Sequences of spanning trees for L_∞-Delaunay triangulations[*]

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──── **Abstract** ────

We extend a known result about L_2-Delaunay triangulations to L_∞-Delaunay. Let $\mathcal{T}_S$ be the set of all non-crossing spanning trees of a planar n-point set S. We prove that for each element T of $\mathcal{T}_S$, there exists a length-decreasing sequence of trees $T_0, \ldots, T_k$ in the L_∞-metric such that $T_0 = T, T_k = MST_\square(S)$ and T_i does not cross T_{i-1} for all $i = 1, \ldots, k$, where $MST_\square(S)$ denotes the minimum spanning tree of S in the L_∞ metric. We also give an $\Omega(\log n)$ lower bound for the length of the sequence.

1   Introduction

The Delaunay triangulation is one of the most studied objects in computational geometry. In its most common form, L_2 or *circle-based*, it can be defined for a set of points $S \subset \mathbb{R}^2$ as a graph $DT_\bigcirc(S)$ where the vertex set is S, and an edge between two vertices u and v exists if and only if there exists a circle with u and v on its boundary containing no point of S in its interior. It is well-known that $DT_\bigcirc(S)$ is a triangulation if no four points of S are co-circular. The L_2-Delaunay triangulation has received a vast amount of attention. The aspects studied range from graph-theoretic properties like Hamiltonicity [5], or functionals for which it is optimal [6], to recent results on their spanning and routing properties [2,7].

The definition of Delaunay graphs is easy to generalize by using some shape other than the circle that is required to be empty, such as a triangle, a square, or in fact any convex shape, leading to so-called *convex*-Delaunay graphs [4]. However, there are relatively few results on properties of convex Delaunay graphs for shapes other than circles, despite the fact that different shapes result in different properties of the Delaunay graphs. A recent result by Bonichon et al. [3] shows that, when the shape is a triangle, every plane triangulation can be realized as a triangle-based Delaunay triangulation, for some point set in the plane. Whereas, it is known that there exist triangulations that cannot be $DT_\bigcirc$-realizable.

For this reason, it is interesting to understand which properties of the Delaunay graphs depend on the circular shape, and what properties do not. In this paper we explore one of the properties of $DT_\bigcirc(S)$, and show that it also holds for the square or L_∞-Delaunay graph, denoted $DT_\square(S)$. Aichholzer et al. [1] showed that the minimum spanning tree of S can be obtained by repeatedly computing minimum spanning trees of constrained Delaunay triangulations. More precisely, the following iterative procedure converges to the minimum spanning tree of S. Start by computing an arbitrary spanning tree T_0 of S. Compute

the Delaunay triangulation of S taking the edges of T_0 as constraints. Next, compute the minimum spanning tree T_1 of that constrained triangulation, and repeat.

In this paper we show that the same occurs with squares, i.e., when the L_∞ metric is used, this procedure converges to the L_∞-minimum spanning tree of S. The main ingredient of our result is, as in [1], a fixed tree theorem (Theorem 3.1) that states that once one iteration of the above procedure does not produce a change, then it must have reached the minimum spanning tree of S. We note, however, that our proofs are different from those used for the L_2-metric, since several key lemmas in [1] rely on properties of circles.

1.1 Definitions

Let G be a plane geometric graph on S and let Q be an axis-aligned square. In the L_∞ metric, the set of points within a fixed distance from a given point is an axis-aligned square centered on the point. Recall that in $\mathbb{R}^n$, $||x||_\infty = \sup\{|x_i| : i \in \{1, \ldots, n\}\}$. The Delaunay graph of S with respect to the L_∞ metric, denoted as $DT_\square(S)$, contains an edge between p and q if and only if there is some homothet of Q with p and q on its boundary without any other point of S in its interior.

Given two vertices p and q, we denote by $Q(p, q)$ any homothet of Q with p and q on its boundary. Let $G = (S, E)$ where $E(G)$ is called a set of *constraints* and each element of $E(G)$ is called a *constraint*. We say that a point p is visible to q if no constraint crosses the line segment pq, and in that case we say that (p, q) is a *visible* edge in G. We define the *visibility graph* of G as the graph with vertices S and the set of all visible edges in G as the edge set. Notice that the visibility graph is always a connected graph. Also, that every constraint is a visible edge. We define $CDT_\square(G)$, the constrained L_∞-Delaunay graph of G, as follows. The constrained L_∞-Delaunay graph contains an edge between p and q if and only if pq is a constraint or there exists a $Q(p, q)$ such that there are no vertices of S in the interior of $Q(p, q)$ visible to both p and q. It is known that any *convex shape*-$CDT(G)$ is a plane graph.

