulo 2.

The complex $\left(C_*(\tilde{F}), \partial_{\tilde{X}}\right)$ is well defined, so, in particular, we have that $\partial_{\tilde{X}}^2 = 0$. In the component in the first column of the second row, this reads as

$$\Phi^F \circ \partial_{X_0} + \partial_{X_1} \circ \Phi^F = 0 \Rightarrow \Phi^F \circ \partial_{X_0} = \partial_{X_1} \circ \Phi^F,$$

which implies that Φ^F is indeed a morphism of complexes, as we wanted to see.

Proof of (2): Consider the constant homotopy I extended to $\left[-\frac{1}{3}, \frac{4}{3}\right]$ in the obvious way, and take g as in the previous proof.

Then, the vector field $X := X_0 - \text{grad}g$ is a pseudogradient adapted to I and satisfies the Palais-Smale condition. Moreover, for each $a \in \text{Crit}(f_0)$ there is a unique trajectory of X that connects $(a, 0)$ to the section $M \times \{1\}$, which is the constant (in the projection to M) trajectory $\gamma(u) := (a, u)$. Therefore, $\Phi^I = \text{Id}$.

Remark 1.46 There a unique trajectory from $(a,0)$ to $(a,1)$ because we are not accepting homoclinic trajectories, so $\mathcal{L}(a,a)=\emptyset$, because of the Palais-Smale transversality condition.

Proof of (3): Consider the homotopies $F: f_0 \to f_1$, $G: f_1 \to f_2$ and $H: f_0 \to f_2$ extended with a constant map to the interval $\left[-\frac{1}{3},\frac{4}{3}\right]$. Choose a "double" homotopy

$$\begin{array}{rccc} K: & M\times\left[-\frac{1}{3},\frac{4}{3}\right]\times\left[-\frac{1}{3},\frac{4}{3}\right] & \longrightarrow & \mathbb{R} \\ & (x,s,t) & \longmapsto & K_{s,t}(x) \end{array}$$

satisfying

- $K_{s,t}=H_t\ \forall s\in\left[-\frac{1}{3},\frac{1}{3}\right]$.
- $K_{s,t}=G_t\ \forall s\in\left[\frac{2}{3},\frac{4}{3}\right]$.
- $K_{s,t}=F_s\ \forall t\in\left[-\frac{1}{3},\frac{1}{3}\right]$.
- $K_{s,t}=f_2\ \forall t\in\left[\frac{2}{3},\frac{4}{3}\right]$.

Choose g a Morse function as in the first proof, but with the requirement that

$$\frac{\partial K}{\partial s}(x,s,t)+g'(s)<0\ \ \forall (x,s,t)\in M\times(0,1)\times\left[\frac{1}{3},\frac{4}{3}\right],$$

and

$$\frac{\partial K}{\partial t}(x,s,t)+g'(t)<0\ \ \forall (x,s,t)\in M\times\left[\frac{1}{3},\frac{4}{3}\right]\times(0,1).$$

Then, take $\tilde{K}(x,s,t)=K(x,s,t)+g(s)+g(t)$. With this choice, we get that

$$\begin{aligned}\mathrm{Crit}(\tilde{K}) = {} & (\mathrm{Crit}(f_0)\times\{0\}\times\{0\})\cup(\mathrm{Crit}(f_1)\times\{1\}\times\{0\})\cup \\ & (\mathrm{Crit}(f_2)\times\{0\}\times\{1\})\cup(\mathrm{Crit}(f_2)\times\{1\}\times\{1\}).\end{aligned}$$

And

- For all $a\in\mathrm{Crit}(f_0)$, $\mathrm{Ind}_{\tilde{K}}((a,0,0))=\mathrm{Ind}_{f_0}(a)+2$.
- For all $b\in\mathrm{Crit}(f_1)$, $\mathrm{Ind}_{\tilde{K}}((b,1,0))=\mathrm{Ind}_{f_1}(b)+1$.
- For all $c\in\mathrm{Crit}(f_2)$, $\mathrm{Ind}_{\tilde{K}}((c,0,1))=\mathrm{Ind}_{f_2}(c)+1$ and $\mathrm{Ind}_{\tilde{K}}((c,1,1))=\mathrm{Ind}_{f_2}(c)$.

To connect the critical points, consider X a pseudogradient adapted to F, Y a pseudogradient adapted to G, and Z adapted to $H+g$. Then, consider a pseudogradient $\mathcal{X}$ adapted to $\tilde{K}$ such that

1. For all $s\in\left[-\frac{1}{3},\frac{1}{3}\right]$, $\mathcal{X}(x,s,t)=Z(x,t)-\mathrm{grad}_s g$.
2. For all $s\in\left[\frac{2}{3},\frac{4}{3}\right]$, $\mathcal{X}(x,s,t)=Y(x,t)-\mathrm{grad}_t g$.
3. For all $t\in\left[-\frac{1}{3},\frac{1}{3}\right]$, $\mathcal{X}(x,s,t)=X(x,s)-\mathrm{grad}_t g$.
4. For all $t\in\left[\frac{2}{3},\frac{4}{3}\right]$, $\mathcal{X}(x,s,t)=X_2(x)-\mathrm{grad}_t g-\mathrm{grad}_s g$, where X_2 is the pseudogradient adapted to f_2.

5. It satisfies the Palais-Smale condition.

In this case, we have that

$$C_{k+2}(\tilde{K}) = C_k(f_0) \oplus C_{k+1}(f_1) \oplus C_{k+1}(f_2) \oplus C_{k+2}(f_2),$$

and the differential map satisfies that

$$\partial_{\mathcal{X}} = \begin{pmatrix} \partial_{X_0} & 0 & 0 & 0 \\ \Phi^F & \partial_{X_1} & 0 & 0 \\ \Phi^H & 0 & \partial_{X_2} & 0 \\ S & \Phi^G & \mathrm{Id} & \partial_{X_2} \end{pmatrix},$$

where $S : C_{k-1}(f_0) \longrightarrow C_k(f_2)$ can be defined over the generators as

$$S(a) = \sum_{b \in \mathrm{Crit}_k(f_2)} n_{\mathcal{X}}(a, b) b,$$

where $n_{\mathcal{X}}(a, b)$ counts the trajectories of $\mathcal{X}$ from $(a, 0, 0)$ to $(b, 1, 1)$ modulo 2.

Therefore, the condition $\partial_{\mathcal{X}}^2 = 0$ implies that

$$\begin{aligned} S \circ \partial_{X_0} + \Phi^G \circ \Phi^F + \Phi^H + \partial_{X_2} \circ S \Rightarrow \\ \Phi^G \circ \Phi^F - \Phi^H = S \circ \partial_{X_0} + \partial_{X_2} \circ S \Rightarrow \\ \widetilde{\Phi^G} \circ \widetilde{\Phi^F} = \widetilde{\Phi^H}. \end{aligned}$$

as we wanted to see.

From here it is clear that the Morse homology does not depend on the Morse function nor the pseudogradient chosen to define it. To prove it, let (f_0, X_0) and (f_1, X_1) two pairs of a Morse functions with their adapted pseudogradients, and let F be a deformation from f_0 to f_1 and G a deformation from f_1 to f_0. Also, take H to be the constant interpolation from f_0 to itself. Then, after we apply the properties, it is clear that F and G induce the morphisms Φ^F and Φ^G, and that they are each others inverses. □

Remark 1.47 This proof justifies that we always denote the Morse homology of a manifold by $H_*(M)$, independently of the choice of the function and pseudogradient.

This last remark can be understood as that two diffeomorphic manifolds have the same homology. However, there is a much stronger fact: the homology is invariant in the same homoeomorphism class. This is so because of the following result:

Theorem 1.48 *The Morse homology of a manifold is isomorphic to its cellular homology. In particular, there is an isomorphism that sends each of the critical points to the cell defined by its unstable manifold.*

The proof of this result can be found at the end of Chapter 4 of [AD14].

Chapter 2

Symplectic manifolds and the Arnold conjecture

In this chapter we are going to present the context in which Floer homology arises, which is the one of symplectic manifolds. Roughly speaking, the structure of a symplectic manifold emerges when trying to generalise the dynamics of Hamiltonian systems from $\mathbb{R}^{2n}$ to a $2n$-dimensional manifold. A thorough description of the theory can be found in [DS01], and we will assume that the reader is familiar with the basic concepts.

We will also present the first motivation for the development of Floer theory, which is the Arnold conjecture on fixed points of symplectomorphisms.

2.1 A toolkit of symplectic geometry

In this section we will recall the main constructions in symplectic geometry, focusing in the ones that will be relevant to define the Floer complex. We will announce several theorems, but in general we are not providing a proof for any of them. All the results explained in this chaper are proved in [DS01]. Let us begin with the definition of a symplectic manifold.

Definition 2.1 Let M be a smooth manifold. We say that a form $\omega \in \Omega^2(M)$ is **symplectic** if

- It is closed, so $d\omega = 0$,
- It is non-degenerate, so for each $p \in M$, $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate, skew-symmetric and biliniear map.

In particular, non-degeneracy implies that ω induces an isomorphism between 1-forms and vector fields. This way, from any function $H \in \mathcal{C}^\infty(M)$ we can retrieve a vector field $X_H \in \mathfrak{X}(M)$ such that

$$\omega(X_H, Y) = -dH \cdot Y$$

for any $Y \in \mathfrak{X}(M)$. Sometimes, the vector field X_H is called the **symplectic gradient** of H.

In particular, if we take the coordinates $(\mathbf{q}, \mathbf{p})$ in $\mathbb{R}^{2n}$ and consider the symplectic structure induced by $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$, then for any function $H \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ we have

that
$$X_H = -\frac{\partial H}{\partial q_1}\frac{\partial}{\partial p_1} - \cdots - \frac{\partial H}{\partial q_n}\frac{\partial}{\partial p_n} + \frac{\partial H}{\partial p_1}\frac{\partial}{\partial q_1} + \cdots + \frac{\partial H}{\partial p_n}\frac{\partial}{\partial q_n}.$$

This leads to the equations of the flow of a Hamiltonian system, this means, the system of differential equations that an integral curve $(\mathbf{q}(t), \mathbf{p}(t))$ of X_H must satisfy:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Thus, symplectic geometry does generalize the notion of a Hamiltonian system, as we claimed before.

Definition 2.2 A diffeomorphism $\varphi : M \to M$ is called a **symplectomorphism** if $\varphi^*\omega = \omega$, this means, if, for any $p \in M$ and $v, w \in T_pM$,

$$\omega_{\varphi(p)}(\mathrm{T}_p\,\varphi \cdot v, \mathrm{T}_p\,\varphi \cdot w) = \omega_p(v, w).$$

Symplectomorphisms are crucial to understand the nature of a symplectic manifold, as they define its group of structure-preserving transformations.

The first natural question that arises is if there is a relation between symplectomorphisms and the vector fields from Hamiltonian functions, and the answer is affirmative:

Proposition 2.3 *For any $H \in \mathcal{C}^\infty(M)$, the flow of X_H is a symplectomorphism at any time.*

Proof. Let $\varphi^t_{X_H}$ denote the flow of X_H at time t. It is clear that $\varphi^0_{X_H} = \mathrm{Id}$, so it is a symplectomorphism. On the other hand, we can see that

$$\frac{d}{dt}(\varphi^t_{X_H})^*\omega = \mathcal{L}_{X_H}\omega,$$

and, by Cartan's formula,

$$\mathcal{L}_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = d(-dH) = 0,$$

where we used both the fact that ω is closed and the definition of X_H. Therefore, $(\varphi^t_{X_H})^*\omega$ does not depend on t, so $(\varphi^t_{X_H})^*\omega = (\varphi^0_{X_H})^*\omega = \omega\ \forall t$. □

Going back to the definition of symplectic manifold, we may want to understand the relationship between the topology of a manifold and the possible symplectic structures on it. To this end, the first result is the following:

Theorem 2.4 (Darboux): *Let (M, ω) be a symplectic manifold, and take $p \in M$. There is a chart $(U; x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that*

$$\omega|_U = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

From this theorem we deduce that all symplectic structures on a manifold are locally essentially the same.

However, the symplectic structure may give global information about the manifold M. On the topology constraints, we can notice the most immediate ones:

1. ω has to be non-degenerate, so $\dim M = 2n$.
2. ω being non-degenerate is equivalent to ω^n being a volume form in M. Thus, M must be orientable.
3. If M is compact and without boundary, then ω cannot be exact. This means, ω has a non-trivial representative class in $H^2(M)$ (in the De Rham cohomology). In particular, the second cohomology group cannot be trivial.

This last result can be further strenghtened into the following theorem in the particular case of dimension 2:

Theorem 2.5 (Moser): *Let S be an orientable surface. Two symplectic structures (S, ω_1) and (S, ω_2) are equivalent if, and only if, $[\omega_1] = [\omega_2]$ in the second De Rham cohomology group.*

The next thing that interests us is the relationship between the symplectic structure on a manifold and other geometric structures. In particular, we are thinking about Riemannian structures and almost-complex structures.

Definition 2.6 A section of $J \in \Gamma(TM \otimes T^*M)$ induces an **almost complex structure** on M if $J \circ J = -\mathrm{Id}$.

As in the case of Riemannian manifolds, this construction generalizes a concept in real vector spaces, namely the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, where the isomorphism J may be regarded as multiplication with the imaginary unit i. We did not specify the condition that M has even dimension because, as it is the case with symplectic geometry, it becomes a consequence of the construction.

The compatibility condition between these three structures can be defined as follows:

Definition 2.7 Let (M, ω) a symplectic manifold, and let J be an almost complex structure. We say that J **is calibrated** by ω if

1. For all $p \in M$, $v, w \in T_pM$, $\omega(Jv, Jw) = \omega(v, w)$ (so J behaves as a symplectomorphism).
2. For all $p \in M$, the bilinear map in T_pM defined by $(v, w) \mapsto \omega_p(v, Jw)$ is positive definite.

It can be easily checked that, under these conditions, the bilinear map $(v, w) \mapsto \omega_p(v, Jw)$ is symmetric. Therefore, it defines a Riemannian metric g over M, such that J is an isometry and $J^* = -J$.

Of course, the first question one may ask is whether these calibrated almost-complex structures do exist for any symplectic manifold. The answer, as we are going to enounce, is affirmative.

Proposition 2.8 *Let (M, ω) be a symplectic manifold. Then,*

1. *The space of almost complex structures calibrated by ω is nonempty and contractible.*
2. *For any Riemannian metric g defined on M, there is a calibrated complex structure J such that the metric induced by ω and J coincide.*

Recall that, in a Riemannian manifold (M, g), any function $H \in \mathcal{C}^\infty(M)$ has a vector field associated to it, called its gradient, defined by

$$g(\text{grad}H, X) = dH \cdot X \quad \forall X \in \mathfrak{X}(M).$$

In the case of a symplectic manifold, the mapping $H \mapsto X_H$ follows the same idea of associating a vector field to a function. A relationship between these two correspondences can be obtained, using the calibrated almost complex structure:

Proposition 2.9 *For any smooth function H defined on M, we have that*

$$X_H = J\text{grad}H.$$

Proof. We have that

$$\omega(X_H, Y) = -dH \cdot Y = -g(\text{grad}H, Y) = -\omega(\text{grad}H, JY) = \omega(J\text{grad}H, Y),$$

where we have used the properties of a calibrated almost complex structure. As ω is nondegenerate and the equality holds for all $Y \in \mathfrak{X}(M)$, we deduce that $X_H = J\text{grad}H$. □

2.2 The Arnold conjecture

In this section we focus on the dynamical aspect of symplectic manifolds. In particular, we are going to be interested in fixed points and periodic orbits of the flow for a given Hamiltonian function. Let us begin with an elementary result.

Proposition 2.10 *A point $p \in M$ is a fixed point for the flow $\varphi^t_{X_H}$ for all time if and only if it is a critical point of H.*

Proof. Let p be a critical point of H, so $dH(p) = 0$. As ω is nondegenerate at each point, we get that $X_H(p) = 0$. Conversely, if $X_H(p) = 0$ then necesarily $dH(p) = 0$ and therefore p is a critical point of H. □

Using Morse theory, it is then quite clear that, if H is a Morse function, we have that

$$\#\{\text{fixed points of } X_H\} \geq \sum_i \text{dim}H_i(M),$$

because the fixed points of X_H coincide with the critical points of H.

Of course, the number of fixed points of a system may be infinite (but only if the manifold is not compact: recall that critical points of a Morse function are isolated by proposition 1.8), but if they are finite there is a constraint on the minimum number of them, and this constraint does not depend on the properties of H but on the topology of the manifold.