We say that p and q are *separated* in a square Q if p and q are in Q and there exists a constraint e crossing Q such that e divides Q into two different convex sets and with p and q in different sets, we refer to Figure 1. For any two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, we define the *width* (respectively *height*) between p and q as $w(p, q) = |x_1 - x_2|$ $(h(p, q) = |y_1 - y_2|)$.

Let G be a graph with vertex set S and p, q be vertices in G and let P be a path from p to q. We will refer to P as a *pq-path*. We denote by $MST(G)$ the minimum spanning tree of the visibility graph of G. An important property of the classic Delaunay triangulation of a point set S is that $MST_\bigcirc(S) \subset DT_\bigcirc(S)$. The same property holds for L_∞-Delauny triangulation, i.e., $MST_\square(S) \subset DT_\square(S)$. We assume throughout that all edges in $S \times S$ have different lengths.

Figure 1 The points p and q are separated in Q by the edge e.

Let $\mathcal{T}_S$ be the set of all crossing-free spanning trees of S. For each element $T \in \mathcal{T}_S$ of S, we define the sequence $T_0, T_1, T_2, \ldots$ where $T_0 = T$, $T_i = MST(CDT_\square(T_{i-1}))$ for all $0 < i$. In this paper we will prove the convergence of this sequence to the $MST_\square(S)$. We begin with some helping lemmas.
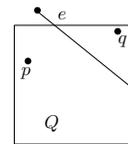
2 Properties of L_∞-minimum spanning trees and L_∞-Delaunay graphs

The following properties will be essential for the proof of the fixed tree theorem.

▶ **Property 2.1.** *An edge $e \in G$ is not present in $MST(G)$ if and only if there is a path in G between e's endpoints that solely consists of edges shorter than e.*

The next two lemmas show that if there exists a visibility path between two points p and q contained in a square Q, then there exists a pq-path of the L_∞-constrained Delaunay triangulation contained in Q as well.

▶ **Lemma 2.2.** *Let p, q be two vertices such that (p, q) is visible in G. Then any $Q(p, q)$ contains a pq-path in $CDT_\square(G)$.*
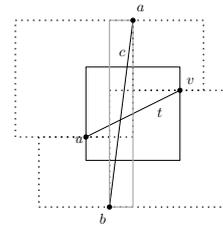
The proof of Lemma 2.2 follows by induction on the number of vertices of S contained in $Q(p, q)$.

▶ **Lemma 2.3.** *Let Q be a square that contains p and q such that there is no edge separating p and q in Q, then there exists a pq-path of $CDT_\square(G)$ contained in Q.*

Proof. Since there is no edge separating p and q in Q there exists a pq-path in the visibility graph of G in Q. The proof follows from Lemma 2.2. ◀

Let $t = (u, v)$ be an edge of the $MST_\square(S)$ crossed by a set of constraints $BE_i(t)$ from T_i. Let Q_t be a smallest empty square containing u and v on its boundary, i.e., $||uv||_\infty = w(Q_t)$ and $u, v \in Q_t$. We say that $c \in BE_i(t)$ is a *diagonal edge* when c crosses t in consecutive sides of Q_t, or call it *vertical edge* otherwise. We say that c is the constraint nearest to u if it is the first constraint crossing t from u to v. Similarly c is the constraint nearest to v if it is the first constraint crossing t from v to u. The weight of a square Q corresponds to its sides length, denoted by $W(Q)$.

The following lemma basically states that if a constraint $c = (a, b)$ crossing t is vertical, then the distances $||av||_\infty, ||au||_\infty, ||bv||_\infty$ and $||bu||_\infty$ are shorter than $||ab||_\infty$, we refer to Figure 2.

▶ **Lemma 2.4.** *Let $T \in \mathcal{T}_S$, and let $t = (u, v) \in MST_\square(T)$ and Q_t defined as above. Let $c = (a, b)$ be a vertical edge crossing t. Then there exist squares $Q(a, v)$, $Q(b, v)$, $Q(a, u)$ and $Q(b, u)$ with weight than $||ab||_\infty$.*

The next lemma shows a similar property fo diagonal edges.

▶ **Lemma 2.5.** *Let $T \in \mathcal{T}_S$, $t = (u, v) \in MST_\square(S)$ and Q_t defined as above. Let $c = (a, b) \in T$ be the nearest edge to v crossing t. If c is a diagonal edge such that c is crossing the side of Q_t containing v, then no edge from $CDT_\square(T)$ crosses t nearer to v than c.*

■ **Figure 2** The dotted squares represent $Q(a, v)$, $Q(b, v)$, $Q(a, u)$ and $Q(b, u)$, which have shorter weight than $||ab||_\infty = h(a, b)$.

Proof. Assume by contradiction that there exists an edge $e = (p, q) \in CDT_\square(T)$ nearer to v than c. Then e is also diagonal and crosses the same sides of Q_t as c. Since e is diagonal and crosses the side of Q_t that contains v, the rectangle with diagonal e contains v in its interior. Hence, any square containing e's endpoints has v in its interior, we refer to Figure 3b. Since c was the nearest constraint to v crossing t, then any square containing e's endpoints has a visible point to both endpoints, which is a contradiction with e being in $CDT_\square(T)$. ◀

▶ **Lemma 2.6.** *Let $T \in \mathcal{T}_S$, $t = (u, v) \in MST_\square(S)$ and Q_t defined as above. Let $c = (a, b) \in T$ be the nearest edge to v crossing t. If c is a vertical edge and there exists an edge $e = (p, q)$ in $CDT(T)$ crossing t nearer to v than c, then e is shorter than c.*

Proof. Using the same arguments in the proof of Lemma 2.5 we notice that e cannot be a diagonal edge. Thus, e is vertical and crosses the same sides of Q_t as c, we refer to

(a) Vertical case. **(b)** Diagonal case.
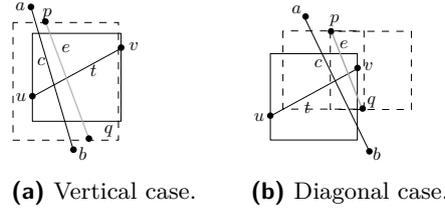
Figure 3 Examples where edge e is a nearer to v than c.

Figure 3a. Without loss of generality, assume that e crosses top and bottom sides of Q_t. Let $Q_e = Q(p, q)$ be a smallest square defining e in $CDT_\square(T)$, then $W(Q_e) = ||e||_\infty$. Notice that since e crosses opposite sides of Q_t, then $||e||_\infty > ||uv||_\infty$. Hence, Q_e contains either u or v in its interior. This point is u otherwise any square containing e would have a visible point to both e's endpoints in its interior. Hence, c blocks u from both p and q. Thus, c crosses Q_e at opposite sides, otherwise Q_e would contain a visible point for both of p and q. Therefore, $||c||_\infty > ||e||_\infty$. ◀

3 Main result

3.1 Fixed tree theorem

▶ **Theorem 3.1.** *Let $T \in \mathcal{T}_S$. $T = MST_\square(CDT_\square(T))$ if and only if $T = MST_\square(S)$.*

Proof. The "if" part is trivial by definition of $MST_\square(S)$. Let us prove the "only if" part. Assume by contradiction that $T \neq MST_\square(S)$ then there exists an edge $t = (u, v) \in MST_\square(S)$ that does not belong to T. Since $t \notin T$ then $t \notin CDT_\square(T)$, otherwise t must be in $MST_\square(CDT_\square(T))$. Hence, there is at least one constraint crossing t in $CDT_\square(T)$.

Let $Q_t = Q(u, v)$ be a smallest square defining t in $DT_\square(S)$. Q_t exists since t is in $MST_\square$. Without loss of generality suppose that u and v belong to the left and right side of Q_t respectively. Thus each edge in $CDT_\square(T)$ crossing t crosses Q_t and has its endpoints outside Q_t. Let $c = (a, b) \in T$ be the nearest constraint to v and let Q_c be a square $Q(a, b)$ with sides of length $||c||_\infty$.

Case 1) c is diagonal. Let $Q_a = Q(a, v)$ be a square with size $||av||_\infty$ and $Q_b = Q(v, b)$ be a square with size $||vb||_\infty$.

If c crosses the right side of Q_t, then $w(a, b) = w(a, v) + w(v, b)$ and $h(a, b) = h(a, v) + h(v, b)$. Hence Q_a and Q_b have smaller size than Q_c. Since c is the nearest constraint to v, then by Lemma 2.5 there is no edge separating a from v in Q_a nor b from v in Q_b. By Lemma 2.3 there is a av-path, P_a and a vb-path, P_b, in Q_a and Q_b respectively. Therefore there is an ab-path in $P_a \cup P_b$ with edges solely shorter than c, contradicting our hypothesis by Property 2.1.