The idea of Arnold's conjecture is to translate this result to the 1-periodic orbits of $\varphi^t_{X_H}$, this means, the solutions $x : [0,1] \to M$ of $\dot{x} = X_H$ such that $x(0) = x(1)$, or, rephrasing,

$$\varphi^1_{X_H}(x(0)) = x(0).$$

It is clear that a fixed point is an orbit, so we can propose the following (already proved) theorem:

Theorem 2.11 *The number of periodic orbits of a Hamiltonian system's solution is greater or equal than the sum of dimensions of the homology groups of M.*

However, we can ask ourselves a far more general question: does this result still hold when the system is not autonomous? This mean, if we take a time dependent Hamiltonian $H_t(x)$, is there a lower bound on the number of periodic solutions to the system

$$\dot{x}(t) = X_{H(t)}(x(t)).$$

Of course, a generalization of the result for fixed points does not make sense anymore: for each symplectic manifold M it is possible to produce some time-dependent Hamiltonian H_t with no fixed points. Nonetheless, the question about the 1-periodic orbits is not so clear. To tackle it, we need to restrict slightly our object of interest to the periodic solutions that are nondegenerate:

Definition 2.12 Let φ^t the flow of a time-dependent Hamiltonian. Consider x a periodic solution of $X_{H(t)}$. We say that x is a **nondegenerate periodic orbit** if

$$\det(\mathrm{T}_{x(0)}\, \varphi^1 - \mathrm{Id}) \neq 0,$$

this means, if $\mathrm{T}_{x(0)}\, \varphi^1$ does not have 1 as an eigenvalue.

Under this condition we do have the following result:

Theorem 2.13 (Arnold's conjecture): *Let (M, ω) be a compact symplectic manifold, and let H_t be a time-dependent Hamiltonian. Then, the number of nondegenerate periodic orbits of its flow is greater or equal than*

$$\sum_i \dim H_i(M).$$

Despite the fact that we call it a conjecture, it has been already proved, although the proof for any compact symplectic manifold M took some time to be found. Floer did a remarkable breakthrough proposing his homology, which we are going to study, to find a proof for aspherical manifolds (a notion that we are going to present in the next section).

As in Morse homology, the most surprising thing about this result is the fact that the lower bound does not depend on H_t, but only on the topology in which the system is defined.

2.3 The conditions on the manifold

As a prelude to the definition of the Floer complex in full rigour, we follow through the essential assumptions that we need to make in order to develop the theory. The reasons why these constrictions are set upon the topology of M will be clear as we define the Floer complex.

The assumptions that we are using are those of asphericality, this means,

Assumption 2.1 *For every smooth map* $\psi : \mathbb{S}^2 \to M$ *we have that*

$$\int_{\mathbb{S}^2} \psi^*\omega = 0.$$

This condition can be understood as that ω is zero over spheres inside the manifold.

Assumption 2.2 *For every smooth map* $\psi : \mathbb{S}^2 \to M$ *the fiber bundle* ψ^*TM *admits a symplectic trivialization.*

These two assumptions are met if we assume the (more restrictive) condition that $\pi_2(M) = 0$, this means, the second homotopy group of M is trivial.

Chapter 3

The Floer equation

In this chapter we are going to define the first constructions needed to define the Floer complex, namely the loop space, the action functional and the flow of the Floer equation. In every step we are going to refer to the Morse homology and the steps proposed at Section 1.3.

3.1 The space of contractible loops

In Section 1.3 we outlined the crucial steps to construct the complex to study the critical points of a given function f in a manifold M. Here, we are to tackle the first step in the list, this means, define our object of study and the space it is contained into.

Let (M, ω) be a compact symplectic manifold, and let H_t a 1-periodic function (in t) defined on $\mathbb{R} \times M$. The object that we are interested in are the 1-periodic solutions of

$$\dot{x}(t) = X_{H_t}(x(t)),$$

this means, the smooth maps $x : \mathbb{S}^1 \to M$ satisfying the previous equation. Thus, it seems that the solutions must be looked for in the space

$$\mathcal{C}^\infty(\mathbb{S}^1, M) = \{x : \mathbb{S}^1 \to M \mid x \text{ smooth}\}.$$

However, this space may not be connected: any two non-homotopic loops belong to different connected components. This means that we need to restrict ourselves to a connected component. As we admit constant solutions (this means, fixed points of X_{H_t}), it is natural that we consider the connected component that contains the trivial loops, this means, the space of contractible loops:

Definition 3.1 The **space of contractible loops** in M is

$$\mathcal{L}M = \{x \in \mathcal{C}^\infty(\mathbb{S}^1, M) \mid x \text{ contractible}\},$$

with the $\mathcal{C}^\infty$-topology[1].

[1] See the second chapter of [Hir12] for more details on the topology of this space.

We will not talk much about the structure of $\mathcal{L}M$ as a manifold, as we mostly will use it in a formal manner. It is important, however, to identify the tangent space $T_x\mathcal{L}M$ at a loop x. This will be a normed space (not necessarily complete, though) of the tangency classes of curves from $(-\varepsilon, \varepsilon)$ to $\mathcal{L}M$. This means, we are interested in the derivative at 0 (with respect to s) of a map

$$u : \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \longrightarrow M.$$

It is clear that this is a section of the vector bundle x^*TM. This way, we can define

Definition 3.2 The **tangent space** of $\mathcal{L}M$ at a loop x is

$$T_x\mathcal{L}M = \Gamma(x^*TM) = \{Y : \mathbb{S}^1 \to TM \mid Y \text{ smooth}, \pi \circ Y = x\},$$

where $\pi : TM \to M$ is the canonical projection of the tangent bundle.

(This is not strictly a definition, but a characterization of the tangent bundle $T\mathcal{L}M$).

3.2 The action functional

Now that we have defined the space of interest, this means, $\mathcal{L}M$, we need a way to emulate the constructions in Morse theory. In Morse theory we were studying a Morse function f on M, graduated by its index and connecting the critical points via a pseudogradient adapted to f. Here, we want to provide a smooth functional[2] from $\mathcal{L}M$ to $\mathbb{R}$ such that its critical points are the 1-periodic solutions of X_{H_t}. This will be precisely the action functional.

Remark 3.3 Take $x \in \mathcal{L}M$ a contractible loop, and let $u : \mathbb{D}^2 \to M$ a smooth map such that $u|_{\mathbb{S}^1} = x$. Then, we can define the quantity

$$\int_{\mathbb{D}^2} u^*\omega.$$

Assumption 2.1 guarantees that this number does not depend on the extension u that we choose for x, but only on the loop. Indeed, if we take two extensions $u, \tilde{u} : \mathbb{D}^2 \to M$, we can glue them along the boundary, so we get a continuous map $\psi : \mathbb{S}^2 \to M$. Then,

$$\int_{\mathbb{D}^2} u^*\omega - \int_{\mathbb{D}^2} \tilde{u}^*\omega = \int_{\mathbb{S}^2} \psi^*\omega = 0.$$

With this remark, we can proceed to define the action functional:

Definition 3.4 Let H_t a 1-periodic Hamiltonian. The **action functional** is the map $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ defined by

$$\mathcal{A}_H(x) = \int_0^1 H_t(x(t))dt - \int_{\mathbb{D}^2} u^*\omega,$$

where u is any extension of x to the disk.

[2] We call *functionals* the functions defined in a space of infinite dimensions

As we just saw, this is well defined, because it does not depend on u.

For instance, if we take $M = \mathbb{R}^{2n}$ (not compact, but good enough for the present example) with the symplectic structure $\omega = \sum dp_i \wedge dq_i$, we know that $\omega = d\alpha$, with $\alpha = \sum p_i dq_i$, so, for any $u : \mathbb{D}^2 \to \mathbb{R}^{2n}$ extending the loop x, we have that

$$\int_{\mathbb{D}^2} u^*\omega = \int_{\mathbb{D}^2} u^* d\alpha = \int_{\mathbb{S}^1} x^*\alpha.$$

Thus, the action functional is

$$\mathcal{A}_H(x) = \int_{\mathbb{S}^1} \left(H_t dt - pdq\right).$$

Going back to the general case, let us state the lemma that underlines the importance of the action functional for our purposes, this means, studying the 1-periodic solutions of X_H:

Proposition 3.5 *A loop $x \in \mathcal{L}M$ is a critical point of $\mathcal{A}_H$ if, and only if, x is a 1-periodic orbit of X_H.*

Proof. Let us compute the differential of $\mathcal{A}_H$ at x along $Y \in T_x\mathcal{L}M$. We need to extend x to a curve through x. More precisely, we need to define $z : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to M$ such that

- $z(0,t) = x(t)$.
- $\frac{d}{ds}\big|_{s=0} z(s,t) = Y(t)$.

Moreover, we need to take $u : \mathbb{D}^2 \to M$ an extension of x to the disk, and take $\tilde{u} : (-\varepsilon, \varepsilon) \times \mathbb{D}^2 \to M$ such that

- $\tilde{u}(s, e^{it}) = z(s, e^{it})$.
- $\tilde{u}(0, p) = u(p)$.

We can then extend Y to $\mathbb{D}^2$ by setting

$$Y(p) = \frac{\partial \tilde{u}}{\partial s}(0, p).$$

Then,

$$d\mathcal{A}_H(x) \cdot Y = \frac{d}{ds}\bigg|_{s=0} \mathcal{A}_H(z(s)) = \frac{d}{ds}\bigg|_{s=0} \left(\int_0^1 H_t(z(s,t)dt - \int_{\mathbb{D}^2} \tilde{u}_s^*\omega \right).$$

Differentiating the second term we get

$$-\int_{\mathbb{D}^2} \left(\frac{d}{ds}\bigg|_{s=0} \tilde{u}_s^*\omega \right) = -\int_{\mathbb{D}^2} u^* \left(\mathcal{L}_{Y(p)}\omega\right) =$$

(applying Cartan's formula, and then Stoke's theorem)

$$= -\int_{\mathbb{D}^2} u^* \left(d i_{Y(p)}\omega\right) = -\int_{\mathbb{S}^1} x^* \left(i_{Y(t)}\omega\right) = \int_0^1 \omega(\dot{x}(t), Y(t))dt.$$

On the other hand, if we differentiate the first term,

$$\int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} H_t(z(s,t))dt = \int_0^1 dH_t(x(t)) \cdot Y(t)dt,$$

and, by the definition of X_H, this is equal to

$$-\int_0^1 \omega(X_{H_t}, Y(t))dt.$$

Therefore,

$$d\mathcal{A}_H(x) \cdot Y = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), Y(t))dt.$$

Then, x is a critical point of $\mathcal{A}_H$ if and only if this expression is 0 for all $Y \in T_x\mathcal{L}M$ and, by the non-degeneracy of ω, this happens if and only if $\dot{x}(t) = X_{H_t}(x(t))$, so it is a periodic solution to the system. □

3.3 The Floer equation

In this section we are going to introduce the analogous concept to the one of a pseudogradient of a function in Morse homology. In particular, it will be enough for us to construct the gradient of the action functional. However, this will not be as easy as the case of finite dimensions.

Take, for the compact symplectic manifold (M, ω), an almost complex structure J calibrated by ω, as defined in 2.7. This induces a Riemannian metric g on M. We can use it to induce an inner product on each fiber of $T\mathcal{L}M$, by

$$\langle X, Y \rangle_x = \int_0^1 g_{x(t)}(X(t), Y(t))dt,$$

for $X, Y \in T_x\mathcal{L}M$.

It is an inner product because $g_{x(t)}$ is at each $t \in \mathbb{S}^1$, so the bilinearity, symmetry and positive definition are preserved.

Take into account that the tangent spaces are not complete, so this inner product does not induce a Hilbert space structure on each fiber $T_x\mathcal{L}M$. Nonetheless, we can define the gradient of a functional without assuming that it exists, and trying to understand under which conditions is it actually defined.

Definition 3.6 Let $F : \mathcal{L}M \to \mathbb{R}$ a functional. The **gradient** of F, if it exists, is the vector field $\text{grad}F \in \mathfrak{X}(\mathcal{L}M)$ such that, for every loop $x \in \mathcal{L}M$ and every $Y \in \mathfrak{X}(\mathcal{L}M)$,

$$\langle \text{grad}F, Y \rangle_x = dF(x) \cdot Y,$$

so

$$\int_0^1 g_{x(t)}(\text{grad}_x F(t), Y(t)) = dF(x) \cdot Y.$$

In the case of the action functional, its gradient (if it exists) satisfies that

$$d\mathcal{A}_H(x)\cdot Y = \langle \mathrm{grad}\mathcal{A}_H, Y\rangle_x = \int_0^1 g_{x(t)}(\mathrm{grad}_x\mathcal{A}_H(t), Y(t))dt,$$

and

$$d\mathcal{A}_H(x)\cdot Y = \int_0^1 \omega(\dot{x}(t) - X_H, Y(t))dt,$$

and, as $g(v,w) = \omega(v, Jw)$, we deduce that

$$\mathrm{grad}_x\mathcal{A}_H = J\dot{x} - JX_H.$$

Let $\mathcal{X}_H$ denote the negative gradient, so $\mathcal{X}_H = -\mathrm{grad}\mathcal{A}_H = JX_H - J\dot{x}$. Recall that, by proposition 2.9, $X_H = J\mathrm{grad}H$. Therefore, we get that

$$\mathcal{X}_H(x)(t) = -\mathrm{grad}H(x(t)) - J_{x(t)}\dot{x}(t).$$

Let us consider $u : (-\varepsilon, \varepsilon) \to \mathcal{L}M$ an integral curve of this vector field. In other words, $u : (-\varepsilon,\varepsilon)\times\mathbb{S}^1 \to M$ is a smooth map with

$$\frac{\partial u}{\partial s}(s,t) = \mathcal{X}_H(u(s,\cdot))(t).$$

From this, we are in position to define the **Floer equation**:

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \mathrm{grad}H(u) = 0. \tag{3.1}$$

If we write all the variables we see that it is a non-linear partial differential equation:

$$\frac{\partial u}{\partial s}(s,t) + J_{u(s,t)}\frac{\partial u}{\partial t}(s,t) + \mathrm{grad}H(u(s,t)) = 0.$$

This equation will be our main focus of study, since its solutions (if there are any) are the flow lines of the negative gradient of the action functional, in the same way that we studied the trajectories of the negative gradient flow in the Morse case.

Let us begin with some remarks on the equation. In particular, its “degenerate” cases:

Remark 3.7 If H is a constant function (so $\mathrm{grad}H = 0$), then the equations are the Cauchy-Riemann equations (for almost complex structures):

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

Remark 3.8 The only stationary solutions (such that $\partial_s u = 0$) of the equation are the 1-periodic orbits of X_H:

$$0 = J\frac{\partial u}{\partial t} + \mathrm{grad}H \Rightarrow \frac{\partial u}{\partial t} = X_H,$$

where we used proposition 2.9 again.

Remark 3.9 If we assume that a solution does not depend on t (this means, $u(s,t) = u(s)$, so all the points are degenerate loops, this means, points in M), then we find ourselves in the case of Morse theory of critical points again, because Floer equation reads

$$\frac{\partial u}{\partial s} = -\mathrm{grad}H.$$

Recall that the negative gradient flow lines in Morse theory are defined in order to connect critical points. However, in the case of the Floer equation, this may not be the case. This means, if u is a solution to the equation, it is not necessarily true that $\lim_{s\to-\infty} u(s,\cdot)$ or $\lim_{s\to\infty} u(s,\cdot)$ exist or are critical points of $\mathcal{A}_H$.

However, we just need to add a condition to our equation in order to have this property. In order to justify it, recall that when we proved proposition 1.23 (which is the analogous result in Morse theory) we used the energy of a trajectory γ in order to prove our result:

$$E(\gamma) = -\int_{\mathbb{R}} \gamma^* df = \int_{+\infty}^{-\infty} \frac{d}{dt}(f \circ \gamma) dt.$$

This can be defined similarly in the case of the solutions of the Floer equation that are defined for all $s \in \mathbb{R}$:

Definition 3.10 Let $u : \mathbb{R} \times \mathbb{S}^1 \to M$ a solution of the Floer equation. Its energy (which may be infinite) is the integral

$$E(u) = -\int_{\mathbb{R}} u^* d\mathcal{A}_H = -\int_{\mathbb{R}} \frac{d}{ds} \left(\mathcal{A}_H \circ u\right) ds.$$

If we expand the last term, we get that

$$E(u) = -\int_{\mathbb{R}} d\mathcal{A}_H(u(s)) \cdot \frac{\partial u}{\partial s} ds = -\int_{\mathbb{R}} g\left(\text{grad}\mathcal{A}_H(u(s)), \frac{\partial u}{\partial s}\right) ds = \int_{\mathbb{R}\times\mathbb{S}^1} \left|\frac{\partial u}{\partial s}\right|^2 dtds.$$

Therefore, the space we are interested in is the one of solutions with finite energy:

Definition 3.11 The **space of solutions with finite energy** is

$$\mathcal{M} = \left\{u : \mathbb{R} \to \mathcal{L}M \mid u \text{ solves 3.1 and } E(u) < \infty\right\}.$$

Remark 3.12 If $u \in \mathcal{M}$ and $E(u) = 0$, then u does not depend on s. By the remark 3.8, in this case u is a loop, and a critical point of $\mathcal{A}_H$.

Remark 3.13 If $u \in \mathcal{M}$ and there are loops $x, y \in \mathcal{L}M$ with

$$\lim_{s\to-\infty} u(s) = y, \quad \lim_{s\to\infty} u(s) = x,$$

then we have that

$$E(u) = \mathcal{A}_H(y) - \mathcal{A}_H(x),$$

just as in the Morse homology case. In Section 3.5 we will see that this is always the case, this means, we always have that u has limiting loops x and y.

3.4 $\mathcal{M}$ is compact

In this section we will tackle the first question about $\mathcal{M}$, in particular, its topology. To prove the results in this section and the next one, we will rely in two important results. The first is well known: the Ascoli-Arzelà theorem (in its general version):

Theorem 3.14 Ascoli-Arzelà: *Let X a localy compact metric space, V a finite dimensional vector space, and $F \subset \mathcal{C}(X, V)$. Then, F is relatively compact in $\mathcal{C}_{loc}(X, V)$ if, and only if, F is equicontinuous and pointwise bounded.*

Recall that F is equicontinuous when, $\forall x \in X, \varepsilon > 0$ there is a $\delta > 0$ such that, $\forall y \in X$ with $d(x, y) < \delta$ and $\forall f \in F$, $\|f(x) - f(y)\| < \varepsilon$. Pointwise boundedness, on the other hand, means that $\forall x \in X$ $\exists K$ such that $\|f(x)\| < M$ $\forall f \in F$.

In our case, we will take $V = \mathbb{R}^m$, where m is such that M can be compactly embedded into V.

The other result that we will use is one about the elliptic regularity of the Floer equation, which can be found in the Chapter 12 of [AD14]:

Proposition 3.15 *If $(u_k)_k$ is a sequence of solutions of the Floer equation and $u_k \xrightarrow[k\to\infty]{} u_0$ in $\mathcal{C}^0_{loc}(\mathbb{R} \times \mathbb{S}^1, M)$, then $u_0 \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^1, M)$, u_0 is also a solution of the Floer equation, and $u_k \xrightarrow[k\to\infty]{} u_0$ in the strong sense in $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^1, M)$.*

With these results in hand, we are able to prove the following theorem:

Theorem 3.16 *The set of solutions of the Floer equation with finite energy, $\mathcal{M}$, is compact.*

The whole content of the proof lies in the following proposition:

Proposition 3.17 *Under the assumption of asphericallity (2.1), there exists some $A > 0$ such that*

$$\|\mathrm{grad}_{(s,t)} u\| \leq A \quad \forall u \in \mathcal{M}, \forall (s, t) \in \mathbb{R} \times \mathbb{S}^1.$$

We will first prove the theorem using the proposition, and then the proposition itself:

Proof. (Theorem): By the proposition 3.17, $\mathcal{M}$ is equicontinuous. This is so because we can apply the mean value theorem to any pair of points x, y in M, and the bound is uniform in the gradients of elements of $\mathcal{M}$. On the other hand, as M is compact, $\|u(s, t)\|$ is uniformly bounded for $u \in \mathcal{M}$. Therefore, by 3.14, $\mathcal{M}$ is relatively compact in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, M)$.

Let $(u_n)_n \subset \mathcal{M}$ a sequence. As we just observed, it must have some subsequence (u_{n_k}) such that $u_{n_k} \xrightarrow[k\to\infty]{} u_0$ for some $u_0 \in \mathcal{C}^0(\mathbb{R}\times\mathbb{S}^1, M)$ in the $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, M)$ sense. However, by 3.15 we deduce that $u_0 \in \mathcal{M}$ and that $u_{n_k} \xrightarrow[k\to\infty]{} u_0$ in the strong sense in $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^1, M)$. □

In order to prove the proposition 3.17 we are going to need the following lemma:

Lemma 3.18 *(Half-max lemma): Let X be a complete metric space, and $g : X \to [0, +\infty)$ a continuous function. Let $x_0 \in X, \varepsilon_0 > 0$. Then, there exist $y \in X$ and $\varepsilon \in (0, \varepsilon_0]$ such that*

1. $d(y, x_0) \leq 2\varepsilon_0$.
2. $\varepsilon_0 g(x_0) \leq \varepsilon g(y)$.
3. $g(x) \leq 2g(y)$ $\forall x \in B(y, \varepsilon)$.

Proof. (Proposition 3.17): First of all, for the purposes of this proof, let us regard the elements of $\mathcal{M}$ as functions $u : \mathbb{R}^2 \to M$ that are periodic in the second component.

We will prove the proposition by contradiction, so we suppose that there are sequences $(u_k)_k \subset \mathcal{M}$, $((s_k, t_k))_k \subset \mathbb{R}$ such that

$$\lim_{k\to\infty} \|\mathrm{grad}_{(s_k,t_k)} u_k\| = +\infty.$$

Under this assumption, we can choose a sequence $(\varepsilon_k)_k \subset (0, +\infty)$ such that

$$\lim_{k\to\infty} \varepsilon_k \|\mathrm{grad}_{(s_k,t_k)} u_k\| = +\infty.$$

(For instance, we can take the sequence $\varepsilon_k = \|\mathrm{grad}_{(s_k,t_k)} u_k\|^{-\frac{1}{2}}$). Let $g_k = \|\mathrm{grad} u_k\|$. Then, we can apply the half-max lemma for each k, taking $x_0 = (s_k, t_k)$ and $\varepsilon_0 = \varepsilon_k$. Then, there exist sequences ε'_k, (s'_k, t'_k) such that

- $\lim\limits_{k\to\infty} \varepsilon'_k \|\mathrm{grad}_{(s'_k,t'_k)} u_k\| = +\infty.$
- $2\|\mathrm{grad}_{(s'_k,t'_k)} u_k\| \geq \|\mathrm{grad}_{(s,t)} u_k\| \ \forall (s, t) \in B((s'_k, t'_k), \varepsilon'_k).$

Take $R_k = \|\mathrm{grad}_{(s'_k,t'_k)} u_k\|$, and consider the sequence of functions $(v_k)_k$ defined by

$$v_k(s, t) = u_k \left(\frac{s}{R_k} + s'_k, \frac{t}{R_k} + t'_k \right).$$

In this case,

$$\mathrm{grad}_{(s,t)} v_k = \frac{1}{R_k} \mathrm{grad}_{\left(\frac{s}{R_k}+s'_k, \frac{t}{R_k}+t'_k\right)} u_k,$$

so, by definition, $\|\mathrm{grad}_{(0,0)} v_k\| = 1$ for all k. Moreover, for all $(s, t) \in B(0, \varepsilon'_k R_k)$ we have that

$$\|\mathrm{grad}_{(s,t)} v_k\| = \frac{1}{R_k} \left\| \mathrm{grad}_{\left(\frac{s}{R_k}+s'_k, \frac{t}{R_k}+t'_k\right)} u_k \right\| \leq \frac{1}{R_k} 2R_k = 2 \Rightarrow$$
$$\Rightarrow \|\mathrm{grad}_{(s,t)} v_k\| \leq 2.$$

As (u_k) are solutions of the Floer equation, we know that

$$\frac{\partial v_k}{\partial s} + J_{v_k} \frac{\partial v_k}{\partial t} + \frac{1}{R_k} \mathrm{grad}_{\frac{t}{R_k}+t'_k} H(v_k) = 0.$$

Given these conditions, we know that $(v_k)_k$ is a pointwise bounded, equicontinuous family. Therefore, by theorem 3.14, we know that there is a (sub)sequence (which we will denote by $(v_k)_k$ in an abuse of notation) that has a limit $v \in \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2, M)$. Moreover, if we use the proposition 3.15 we conclude that $v \in \mathcal{C}^\infty(\mathbb{R}^2, M)$ and

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0,$$

so v is J-holomorphic. In addition,

$$\|\mathrm{grad}_{(0,0)} v\| = 1,$$

and
$$\|\text{grad}_{(s,v)}v\| \leq 2\forall(s,t) \in \mathbb{R}^2,$$
because the balls $B(0, \varepsilon'_k R_k)$ tend to cover the entire plane as $k \to \infty$, and $\|\text{grad}_{(s,t)}v_k\|$ is bounded by 2 in these balls.

Now we will use this limit $v_k \to v$ to construct something forbidden by our asphericallity assumption: a sphere with nonzero symplectic area. To do this, we need to begin by checking that v has finite energy.

Consider $B_k = B((s'_k, t'_k), \varepsilon'_k)$. We have that
$$\int_{B(0,\varepsilon'_k R_k)} \|\text{grad}v_k\|^2 = \int_{B_k} \|\text{grad}u_k\| dtds = \int_{B_k} \left(\left\| \frac{\partial u_k}{\partial s} \right\|^2 + \left\| \frac{\partial u_k}{\partial t} - X_t(u_k) + X_t(u_k) \right\|^2 \right) dtds \leq$$
$$\leq \int_{B_k} \left(\left\| \frac{\partial u_k}{\partial s} \right\|^2 + \left\| \frac{\partial u_k}{\partial t} - X_t(u_k) \right\|^2 + \|X_t(u_k)\|^2 \right) dtds.$$
Now let us study separately each of the terms. By the corollary 3.23 (that we will prove in the next section) we know that the energy of u_k is bounded as $k \to \infty$, so
$$\int_{B_k} \left(\left\| \frac{\partial u_k}{\partial s} \right\|^2 + \left\| \frac{\partial u_k}{\partial t} - X_t(u_k) \right\|^2 \right) dsdt \leq 2E(u_k) \leq 2C,$$
for some $C > 0$. On the other hand,
$$\int_{B_k} \|X_t(u_k)\|^2 \leq |B_k| \sup_{p \in M} \|X_t(p)\|^2 < +\infty,$$
because M is compact and $|B_k| \xrightarrow[k \to \infty]{} 0$. Therefore, by Fatou's Lemma, $E(v) < +\infty$.

Now, we are ready to see that the symplectic area of v is finite and nonzero:
$$\int_{\mathbb{R}^2} v^*\omega = \int_{\mathbb{R}^2} \omega \left(\frac{\partial v}{\partial s}, \frac{\partial v}{\partial t} \right) = \int_{\mathbb{R}^2} \omega \left(-J_v \frac{\partial v}{\partial t}, \frac{\partial v}{\partial t} \right) = \int_{\mathbb{R}^2} \left\| \frac{\partial v}{\partial t} \right\|^2 < +\infty,$$
because the energy is finite (as we just saw). Moreover, the last integral has to be nonzero because $\|\text{grad}_{(0,0)}v\| = 1$.

Moreover, we claim that there exists a sequence $r_k \to \infty$ such that the length of $v(\partial B(0, r_k))$ tends to 0 as $k \to \infty$. To prove this, as $v^*\omega$ is a symplectic form in $\mathbb{R}^2$, we may express it in polar coordinates. Thus, there is a function $f : [0, +\infty) \times [0, 2\pi] \to (0, +\infty)$ such that
$$v^*\omega(\rho, \theta) = f(\rho, \theta)\rho d\theta \wedge d\rho.$$
This induces the Riemannian metric in $\mathbb{R}^2$ defined by $f(\rho, \theta)(d\rho^2 + \rho^2 d\theta^2)$. Thus, the length of $v(\partial B(0, r))$ is
$$l(r) = r \int_0^{2\pi} \sqrt{f(r, \theta)} d\theta.$$
On the other hand, the area function is
$$A(r) = \int_{B(0,r)} v^*\omega = \int_0^{2\pi} \left(\int_0^r f(\rho, \theta) d\rho \right) d\rho.$$

As we just proved, $A(r)$ is a bounded function. Its derivative has the expression

$$A'(r) = r \int_0^{2\pi} f(r,\theta) d\theta.$$

Finally, if we apply the Cauchy-Schwarz inequality to the integral in the definition of $l(r)$, we find that

$$l(r) \leq r\sqrt{\int_0^{2\pi} d\theta \int_0^{2\pi} f(r,\theta)d\theta} = r\sqrt{2\pi \frac{A'(r)}{r}} \Rightarrow$$
$$\Rightarrow l^2(r) \leq 2\pi r A'(r).$$

Therefore, as $A(r)$ is bounded, there exists a sequence $(r_k)_k$ such that $\lim_{k\to\infty} r_k A'(r_k) = 0$. To prove this, for instance, we know that

$$\lim_{k\to\infty} \frac{A(k^2) - A(k)}{\ln k} = 0,$$

and

$$\frac{A(k^2) - A(k)}{\ln k} = \frac{A(k^2) - A(k)}{\ln k^2 - \ln k} = \frac{A'(r_k)}{1/r_k}$$

for some $k \leq r_k \leq k^2$, by the mean value theorem.

With all the facts that we have proved, we are in the situation to explain the phenomenon: v forms a "bubble" inside of M, to the point that it tends towards a sphere.

Let k sufficiently large so that $\gamma = v(\partial B(0, r_k))$ is contained inside a Darboux chart for ω, $U \subset M$. Inside of U, we have that $\omega = d\lambda$ for some 1-form λ. As $\gamma \subset U$, we can take D_k a closed disk with boundary γ inside of U. Therefore, $v(B(0, r_k)) \cup D_k$ is a sphere $S^2_{r_k} \subset M$. By the assumption of asphericallity, we know that

$$\int_{S^2_{r_k}} i^*\omega = 0,$$

so

$$\int_{D_k} i^*\omega + \int_{v(B(0,r_k))} \omega = 0.$$

The first term tends to 0 as k goes to infinity, because

$$\int_{D_k} i^*\omega = \int_{D_k} d\lambda = \int_{v(\partial B(0,r_k))} \lambda,$$

and thus

$$\left| \int_{D_k} \omega \right| \leq \text{length}(\gamma) \sup_U \|\lambda\| \xrightarrow[k\to\infty]{} 0.$$

On the other hand, as $r_k \xrightarrow[k\to\infty]{} \infty$, the second term satisfies that

$$\int_{v(B(0,r_k))} \omega = \int_{B(0,r_k)} v^*\omega \xrightarrow[k\to\infty]{} \int_{\mathbb{R}^2} v^*\omega > 0.$$

Therefore,

$$\lim_{k\to\infty} \int_{S^2_{r_k}} \omega > 0,$$

which contradicts our assumption. Therefore, the assumption made at the beginning of this proof must be wrong: there has to be a bound A such that

$$\|\text{grad}_{(s,t)}u\| \leq A \quad \forall u \in \mathcal{M}, (s,t) \in \mathbb{R} \times \mathbb{S}^1.$$

□

3.5 Limit endpoints of elements of $\mathcal{M}$

We devote this section to prove what we already hinted in the remark 3.13: the elements of $\mathcal{M}$ connect the critical points of the action functional. The precise statement of the theorem is the following:

Theorem 3.19 *If all the trajectories of X_t are non-degenerate, and under the assumption of asphericallity, then for all $u \in \mathcal{M}$ there are $x, y \in \mathcal{L}M$ critical points of $\mathcal{A}_H$ such that*

$$\lim_{s\to-\infty} u(s,\cdot) = x \ , \quad \lim_{s\to+\infty} u(s,\cdot) = y$$

in the $\mathcal{C}^\infty(\mathbb{S}^1, M)$ sense, and

$$\lim_{s\to\pm\infty} \frac{\partial u}{\partial s}(s,t) = 0$$

uniformly in t.

If we prove this theorem, we will have constructed the analogous to the pseudogradient adapted to a function in a manifold M in Morse theory: a flow that connects critical points in a meaningful way. In the Morse case, this flow appears integrating the pseudogradient flow, but in the Floer case we needed to be a little more subtle and avoid the proper definition of the vector field, defining instead the equation that the flow must satisfy.

First of all, we need a basic lemma on what the nondegeneracy of X_t implies for the critical points of the action functional:

Lemma 3.20 *Under the assumption of nondegeneracy, the number of critical points of $\mathcal{A}_H$ (or the number of periodic orbits of X_t) is finite.*

Proof. Let us consider the following submanifolds of $M \times M$:

- The diagonal $\Delta = \{(x,x) \mid x \in M\}$.
- The graph of the flow $\varphi^t_{X_t}$ of X_t at time 1: $F = \{(x, \varphi^1_{X_t}(x)) \mid x \in M\}$.

The fact that the periodic orbits of X_t are nondegenerate is equivalent to $\Delta \pitchfork F$. Moreover, $\dim(\Delta) = \dim(F) = 2n$ and $\dim(M \times M) = 4n$, so $\dim(\Delta \cap F) = 0$, so it is a submanifold of dimension 0 of a compact manifold, and therefore it is finite. □

In order to prove the theorem, we need to begin by proving a certain bound in the energy and the action functional, depending on the critical points of $\mathcal{A}_H$. Let us denote $u_s = u(s,\cdot) \in \mathcal{L}M$.

Proposition 3.21 *Let $u \in \mathcal{M}$. There exist $x, y \in \text{Crit}(\mathcal{A}_H)$ such that*

$$\lim_{s\to-\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(x), \quad \lim_{s\to+\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(y).$$

This has some immediate consequences:

Corollary 3.22 *If $\mathcal{M}$ is nonempty, then* $\operatorname{Crit}(\mathcal{A}_H)$ *is nonempty.*

This does not even require a proof, as it is an immediate consequence of the proposition.