If c crosses the left side of Q_t then let $c' = (a', b')$ be the nearest constraint to u. The edge c' crosses the left side of Q_c as well, otherwise c' and c cross each other which is a contradiction. Thus, $w(a', b') = w(a', u) + w(u, b')$ and $h(a', b') = h(a', u) + h(a, b')$. Analogously as before we get a contradiction.

Case 2) c is vertical. By Lemma 2.4 there exist $Q_a^v = Q(a, v)$ and $Q_b^v = Q(b, v)$ with lower size than Q_c. Since c is the first constraint crossing t from v to u, by Lemma 2.6 there does not exist an edge separating a and v in Q_a^v nor an edge separating b and v in Q_b^v. From Lemma 2.3 there exists an av-paths P_a in Q_a^v and a vb-path P_b in Q_b^v. Hence, there exists

a path in $P_a \cup P_b$ from a to b with edges solely shorter than c which is a contradiction by Property 2.1. Therefore $T = MST_\square(S)$. ◀

Consider an arbitrary tree $T \in \mathcal{T}_S$ and a sequence $T_0, T_1, \ldots$, such that $T_0 = T, T_i = MST(CDT_\square(T_{i-1}))$ for all $i \geq 1$. Notice that this is a length-decreasing sequence, since T_i is shorter than T_{i-1} unless both are identical trees. Also, by definition of T_i, the trees T_i and T_{i-1} do not cross since they belong to the same plane graph, namely $CDT_\square(T_{i-1})$, for all $i > 1$. As a consequence of the Fixed tree theorem we obtain a length-decreasing sequence of trees in $\mathcal{T}_S$ which reaches a fixed point $T_k = MST_\square(S)$ in a finite number of steps.

▶ **Theorem 3.2.** *For any $T \in \mathcal{T}_S$ there exists a sequence $T_0, T_1, \ldots, T_k$ such that $T_0 = T$ and $T_k = MST_\square(S)$.*

Figure 4 shows a sequence $T_0, T_1, \ldots, T_k$ such that $T_0 = T$, $T_i = MST_\square(T_{i-1})$ and $T_k = MST_\square(S)$ for all $1 \leq i \leq k$ which converges in 4 steps.
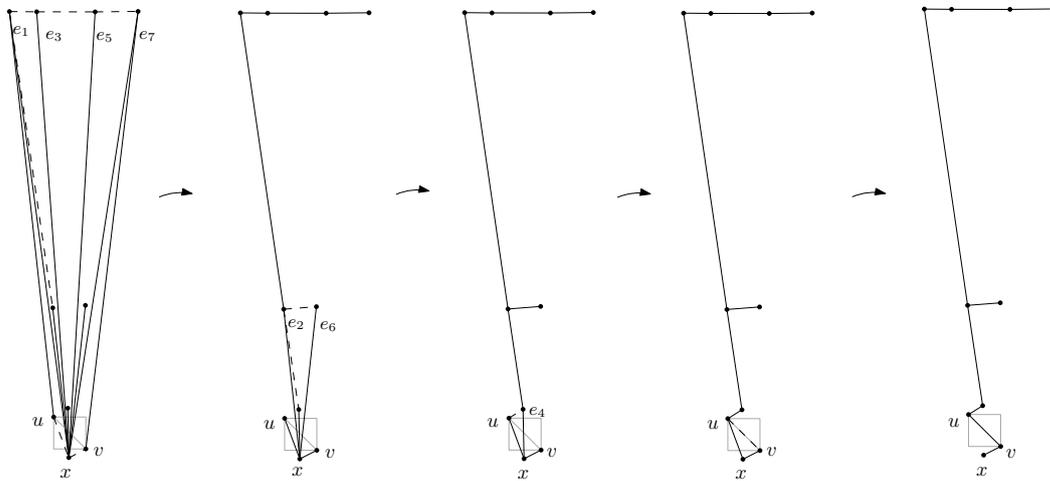

Figure 4 Example of a sequence with a spanning tree of a 10-point set that converges in 4 steps. The dashed edges represent the appearing edges at stage $i + 1$.