Corollary 3.23 *There exist $C_1, C_2 > 0$ such that*

$$|\mathcal{A}_H(u)| < C_1, E(u) < C_2 \qquad \forall u \in \mathcal{M}.$$

Proof. Because of the lemma 3.20, we know that $\operatorname{Crit}(\mathcal{A}_H)$ is finite. Therefore, $\mathcal{A}_H(\operatorname{Crit}(\mathcal{A}_H))$ is a bounded set in $\mathbb{R}$. Moreover, the function $s \mapsto \mathcal{A}_H(u_s)$ is decreasing, because

$$\frac{d}{ds}\mathcal{A}_H(u_s) = d_{u_s}\mathcal{A}_H \cdot \frac{\partial u}{\partial s} = g\left(\operatorname{grad}_{u_s}\mathcal{A}_H, \frac{\partial u}{\partial s}\right) = -\left\|\frac{\partial u}{\partial s}\right\|^2 \leq 0,$$

(because $\operatorname{grad}_{u_s}\mathcal{A}_H = -\frac{\partial u}{\partial s}$).

Therefore,

$$\mathcal{A}_H(y) = \lim_{s\to+\infty}\mathcal{A}_H(u_s) \leq \mathcal{A}_H(u_s) \leq \lim_{s\to-\infty}\mathcal{A}_H(u_s) = \mathcal{A}_H(x),$$

from which follows the bound for $\mathcal{A}_H$. To deduce that the energy must be bounded in $\mathcal{M}$, it is enough to consider that

$$E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$$

because of the remark 3.13, so the energy must also be bounded. □

Proof. (Proposition 3.21): Let us prove the case for $s \to +\infty$, as the case for $-\infty$ is analogous.

As we just showed, the function $s \mapsto \mathcal{A}_H(u_s)$ is decreasing. As it is also continuous, it is enough to prove that there exists some sequence $(s_k)_k$ with $s_k \xrightarrow[k\to\infty]{} +\infty$ and some $y \in \operatorname{Crit}(\mathcal{A}_H)$ with

$$\lim_{k\to\infty}\mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(y).$$

We will prove this in 3 steps:

1. There exists a $(s_k)_k$ tending to infinity such that $u_{s_k} \xrightarrow[k\to\infty]{} y$ in the $\mathcal{C}^0(\mathbb{S}^1, M)$ topology for some $y \in \mathcal{C}^0(\mathbb{S}^1, M)$.
2. y is of class $\mathcal{C}^\infty$, and $y \in \operatorname{Crit}(\mathcal{A}_H)$.
3. $\mathcal{A}_H(u_{s_k}) \xrightarrow[k\to\infty]{} \mathcal{A}_H(y)$.

Step 1: Let $u \in \mathcal{M}$. Its energy is finite, so

$$\int_{-\infty}^{+\infty}\left(\int_0^1\left|\frac{\partial u}{\partial t} - X_t(u)\right|^2 dt\right) ds < \infty.$$

Therefore, there exists some sequence $(s_k)_k$ with $s_k \xrightarrow[k\to\infty]{} +\infty$ such that

$$\lim_{k\to\infty}\left\|\frac{\partial u}{\partial t}(s_k, \cdot) - X_t(u(s_k, \cdot))\right\|_{L^2}^2 = 0.$$

Let $u_k = u(s_k, \cdot)$. Taking an embedding of M into $\mathbb{R}^m$, we can consider that

$$\lim_{k\to\infty} \|\dot{u}_k - X_t(u_k)\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^m)} = 0.$$

M is compact, so X_t is bounded, so there is some $R > 0$ such that $\sup_{p\in M, t\in\mathbb{S}^1} \|X_t(p)\| < R$. Therefore, there is some $B > 0$ with $\|\dot{u}_k\|_{L^2} \leq B$ for all k. This implies that $(u_k)_k$ is equicontinuous:

$$\|u_k(t_1) - u_k(t_0)\| = \left\| \int_{t_0}^{t_1} \dot{u}_k(t)dt \right\| \leq \|\chi_{[t_0,t_1]}\|_{L^2} \|\dot{u}_k\|_{L^2} = \sqrt{t_1 - t_0} B,$$

(where we applied the Cauchy-Schwarz inequality). Therefore, applying the theorem 3.14 (a subsequence of) u_k has a limit y in the $\mathcal{C}^0(\mathbb{S}^1, \mathbb{R}^m)$ topology or, equivalently, in the $\mathcal{C}^0(\mathbb{S}^1, M)$ topology.

Step 2: In order to prove that $y \in \mathcal{C}^\infty(\mathbb{S}^1, M)$, we can show that

$$y(t) - y(0) = \int_0^t X_\tau(y(\tau))d\tau.$$

In order to prove this, we need to use the convergence of u_k:

$$y(t) - y(0) - \int_0^t X_\tau(y(\tau))d\tau =$$

$$= \lim_{k\to\infty} \left(u_k(t) - u_k(0) - \int_0^t X_\tau(y(\tau))d\tau \right) = \lim_{k\to\infty} \left(\int_0^t \dot{u}_k(\tau)d\tau - \int_0^t X_\tau(y(\tau))d\tau \right) =$$

$$= \lim_{k\to\infty} \left(\int_0^t (\dot{u}_k(\tau) - X_\tau(u_k(\tau)))\, d\tau \right) + \lim_{k\to\infty} \left(\int_0^t (X_\tau(u_k(\tau)) - X_\tau(y(\tau)))\, d\tau \right).$$

To prove that the first term converges to zero we can apply the Cauchy-Schwarz inequality and the fact that

$$\|\dot{u}_k - X_t(u_k)\|_{L^2} \xrightarrow[k\to\infty]{} 0.$$

On the other hand, to see that se second term converges to zero too we just need to use the $\mathcal{C}^0$ convergence of $(u_k)_k$, which yields

$$\left| \int_0^t (X_\tau(u_k(\tau)) - X_\tau(y(\tau)))\, d\tau \right| \leq 2\pi \|X_t(u_k) - X_t(y)\|_{\mathcal{C}^0} \leq 2\pi R \|u_k - y\|_{\mathcal{C}^0} \xrightarrow[k\to\infty]{} 0.$$

Therefore,

$$y(t) = y(0) + \int_0^t X_\tau(y(\tau))d\tau,$$

so y is differentiable, and

$$\dot{y}(t) = X_t(y(t)).$$

This allows us to apply a bootstrapping argument, this means, as $y \in \mathcal{C}^0(\mathbb{S}^1, M)$, by this formula it is clear that $y \in \mathcal{C}^1(\mathbb{S}^1, M)$. However, we can apply this argument to show that y is $\mathcal{C}^2$, $\mathcal{C}^3$, and so on, so actually $y \in \mathcal{C}^\infty(\mathbb{S}^1, M)$. With the same argument we can prove that $u_k \xrightarrow[k\to\infty]{} y$ in the $\mathcal{C}^\infty$ topology, because all of its derivatives converge uniformly.

Step 3: We want to prove that $\mathcal{A}_H(u_k) \xrightarrow[k\to\infty]{} \mathcal{A}_H(y)$. First of all, as u_k converges uniformly, $(H_t(u_k))_k$ also converges uniformly, so

$$\int_0^1 H_t(u_k(t))dt \xrightarrow[k\to\infty]{} \int_0^1 H_t(y(t))dt.$$

On the other hand, we need to prove that, if $\widetilde{u_k}$ and $\widetilde{y}$ are extensions to the disk of u_k and y respectively, then

$$\lim_{k\to\infty}\left(\int_{\mathbb{D}^2} \widetilde{u_k}^*\omega - \int_{\mathbb{D}^2} \widetilde{y}^*\omega\right) = 0.$$

To compute this limit we need to use the local exactness of ω together with the asphericallity condition (2.1) and the $\mathcal{C}^1$ convergence of $(u_k)_k$. Consider $U \subset M$ an open retractible neighbourhood of $y(\mathbb{S}^1)$, so that $\omega|_U$ is exact, so $\omega|_U = d\lambda$ for some $\lambda \in \Omega^2(U)$. For k large enough, as u_k converges uniformly, $u_k(\mathbb{S}^1) \subset U$. This way, we can construct a sphere gluing together three surfaces:

1. A cylinder $C \subset U$ defining an homotopy between u_k and y. We may parametrize it by a smooth map $\varphi : [0,1] \times \mathbb{S}^1 \to M$ with $\varphi|_0 = u_k$ and $\varphi|_1 = y$.
2. The disk $\widetilde{u_k}$ with boundary u_k.
3. The disk $\widetilde{y}$ with boundary y.

If we denote by S the sphere obtained by gluing these surfaces by the boundary, and applying the assumption 2.1, we know that

$$\int_S i^*\omega = 0,$$

so

$$\int_{\mathbb{D}^2} \widetilde{u_k}^*\omega - \int_{\mathbb{D}^2} \widetilde{y}^*\omega = -\int_0^1 \int_{\mathbb{S}^1} \varphi_s^*\omega ds.$$

Applying the Stokes theorem, and knowing that ω is exact in U, we deduce that

$$\int_0^1 \int_{\mathbb{S}^1} \varphi_s^*\omega ds = \int_{\mathbb{S}^1} u_k^*\lambda - \int_{\mathbb{S}^1} y^*\lambda = \int_{\mathbb{S}^1} (\lambda(\dot{u}_k) - \lambda(\dot{y}))\, dt =$$

$$= \int_{\mathbb{S}^1} \lambda\left(\dot{u}_k - X_t(u_k)\right) dt + \int_{\mathbb{S}^1} \lambda\left(X_t(u_k) - X_t(y)\right) dt,$$

(where we have applied that $\dot{y} = X_t(y)$).

Let $L = \sup_{p\in C} \|\lambda\|$. Then, the first term of the last expression converges to 0, because $\dot{u}_k - X_t(u_k)$ converges to 0. On the other hand,

$$\left|\int_{\mathbb{S}^1} \lambda\left(X_t(u_k) - X_t(y)\right) dt\right| \leq L\|X_t(u_k) - X_t(y)\|_{L^1} \leq LR\|u_k - y\|_{L^1} \xrightarrow[k\to\infty]{} 0,$$

where the last convergence is due to the fact that u_k converges to y uniformly. □

As the results we have used so far were necessary to prove theorem 3.16 we have not alluded to the results in the previous section in our proofs. From now on, however, we can (and will) use the compacity theorem in order to prove theorem 3.19.

Remark 3.24 The additive group $(\mathbb{R}, +)$ acts on the right on $\mathcal{M}$, as $(u \cdot \sigma)(s,t) = u(s+\sigma, t)$ for $s, t, \sigma \in \mathbb{R}$.

Lemma 3.25 *Let $u \in \mathcal{M}$, and $(s_k)_k$ a sequence of real numbers with $s_k \xrightarrow[k\to\infty]{} +\infty$. Then, there exists a subsequence $(k_l)_l$ and some $y \in \mathrm{Crit}(\mathcal{A}_H)$ such that*

$$u \cdot k_l \xrightarrow[l\to\infty]{} y.$$

Proof. Let $u_k = u \cdot s_k$, which is a sequence in $\mathcal{M}$. As this set is compact (by theorem 3.16), there is a subsequence (by abuse of notation we will denote it by u_k as the original sequence) and some element $v \in \mathcal{M}$ such that $u_k \to v$ in the $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ topology. Therefore, for $s \in \mathbb{R}$, and $t \in \mathbb{S}^1$,

$$\lim_{k\to\infty} u(s+s_k, t) = v(s,t).$$

However, by proposition 3.21 there exists a $y \in \mathrm{Crit}(\mathcal{A}_H)$ such that

$$\mathcal{A}_H(v(s)) = \lim_{k\to\infty} \mathcal{A}_H(u(s+s_k,t)) = \lim_{s\to\infty} \mathcal{A}_H(u(s,t)) = \mathcal{A}_H(y).$$

This implies that the function $s \mapsto \mathcal{A}_H(v(s,\cdot))$ is constant. Therefore, by definition, $E(v) = 0$. But, as we saw in the remark 3.12, v has to be a critical point of $\mathcal{A}_H$. Therefore, $v = y$. □

With this lemma, we can prove the theorem 3.19.

Proof. (Theorem 3.19): The topology of $\mathcal{L}M$ is metrizable (see [Hir12] at Chapter 2, Section 4 for more information on this fact), so we can choose a metric d_∞ in $\mathcal{L}M$ defining its topology. For each $x \in \mathrm{Crit}(\mathcal{A}_H)$ we consider the open ball

$$B(x,\varepsilon) = \{\gamma \in \mathcal{L}M \mid d_\infty(x,\gamma) < \varepsilon\}.$$

As $\mathrm{Crit}(\mathcal{A}_H)$ is finite, we can choose $\varepsilon > 0$ such that these balls are disjoint. In this case, let

$$U_\varepsilon = \bigcup_{x \in \mathrm{Crit}(\mathcal{A}_H)} B(x,\varepsilon) \subset \mathcal{L}M.$$

Take some $u \in \mathcal{M}$. For each $\varepsilon > 0$ there exists s_ε such that $u(s,\cdot) \in U_\varepsilon$ for all $s > s_\varepsilon$. If this were not true, then would be able to build a sequence s_k with $u \cdot s_k \notin U_\varepsilon$ for all k, getting a contradiction with lemma 3.25.

Moreover, again by lemma 3.25, there exists some $y \in \mathrm{Crit}(\mathcal{A}_H)$ such that $u(s,\cdot) \in B(y,\varepsilon)$ for all $s > s_\varepsilon$. However, this is precisely the definition of convergence in a metric space, so

$$\lim_{s\to\infty} u(s,\cdot) = y \in \mathrm{Crit}(\mathcal{A}_H).$$

In addition, by the proposition 3.15, we know that

$$\lim_{s\to\infty} \frac{\partial u}{\partial t}(s,\cdot) = \dot{y}.$$

Therefore,

$$\lim_{s\to\infty} \frac{\partial u}{\partial s}(s,\cdot) = \lim_{k\to\infty} \left(-J\frac{\partial u}{\partial t}(s,\cdot) + JX_t(u(s,\cdot)) \right) = -J \cdot y + JX_t(y) =$$
$$= J(X_t(y) - \dot{y}) = 0.$$

□

Corollary 3.26 *If $u \in \mathcal{M}$ and $1 \leq k \leq m$, then*

$$\lim_{s\to\infty} \frac{\partial^m u}{\partial^k s \partial^{m-k} t}(s,t) = 0$$

uniformly in t.

Chapter 4

The Floer complex

This chapter will give the guidelines of how the Floer complex is constructed using the tools introduced in Chapter 3. In particular, we will define the differential on the chain complex, and comment on the invariance of the resulting homology.

We will not prove the results cited in this chapter, but the proofs can be found in chapters 8 and 9 of [AD14].

4.1 Linearization of the Floer equation

We will use the Conley-Zehnder index throughout this chapter. For the purposes of this chapter, it is enough to say that it is a map that associates an integer number to each path of symplectic matrices starting from Id and ending to a non-degenerate symplectic matrix (by this we mean that it does not have the eigenvalue 1). In an abuse of notation, for $x \in \mathcal{L}M$ we will denote by $\mu_{CZ}(x)$ the Conley-Zehnder index of the differential of the flow along x, $d\varphi^t_{X_H}(x(t))$. There is an introduction to how to define this map and its properties in the Appendix B.

Let us begin with the definition of the Floer complex:

Definition 4.1 Let H_t be a non-degenerate Hamiltonian, and J an almost complex structure calibrated by ω. The ***k*-th group of the Floer complex** is the $\mathbb{Z}_2$-module generated by the periodic solutions of X_H with Conley-Zehnder index k:

$$CF_k(M, H, J) = \langle \{ \gamma : \mathbb{S}^1 \to M \mid \dot{\gamma} = X_H(\gamma) \text{ and } \mu_{CZ}(\gamma) = k \} \rangle_{\mathbb{Z}_2}.$$

In order to define a complex chain we need to provide a differential. In the case of the Morse complex we used the flow of a pseudogradient vector field to connect critical points. In the case of the Floer complex, we will use the solutions to the Floer equation (3.1). We already know (by Theorem 3.19) that the solutions to this equation connect the critical points of the action functional, this means, the periodic orbits of X_H. Moreover, we have seen (in Theorem 3.16) that the set of solutions of finite energy $\mathcal{M}$ is compact. Therefore, we just need to find the way to count these solutions in the case of consecutive indices. In order to do this, we must look at the linearized version of the Floer equation. We need to begin by understanding in which space we need to define it.

Definition 4.2 The **Floer operator** is the function

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{C}^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{S}^1, M) & \longrightarrow & \mathcal{C}^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{S}^1, M) \\ & u & \longmapsto & \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \text{grad}H(u) \end{array}.$$

In order to linearize this operator, we have to look at a space of perturbations of the solutions $u : \mathbb{R}\times\mathbb{S}^1 \to M$. In particular, we will look for the Sobolev-kind of functions that result from perturbing u. The resulting space will be a Banach manifold, this means, a topological space that is locally diffeomorphic to a Banach space. A good introduction to the theory of Banach manifolds can be found at [AMR12], although the general idea is sufficient for the purposes of this chapter.

Definition 4.3 Let $\mathcal{M}(x, y)$ denote the space of solutions of the Floer equation such that

$$\lim_{s\to-\infty} u(s,t) = x, \lim_{s\to+\infty} u(s,t) = y.$$

Then, we define the Banach manifold

$$\mathcal{P}^{1,p}(x,y) = \left\{P : (s,t) \mapsto \exp_{w(s,t)} Y(s,t) \mid w \in \mathcal{M}(x,y) \text{ and } Y \in W^{1,p}(w^*(TM))\right\},$$

where $W^{1,p}(w^*(TM))$ denotes the space of fibers of $w^*(TM)$ that belong to the Sobolev space $W^{1,p}$, for $p > 2$. By this, we mean that Y is a $W^{1,p}$-map

$$Y: \quad \mathbb{R}\times\mathbb{S}^1 \quad \to \quad TM$$

with $\pi \circ Y = w$.

Remark 4.4 The fact that $p > 2$ implies that Y are continuous maps, so the elements of $\mathcal{P}^{1,p}(x, y)$ are in turn continuous.

In this Banach manifold, the Floer operator has a natural extension

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{P}^{1,p}(x,y) & \to & L^p(\mathbb{R}\times\mathbb{S}^1, TM) \\ & u & \longmapsto & \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \text{grad}H(u) \end{array},$$

and we can compute its linearization (assuming that we are inside some local chart of $\mathcal{P}^{1,p}(x, y)$):

$$\mathcal{F}(u+Y) = \frac{\partial(u+Y)}{\partial s} + J(u)\frac{\partial(u+Y)}{\partial t} + \text{grad}H(u+Y),$$

which leads to

$$d\mathcal{F}_u(Y) = \frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} + dJ_u(Y)\frac{\partial u}{\partial t} + d(\text{grad}H(u))Y$$

Proposition 4.5 *Let us denote by $S(s,t)$ the linear operator such that $S(s,t)Y = dJ_u(Y)\frac{\partial u}{\partial t} + d(\text{grad}H(u))Y$. Then, the limits*

$$\lim_{s\to\pm\infty} S(s,t) = S^\pm(t),$$

exist, $S^\pm(t)$ are symmetric matrices, and

$$\lim_{s\to\pm\infty} \frac{\partial S}{\partial s}(s,t) = 0.$$

Moreover, the equations $\frac{\partial Y}{\partial t} = JS^\pm(t)Y$ are the linearizations of $\dot{x} = X_H(x)$ and $\dot{y} = X_H(y)$, respectively.

Therefore, the matrices $S^{\pm}$ carry all the relevant information in order to compute the Conley-Zehnder indexes of x and y. In particular, we will be able to use the values of the indices to compute the dimension of $\mathcal{M}(x, y)$.

4.2 Fredholm operators. Regularity

Definition 4.6 A linear map $L : E \to F$ between Banach spaces is a **Fredholm operator** if its kernel is finite dimensional and its image has finite codimension. In this case, its index is

$$\mathrm{Ind}(L) = \dim \mathrm{Ker}L - \dim \mathrm{CoKer}L.$$

If $\mathcal{F} : E \to F$ is a smooth map between Banach manifolds, we say that it is Fredholm if $d\mathcal{F}_x$ is Fredholm for every $x \in E$, and define its index as the index of its linearization.

Fredholm operators are essential in the theory of partial differential equations. In our context, we want to use a specific result about Fredholm maps:

Theorem 4.7 *(Submersion theorem)*[1]*: Let $\mathcal{F} : E \to F$ a Fredholm map between Banach manifolds and let $y \in F$ such that $d\mathcal{F}_x$ is surjective $\forall x \in \mathcal{F}^{-1}(y)$. Then, $\mathcal{F}^{-1}(y)$ is a smooth manifold of dimension* $\mathrm{Ind}(\mathcal{F})$*, and its tangent space at x is* $\mathrm{Ker}(d\mathcal{F}_x)$*.*

This theorem gives us the link that we wanted: if the Floer map is a Fredholm operator, we are able to compute its index, and we are able to show that $d\mathcal{F}_u$ is surjective, then we know the dimension of $\mathcal{M}(x, y)$.

To meet the first requirement, we have the following theorem, which is proved in [AD14], Chapter 8:

Theorem 4.8 *For all $u \in \mathcal{M}(x, y)$ the operator $d\mathcal{F}_u$ is Fredholm, and*

$$\mathrm{Ind}(d\mathcal{F}_u) = \mu_{CZ}(x) - \mu_{CZ}(y).$$

For the second requirement, we have to refer to the notion of regularity.

Definition 4.9 A pair (H, J) of a time-dependent Hamiltonian and an almost complex structure over M is said to be **regular** if, for all $u \in \mathcal{M}$, the linearized Floer operator

$$d\mathcal{F}_u : \quad \mathrm{T}_u\, \mathcal{P}^{1,p}(x, y) \quad \to \quad L^p(\mathbb{R} \times \mathbb{S}^1, M)$$

is surjective.

Theorem 4.10 *For any non-degenerate Hamiltonian H_0 and for any almost complex structure J, there is a dense Banach space $\mathcal{H} \subset \mathcal{C}^{\infty}(\mathbb{S}^1 \times M, \mathbb{R})$ (with the $\mathcal{C}^1$ topology) and subset $\mathcal{H}_{\mathrm{reg}}$ such that:*

1. *$\mathcal{H}_{\mathrm{reg}}$ is the intersection of a countable family of open dense subsets of $\mathcal{H}$.*

2. *$\mathcal{H}_{\mathrm{reg}}$ contains an open neighbourhood of* 0*.*

3. *For all $h \in \mathcal{H}_{\mathrm{reg}}$, $H = H_0 + h$ is non-degenerate, and (H, J) is regular.*

[1]See [AMR12] Theorem 3.5.4, for more details

Therefore, we can guarantee that for any pair (H, J) of non-degenerate Hamiltonian and almost complex structure there is another pair (H', J') arbitrarily close (in the $\mathcal{C}^1$ sense) such that it is regular.

With all these results we are able to construct the differential of the Floer complex. If we define the space of trajectories

$$\mathcal{L}(x, y) = \mathcal{M}(x, y)\,/\mathbb{R}$$

(where we are taking the quotient by the translation action in the s coordinate), then we know that it is compact and, by Theorem 4.8,

$$\dim(\mathcal{L}(x, y)) = \mu_{CZ}(x) - \mu_{CZ}(y) - 1.$$

Therefore,

Lemma 4.11 *If $\mu_{CZ}(x) = \mu_{CZ}(y)+1$, then $\mathcal{L}(x, y)$ is a finite set. Moreover, if $\mu_{CZ}(x) = \mu_{CZ}(y)+2$, then $\mathcal{L}(x, y)$ is a finite union of compact and connected 1-dimensional manifolds.*

With this in mind, if we denote by $n(x, y) = \#\mathcal{L}(x, y)$ (when x and y have consecutive indices), we can define the differential of the Floer complex.

Definition 4.12 The k-th differential of the Floer complex $\partial_k : CF_k(M, H, J) \to CF_{k-1}(M, H, J)$ can be defined over the generators of $CF_k(M, H, J)$ as

$$\partial_k x = \sum_{\mu_{CZ}(y)=k-1} n(x, y)y.$$

As a consequence of 4.11, $\partial_k \circ \partial_{k+1} = 0$.

Let $HF_k(M, H, J)$ denote the k-th homology group resulting from the Floer complex, this means,

$$HF_k(M, H, J) = \mathrm{Ker}\partial_k \,/\mathrm{Im}\partial_{k+1}.$$

As it was the case with Morse homology, the Floer homology is a topological invariant. In fact, we have the following two theorems:

Theorem 4.13 *Let (H_a, J_a) and (H_b, J_b) be two pairs of non-degenerate Hamiltonians and almost complex structures over the same symplectic manifold M. Then, there is a morphism of complexes*

$$F_\bullet : \quad CF_\bullet(M, H_a, J_a) \quad \longrightarrow \quad CF_\bullet(M, H_b, J_b)$$

that induces an isomorphism on the homology.

Finally, the key result that yields the most information about the Floer homology the following:

Theorem 4.14 *If H is an autonomous Hamiltonian small enough in the $\mathcal{C}^2$ norm so that all of its periodic orbits are its fixed points, and if (H, J) is regular and X is a pseudogradient adapted to H satisfying the Smale condition, then there is an isomorphism of chain complexes*

$$CF_*(M, H, J) = C_{*+n}(M, H, X).$$

Therefore, the Floer homology of a manifold is isomorphic to its Morse homology.

Conclusions

In Chapter 1 we presented the Morse homology, stressing out how we obtain a connection between Morse functions and the topology of the manifold. In particular, in Section 1.3 we saw a layout of how to construct this kind of homology. We have tried to keep that layout in mind all the time while doing Morse theory, because in the context of Floer homology we reproduced the analogous steps. Then, we moved on to a review on symplectic geometry and the Arnold conjecture. A special case of the Arnold conjecture, when the Hamiltonian does not depend on time (this means, in an autonomous system), came as a corollary of the Morse theory.

To tackle the general case we studied, in Chapter 3, the basic ingredients of Floer homology: the action functional, the Floer equation, the energy of a solution of the Floer equation, and the space $\mathcal{M}$ of solutions of finite energy. Recall that the Floer equation is a perturbed version of the Cauchy-Riemann equation, this means, its solutions have a similar behaviour that the one of the pseudoholomorphic curves.

Finally, the properties of $\mathcal{M}$, together with the notion of transversality in the context of infinite dimensions (which corresponds to the notion of regularity, which we talked about in Chapter 4), allow us to connect properly the periodic orbits between them, in the same way as how we connected the critical points of a Morse function using the negative gradient (or some other pseudogradient). We sketched the final steps of the definition of Floer homology using this connections, in a way completely analogous to the final steps of the Morse theory.

The first main result derived from the Floer homology is the one it was intended for in the first place:

Remark 4.15 Let M be a compact, symplectic manifold. Suppose that it is aspherical, this means, that it satisfies the assumptions 2.1 and 2.2. Then, the conjecture 2.13 is true: for any time-dependent Hamiltonian H_t, the number of non-degenerate periodic orbits of the flow X_H is greater or equal than

$$\sum_k \dim(H_k(M)).$$

It would be wrong to assume that Floer's contribution is limited to a particular case of the Arnold conjecture. On one hand, his proof broadened remarkably the class of symplectic manifolds for which the result was known to be true: previously Eliashberg had proven the conjecture in dimension 2, and Conley and Zehnder proved it for the tori. With his homology, Floer was able to prove the conjecture for manifolds with $\pi_2(M) = 0$, and later for monotone manifolds. On the other hand, the tools introduced by Floer paved the way for other mathematicians to extend the proof for weakly monotone manifolds (Hofer, Salamon and Ono), and finally for the general case (Fukaya and Ono, Liu and Tian, Hofer and Salamon).

Furthermore, Floer theory has proved to be a versatile tool in symplectic topology, and investigation in this field is still going on. There are still many things to be understood about the theory and its implications.

There are several ways in which it is possible to continue the work of this master thesis:

- Find a way to extend the tools of Floer theory to non aspherical manifolds, this means, finding a way to go around the assumptions 2.1 and 2.2. These were important in order to define the action functional $\mathcal{A}_H$ and prove that its critical points are the periodic orbits of X_H. We also needed them to guarantee that $\mathcal{M}$ is compact (to prevent the formation of a "bubble" in the proof of Proposition 3.17), and it is necessary once again (although we did not mention it here) to define properly the Banach manifold $\mathcal{P}^{1,p}(x, y)$. Finding a way around these obstacles would allow us to generalize the Floer homology for a broader class of manifolds.

- Extend the action functional to non-contractible loops. We needed the loops to be contractible in order to define $\mathcal{A}_H$ on the first place, but if it were possible to study all the possible loops, we would obtain a more detailed description of the periodic orbits of the system. An introduction to this approach can be found at the section 6.7 of [AD14].

- Extend the Floer homology to broader classes of manifolds beyond the symplectic case.

Appendix A

Additional results on Morse theory

In this chapter we are going to provide some proofs that were not included in Chapter 1 because they were rather too technical, and including them would not give any additional insight into the theory of Morse homology. However, these results are important enough to be written in this appendix

A.1 The Morse functions are generic

In section 1.1 we asked ourselves if any manifold M admits Morse functions, and if this functions are generic in some sense. In this section we will provide an affirmative answer to both questions. To prove the existence, we are going to need two fundamental theorems of differential geometry:

Theorem A.1 (Whitney embedding theorem): *Any smooth manifold of dimension n can be smoothly embedded in $\mathbb{R}^{2n}$, if $n > 0$.*

Theorem A.2 (Sard's theorem): *Let $f : M \to N$ a smooth map. Then, the set of critical values of f has measure zero.*

Therefore, we can think of any manifold M as a smooth submanifold of $\mathbb{R}^N$, for some N. This allows us to state the following proposition:

Proposition A.3 *Let $M \subset \mathbb{R}^N$ a submanifold. For almost every point $p \in \mathbb{R}^N$, the function*

$$\begin{array}{rccc} f_p : & M & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \|x-p\|^2 \end{array}$$

is a Morse function.

Proof. Let $(u_1, ..., u_d) \mapsto x(u_1, ..., u_d)$ a local parametrization from $\mathbb{R}^d$ into $M \subset \mathbb{R}^N$ in a neighbourhood of a point $p \in M$. In this coordinates, the partial derivatives of the function f_p are

$$\frac{\partial f_p}{\partial u_i} = 2(x-p) \cdot \frac{\partial x}{\partial u_i},$$

and

$$\frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + (x-p) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right).$$

The point x is therefore a non-degenerate critical point if and only if $(x-p)$ is orthogonal to $\mathrm{T}_x M$ and the matrix $\frac{\partial^2 f_p}{\partial u_i \partial u_j}$ has rank d.

To show that f_p is a Morse function for almost all $p \in \mathbb{R}^N$ it suffices to show that the points that do not satisfy the condition are the critical values of a smooth map, and then apply Sard's theorem. Consider the normal fiber bundle of the embedding of M into $\mathbb{R}^N$, which is a smooth manifold:

$$N = \{(x, v) \in M \times \mathbb{R}^N \mid v \perp \mathrm{T}_x M\},$$

and the map

$$\begin{array}{rccc} E: & N & \longrightarrow & \mathbb{R}^N \\ & (x,v) & \longmapsto & x+v \end{array}.$$

Then, it sufices to proof the following lemma:

Lemma A.4 *The point $p = x + v$ is a critical value of E if, and only if, the matrix with components*

$$\frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} - v \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right)$$

is not invertible.

Proof. Consider, for each point of the local chart $(u_1, ..., u_d)$, an orthonormal basis of $(\mathrm{T}_x M)^{\perp}$ by the $N-d$ vectors $v_1, ..., v_{N-d}$. Then, we have a local parametrization for N, given by the map

$$(u_1, ..., u_d, t_1, ..., t_{N-d}) \longmapsto \left(x(u_1, ..., u_d), \sum_{i=1}^{N-d} t_i v_i(u_1, ..., u_d)\right).$$

In these coordinates, the partial derivatives of E are

$$\begin{cases} \frac{\partial E}{\partial u_i} = \frac{\partial x}{\partial u_i} + \sum_{k=1}^{N-d} t_k \frac{\partial v_k}{\partial u_i} \\ \frac{\partial E}{\partial t_j} = v_j \end{cases}.$$

If we compute the inner products of these N vectors with the N independent vectors $\frac{\partial x}{\partial u_1}, ..., \frac{\partial x}{\partial u_d}, v_1, ..., v_{N-d}$, we get a square matrix that has the same rank as the Jacobian of E, and this matrix has the form

$$\begin{pmatrix} \left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + \sum_k t_k \frac{\partial v_k}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}\right) & \left(\sum_k \frac{\partial v_k}{\partial u_i} \cdot v_l\right) \\ 0 & \mathrm{Id} \end{pmatrix}.$$

But v_k are orthogonal to $\frac{\partial x}{\partial u_j}$, so

$$0 = \frac{\partial}{\partial u_i}\left(v_k \cdot \frac{\partial x}{\partial u_j}\right) = \frac{\partial v_k}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + v_k \cdot \frac{\partial^2 x}{\partial u_i \partial u_j},$$

so

$$\sum_k t_k \frac{\partial v_k}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} = -\sum_k t_k v_k \frac{\partial^2 x}{\partial u_i \partial u_j} = v \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}.$$

And, thus, the lemma is proved. □

So the proposition is proved. □

To answer the second question, about the genericness of Morse functions on a given manifold, we have the following proposition:

Proposition A.5 *Let M be a smooth manifold, and let $f : M \to \mathbb{R}$ a smooth function. Let k be a positive integer. Then, f and all its derivatives of order $\leq k$ can be uniformly approximated by Morse functions on every compact subset.*

Proof. Choose an embedding of M into $\mathbb{R}^N$ such that its first coordinate is the function f,

$$h(x) = (f(x), h_2(x), ..., h_N(x)).$$

Choose c a (large) real number. By the proposition A.3, for almost all point $p = (-c + \varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$ the function f_p is a Morse function, so the function

$$g_{c,\varepsilon}(x) = \frac{f_p(x) - c^2}{2c}$$

is also a Morse function. Moreover, notice that

$$g_{c,\varepsilon}(x) = \frac{1}{2c}\left((f(x) + c - \varepsilon_1)^2 + (h_2(x) - \varepsilon_2)^2 + \cdots + (h_N(x) - \varepsilon_N)^2 - c^2\right) =$$

$$= f(x) + \frac{f(x)^2 + \sum h_i(x)^2}{2c} - \frac{\varepsilon_1 f(x) + \sum \varepsilon_i h_i(x)}{c} + \sum \varepsilon_i^2 - \varepsilon_1,$$

so, as claimed, on any compact subset of M, $g_{c,\varepsilon}$ tends to f and its first k derivatives also tend to the first k derivatives of f as $\varepsilon \to 0$ and $c \to \infty$. □

Therefore, the Morse functions are actually generic, in the sense that we can approximate any smooth function with Morse functions over compact subsets of M.