3.2 Lower bound

A natural question, once we know that this sequence converges, is how fast it reaches the $MST_\square(S)$. As a first step for answering this question we give a lower bound based on a construction shown in Figure 4, similar to the one given in [1], which has length $\Theta(\log n)$. The edge $t = (u, v)$ is an edge of the $MST_\square(S)$ where t is a diagonal of the square Q that defines the edge t in $DT_\square(S)$. Let $n = 2^m + 2$ for some $m \in \mathbb{N}$. Let x be a point below Q such that all the $n - 1$ edges of the initial tree T are incident to x and the edges that are not incident to u and v are vertical edges, i.e., edges crossing the top and bottom sides of Q_t. We order the vertical edges $e_1, e_2, \ldots e_{n-3}$ from left to right, refer to Figure 4. The edges with odd index have length ℓ. For $i = 2, \ldots, m$, the length of the edge e_j where $j \equiv 2i$ modulo 2^{i+1} is $\frac{\ell}{3^{i-1}}$. Notice that at step $i - 1$, the longest edges of T_{i-1} are the edges with length $\frac{\ell}{3^{i-1}}$ and these edges are the only ones of T_{i-1} not present at T_i for $1 \leq i < m$. Indeed, let $e_k = (a, x)$ be a vertical edge crossing t in T_{i-1} where $k \not\equiv 2i$ modulo 2^{i+1}, then $||e||_\infty \leq \frac{\ell}{3^{i-1}}$. Then any xa-path different from (x, a) would contain an edge with length $\frac{2\ell}{3^{i-1}}$, refer to Figure 5. Also, the edges e_j where $j \equiv 2i$ modulo 2^{i+1} disappear at stage $i - 1$, since there

exists a path between e_j's endpoints with edges shorter than e_j, refer to Figure 5. Hence the tree undergoes $m + 1 = \log_2(n-2) + 1$ iterations.
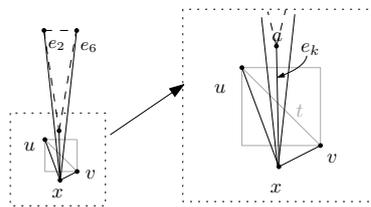
■ **Figure 5** The dashed edges have length $||e||_\infty = \frac{2\ell}{3^{i-1}}$ and all the xa-paths different from (x, a) contain one of the endpoint different from x of e_2 or e_6.

▶ **Theorem 3.3.** *There exists a point set S and $T \in \mathcal{T}_S$, such that the sequence $T_0, T_1, \ldots, T_k$ with $T_0 = T, T_i = MST_\square(CDT_\square(T_{i-1}))$ and $T_k = MST_\square(S)$ has length $\Theta(\log n)$.*

4 Conclusions

We have extended the convergence of sequences of crossing-free spanning trees for the L_2 metric [1] to the L_∞ metric. We have also given an $\Omega(\log n)$ lower bound for the maximum length of these sequences. In the full version of this paper, which is in preparation, we show that every sequence $T_0, \ldots, T_k$ has length $O(\log n)$, but due to space limitations we cannot include more details here. We also believe that the same techniques can be applied to other Delaunay triangulations defined by regular polygonal shapes. However, the difficulty is that we have to consider many different cases because of the different variations of edges crossing such polygons (for squares we only had to consider *vertical* and *diagonal* edges).

───── **References** ─────

1 Oswin Aichholzer, Franz Aurenhammer, and Ferran Hurtado. Sequences of spanning trees and a fixed tree theorem. *Computational Geometry*, 21(1):3–20, 2002.

2 Nicolas Bonichon, Prosenjit Bose, Jean-Lou De Carufel, Ljubomir Perkovic, and André van Renssen. Upper and lower bounds for online routing on Delaunay triangulations. *Discrete & Computational Geometry*, 58(2):482–504, 2017.

3 Nicolas Bonichon, Cyril Gavoille, Nicolas Hanusse, and David Ilcinkas. Connections between Theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 266–278. Springer, 2010.

4 Prosenjit Bose, Paz Carmi, Sébastien Collette, and Michiel Smid. On the stretch factor of convex Delaunay graphs. *Journal of computational geometry*, 1(1):41–56, 2010.

5 Michael B Dillencourt. Toughness and Delaunay triangulations. *Discrete & Computational Geometry*, 5(6):575–601, 1990.

6 Oleg R Musin. Properties of the Delaunay triangulation. In *Proceedings of the thirteenth annual symposium on Computational geometry*, pages 424–426. ACM, 1997.

7 Ge Xia. The stretch factor of the Delaunay triangulation is less than 1.998. *SIAM Journal on Computing*, 42(4):1620–1659, 2013.