A.2 The broken trajectories are the boundary in dimension 1

In this section, we are going to focus on the proof of theorem 1.43. This theorem is fundamental to the construction of the Morse complex, because it is, ultimately, the proof that it is indeed a complex, this means, that $\partial_X^2 = 0$. Let us restate it here, with more precision.

Recall that we have a triple (M, f, X), where M is a compact smooth manifold, f is a Morse function defined on M, and X is a pseudogradient adapted to f and satisfying the Smale condition.

Theorem A.6 *Let* $a \in \mathrm{Crit}_{k+1}(f), b \in \mathrm{Crit}_k(f), c \in \mathrm{Crit}_{k-1}(f)$. *For all* $(\lambda_1, \lambda_2) \in \mathcal{L}(a,b) \times \mathcal{L}(b,c)$, $\exists \psi : [0,\delta) \to \overline{\mathcal{L}}(a,c)$ *(for some* $\delta > 0$*) such that*

1. $\psi(0) = (\lambda_1, \lambda_2)$.
2. $\psi(t) \in \mathcal{L}(a,c)\ \forall t > 0$.
3. $\psi|_{\{t>0\}} : (0,\delta) \to \mathcal{L}(a,c)$ *is an embedding.*
4. *For all* $(l_n)_n \subset \mathcal{L}(a,c)$ *with* $l_n \xrightarrow[n\to\infty]{} (\lambda_1,\lambda_2)$ *in* $\overline{\mathcal{L}}(a,c)$, $l_n \in \psi((0,\delta))\ \forall n$ *(at least, for* n *large enough).*

This means that $\overline{\mathcal{L}}(a,c)$ *is a finite union of compact and connected smooth manifolds with boundary.*

Proof. Take $\alpha = f(b)$. Let $\Omega(b) \subset M$ be a Morse chart for (f,X), such that $\partial\Omega(b)$ coincides with $f^{-1}(\alpha+\varepsilon)$ in a neighbourhood of $W^s(b)\cap\partial\Omega(b)$ and with $f^{-1}(\alpha-\varepsilon)$ in a neighbourhood of $W^u(b) \cap \partial\Omega(b)$ for some $\varepsilon > 0$.

Let $b^- = \partial\Omega(b) \cap \lambda_1$ be the entry point of λ_1 into $\Omega(b)$, and $b^+ = \partial\Omega(b) \cap \lambda_2$ be the exit point of λ_2. Take $U \subset \partial\Omega(b)$ a neighbourhood of b^-. Notice that U is diffeomorphic to an open disk of dimension $n-1$. Let us define the following sets, all of them in $\partial\Omega(b)$:

- $P := U \cap W^u(a)$. The unstable manifold meets $f^{-1}(\alpha+\varepsilon)$ transversally, so P is diffeomorphic to an open disk of dimension $\dim(P) = \dim(U) + \dim(W^u(a)) - \dim(M) = k$.
- $S_+(b) = U \cap W^s(b)$. For the same reason, $\dim(S_+(b)) = n-k-1$, and it is diffeomorphic to a sphere.
- $S_-(b) = W^u(b) \cap f^{-1}(\alpha - \varepsilon)$. It is diffeomorphic to a sphere of dimension $k-1$.

As X satisfies the Smale condition, we know that $W^u(a) \pitchfork W^s(b)$. Therefore, $P \pitchfork S_+(b)$ in U, so $\dim(P \cap S_+(b)) = 0$. Therefore, (shrinking U if necessary) we can assume that $P \cap S_+(b) = \{b^-\}$.

Take $D = P\backslash\{b^-\}$. It is diffeomorphic to $\{x \in \mathbb{R}^{k-1} \mid 0 < \|x\| < 1\}$, a punctured open disk. Notice that, by definition, $D \cap W^s(b) = \emptyset$. Therefore, the flux of X starting at any point of D will eventually leave $\Omega(b)$, so we can define the embedding

$$\Phi : \quad D \longrightarrow \partial\Omega(b)$$

induced by this flux.

Let us consider the set $Q = \mathrm{Im}\Phi \cup S_-(b) \subset f^{-1}(\alpha-\varepsilon)$. The key to prove this theorem is contained in the following proposition:

Proposition A.7 *Q is a k-dimensional manifold with boundary, and* $\partial Q = S_-(b)$.

Let us suppose that it is true. Using the Smale transversality condition of X in $W^u(a) \cap W^s(c)$ and in $W^u(b) \cap W^s(c)$, we can see that, as $\mathrm{Im}\Phi \subset W^u(a) \cap f^{-1}(\alpha-\varepsilon)$ and $S_-(b) = W^u(b) \cap f^{-1}(\alpha - \varepsilon)$,

$$\dim(\mathrm{Im}\Phi \cap W^s(c)) = 1,$$

$$\dim(S_-(b) \cap W^s(c)) = 0.$$

Therefore, $Q \cap W^s(c)$ is a 1-dimensional manifold with boundary, and its boundary is, by the proposition A.7,

$$\partial Q \cap W^s(c) = S_-(b) \cap W^s(c) = W^u(b) \cap W^s(c) \cap f^{-1}(\alpha - \varepsilon) \cong \mathcal{L}(b, c),$$

and $b^+ \in \partial Q \cap W^s(c)$. Now, consider χ a local parametrization of this manifold in a neighbourhood of b^+, this means, an embedding

$$\chi : \quad [0, \delta) \quad \longrightarrow \quad Q \cap W^s(c) \ ,$$

with $\chi(0) = b^+$. Then, we can consider the diffeomorphism

$$\Phi^{-1} \circ \chi : \quad (0, \delta) \quad \longrightarrow \quad W^s(c) \cap D$$

(it is well defined because Φ is defined following the flow lines of X, so $W^s(c)$ is invariant under its action). We can use the lemma 1.41 to deduce that

$$\lim_{t \searrow 0} (\Phi^{-1} \circ \chi)(t) = b^-,$$

so we can extend $\Phi^{-1} \circ \chi$ continuously to a map

$$\psi : \quad [0, \delta) \quad \longrightarrow \quad (W^s(c) \cap P) \cup \{b^-\}$$

with $\psi(0) = b^-$. We can see that

$$W^s(c) \cap P = W^s(c) \cap W^u(a) \cap f^{-1}(\alpha + \varepsilon) \cong \mathcal{L}(a, c),$$

and b^- is a natural representation of $(\lambda_1, \lambda_2) \in \mathcal{L}(a, b) \times \mathcal{L}(b, c)$, so we can rewrite the definition of ψ as

$$\psi : \quad [0, \delta) \quad \longrightarrow \quad \overline{\mathcal{L}}(a, c) \ .$$

Moreover, we see that:

- $\psi(0) = (\lambda_1, \lambda_2)$.
- $\psi(t) \in \mathcal{L}(a, c)$ for $t > 0$.
- $\psi|_{\{t>0\}}$ is an embedding.

We just need to prove the property 4 to conclude the proof of the theorem.

Consider $l_n \xrightarrow[n\to\infty]{} (\lambda_1, \lambda_2)$, with $l_n \in \mathcal{L}(a, c)$ $\forall n$. For n sufficiently large, l_n enters $\Omega(b)$ through U, and exits it through a neighbourhood of b^+ in $f^{-1}(\alpha - \varepsilon)$. Let l_n^- denote the entry points, and l_n^+ the exit points. By the lemma 1.41, $l_n^- \xrightarrow[n\to\infty]{} b^-$, and $l_n^+ \xrightarrow[n\to\infty]{} b^+$.

For n large enough, $l_n^- \in D$, so it is in the domain of Φ. Thus, $l_n^+ = \Phi(l_n^-) \in Q \cap W^s(c)$, and therefore $l_n^+ \in \mathrm{Im}\chi$. Then, it is clear that $l_n \in \mathrm{Im}\psi$, as we wanted to see. □

Now let us prove the proposition A.7. We want to conclude that Q is a k-dimensional manifold with boundary, and its boundary is precisely $S_-(b)$.

Proof. (Proposition A.7): Recall that $Q = \mathrm{Im}\Phi \cup S_-(b)$, with $\Phi : D \to f^{-1}(\alpha - \varepsilon)$ an embedding. Also, $D = P \backslash \{b^-\}$ is diffeomorphic to a punctured open disk of dimension $k-1$. This means that we can take $D \cong (0,1) \times \mathbb{S}^{k-1}$. Therefore, we can take coordinates (t,z) in D, with $t \in (0,1)$ and $z \in \mathbb{S}^{k-1}$. We can construct this coordinates in a way that $\lim\limits_{t \searrow 0}(t,z) = b^-$ in P for all $z \in \mathbb{S}^{k-1}$.

We want to show that, on the other hand,

$$\lim_{t \searrow 0} \Phi((t,z)) \in S_-(b) \ \forall z \in \mathbb{S}^{k-1}.$$

Let $d^+ = \lim\limits_{s \searrow 0}(s,w)$ for some $w \in \mathbb{S}^{k-1}$. It is clear that $f(d^+) = \alpha - \varepsilon$ because f is continuous, so we need to show that $d^+ \in W^u(b)$.

Suppose that it is false, so $d^+ \notin W^u(b)$. Therefore, if we take the flow of $-X$ starting at d^+, we must exit the chart $\Omega(b)$ at some point d^- with $f(d^-) = \alpha + \varepsilon$. Let $(x_n)_n$ a sequence in $\mathrm{Im}\Phi$ such that $\lim\limits_{n\to\infty} x_n = d^+$. Take $y_n = \Phi^{-1}(x_n)$ for each n. We know that $f(y_n) = \alpha + \varepsilon \ \forall n$, so, applying the lemma 1.41, we deduce that $y_n \xrightarrow[n\to\infty]{} d^-$. However, we can see that $y_n \xrightarrow[n\to\infty]{} b^-$ because the first component tends to 0. Therefore, we conclude that $b^- = d^-$. But we can see that, if φ_X^t denotes the flow of X,

$$f(\varphi_X^t(b^-)) \xrightarrow[t\to\infty]{} \alpha,$$

so $f(\varphi_X^t(b^-)) \geq \alpha \ \forall t > 0$. On the other hand, $f(d^+) = \alpha - \varepsilon < \alpha$ and $d^+ = \varphi_X^{t_0}(d^-)$ for some $t_0 > 0$. Thus, we reach a contradiction.

We know that $S_-(b) \cong \mathbb{S}^{k-1}$, so we can identify Q with $[0,1) \times \mathbb{S}^{k-1}$ and $S_-(b)$ with $\{0\} \times \mathbb{S}^{k-1}$ with the same coordinates as before. We just showed that we can define the map

$$\begin{array}{rccc} \rho : & \mathbb{S}^{k-1} & \longrightarrow & S_-(b) \\ & z & \longmapsto & \lim\limits_{t \searrow 0} \Phi((t,z)) \end{array}$$

and it is indeed well defined. If we show that ρ is bijective, we will be able to conclude that $S_-(b)$ is precisely the boundary of the manifold Q, and we will be done. In fact, as $S_-(b) \cong \mathbb{S}^{k-1}$, it sufices to show that ρ is injective.

To do this, we need to use the fact that $\Omega(b)$ is a Morse chart and, by the Morse Lemma 1.8, it admits a coordinate system (x_-, x_+)[1] such that $f(x_-, x_+) = c - \|x_-\|^2 + \|x_+\|^2$. Then, it can be easily proved that the flow of X (which coincides with that of $-\mathrm{grad} f$ in $\Omega(b)$ because of the definition of a pseudogradient adapted to f) takes the form

$$\varphi_X^s(x_-, x_+) = (e^{2s}x_-, e^{-2s}x_+).$$

On the other hand, we can see that, for some $\delta > 0$,

$$Q = \{(x_-, x_+) \mid \|x_-\|^2 = \varepsilon, 0 \leq \|x_+\| < \delta\},$$

[1]This means, with $(x_-, x_+) = (x_-^1, ..., x_-^k, x_+^1, ..., x_+^{n-k})$.

and that

$$D = \{(x_-, x_+) \mid \|x_+\|^2 = \varepsilon, 0 < \|x_-\|^2 < \delta'\}$$

for some δ' that depends on δ, as we are going to show. With all of this, it is possible to compute the s such that $\varphi_X^s(x) \in \mathrm{Im}\Phi$ for a given $x \in D$, so we can deduce the form of Φ in these coordinates:

$$\Phi((x_-, x_+)) = \left(\frac{\|x_+\|}{\|x_-\|}x_-, \frac{\|x_-\|}{\|x_+\|}x_+\right).$$

This way, if we let $(x_-, x_+) \in D$, we have that $(x_-, x_+) = ((t, z), \sqrt{\varepsilon})$ (where $\sqrt{\varepsilon}$ denotes the vector with all the coordinates equal to $\sqrt{\varepsilon}$, and (t, z) are the polar coordinates that we introduced before). Then, we conclude that

$$\Phi((t, z)) = (\sqrt{\varepsilon}z, t).$$

Therefore, $\rho(z) = \lim_{t \searrow 0} \Phi((t, z)) = (\sqrt{\varepsilon}z, 0)$, so the map ρ is injective, as we wanted to prove. □

Appendix B

The Conley-Zehnder index

In this chapter we will give a basic definition of the Conley-Zehnder index of a path of symplectic matrices. In the first section we will give a justification of why we are interested in such a map and how do the principal properties we ask from it arise. In the second section we will study the topology of the Lie group $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$, which will provide us with the tools needed to define the rotation map. Finally, we will use all these tools to define the Conley-Zehnder index and to prove its most essential properties.

In this chapter we follow the scheme of [Gut12].

B.1 Introduction

Let us recall the program sketched in section 1.3. In order to define the Floer homology, we need to define some sort of index that allows us to classify the critical points of the action functional, this means, the periodic orbits of a given Hamiltonian system with some Hamiltonian H_t depending on time.

Let x be such a solution, so

$$\dot{x}(t) = X_t(x(t)),$$

and let us assume that it is non-degenerate, this means, that

$$\det(\mathrm{T}_{x(0)}\, \varphi^1_{X_t} - \mathrm{Id}) \neq 0.$$

Let $Z(t)$ a symplectic frame of TM along x, this means, that $Z(t)$ is a basis of $T_{x(t)}M$ for each $t \in \mathbb{S}^1$ varying smoothly. Then, we can define the map

$$A: \quad [0,1] \quad \longrightarrow \quad \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$$

such that $A(t)$ is $\mathrm{T}_{x(t)}\, \varphi^t_{X_t}$ expressed in the basis $Z(t)$ (where $\varphi^t_{X_t}$ denotes the flow of the vector field X_t). Under these conditions, it is clear that $A(0) = \mathrm{Id}$ and that $A(1)$ does not have 1 as an eigenvalue. Therefore, we can define the set of paths of matrices that will interest us in the following sections:

$$\mathrm{SP}(n) = \left\{ \psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \;\middle|\; \begin{array}{l} \psi(0) = \mathrm{Id} \\ \psi(1) \text{ does not have 1 as an eigenvalue} \end{array} \right\}.$$

The goal of this chapter, therefore, is to define some map

$$\mu : \quad \mathrm{SP}(n) \quad \longrightarrow \quad \mathbb{Z}$$

that allows us to classify the periodic orbits.

B.2 The rotation map

The index that we define in this chapter has an intuitive interpretation in terms of the topology of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$. In particular, it relies in the fact (that we will prove) that $\pi_1(\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)) \cong \mathbb{Z}$. In particular, we will see that $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ can always be regarded as the topological product of $\mathbb{S}^1$ with a simply connected space. From this we will produce a map $\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to \mathbb{S}^1$, the rotation map, that will allow us to define the Conley-Zehnder index in terms of the degree of a map from $\mathbb{S}^1$ to itself.

B.2.1 The topology of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$

Recall that the symplectic group of dimension $2n$ is the subgroup of $GL(\mathbb{R}^{2n})$ defined by

$$\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) = \{A \in GL(\mathbb{R}^{2n} \mid A^T \Omega_0 A = \Omega_0\},$$

where

$$\Omega_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

We have the following result about the structure of this group:

Theorem B.1 *Let $A \in GL(\mathbb{R}^{2n})$ be an invertible matrix. Then, there exists a unique decomposition $A = OP$ such that O is orthogonal and P is symmetric and positive definite. Moreover, if A is symplectic, then both O and P are symplectic, so $O \in U(n)$ (the complex unitary matrix group of dimension n), and $P = \exp(S)$ for some symmetric matrix in the Lie algebra of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$.*

If we denote $V = \mathfrak{sp}(\mathbb{R}^{2n}, \Omega_0) \bigcap Sym(\mathbb{R}^{2n})$, then we deduce that this decomposition induces a topological factorization

$$\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \cong U(n) \times V,$$

where V is a vector space, so $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \simeq U(n)$.

Proof. The matrix $A^T A$ is symmetric and positive definite, so there is an orthogonal matrix Q such that $QA^T AQ^T = \mathrm{diag}(a_1, ..., a_{2n})$ with all the a_i real and strictly positive. Let us define the matrices

$$P = Q^T \mathrm{diag}(\sqrt{a_1}, ..., \sqrt{a_{2n}})Q,$$
$$S = Q^T \mathrm{diag}(\ln(a_1), ..., \ln(a_{2n}))Q,$$

so they are the unique matrices such that $P^2 = A^T A$ and $\exp(S) = A^T A$. Then, we define $O = AP^{-1}$. It is an orthogonal matrix, because

$$O^T O = P^{-1} A^T A P^{-1} = P^{-1} P^2 P^{-1} = \mathrm{Id}.$$

Moreover, it is a symplectic matrix, because, as A is symplectic,

$$A = \Omega_0^{-1}(A^T)^{-1}\Omega_0,$$

so[1]

$$OP = \Omega_0^{-1}((OP)^T)^{-1}\Omega_0 = \Omega_0^{-1}(O^T)^{-1}(P^T)^{-1}\Omega_0 = \Omega_0^{-1}(O^T)^{-1}\Omega_0\Omega_0^{-1}(P^T)^{-1}\Omega_0$$

and, by the uniqueness of the decomposition, $O = \Omega_0^{-1}O\Omega_0$ and $P = \Omega_0^{-1}P\Omega_0$.

On the other hand, we have that $P = \exp(\frac{1}{2}S)$ and, as P is symplectic, we know that $S \in \mathfrak{sp}(\mathbb{R}^{2n}, \Omega_0)$.

Therefore, we have shown that $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \cong U(n) \times V$. □

Proposition B.2 *The complex unitary group is homeomorphic to the product* $\mathbb{S}^1 \times SU(n)$ *(the special unitary group). Moreover,* $SU(n)$ *is simply connected. Therefore,* $\pi_1(U(n)) = \mathbb{Z}$*, so* $\pi_1(\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)) = \mathbb{Z}$.

Proof. The bijection between $U(n)$ and $\mathbb{S}^1 \times SU(n)$ is easily stablished by the map

$$\begin{array}{ccc} \mathbb{S}^1 \times SU(n) & \longrightarrow & U(n) \\ (e^{i\varphi}, O) & \longmapsto & \mathrm{diag}(e^{i\varphi}, 1, ..., 1)O \end{array}$$

whose inverse associates to a unitary matrix O its determinant $\det(O) = e^{i\varphi}$ and the special unitary matrix $\mathrm{diag}(e^{-i\varphi}, 1, ..., 1)O$.

To prove that $SU(n)$ is simply connected we can use its action on the sphere $\mathbb{S}^{2n-1} = \{z \in \mathbb{C} \mid |z|^2 = 1\}$. This action is transitive, and the isotropy group at $(1, 0, ..., 0)$ is $SU(n-1)$. Therefore, we get that

$$SU(n) \big/ SU(n-1) \cong \mathbb{S}^{2n-1}.$$

The long exact sequence of this quotient gives that

$$\cdots \to \pi_2(\mathbb{S}^{2n-1}) \to \pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(\mathbb{S}^{2n-1}).$$

If $n > 1$ then $\pi_2(\mathbb{S}^{2n-1}) = 0$ and $\pi_1(\mathbb{S}^{2n-1}) = 0$. Moreover, $SU(1) = \{1\}$. Therefore,

$$\pi_1(SU(n)) \cong \pi_1(SU(n-1)) \cong \cdots \cong \pi_1(SU(1)) = \pi_1(\{1\}) = 0,$$

and therefore $SU(n)$ is simply connected $\forall n \geq 1$. □

Remark B.3 This proposition provides an idea of how to proceed: we can define a map $f : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to \mathbb{S}^1$ such that $f_* : \pi_1(\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)) \to \pi_1(\mathbb{S}^1)$ is an isomorphism. When we restrict to $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \cap O(2n) = U(n)$, we can simply use the complex determinant $\det_{\mathbb{C}}$ of the matrix, so we just need to extent it in a way to the whole of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ to get the desired map.

On the other hand, to each path $\gamma : [0, 1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ we can associate the path $f \circ \gamma : [0, 1] \to \mathbb{S}^1$ which, under the appropriate conditions, may be regarded as a closed path, so we get $\tilde{f} \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1$. Therefore, we can compute the degree of this path, which in turn will allow us to graduate the original path.

[1] Here we use several times the fact that $\Omega_0^{-1} = -\Omega_0$.

B.2.2 The rotation map

In this section we will describe the map representing the "rotation" part of the symplectic group as we hinted previously. Rather than constructing the map (which is a cumbersome process conducted in [Gut12], where the uniqueness of such a map is also proved), we will present the properties that this map satisfies. Using them, we will then construct it in the particular case of $\mathbb{R}^2$ as an example.

Theorem B.4 *There exists a continuous map $\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to \mathbb{S}^1$ satisfying the following conditions:*

1. **Naturality:** *If $A, T \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$, then*
$$\rho(TAT^{-1}) = \rho(A).$$
2. **Product property:** *If $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_1)$ and $B \in \mathrm{Sp}(\mathbb{R}^{2m}, \Omega_2)$ (for some Ω_1, Ω_2 defining symplectic structures in $\mathbb{R}^{2n}, \mathbb{R}^{2m}$ respectively), then*
$$\rho\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \rho(A)\rho(B).$$
3. **Determinant property:** *If $A \in U(n) = \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \cap O(2n)$, then*
$$\rho(A) = \det{}_{\mathbb{C}}(X + iY),$$
where X, Y are the matrices such that
$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$
In particular, as we required in remark B.3, this implies that ρ induces an isomorphism on the first homotopy groups:
$$\rho_* : \quad \pi_1(\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)) \quad \longrightarrow \quad \pi_1(\mathbb{S}^1) \ .$$
4. **Normalization:** *If A has no eigenvalues in the unit circle, then $\rho(A) = \pm 1$.*
5. **Conjugation:** *For all $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$, we have that*
$$\rho(A^T) = \rho(A^{-1}) = \overline{\rho(A)}.$$

We will not prove this theorem in general, but we will show the proof for the case of dimension 2. The general process, as we said earlier, can be seen at [Gut12].

Construction of the rotation map in dimension 2:
Consider $A \in \mathrm{Sp}(\mathbb{R}^2, \Omega_0)$. By the properties of the rotation map, we see that the eigenvalues play a crucial role to understand its behaviour, and, in dimension 2, the eigenvalues of A may simply be understood via the characteristic polynomial of A, that has the form
$$p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A),$$
and, as A is symplectic, $\det(A) = 1$. Therefore, the discriminant of $p_A(\lambda)$ is
$$\Delta = \mathrm{tr}(A)^2 - 4.$$
This way, we can classify the behaviour in the following cases:

1. If $A = \pm\mathrm{Id}$, it is particularly easy to compute the rotation map, because they are both unitary matrices. Applying the determinant property, we find that

$$\rho(\pm\mathrm{Id}) = \det{}_{\mathbb{C}}(\pm 1) = \pm 1.$$

2. $|\mathrm{tr}(A)| > 2$: this implies that $\Delta > 0$, so the matrix has two distinct real eigenvalues, α and α^{-1}, which must have the same sign. Applying the normalization property we see that $\rho(A) = \pm 1$. Using the computation that we just did for $\pm\mathrm{Id}$ and the continuity of ρ, we deduce that

$$\rho\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0 \end{cases}.$$

If we also apply the naturality property of ρ, we deduce that

$$\rho(A) = \begin{cases} 1 & \text{if } \mathrm{tr}(A) > 2 \\ -1 & \text{if } \mathrm{tr}(A) < -2 \end{cases}.$$

3. $|\mathrm{tr}(A)| < 2$: this implies that $\Delta < 0$, so the matrix has two distinct complex eigenvalues, α and $\overline{\alpha}$, with $\alpha\overline{\alpha} = 1$. Therefore, $\alpha = e^{i\varphi}$ for some $\varphi \in \mathbb{R}$. We may therefore choose coordinates such that

$$A = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix},$$

so

$$\rho(A) = e^{\pm i\varphi}.$$

To determine the sign in the above expression, one must take into account the orientation of the basis (v, w) that one chooses to represent A as stated earlier. The one that is consistent with the naturality condition is the choice of φ such that $Az = e^{i\varphi}z$, where $z = v - iw$ with $\Omega_0(v, w) > 0$.

4. $\mathrm{tr}(A) = \pm 2$: this implies that $\Delta = 0$ and that A has a double eigenvalue $\alpha = \pm 1$. In consequence, we can choose coordinates such that

$$A = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}.$$

By the continuity of ρ, $\rho(A)$ is the same as the limit of $\rho(A_t)$ for any continuous path that tends towards A. Therefore, we may choose the path of symplectic matrices

$$A_t = \begin{pmatrix} \pm e^{t} & e^{t} \\ 0 & \pm e^{-t} \end{pmatrix},$$

so

$$\rho(A) = \lim_{t \to 0} \rho(A_t) = \pm 1,$$

and we conclude that

$$\rho(A) = \begin{cases} 1 & \text{if } \mathrm{tr}(A) = 2 \\ -1 & \text{if } \mathrm{tr}(A) = -2 \end{cases}.$$

In the general case, one must use the product property and appropriate forms of A to construct the rotation map for any symplectic matrix.

To end this section we provide a formula for ρ that will be useful in the following results.

Lemma B.5 *Let $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$, and let $\lambda_1, ..., \lambda_{2n}$ be its eigenvalues, repeated according to their multiplicity. Then, the rotation map can be computed as*

$$\rho(A) = \prod \frac{\lambda_i}{|\lambda_i|},$$

where the only eigenvalues contributing to the product are half of the eigenvalues, particularly

1. *Eigenvalues with $|\lambda_i| < 1$.*

2. *Half of the eigenvalues equal to 1 (we can do so because this number has to be even for the determinant to be 0).*

3. *Half of the eigenvalues equal to -1 (as in the last point, there must be an even number of such eigenvalues).*

4. *If $\lambda_i = e^{i\varphi} \neq \pm 1$, then $\lambda_i^{-1}, \overline{\lambda_i}$ and $\overline{\lambda_i^{-1}}$ are also eigenvalues of A. Let E_λ denote the sum of their (complex) eigenspaces, and let Q be the quadratic form defined by*

$$\begin{array}{cccc} Q: & E_\lambda \times E_\lambda & \longrightarrow & \mathbb{C} \\ & (v, w) & \longmapsto & \mathrm{Im}(\Omega_0(v, \overline{w})) \end{array}.$$

The signature of Q is $(2r, 2s)$, and we take r times the eigenvalue λ_i as of the first kind.

B.3 The Conley-Zehnder index

As we said in the introduction, the ultimate goal of this chapter is defining a map

$$\mu : \quad \mathrm{SP}(n) \longrightarrow \mathbb{Z}$$

that allows us to classify the paths of symplectic matrices according to their rotation map.

Definition B.6 The positive (resp. negative) components of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ are

$$\mathrm{Sp}(2n)^{\pm} = \{A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \det(A - \mathrm{Id}) \gtrless 0\}.$$

It can be seen that both $\mathrm{Sp}(2n)^+$ and $\mathrm{Sp}(2n)^-$ are path-connected. Therefore, it is possible for us to choose a "representative" element in each connected component,

$$W^+ = -\mathrm{Id}, \quad W^- = \left(\begin{array}{cc|c} 2 & 0 & \\ 0 & \frac{1}{2} & 0 \\ \hline \multicolumn{2}{c|}{0} & -\mathrm{Id} \end{array}\right).$$

We can easily compute ρ in this matrices, as they are diagonal:

$$\rho(W^+) = \rho(-\mathrm{Id}) = \rho\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (-1)^n,$$

$$\rho(W^-) = \rho\left(\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)\rho\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)^{n-1} = (-1)^{n-1}.$$

The idea is to associate to each $\psi \in \mathrm{SP}(n)$ an extended path $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ such that $\widetilde{\psi}\big|_{[1,2]} \subset \mathrm{Sp}(2n)^{\pm}$ and $\widetilde{\psi}(2) = W^{\pm}$. The main issue with this is that we can extend the same path ψ in a lot of ways (even though the endpoint must always be W^+ or W^- if $\psi(1) \in \mathrm{Sp}(2n)^+$ or $\psi(1) \in \mathrm{Sp}(2n)^-$, respectively). However, it can be proved that all the extensions of the same path are homotopically equivalent. To show this, we will prove the following theorem:

Theorem B.7 *Any continuous loop in* $\mathrm{Sp}(2n)^+$ *or* $\mathrm{Sp}(2n)^-$ *is contractible.*

Proof. A loop $\gamma : \mathbb{S}^1 \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is contractible if, and only if, its image by ρ is contractible. It is because ρ induces an isomorphism ρ_* between the first homotopy groups. Therefore, it suffices to see that $\rho \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1$ is contractible.

In the last section we introduced the lemma B.5 for ρ. Let us consider the map that sends each symplectic matrix A to the set of its eigenvalues of first kind:

$$\begin{array}{ccc} \mathrm{Sp}(2n)^{\pm} & \longrightarrow & \Lambda_n = \mathbb{C}^n \,/\text{permutations of the elements} \\ A & \longmapsto & \{\lambda_1, ..., \lambda_n\} \end{array},$$

which is a continuous map. In other words, we can define the continuous maps $\Lambda_1, ..., \Lambda_n : [0,1] \to \mathbb{C}$ such that $\Lambda_i(t)$ is the i-th eigenvalue of first kind of $\gamma(t)$. We want to define continuous maps $\alpha_i : [0,1] \to [0, 2\pi]$ such that

$$e^{i\alpha_i(t)} = \frac{\Lambda_i(t)}{|\Lambda_i(t)|},$$

which is well defined whenever $\frac{\Lambda_i(t)}{|\Lambda_i(t)|} \neq 1$. In the case that $\frac{\Lambda_i(t)}{|\Lambda_i(t)|} = 1$, this means, that $\Lambda_i(t)$ is real and positive, we need to provide a consistent definition of $\alpha_i(t)$.

It can be seen that, if $\gamma(t) \in \mathrm{Sp}(2n)^+$, then the number of $\Lambda_i(t)$ that are real and positive is even, and, conversely, it is odd when $\gamma(t) \in \mathrm{Sp}(2n)^-$. Therefore, if the number is $2k$ (resp. $2k+1$), we can take k (resp. $k+1$) of the $\alpha_i(t)$ to be $\alpha_i(t) = 2\pi$, and the remaining k to be $\alpha_i(t) = 0$. It can be proved that this results in a set of continuous maps $\alpha_1, ..., \alpha_n$.

Thus, if $\gamma : \mathbb{S}^1 \to \mathrm{Sp}(2n)^{\pm}$, its image by each of the α_i is a loop $\alpha_i \circ \gamma : \mathbb{S}^1 \to [0, 2\pi]$ and therefore contractible. This means that

$$\rho(\gamma(t)) = e^{i\sum_{j=1}^{n} \alpha_j(t)}$$

is a contractible map, and, by our first observation in this proof, γ is contractible. □

Therefore, if $\widetilde{\psi_1}$ and $\widetilde{\psi_2}$ are extensions of the same path $\psi \in \mathrm{SP}(n)$, then $\widetilde{\psi_1} \simeq \widetilde{\psi_2}$.

Now we are ready to define the Conley-Zehnder index and to prove its most interesting properties.

Definition B.8 Let $\psi \in \mathrm{SP}(n)$. Its **Conley-Zehnder index** is

$$\mu_{CZ}(\psi) = \deg(\rho^2 \circ \widetilde{\psi}),$$

where ρ^2 means the product of ρ with itself. The reason for this square is the need of having a map from $\mathbb{S}^1$ to itself. We know that $\rho \circ \widetilde{\psi} : [0,2] \to \mathbb{S}^1$ and that $(\rho \circ \widetilde{\psi})(2) = \pm 1$. As we need to have a periodic map to compute its degree, we square the result.

Remark B.9 The map is well defined, this means, it does not depend on the extension $\widetilde{\psi}$ chosen for ψ. This is a consequence of theorem B.7, because any two such extensions are homotopic, so their compositions with ρ^2 are also homotopic, and therefore they have the same degree.

Proposition B.10 *The Conley-Zehnder index has the following properties:*

1. **Naturality:** *For any path* $\phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$*,*
$$\mu_{CZ}(\phi\psi\phi^{-1}) = \mu_{CZ}(\psi).$$
2. **Homotopy:** *If* $\psi_0 \simeq \psi_1$*, then* $\mu_{CZ}(\psi_0) = \mu_{CZ}(\psi_1)$*.*
3. **Zero property:** *If* $\psi(s)$ *has no eigenvalue on the circle for* $s > 0$*, then*
$$\mu_{CZ}(\psi) = 0.$$
4. **Product property:** *If*
$$\psi_1 \oplus \psi_2 = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0),$$
then
$$\mu_{CZ}(\psi_1 \oplus \psi_2) = \mu_{CZ}(\psi_1) + \mu_{CZ}(\psi_2).$$
5. **Loop:** *If* $\phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *with* $\phi(0) = \phi(1) = \mathrm{Id}$*, then*
$$\mu_{CZ}(\phi\psi) = \mu_{CZ}(\psi) + 2\,deg(\rho \circ \phi).$$
6. **Signature:** *If* S *is a symmetric non-degenerate* $2n \times 2n$ *matrix with all eigenvalues satisfying* $|\lambda_i| < 2\pi$ *and* $\psi(t) = \exp(J_0 S t)$*, then*
$$\mu_{CZ}(\psi) = \frac{1}{2}\mathrm{Sign}(S),$$
where Sign *denotes the signature of* S*.*
7. **Determinant:** *If* $\psi \in SP(n)$*,*
$$(-1)^{n-\mu_{CZ}(\psi)} = \mathrm{sign}\left(\det(\mathrm{Id} - \psi(1))\right).$$
8. **Inverse:**
$$\mu_{CZ}(\psi^{-1}) = \mu_{CZ}(\psi^T) = -\mu_{CZ}(\psi).$$

Proof. (Naturality): Let $\widetilde{\phi}$ be an extension of ϕ to $[0,2]$ such that $\phi(t) \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ for all t and $\phi(2) = \mathrm{Id}$. Then, $\widetilde{\psi}' = \widetilde{\phi}\widetilde{\psi}\widetilde{\phi}^{-1}$ is a path connecting the identity with $W^{\pm}$, so we can compute
$$\deg(\rho^2 \circ \widetilde{\psi}'),$$
and, by the naturality property of ρ, $\rho(\widetilde{\psi}'(t)) = \rho(\widetilde{\psi}(t))$ for all t, so
$$\mu_{CZ}(\phi\psi\phi^{-1}) = \deg(\rho^2 \circ \widetilde{\psi}') = \deg(\rho^2 \circ \widetilde{\psi}) = \mu_{CZ}(\psi).$$

□

Proof. (Homotopy): We consider that $\psi_0 \simeq \psi_1$ in $\mathrm{SP}(n)$, so there is a map $\psi : [0,1]\times[0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ with $\psi_s(1) \in \mathrm{Sp}(2n)^{\pm}$ for all $s \in [0,1]$. Let $\widetilde{\psi_0}$ be an extension of ψ_0, and consider the extension $\widetilde{\psi_s}$ of ψ_s defined by

$$\widetilde{\psi_s}(t) = \begin{cases} \psi_s(t) & 0 \le t \le 1 \\ \psi_{s(3-2t)}(1) & 1 \le t \le \frac{3}{2} \\ \widetilde{\psi_0}(2t-2) & \frac{3}{2} \le t \le 2 \end{cases},$$

which is continuous and defines a homotopy between $\widetilde{\psi_0}$ and $\widetilde{\psi_1}$. Therefore,

$$\mu_{CZ}(\psi_0) = \mu_{CZ}(\psi_1).$$

□

Proof. (Zero property): By the normalization property and the continuity of ρ we deduce that $\rho(\psi(t)) = 1\ \forall t \in [0,1]$. Moreover, it is possible to construct an extension of ψ such that its eigenvalues are outside of the circle for $t < 2$. Thus, we get that $\rho^2(\widetilde{\psi}(t)) = 1\ \forall t \in [0,2]$. We conclude that $\mu_{CZ}(\psi) = \deg(\rho^2 \circ \widetilde{\psi}) = 0$. □

Proof. (Product): We will use the product property of ρ. Consider $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ the extensions of ψ_1 and ψ_2. If one (or both) of $\widetilde{\psi}_i(2)$ is equal to W^-, then $(\widetilde{\psi}_1 \oplus \widetilde{\psi}_2)(2) = W^{\pm}$. On the other hand, if $(\widetilde{\psi}_1 = W^+$ and $\widetilde{\psi}_2)(2) = W^+$ then $(\widetilde{\psi}_1 \oplus \widetilde{\psi}_2)(2)$ is homotopically equivalent to W^+, so we can find some path $\widetilde{\psi}' \simeq \widetilde{\psi}_1 \oplus \widetilde{\psi}_2$ with $\widetilde{\psi}'(2) = W^+$ and such that $\widetilde{\psi}'\Big|_{[0,1]} = \psi_1 \oplus \psi_2$. Therefore,

$$\mu_{CZ}(\psi_1 \oplus \psi_2) = \deg(\rho^2 \circ \widetilde{\psi}') = \deg(\rho^2 \circ (\widetilde{\psi}_1 \oplus \widetilde{\psi}_2)) =$$
$$= \deg((\rho^2 \circ \widetilde{\psi}_1)\cdot(\rho^2 \circ \widetilde{\psi}_1)) = \deg(\rho^2 \circ \psi_1) + \deg(\rho^2 \circ \psi_2) = \mu_{CZ}(\psi_1) + \mu_{CZ}(\psi_2).$$

□

Lemma B.11 *Consider $\varphi, \psi : [0,T] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ two paths of symplectic matrices with $\varphi(0) = \psi(0) = \mathrm{Id}$. Then, the product path $\psi\varphi$ is homotopic to the path*

$$(\psi \diamond \varphi)(t) = \begin{cases} \varphi(2t) & t \le \frac{T}{2} \\ \psi\left(2\left(t - \frac{T}{2}\right)\right)\varphi(T) & t \ge \frac{T}{2} \end{cases},$$

so $\psi\varphi \simeq \psi \diamond \varphi$.

Proof. (Lemma): Consider the homotopy $\chi : [0,1]\times[0,T] \to \mathrm{Sp}(\mathbb{R}^2, \Omega_0)$ defined by

$$\chi_s(t) = \begin{cases} \varphi(2t) & t \le \frac{sT}{2} \\ \psi\left(\frac{2}{2-s}\left(t - \frac{sT}{2}\right)\right)\varphi\left(sT + \frac{2(1-s)}{2-s}\left(t - \frac{sT}{2}\right)\right) & t \ge \frac{sT}{2} \end{cases}.$$

It is continuous, because $\chi_s\left(\frac{sT}{2}\right) = \varphi(sT)\ \forall s$, and

$$\chi_0(t) = \psi(t)\varphi(t),$$

$$\chi_1(t) = \begin{cases} \varphi(2t) & t \le \frac{T}{2} \\ \psi\left(2\left(t - \frac{T}{2}\right)\right)\varphi(T) & t \ge \frac{T}{2} \end{cases}.$$

□

Proof. (Loop): As $\phi(0) = \phi(1) = \mathrm{Id}$ and $\psi \in \mathrm{SP}(n)$, we can apply the lemma, so $\phi\psi \simeq \phi\diamond\psi$. Thus,

$$\mu_{CZ}(\phi\psi) = \deg(\rho^2 \circ (\widetilde{\psi} \diamond \phi)) = \deg(\rho^2 \circ \widetilde{\psi}) + \deg(\rho^2 \circ \phi) = \mu_{CZ}(\psi) + 2\deg(\rho \circ \phi).$$

□

Proof. (Signature): Since S is a symmetric matrix, there is an orthogonal matrix P of determinant 1 such that $PSP^T = \mathrm{diag}(a_1, ..., a_{2n})$. Our assumptions imply that $a_i \neq 0$ and $|a_i| < 2\pi$ $\forall i$, and therefore the eigenvalues of J_0S all have norm smaller than 2π. This implies that $\exp(J_0S)$ does not admit 1 as an eigenvalue, so $\exp(tJ_0S) \in \mathrm{SP}(n)$.

Consider P_s a path of orthogonal matrices with $P_0 = P$ and $P_1 = \mathrm{Id}$ with $\|P_sSP_s^T\| < 2\pi$. In this case, $\exp(J_0P_sSP_s^T)$ does not admit the eigenvalue 1 for any s, so $\exp(tJ_0PSP^T) \simeq \exp(tJ_0S)$ in $\mathrm{SP}(n)$. Therefore, it is enough to prove the property for the case when S is diagonal.

In this case,

$$J_0S = \begin{pmatrix} 0 & \mathrm{diag}(-a_{n+1}, ..., -a_{2n}) \\ \mathrm{diag}(a_1, ..., a_n) & 0 \end{pmatrix}, \text{ with } |a_i| < 2\pi.$$

We can decompose $(\mathbb{R}^{2n}, \Omega_0)$ into a sum of n symplectic planes in a way that ψ decomposes in ψ_i in each of the planes, and such that $\mu_{CZ}(\psi)$ is the sum of all the $\mu_{CZ}(\psi_i)$, for

$$\psi_i(t) = \exp\left(t\begin{pmatrix} 0 & -a_{n+i} \\ a_i & 0 \end{pmatrix}\right),$$

by the product property. The closed formula of ψ_i can be computed from the sign of a_i, a_{n+i} and a_ia_{n+i}:

1. If $a_ia_{n+i} > 0$,

$$\psi_i(t) = \begin{pmatrix} \cos(\sqrt{a_ia_{n+i}}t) & \mp\sqrt{\frac{a_{n+i}}{a_i}}\sin(\sqrt{a_ia_{n+i}}t) \\ \pm\sqrt{\frac{a_i}{a_{n+i}}}\sin(\sqrt{a_ia_{n+i}}t) & \cos(\sqrt{a_ia_{n+i}}t) \end{pmatrix} \text{ if } a_i \gtrless 0.$$

2. If $a_ia_{n+i} < 0$,

$$\psi_i(t) = \begin{pmatrix} \cosh(\sqrt{-a_ia_{n+i}}t) & \pm\sqrt{-\frac{a_{n+i}}{a_i}}\sinh(\sqrt{-a_ia_{n+i}}t) \\ \pm\sqrt{-\frac{a_i}{a_{n+i}}}\sinh(\sqrt{-a_ia_{n+i}}t) & \cosh(\sqrt{-a_ia_{n+i}}t) \end{pmatrix} \text{ if } a_i \gtrless 0.$$

In the second case, $\psi_i(t)$ has no eigenvalues on the circle for $t > 0$, so $\mu_{CZ}(\psi_i) = 0$ by the zero property.

Otherwise, the eigenvalues of $\psi_i(t)$ are $\cos(\sqrt{a_ia_{n+i}}t) \pm i\sin(\sqrt{a_ia_{n+i}}t)$ so, rotating the plane if needed, we get that

$$\rho(\psi_i(t)) = \begin{cases} e^{i\sqrt{a_ia_{n+i}}t} & \text{if } a_i > 0 \\ e^{-i\sqrt{a_ia_{n+i}}t} & \text{if } a_i < 0 \end{cases}.$$

Either way, $\det(\mathrm{Id} - \psi(1)) = 2(1 - \cos(a_i a_{n+i})) > 0$, so the extension $\widetilde{\psi}_i$ must end in $W^+ = -\mathrm{Id}$, so $\rho(\widetilde{\psi}_i(2)) = e^{\pm i\pi}$. In the case when $a_i > 0$ $\rho(\widetilde{psi_i}(t))$ goes from $1 = e^0$ to $e^{i\pi}$ counterclockwise, so $\mu_{CZ}(\psi_i) = 1$. On the other hand, when $a_i < 0$ $\rho(\widetilde{psi_i}(t))$ goes from $1 = e^0$ to $e^{-i\pi}$ clockwise, so $\mu_{CZ}(\psi_i) = -1$. Rephrasing, $\mu_{CZ}(\psi_i) = \mathrm{sign}(a_i)$. Therefore,

$$\mu_{CZ}(\psi) = \#\{i \leq n \mid a_i > 0, a_i a_{n+i} > 0\} - \#\{i \leq n \mid a_i < 0, a_i a_{n+i} > 0\} = \frac{1}{2}\mathrm{Sign}(S).$$

□

Proof. (Determinant): If $\det(\mathrm{Id} - \psi(1)) > 0$, then $\widetilde{\psi}(2) = W^+$, so $\rho(\widetilde{\psi}(2)) = (-1)^n$. Otherwise, if $\det(\mathrm{Id} - \psi(1)) < 0$, then $\widetilde{\psi}(2) = W^-$ and $\rho(\widetilde{\psi}(2)) = (-1)^{n-1}$. The degree of $\rho^2 \circ \widetilde{\psi}$ is even when $\rho(\widetilde{\psi}) = 1$, and odd when $\rho(\widetilde{\psi}(2)) = -1$. Therefore,

$$(-1)^{n-\mu_{CZ}(\psi)} = \mathrm{sign}(\det(\mathrm{Id} - \psi(1))).$$

□

Proof. (Inverse): Recall that $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ implies that $A^T = \Omega_0 A^{-1} \Omega_0^T = \Omega_0^{-1} A^{-1} \Omega_0$ so, by the naturality property of μ_{CZ},

$$\mu_{CZ}(\psi^T) = \mu_{CZ}(\psi^{-1}).$$

Moreover, by the conjugation property of ρ we know that $\rho(A^{-1}) = \overline{\rho(A)} = \rho(A)^{-1}$ for any symplectic matrix. On the other hand, it can be easily checked that $\widetilde{\psi^{-1}} \simeq \widetilde{\psi}^{-1}$. Therefore,

$$\mu_{CZ}(\psi^{-1}) = \deg(\rho^2 \circ \widetilde{\psi^{-1}}) = \deg(\rho^2 \circ \widetilde{\psi}^{-1}) = \deg((\rho^2 \circ \widetilde{\psi})^{-1}) = -\deg(\rho^2 \circ \widetilde{\psi}) = -\mu_{CZ}(\psi).$$

□

Now that we have seen quite in detail the properties of the Conley-Zehnder index, a natural question that arises is if there are other maps satisfying similar properties. However, the answer is that the Conley-Zehnder index is completely determined by (some of) its properties:

Theorem B.12 *Let $\mu' : SP(n) \to \mathbb{Z}$ satisfying the homotopy, loop and signature properties as in B.10. Then, $\mu'(\psi) = \mu_{CZ}(\psi)$ $\forall \psi \in SP(n)$.*

Proof. Take $\psi \in \mathrm{SP}(n)$. As we observed earlier, there exists $\widetilde{\psi} \in \mathrm{SP}(n)$ with $\widetilde{\psi} \simeq \psi$ and $\widetilde{\psi}(1) = W^{\pm}$. By the homotopy property, $\mu'(\psi) = \mu'(\widetilde{\psi})$.

Let $S^{\pm}$ be such that $W^{\pm} = \exp(\pi J_0 S^{\pm})$. In this case, $S^+ = \mathrm{Id}$, and

$$S^- = \begin{pmatrix} 0 & 0 & -\frac{\ln 2}{\pi} & 0 \\ 0 & \mathrm{Id}_{n-1} & 0 & 0 \\ -\frac{\ln 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-1} \end{pmatrix}.$$

Consider $\xi(t) = \exp(t\pi J_0 S^+)$ (resp with S^- if $\psi(1) = W^-$), and let $\xi^-(t) = \xi(1-t)$. Take $\phi = \widetilde{\psi} \diamond \xi^-$, which is a continuous loop, because $\phi(0) = \phi(1) = W^{\pm}$.

On the other hand, we know that $\widetilde{\psi} \simeq \phi \diamond \xi$. By the lemma B.11, we know also that $\phi \diamond \xi \simeq \phi\xi$. Therefore, $\mu'(\widetilde{\psi}) = \mu'(\phi\xi)$. If we apply now the loop condition, we see that

$$\mu'(\phi\xi) = 2\text{deg}(\rho \circ \phi) + \mu'(\xi),$$

and applying the signature condition we see that

$$\mu'(\xi) = \frac{1}{2}\text{Sign}(S^{\pm}).$$

Therefore, we conclude that, for any map μ' satisfying the homotopy, loop and signature conditions,

$$\mu'(\psi) = 2\text{deg}(\rho \circ \phi) + \frac{1}{2}\text{Sign}(S^{\pm}).$$

However, we would get to this very same expression if $\mu' = \mu_{CZ}$. Therefore,

$$\mu'(\psi) = \mu_{CZ}(\psi).$$

□

This is why we can say that the homotopy, loop and signature conditions determine completely the Conley-Zehnder index.

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